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Iterative Non-iterative Integrals in Quantum Field Theory

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Abstract Single scale Feynman integrals in quantum field theories obey difference or differential equations with respect to their discrete parameter N or continuous parameter x. The analysis of these equations reveals to which order they factorize, which can be different in both cases. The simplest systems are the ones which factorize to first order. For them complete solution algorithms exist. The next interesting level is formed by those cases in which also irreducible second order systems emerge. We give a survey on the latter case. The solutions can be obtained as general ${}_2F_1$ solutions. The corresponding solutions of the associated inhomogeneous differential equations form so-called iterative non-iterative integrals. There are known conditions under which one may represent the solutions by complete elliptic integrals. In this case one may find representations in terms of meromorphic modular functions, out of which special cases allow representations in the framework of elliptic polylogarithms with generalized parameters. These are in general weighted by a power of $1/\eta(\tau)$, where $\eta(\tau)$ is Dedekind's η-function. Single scale elliptic solutions emerge in the ρ-parameter, which we use as an illustrative example. They also occur in the 3-loop QCD corrections to massive operator matrix elements and the massive 3-loop form factors.

1 Introduction

In this paper a survey is presented on the classes of special functions, represented by particular integrals, to which presently known single scale Feynman-integrals evaluate. Zero-scale integrals, also playing an important role in elementary particle physics, are given by special numbers, see e.g. [1–5]. To this class the expansion coefficients of the β-functions [6–8] and the renormalized masses, as well as

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$(g-2)$ [5], do belong. Single scale quantities depend on one additional parameter as e.g. the Mellin variable N, a momentum fraction or scale-ratio $x \in [0,1]$ and similar quantities. To this class contribute e.g. the massless Wilson coefficients [9], the anomalous dimensions [10–12], and the massive Wilson coefficients at large virtualities Q^2 [13–17].

It is now interesting to see which function spaces span the analytic results of these quantities. Traditionally two representations are studied: *i)* the Mellin space representation following directly from the light cone expansion [18] and *ii)* its Mellin inversion, the x-space representation, with x the Bjorken variable or another ratio of invariants, which in particular has phenomenological importance.

In the first case the quantities considered obey difference equations, while in the second case the corresponding equations are differential equations which are related to the former ones [19]. In all the cases quoted above either the recurrences or the differential operators or both factorize at *first order* after an appropriate application of decoupling formalisms [20–22]. Due to this all these cases can be solved algorithmically in any basis of representation, as has been shown in Ref. [23]. In N-space the solution is then possible using C. Schneider's packages `Sigma` [24, 25], `EvaluateMultiSum` and `SumProduction` [26]. Corresponding solutions in x-space can be obtained using the method of differential equations [23, 27]. This applies both to the direct calculation of the Feynman diagrams as well as to the calculation of their master integrals which are obtained using the integration by parts relations [28].

The above class of problems is the first one in a row. In general, the difference and differential equation systems do not decouple at first order, but will have higher order subsystems, i.e. of second, third, fourth order etc., cf. [29]. Since the first order case is solved completely [23], it is interesting to see which mathematical spaces represent the solution. In N-space next to pure rational function representations the nested harmonic sums emerge [30, 31]. They correspond to the harmonic polylogarithms in x-space [32]. At the next level generalized harmonic sums and iterated integrals of the Kummer-Poincaré type appear [3, 33, 34]. These are followed by iterated integrals over cyclotomic letters [2] and further by square-root valued letters, cf. [4] and their associated sums and special constants, cf. also [29, 35, 36]. This chain of functions is probably not complete yet, as one might think of more general Volterra-iterated integrals and their associated nested sums, which are also obeying first order factorization. The main properties of these functions, such as their shuffling relations [37, 38] and certain general transformations are known. Most of the corresponding mathematical properties to effectively handle these special functions are implemented in the package `HarmonicSums` [2–4, 39, 40].

The next important problem is, how to deal with cases in which neither recurrences in N-space nor differential equations in x-space factorize at first order. Here, the general solution can be given by so-called iterative non-iterative integrals[1], implied by the representation of the solution through the variation of constant [42] at

[1] Iterative non-iterative integrals have been introduced by the author in a talk on the 5th International Congress on Mathematical Software, held at FU Berlin, July 11-14, 2016, with a series of colleagues present, cf. [41].

any order of non-decoupling. This, of course, is a quite general statement, calling for refinement w.r.t. the corresponding special functions at non-decoupling to 2nd, 3rd, etc. order. In this article we will deal with the 2nd order case, discussing results, which have been obtained in Refs. [43,44] and by other authors recently. At present, in the singly variate case, the highest order of non-decoupling being observed is 2nd order, see e.g. Refs. [43–66].

2 Second order differential equations and ${}_2F_1$ Solutions

We consider the non-factorizable problem of order two in x-space. It is given by a corresponding differential equation of second order, usually with more than three singularities. Below we will give illustrations for equations which emerge in the calculation of the ρ-parameter [44, 67]. These are Heun differential equations [68]. A second order differential equation with three singularities can be mapped into a Gauß' differential equation [69]. In case of more singularities, this is possible too, however, the argument if the ${}_2F_1$ function is a rational function through which the other singularities are described. It is of advantage to look for the latter type solutions, since the properties of the ${}_2F_1$ function are very well known [70–74].

We consider the non-factorizable linear differential equations of second order

$$\left[\frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)\right]\psi(x) = N(x)\ , \tag{1}$$

with rational functions $r(x) = p(x), q(x)$, which may be decomposed into[2]

$$r(x) = \sum_{k=1}^{n_r} \frac{b_k^{(r)}}{x - a_k^{(r)}}, \qquad a_k^{(r)}, b_k^{(r)} \in \mathbb{Z}\ . \tag{2}$$

The homogeneous equation is solved by the functions $\psi_{1,2}^{(0)}(x)$, which are linearly independent, i.e. their Wronskian W obeys

$$W(x) = \psi_1^{(0)}(x)\frac{d}{dx}\psi_2^{(0)}(x) - \psi_2^{(0)}(x)\frac{d}{dx}\psi_1^{(0)}(x) \neq 0\ . \tag{3}$$

The homogeneous Eq. (1) determines the well-known differential equation for $W(x)$

$$\frac{d}{dx}W(x) = -p(x)W(x)\ , \tag{4}$$

which, by virtue of (2), has the solution

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[2] In the present case only single poles appear; for Fuchsian differential equations $q(x)$ may have double poles.

$$W(x) = \prod_{k=1}^{n_p} \left(\frac{1}{x - a_k^{(p)}}\right)^{b_k^{(p)}} , \tag{5}$$

normalizing the functions $\psi_{1,2}^{(0)}$ accordingly. A particular solution of the inhomogeneous equation (1) is then obtained by Euler-Lagrange variation of constants [42]

$$\psi(x) = \psi_1^{(0)}(x)\left[C_1 - \int dx\, \psi_2^{(0)}(x) n(x)\right] + \psi_2^{(0)}(x)\left[C_2 + \int dx\, \psi_1^{(0)}(x) n(x)\right], \tag{6}$$

with

$$n(x) = \frac{N(x)}{W(x)} \tag{7}$$

and two constants $C_{1,2}$ to be determined by special physical requirements. As examples we consider the systems of differential equations given in [67] for the $O(\varepsilon^0)$ terms in the dimensional parameter. These are master integrals determining the ρ-parameter at general fermion mass ratio at 3-loop order. The corresponding equations read

$$\begin{aligned} 0 = {} & \frac{d^2}{dx^2} f_{8a}(x) + \frac{9 - 30x^2 + 5x^4}{x(x^2-1)(9-x^2)} \frac{d}{dx} f_{8a}(x) - \frac{8(-3+x^2)}{(9-x^2)(x^2-1)} f_{8a}(x) \\ & - \frac{32x^2}{(9-x^2)(x^2-1)} \ln^3(x) + \frac{12(-9+13x^2+2x^4)}{(9-x^2)(x^2-1)} \ln^2(x) \\ & - \frac{6(-54+62x^2+x^4+x^6)}{(9-x^2)(x^2-1)} \ln(x) + \frac{-1161+251x^2+61x^4+9x^6}{2(9-x^2)(x^2-1)} \end{aligned} \tag{8}$$

$$\begin{aligned} f_{9a}(x) = {} & -\frac{5}{8}(-13-16x^2+x^4) + \frac{x^2}{2}(-24+x^2)\ln(x) + 3x^2\ln^2(x) - \frac{2}{3} f_{8a}(x) \\ & + \frac{x}{6}\frac{d}{dx} f_{8a}(x). \end{aligned} \tag{9}$$

There are more equations contributing to the problem, cf. [43], in which in the inhomogeneity more harmonic polylogarithms $H_{\mathbf{a}}(x)$ [32] contribute. Eq. (8) is an Heun equation in x^2. Its homogeneous solutions, [43], are:

$$\psi_{1a}^{(0)}(x) = \sqrt{2\sqrt{3}\pi} \frac{x^2(x^2-1)^2(x^2-9)^2}{(x^2+3)^4} {}_2F_1\left[\begin{matrix} \frac{4}{3} \ \frac{5}{3} \\ 2 \end{matrix}; z\right] \tag{10}$$

$$\psi_{2a}^{(0)}(x) = \sqrt{2\sqrt{3}\pi} \frac{x^2(x^2-1)^2(x^2-9)^2}{(x^2+3)^4} {}_2F_1\left[\begin{matrix} \frac{4}{3} \ \frac{5}{3} \\ 2 \end{matrix}; 1-z\right], \tag{11}$$

with

$$z = z(x) = \frac{x^2(x^2-9)^2}{(x^2+3)^3} . \tag{12}$$

The Wronskian for this system is

$$W(x) = x(9-x^2)(x^2-1). \tag{13}$$

These are single-${}_2F_1$ solutions, however, they are not given by single elliptic integrals. One first uses contiguous relations and then mappings according to the triangle group [75–77] and the algorithm described in appendix A of [43] to obtain the solutions

$$\begin{aligned}\psi_{1b}^{(0)}(x) &= \frac{\sqrt{\pi}}{4\sqrt{6}}\Biggl\{-(x-1)(x-3)(x+3)^2\sqrt{\frac{x+1}{9-3x}}\,{}_2F_1\left[\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ & 1\end{matrix};z\right]\\ &\quad +(x^2+3)(x-3)^2\sqrt{\frac{x+1}{9-3x}}\,{}_2F_1\left[\begin{matrix}\frac{1}{2} & -\frac{1}{2}\\ & 1\end{matrix};z\right]\Biggr\}\end{aligned} \tag{14}$$

$$\begin{aligned}\psi_{2b}^{(0)}(x) &= \frac{2\sqrt{\pi}}{\sqrt{6}}\Biggl\{x^2\sqrt{(x+1)(9-3x)}\,{}_2F_1\left[\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ & 1\end{matrix};1-z\right]\\ &\quad +\frac{1}{8}\sqrt{(x+1)(9-3x)}(x-3)(x^2+3)\,{}_2F_1\left[\begin{matrix}\frac{1}{2} & -\frac{1}{2}\\ & 1\end{matrix};1-z\right]\Biggr\},\end{aligned} \tag{15}$$

where

$$z(x) = -\frac{16x^3}{(x+1)(x-3)^3}\ . \tag{16}$$

and

$${}_2F_1\left[\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ & 1\end{matrix};z\right] = \frac{2}{\pi}\mathbf{K}(z) \tag{17}$$

$${}_2F_1\left[\begin{matrix}\frac{1}{2} & -\frac{1}{2}\\ & 1\end{matrix};z\right] = \frac{2}{\pi}\mathbf{E}(z)\ , \tag{18}$$

cf. [78]. Here $\mathbf{K}$ denotes the elliptic integral of the first and $\mathbf{E}$ the elliptic integral of the second kind.

Analyzing the criteria given in [79, 80] one finds, that the solution (14,15) cannot be rewritten such, that the elliptic integral of the second kind, $\mathbf{E}(z)$, does not emerge in the solution. The corresponding inhomogeneous solution is now obtained be Eq. (6).

We would like to end this section by a remark on simple elliptic solutions, which are sometimes also obtained in x-space. They are given by complete elliptic integrals $\mathbf{K}$ and $\mathbf{E}$ of the argument $1-x$ or x. In Mellin space, they correspond to a *first order* factorizable problem, cf. [64] for an example. The Mellin transform

$$\mathbf{M}[f(x)](N) = \int_0^1 dx x^{N-1} f(x) \tag{19}$$

yields

$$\mathbf{M}[\mathbf{K}(1-z)](N) = \frac{2^{4N+1}}{(1+2N)^2 \binom{2N}{N}^2} \tag{20}$$

$$\mathbf{M}[\mathbf{E}(1-z)](N) = \frac{2^{4N+2}}{(1+2N)^2(3+2N)\binom{2N}{N}^2}, \tag{21}$$

since

$$\mathbf{K}(1-z) = \frac{1}{2}\frac{1}{\sqrt{1-z}} \otimes \frac{1}{\sqrt{1-z}} \tag{22}$$

$$\mathbf{E}(1-z) = \frac{1}{2}\frac{z}{\sqrt{1-z}} \otimes \frac{1}{\sqrt{1-z}} . \tag{23}$$

The Mellin convolution is defined by

$$A(x) \otimes B(x) = \int_0^1 dz_1 \int_0^1 dz_2 \delta(x - z_1 z_2) A(z_1) B(z_2). \tag{24}$$

Eqs. (20) and (21) are hypergeometric terms in N, which has been shown already in Ref. [35] for $\mathbf{K}(1-x)$, see also [4]. As we outlined in Ref. [23] the solution of systems of differential equations or difference equations can always be obtained algorithmically in the case either of those factorizes to first order. The transition to x-space is then straightforward.

3 Iterative non-iterative integrals

Differential operators factorizing at first order have iterative integral solutions of the kind

$$F_{a_1,\ldots,a_k}(x) = \int_0^x dy_1 f_{a_1}(y_1) \int_0^{y_1} dy_2 f_{a_2}(y_2) \ldots \int_0^{y_{k-1}} dy_k f_{a_k}(y_k), \tag{25}$$

where $\mathfrak{A}$ is a certain alphabet and $\forall f_l(x) \in \mathfrak{A}$. In particular, the spaces of iterative integrals discussed in Refs. [2,4,32,34,35] are examples for this.

As well-known, the integral representation of the ${}_2F_1$-function in the cases having been discussed above

$${}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; z\right] = \frac{\Gamma(x)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} \tag{26}$$

cannot be rewritten as an integral in which the z dependence is just given by its boundaries.[3] Therfore Eq. (6) contains *definite* integrals, over which one integrates *iteratively*. We have called these *iterative non-iterative integrals* in [41, 43]. They will also occur in case the degree of non-factorization is larger than one by virtue of the corresponding formula of the variation of the constant; the corresponding solutions of the homogeneous equations will have (multiple) integral representations with the same property like for Eq. (26).

The new iterative integrals are given by

$$\mathbb{H}_{a_1,\ldots,a_{m-1};\{a_m;F_m(r(y_m))\},a_{m+1},\ldots,a_q}(x) = \int_0^x dy_1 \hat{f}_{a_1}(y_1) \int_0^{y_1} dy_2 \ldots \int_0^{y_{m-1}} dy_m \hat{f}_{a_m}(y_m) \times F_m[r(y_m)] H_{a_{m+1},\ldots,a_q}(y_m), \tag{27}$$

and cases in which more than one definite integral F_m appears. Here the $\hat{f}_{a_i}(y)$ are the usual letters of the different classes considered in [2–4, 32] multiplied by hyper-exponential pre-factors

$$r(y) y^{r_1} (1-y)^{r_2}, \quad r_i \in \mathbb{Q},\ r(y) \in \mathbb{Q}[y] \tag{28}$$

and $F[r(y)]$ is given by

$$F[r(y)] = \int_0^1 dz g(z, r(y)), \quad r(y) \in \mathbb{Q}[y], \tag{29}$$

such that the y-dependence cannot be transformed into one of the integration boundaries completely. We have chosen here $r(y)$ as a rational function because of concrete examples in this paper, which, however, is not necessary.

The further analytic representation of the functions $\mathbb{H}$ will be subject to the iterated functions $\hat{f}_l$ and F_m. We will turn to this in the case of the examples (6) for $\psi_{1(2)b}$ in Section 5.

4 Numerical representation

For physical applications numerical representations of the Feynman integrals have to be given. The use of integral-representations in `Mathematica` or `Maple` is possible, but usually to slow. One aims on efficient numerical implementations. In case of multiple polylogarithms it is available in `Fortran` [81, 82], for cyclotomic polylogarithms in [82], where in both cases the method of Bernoulli-improvement is used [83]. For generalized polylogarithms a numerical implementation was given in [84]. All these representations are series representations. Furthermore, there ex-

[3] This will not apply to simpler cases like ${}_2F_1\left[\begin{matrix}1,1\\2\end{matrix};-z\right] = \ln(1+z)/z$ or ${}_2F_1\left[\begin{matrix}\frac{1}{2},1\\\frac{3}{2}\end{matrix};z\right] = \arctan(z)/z$, however.

ist numerical implementations for the efficient use of harmonic sums in complex contour integral calculations [85].

Also in case of the solutions (6) analytic series representations can be given. This has been already the solution-strategy in [67], using power-series Ansätze, without further reference to the expected mathematical structure. It turns out, that series expansions around $x=0,1$ are not convergent in the whole interval $x\in[0,1]$. However, they have a sufficient region of overlap. Some series expansions of the inhomogeneous solution even exhibit a singularity, cf. [43], although this singularity is an artefact of the series expansion only. Yet these solutions can be obtained analytically and they evaluate very fast numerically.

The first terms of the expansion of f_{8a} around $x=0$ read

$$\begin{aligned}
&f_{8a}(x) = \\
&-\sqrt{3}\Bigg[\pi^3\left(\frac{35x^2}{108}-\frac{35x^4}{486}-\frac{35x^6}{4374}-\frac{35x^8}{13122}-\frac{70x^{10}}{59049}-\frac{665x^{12}}{1062882}\right)+\Bigg(12x^2-\frac{8x^4}{3}\\
&-\frac{8x^6}{27}-\frac{8x^8}{81}-\frac{32x^{10}}{729}-\frac{152x^{12}}{6561}\Bigg)\mathrm{Im}\left[\mathrm{Li}_3\left(\frac{e^{-\frac{i\pi}{6}}}{\sqrt{3}}\right)\right]\Bigg]-\pi^2\Bigg(1+\frac{x^4}{9}-\frac{4x^6}{243}-\frac{46x^8}{6561}\\
&-\frac{214x^{10}}{59049}-\frac{5546x^{12}}{2657205}\Bigg)+\left(\frac{3}{2}+\frac{x^4}{6}-\frac{2x^6}{81}-\frac{23x^8}{2187}-\frac{107x^{10}}{19683}-\frac{2773x^{12}}{885735}\right)\psi^{(1)}\!\left(\frac{1}{3}\right)\\
&-\sqrt{3}\pi\left(\frac{x^2}{4}-\frac{x^4}{18}-\frac{x^6}{162}-\frac{x^8}{486}-\frac{2x^{10}}{2187}-\frac{19x^{12}}{39366}\right)\ln^2(3)-\Bigg[33x^2-\frac{5x^4}{4}-\frac{11x^6}{54}\\
&-\frac{19x^8}{324}-\frac{751x^{10}}{29160}-\frac{2227x^{12}}{164025}+\pi^2\left(\frac{4x^2}{3}-\frac{8x^4}{27}-\frac{8x^6}{243}-\frac{8x^8}{729}-\frac{32x^{10}}{6561}-\frac{152x^{12}}{59049}\right)\\
&+\left(-2x^2+\frac{4x^4}{9}+\frac{4x^6}{81}+\frac{4x^8}{243}+\frac{16x^{10}}{2187}+\frac{76x^{12}}{19683}\right)\psi^{(1)}\left(\frac{1}{3}\right)\Bigg]\ln(x)+\frac{135}{16}+19x^2\\
&-\frac{43x^4}{48}-\frac{89x^6}{324}-\frac{1493x^8}{23328}-\frac{132503x^{10}}{5248800}-\frac{2924131x^{12}}{236196000}-\left(\frac{x^4}{2}-12x^2\right)\ln^2(x)\\
&-2x^2\ln^3(x)+O\left(x^{14}\ln(x)\right). \qquad (30)
\end{aligned}$$

Likewise, one may expand around $y=1-x=0$ and obtains

$$\begin{aligned}
f_{8a}(x) = &\ \frac{275}{12}+\frac{10}{3}y-25y^2+\frac{4}{3}y^3+\frac{11}{12}y^4+y^5+\frac{47}{96}y^6+\frac{307}{960}y^7+\frac{19541}{80640}y^8\\
&+\frac{22133}{120960}y^9+\frac{1107443}{7741440}y^{10}+\frac{96653063}{851558400}y^{11}+\frac{3127748803}{34062336000}y^{12}\\
&+7\Bigg(2y^2-y^3-\frac{1}{8}y^4-\frac{1}{64}y^6-\frac{1}{128}y^7-\frac{3}{512}y^8-\frac{1}{256}y^9-\frac{47}{16384}y^{10}
\end{aligned}$$

$$-\frac{69}{32768}y^{11}-\frac{421}{262144}y^{12}\Bigg)\zeta_3+O(y^{13})\ . \tag{31}$$

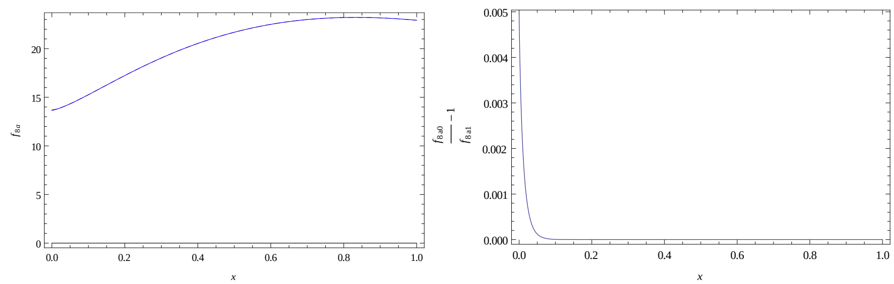

Fig. 1 *The inhomogeneous solution of Eq. (8) as a function of x. Left panel: Red dashed line: expansion around* $x=0$*; blue line: expansion around* $x=1$*. Right panel: illustration of the relative accuracy and overlap of the two solutions* $f_{8a}(x)$ *around 0 and 1.*

In Figure 1 a numerical illustration for the function $f_{8a}(x)$ is given together with the validity of the two expansions taking into account 50 terms. For many physics applications one would proceed in the above way and stop here. However, from the point of view of mathematics further interesting aspects arise to which we tun now.

5 Representation in terms of modular forms

The iterative non-iterative integral (6) is non-iterative by virtue of the emergence of the two complete elliptic integrals $\mathbf{K}(z)$ and $\mathbf{E}(z)$, with the modulus squared $k^2=z(x)$ the rational function (16). Accordingly, the second solution depends on the functions $\mathbf{K}'(z)=\mathbf{K}(1-z)$ and $\mathbf{E}'(z)=\mathbf{E}(1-z)$. One may re-parameterize the problem referring to the nome

$$q=\exp(i\pi\tau), \tag{32}$$

as the new variable with

$$\tau=i\frac{\mathbf{K}(1-z(x))}{\mathbf{K}(z(x))}\quad \text{with}\quad \tau\in\mathbb{H}=\{z\in\mathbb{C},\mathsf{Im}(z)>0\}\,. \tag{33}$$

All functions contributing to the solutions (6, 14, 15) have now to be translated from x to q.

5.1 The mathematical framework

For the further discussion, a series of definitions is necessary, see also Refs. [86–108]. We will use Dedekind's η-function [109]

$$\eta(\tau) = \frac{q^{\frac{1}{12}}}{\phi(q^2)}, \quad \phi(q) = \prod_{k=1}^{\infty} \frac{1}{1-q^k}, \tag{34}$$

to express all quantities in the following. Here $\phi(q)$ denotes Euler's totient function [110].

Definition 1. Let $r = (r_\delta)_{\delta|N}$ be a finite sequence of integers indexed by the divisors δ of $N \in \mathbb{N}\backslash\{0\}$. The function $f_r(\tau)$

$$f_r(\tau) := \prod_{\delta|N} \eta(\delta\tau)^{r_\delta}, \quad \delta, N \in \mathbb{N}\backslash\{0\}, \;\; r_\delta \in \mathbb{Z}, \tag{35}$$

is called η*-ratio*. The η-ratios, up to differential operators in q, will represent all expressions in the following.

Let

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; a,b,c,d \in \mathbb{Z}, \; \det(M) = 1 \right\}.$$

$\mathrm{SL}_2(\mathbb{Z})$ is the *modular group*.

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathbb{C} \cup \infty$ one defines the Möbius transformation

$$gz \mapsto \frac{az+b}{cz+d}.$$

Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S, T \in \mathrm{SL}_2(\mathbb{Z}).$$

The polynomials of S and T span $\mathrm{SL}_2(\mathbb{Z})$.

For $N \in \mathbb{N}\backslash\{0\}$ one considers the *congruence subgroups* of $\mathrm{SL}_2(\mathbb{Z})$, $\Gamma_0(N), \Gamma_1(N)$ and $\Gamma(N)$, defined by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), c \equiv 0 \pmod{N} \right\},$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), a \equiv d \equiv 1 \pmod{N}, \;\; c \equiv 0 \pmod{N} \right\},$$

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), a \equiv d \equiv 1 \pmod{N}, \;\; b \equiv c \equiv 0 \pmod{N} \right\},$$

with $\mathrm{SL}_2(\mathbb{Z}) \supseteq \Gamma_0(N) \supseteq \Gamma_1(N) \supseteq \Gamma(N)$ and $\Gamma_0(N) \subseteq \Gamma_0(M),\ M|N$.
If $N \in \mathbb{N}\backslash\{0\}$, then the *index* of $\Gamma_0(N)$ in $\Gamma_0(1)$ is

$$\mu_0(N) = [\Gamma_0(1) : \Gamma_0(N)] = N \prod_{p|N} \left(1 + \frac{1}{p}\right).$$

The product is over the prime divisors p of N.

Definition 2. Let $x \in \mathbb{Z}\backslash\{0\}$. The analytic function $f : \mathbb{H} \to \mathbb{C}$ is a *holomorphic modular form* of *weight* $w = k$ for $\Gamma_0(N)$ and *character* $a \mapsto \left(\frac{x}{a}\right)$ if

1. $$f\left(\frac{az+b}{cz+d}\right) = \left(\frac{x}{a}\right)(cz+d)^k f(z), \quad \forall z \in \mathbb{H},\ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$
2. $f(z)$ is holomorphic in $\mathbb{H}$
3. $f(z)$ is holomorphic at the cusps of $\Gamma_0(N)$.

Here $\left(\frac{x}{a}\right)$ denotes the Jacobi symbol. A modular form is called a *cusp form* if it vanishes at the cusps.

For any congruence subgroup G of $\mathrm{SL}_2(\mathbb{Z})$ a *cusp* of G is an equivalence class in $\mathbb{Q} \cup \infty$ under the action of G.

Definition 3. A *meromorphic modular function* f for $\Gamma_0(N)$ and weight $w = k$ obeys

1. $f(\gamma z) = (cz+d)^k f(z), \quad \forall z \in \mathbb{H}$ and $\forall \gamma \in \Gamma_0(N)$
2. f is meromorphic in $\mathbb{H}$
3. f is meromorphic at the cusps of $\Gamma_0(N)$.

The q-expansion of a meromorphic modular form has the form

$$f^*(q) = \sum_{k=-N_0}^{\infty} a_k q^k, \quad \text{for some } N_0 \in \mathbb{N}.$$

Lemma 1. *The set of functions $\mathscr{M}(k;N;x)$ for $\Gamma_0(N)$ and character x, defined above, forms a finite dimensional vector space over $\mathbb{C}$. In particular, for any non-zero function $f \in \mathscr{M}(k;N;x)$ we have*

$$\mathrm{ord}(f) \le b = \frac{k}{12}\mu_0(N),$$

cf. e.g. [89, 92, 107]. The bound (36) on the dimension can be refined, see e.g. [103–106] for details.[4]. The number of independent modular forms $f \in \mathscr{M}(k;N;x)$ is $\le b$, allowing for a basis representation in finite terms.

For any η-ratio f_r one can prove that there exists a minimal integer $l \in \mathbb{N}$, an integer $N \in \mathbb{N}$ and a character x such that

[4] The dimension of the corresponding vector space can be also calculated using the `Sage` program by W. Stein [111].

$$\bar{f}_r(\tau) = \eta^l(\tau) f_r(\tau) \in \mathscr{M}(k;N;x)$$

is a holomorphic modular form. All quantities which are expanded in q-series below will be first brought into the above form. In some cases one has $l = 0$. This form is of importance to obtain Lambert-Eisenstein series [112, 113], which can be rewritten in terms of elliptic polylogarithms [114].

A basis of the vector space of holomorphic modular forms is given by the associated Lambert-Eisenstein series with character and binary products thereof [87, 107].

The Lambert-Eisenstein series are given by

$$\sum_{k=1}^{\infty} \frac{k^\alpha q^k}{1-q^k} = \sum_{k=1}^{\infty} \sigma_\alpha(k) q^k, \quad \sigma_\alpha(k) = \sum_{d|k} d^\alpha, \quad \alpha \in \mathbb{N}. \tag{36}$$

They can be rewritten in terms of elliptic polylogarithms, which we will use rather as a frame in the following,

$$\mathrm{ELi}_{n;m}(x;y;q) := \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{x^k}{k^n} \frac{y^l}{l^m} q^{kl} \tag{37}$$

by

$$\sum_{k=1}^{\infty} \frac{k^\alpha q^k}{1-q^k} = \sum_{k=1}^{\infty} k^\alpha \mathrm{Li}_0(q^k) = \sum_{k,l=1}^{\infty} k^\alpha q^{kl} = \mathrm{ELi}_{-\alpha;0}(1;1;q), \tag{38}$$

with $\mathrm{Li}_0(x) = x/(1-x)$. It also appears useful to define [61],

$$\overline{E}_{n;m}(x;y;q) = \begin{cases} \frac{1}{i}[\mathrm{ELi}_{n;m}(x;y;q) - \mathrm{ELi}_{n;m}(x^{-1};y^{-1};q)], & n+m \ \text{even} \\ \mathrm{ELi}_{n;m}(x;y;q) + \mathrm{ELi}_{n;m}(x^{-1};y^{-1};q), & n+m \ \text{odd}. \end{cases} \tag{39}$$

The multiplication relation of elliptic polylogarithms is given by [114]

$$\begin{aligned} &\mathrm{ELi}_{n_1,\dots,n_l;m_1,\dots,m_l;0,2o_2,\dots,2o_{l-1}}(x_1,\dots,x_l;y_1,\dots,y_l;q) = \\ &\mathrm{ELi}_{n_1;m_1}(x_1;y_1;q)\mathrm{ELi}_{n_2,\dots,n_l;m_2,\dots,m_l;2o_2,\dots,2o_{l-1}}(x_2,\dots,x_l;y_2,\dots,y_l;q), \end{aligned} \tag{40}$$

with

$$\begin{aligned} &\mathrm{ELi}_{n,\dots,n_l;m_1,\dots,m_l;2o_1,\dots,2o_{l-1}}(x_1,\dots,x_l;y_1,\dots y_l;q) \\ &= \sum_{j_1=1}^{\infty} \dots \sum_{j_l=1}^{\infty} \sum_{k_1=1}^{\infty} \dots \sum_{k_l=1}^{\infty} \frac{x_1^{j_1}}{j_1^{n_1}} \dots \frac{x_l^{j_l}}{j_l^{n_l}} \frac{y_1^{k_1}}{k_1^{m_1}} \frac{y_l^{k_l}}{k_l^{m_l}} \frac{q^{j_1k_1+\dots+q_lk_l}}{\prod_{i=1}^{l-1}(j_ik_i+\dots+j_lk_l)^{o_i}}, l > 0. \end{aligned} \tag{41}$$

The logarithmic integral of an elliptic polylogarithm is given by

$$\mathrm{ELi}_{n_1,\dots,n_l;m_1,\dots,m_l;2(o_1+1),2o_2,\dots,2o_{l-1}}(x_1,\dots,x_l;y_1,\dots,y_l;q) =$$

$$\int_0^q \frac{dq'}{q'} \mathrm{ELi}_{n_1,\ldots,n_l;m_1,\ldots,m_l;2o_1,\ldots,2o_{l-1}}(x_1,\ldots,x_l;y_1,\ldots,y_l;q'). \tag{42}$$

Similarly, cf. [61],

$$\overline{E}_{n_1,\ldots,n_l;m_1,\ldots,m_l;0,2o_2,\ldots,2o_{l-1}}(x_1,\ldots,x_l;y_1,\ldots,y_l;q) = \overline{E}_{n_1;m_1}(x_1;y_1;q)\overline{E}_{n_2,\ldots,n_l;m_2,\ldots,m_l;2o_2,\ldots,2o_{l-1}}(x_1,\ldots,x_l;y_1,\ldots,y_l;q) \tag{43}$$

$$\overline{E}_{n_1,\ldots,n_l;m_1,\ldots,m_l;2(o_1+1),2o_2,\ldots,2o_{l-1}}(x_1,\ldots,x_l;y_1,\ldots,y_l;q) = \int_0^q \frac{dq'}{q'} \overline{E}_{n_1,\ldots,n_l;m_1,\ldots,m_l;2o_1,\ldots,2o_{l-1}}(x_1,\ldots,x_l;y_1,\ldots,y_l;q') \tag{44}$$

holds.

The integral over the product of two more general elliptic polylogarithms is given by

$$\int_0^q \frac{d\bar{q}}{\bar{q}} \mathrm{ELi}_{m,n}(x,\bar{q}^a,\bar{q}^b)\mathrm{ELi}_{m',n'}(x',\bar{q}^{a'},\bar{q}^{b'}) = \sum_{k=1}^\infty \sum_{l=1}^\infty \sum_{k'=1}^\infty \sum_{l'=1}^\infty \frac{x^k}{k^m} \frac{x'^k}{k'^{m'}} \frac{q^{al}}{l^n} \frac{q^{a'l'}}{l'^n} \times \frac{q^{bkl+b'k'l'}}{al+a'l'+bkl+bk'l'}. \tag{45}$$

Integrals over other products are obtained accordingly.
In the derivation often the argument $q^m,\quad m \in \mathbb{N}, m > 0$, appears, which shall be mapped to the variable q. We do this for the Lambert series using the replacement

$$\mathrm{Li}_0(x^m) = \frac{x^m}{1-x^m} = \frac{1}{m}\sum_{k=1}^m \frac{\rho_m^k x}{1-\rho_m^k x} = \frac{1}{m}\sum_{k=1}^m \mathrm{Li}_0(\rho_m^k x), \tag{46}$$

with

$$\rho_m = \exp\left(\frac{2\pi i}{m}\right). \tag{47}$$

One has

$$\sum_{k=1}^\infty \frac{k^\alpha q^{mk}}{1-q^{mk}} = \mathrm{ELi}_{-\alpha;0}(1;1;q^m) = \frac{1}{m^{\alpha+1}} \sum_{n=1}^m \mathrm{ELi}_{-\alpha;0}(\rho_m^n;1;q) . \tag{48}$$

Relations like (46, 48) and similar ones are the sources of the mth roots of unity, which correspondingly appear in the parameters of the elliptic polylogarithms through the Lambert series.

Furthermore, the following sums occur

$$\sum_{m=1}^\infty \frac{(am+b)^l q^{am+b}}{1-q^{am+b}} = \sum_{n=1}^l \binom{l}{n} a^n b^{l-n} \sum_{m=1}^\infty \frac{m^n q^{am+b}}{1-q^{am+b}}, \quad a,l \in \mathbb{N}, \quad b \in \mathbb{Z} \tag{49}$$

and

$$\sum_{m=1}^{\infty} \frac{m^n q^{am+b}}{1-q^{am+b}} = \mathrm{ELi}_{-n;0}(1;q^b;q^a) = \frac{1}{a^{n+1}} \sum_{\nu=1}^{a} \mathrm{ELi}_{-n;0}(\rho_a^\nu;q^b;q) \, . \tag{50}$$

Likewise, one has

$$\begin{aligned}\sum_{m=1}^{\infty} \frac{(-1)^m m^n q^{am+b}}{1-q^{am+b}} &= \mathrm{ELi}_{-n;0}(-1;q^b;q^a) \\ &= \frac{1}{a^{n+1}} \left\{ \sum_{\nu=1}^{2a} \mathrm{ELi}_{-n;0}(\rho_{2a}^\nu;q^b;q) - \sum_{\nu=1}^{a} \mathrm{ELi}_{-n;0}(\rho_a^\nu;q^b;q) \right\}.\end{aligned} \tag{51}$$

In intermediate representations also Jacobi symbols appear, obeying the identities

$$\left(\frac{-1}{(2k)\cdot n + (2l+1)}\right) = (-1)^{k+l}; \quad \left(\frac{-1}{ab}\right) = \left(\frac{-1}{a}\right)\left(\frac{-1}{b}\right). \tag{52}$$

In the case of an even value of the denominator one may factor $\left(\frac{-1}{2}\right) = 1$ and consider the case of the remaining odd-valued denominator.

We found also Lambert series of the kind

$$\sum_{m=1}^{\infty} \frac{q^{(c-a)m}}{1-q^{cm}} = \mathrm{ELi}_{0;0}(1;q^{-a};q^c) = \frac{1}{c} \sum_{n=1}^{c} \mathrm{ELi}_{0;0}(\rho_c^n;q^{-a};q) \tag{53}$$

$$\sum_{m=1}^{\infty} (-1)^m \frac{q^{(c-a)m}}{1-q^{cm}} = \mathrm{ELi}_{0;0}(1;-q^{-a};q^c) = \frac{1}{c} \sum_{n=1}^{c} \mathrm{ELi}_{0;0}(\rho_c^n;-q^{-a};q), \quad a,c \in \mathbb{N}\backslash\{0\} \tag{54}$$

in intermediate steps of the calculation.

Also the functions

$$\begin{aligned}Y_{m,n,l} &:= \sum_{k=0}^{\infty} \frac{(mk+n)^{l-1} q^{mk+n}}{1-q^{mk+n}} \\ &= n^{l-1} \mathrm{Li}_0(q^n) + \sum_{j=0}^{l-1} \binom{l-1}{j} n^{l-1-j} m^j \mathrm{ELi}_{-j;0}(1;q^n;q^m)\end{aligned} \tag{55}$$

$$Z_{m,n,l} := \sum_{k=1}^{\infty} \frac{k^{m-1} q^{nk}}{1-q^{lk}} = \mathrm{ELi}_{0;-(m-1)}(1;q^{n-l};q^l) \tag{56}$$

$$\begin{aligned}T_{m,n,l,a,b} &:= \sum_{k=0}^{\infty} \frac{(mk+n)^{l-1} q^{a(mk+n)}}{1-q^{b(mk+n)}} = n^{l-1} q^{n(a-b)} \mathrm{Li}_0\left(q^{nb}\right) \\ &\quad + q^{n(a-b)} \sum_{j=0}^{l-1} \binom{l-1}{j} m^j n^{l-1-j} \mathrm{ELi}_{-j;0}\left(q^{m(a-b)};q^{nb};q^{mb}\right)\end{aligned} \tag{57}$$

contribute. Note that (part of) the parameters $(x;y)$ of the elliptic polylogarithms can become q-dependent, unlike the case in [54, 61]. The elliptic polylogarithms rather form a suitable frame here, while we give preference to the Lambert-Eisenstein series. The q-dependence of $x(y)$ does not spoil the integration relations, which can be generalized in case factors $1/\eta(\tau)$ do not occur in addition.

5.2 The q-representation of the inhomogeneous solution

Now we turn to (6) again and express all quantities in terms of the variable q.

The modulus is given by

$$k = \frac{4\eta^8(2\tau)\eta^4\left(\frac{\tau}{2}\right)}{\eta^{12}(\tau)}, \qquad k' = \frac{\eta^4(2\tau)\eta^8\left(\frac{\tau}{2}\right)}{\eta^{12}(\tau)}, \tag{58}$$

which implies the following relation by $k' = \sqrt{1-k^2}$ for η functions

$$1 = \frac{\eta^8\left(\frac{\tau}{2}\right)\eta^8(2\tau)}{\eta^{24}(\tau)}\left[16\eta^8(2\tau) + \eta^8\left(\frac{\tau}{2}\right)\right] . \tag{59}$$

The elliptic integral of the first kind has the representation [78], sometimes also written using Jacobi's ϑ_i-functions [115],

$$\mathbf{K}(k^2) = \frac{\pi}{2}\frac{\eta^{10}(\tau)}{\eta^4\left(\frac{1}{2}\tau\right)\eta^4(2\tau)}, \qquad \mathbf{K}'(k^2) = -\frac{1}{\pi}\mathbf{K}(k^2)\,\ln(q)\,. \tag{60}$$

The elliptic integrals of the 2nd kind, $\mathbf{E}$ and $\mathbf{E}'$ are given by [116, 117]

$$\mathbf{E}(k^2) = \mathbf{K}(k^2) + \frac{\pi^2 q}{\mathbf{K}(k^2)}\frac{d}{dq}\ln\left[\vartheta_4(q)\right] \tag{61}$$

and the Legendre identity [118]

$$\mathbf{K}(z)\mathbf{E}(1-z) + \mathbf{E}(z)\mathbf{K}(1-z) - \mathbf{K}(z)\mathbf{K}(1-z) = \frac{\pi}{2}, \tag{62}$$

to express $\mathbf{E}'$,

$$\mathbf{E}'(k^2) = \frac{\pi}{2\mathbf{K}(k^2)}\left[1 + 2\ln(q)\, q\frac{d}{dq}\ln\left[\vartheta_4(q)\right]\right], \tag{63}$$

where the Jacobi ϑ functions are given by

$$\vartheta_2(q) = \frac{2\eta^2(2\tau)}{\eta(\tau)}, \quad \vartheta_3(q) = \frac{\eta^5(\tau)}{\eta^2\left(\frac{1}{2}\tau\right)\eta^2(2\tau)}, \quad \vartheta_4(q) = \frac{\eta^2\left(\frac{\tau}{2}\right)}{\eta(\tau)}. \tag{64}$$

We have now to determine the kinematic variable $x = x(q)$ analytically. This is not always possible for other choices of the definition of q, cf. [59]. In the present case, however, a cubic Legendre-Jacobi transformation [119, 120][5] allows the solution. Following [51, 52, 124, 125]

$$\frac{16y}{(1-y)(1+3y)^3} = \frac{\vartheta_2^4(q)}{\vartheta_3^4(q)} \tag{65}$$

is solved by

$$y = \frac{\vartheta_2^2(q^3)}{\vartheta_2^2(q)} \equiv -\frac{1}{3\bar{x}} = \frac{1}{3x}. \tag{66}$$

Both the expressions (65, 66) are modular functions. For definiteness, we consider the range in q

$$q \in [-1,1] \quad \text{which corresponds to} \quad y \in \left[0, \tfrac{1}{3}\right], \quad x \in [1, +\infty[\tag{67}$$

in the following. Here the variable x lies in the unphysical region. However, the nome q has to obey the condition (67). Other kinematic regions can be reached performing analytic continuations.

One obtains

$$x = \frac{1}{3}\frac{\eta^4(2\tau)\eta^2(3\tau)}{\eta^2(\tau)\eta^4(6\tau)}. \tag{68}$$

By this all ingredients of the inhomogeneous solution (6) can now be rewritten in q. Using the on-line encyclopedia of integer sequences [126] one finds in particular for entry `A256637`

$$\sqrt{(1-3x)(1+x)} = \frac{1}{i\sqrt{3}} \left.\frac{\eta\left(\frac{\tau}{2}\right)\eta\left(\frac{3\tau}{2}\right)\eta(2\tau)\eta(3\tau)}{\eta(\tau)\eta^3(6\tau)}\right|_{q\to -q} \tag{69}$$

and for terms in the inhomogeneity and the Wronskian `A187100, A187153` [126]

$$\frac{1}{1-x} = -3\frac{\eta^2(\tau)\eta\left(\frac{3}{2}\tau\right)\eta^3(6\tau)}{\eta^3\left(\frac{1}{2}\tau\right)\eta(2\tau)\eta^2(3\tau)} \tag{70}$$

$$\frac{1}{1-3x} = -\frac{\left[\eta(\tau)\eta\left(\frac{3}{2}\tau\right)\eta^2(6\tau)\right]^3}{\eta\left(\frac{1}{2}\tau\right)\eta^2(2\tau)\eta^9(3\tau)}. \tag{71}$$

[5] This is, besides the well-know Landen transformation [78, 121], the next higher modular transformation; for a survey cf. [122]. Also for the hypergeometric function ${}_2F_1\left[\begin{matrix}\frac{1}{r}, 1-\frac{1}{r}\\ 1\end{matrix}; z(x)\right]$ there are rational modular transformations [123].

This method can be applied since the q-series of the associated holomorphic modular form to these expressions factoring off a power of $1/\eta(\tau)$ is determined by a finite number of expansion coefficients.

Next we would like to investigate which kind of modular form the solution $\psi(x)$ is. Some of its building blocks, like $\mathbf{K}$, are holomorphic modular forms [86, 87], while others, like $\mathbf{E}$, are meromorphic modular forms. In case a solution can be thoroughly expressed by holomorphic modular forms, as e.g. in case of the sun-rise graph studied in Refs. [54, 59], one has then the possibility to express the result in terms of polynomials of Lambert–Eisenstein series [112, 113], which are given by elliptic polylogarithms [114] and their generalizations, cf. e.g. [61] an references therein.

The elliptic integral of the first kind can be expressed by E or $\overline{E}$-functions only.

$$\mathbf{K}(z) = \frac{\pi}{2}\left[1+2\overline{E}_{0;0}(i;1;q)\right], \tag{72}$$

On the other hand, this is not the case for $1/\mathbf{K}(z)$, a function needed to represent $\mathbf{E}$:

$$\begin{aligned}
\frac{1}{\mathbf{K}(z)} = \frac{2}{\pi\eta^{12}(\tau)}\Biggl\{\frac{5}{48}\Biggl\{&1-24\mathrm{ELi}_{0;-1}(1;1;q)-4\Bigl[1-\frac{3}{2}\Bigl[\mathrm{ELi}_{0;-1}(1;1;q)\\
&+\mathrm{ELi}_{0;-1}(1;i;q)+\mathrm{ELi}_{0;-1}(1;-1;q)+\mathrm{ELi}_{0;-1}(1;-i;q)\Bigr]\Bigr]\Biggr\}\Biggl\{-1\\
&+4\Bigl[-\frac{1}{2}\Bigl[\mathrm{ELi}_{-2;0}(i;1/q;q)+\mathrm{ELi}_{-2,0}(-i;1/q;q)\Bigr]+\Bigl[\mathrm{ELi}_{-1;0}(i;1/q;q)\\
&+\mathrm{ELi}_{-1;0}(-i;1/q;q)\Bigr]-\frac{1}{2}\Bigl[\mathrm{ELi}_{0,0}(i;1/q;q)+\mathrm{ELi}_{0,0}(-i;1/q;q)\Bigr]\Bigr]\Biggr\}\\
&-\frac{1}{16}\Biggl\{5+4\Bigl[-\frac{1}{2}\Bigl[\mathrm{ELi}_{-4;0}(i;1/q;q)+\mathrm{ELi}_{-4;0}(-i;1/q;q)\Bigr]\\
&+2\Bigl[\mathrm{ELi}_{-3;0}(i;1/q;q)+\mathrm{ELi}_{-3,0}(-i;1/q;q)\Bigr]-3\Bigl[\mathrm{ELi}_{-2;0}(i;1/q;q)\\
&+\mathrm{ELi}_{-2,0}(-i;1/q;q)\Bigr]+2\Bigl[\mathrm{ELi}_{-1;0}(i;1/q;q)+\mathrm{ELi}_{-1;0}(-i;1/q;q)\Bigr]\\
&-\frac{1}{2}\Bigl[\mathrm{ELi}_{0;0}(i;1/q;q)+\mathrm{ELi}_{0,0}(-i;1/q;q)\Bigr]\Biggr\}\Biggr\}.
\end{aligned} \tag{73}$$

Here and in a series of other building blocks the factor $1/\eta^{12}(\tau)$ emerges through which the corresponding quantity becomes a meromorphic modular form [43].

Still one has to express the inhomogeneities of the corresponding differential equations. They are given by harmonic polylogarithms $H_{\mathbf{a}}(x)$ and rational prefactors in x. In the variable $q = q(x)$ they will be different, cf. [43, 59], depending on the definition of q.

Since for the q-series of $1/\eta(\tau)$ no closed form expression of the expansion coefficients is known, one cannot write down a closed form integration relation for polynomials out of quantities like this, unlike the case for polynomials out of Lambert-Eisenstein series, see Ref. [43] for details. Therefore, a closed analytic solution of the inhomogeneous solution using structures like elliptic polylogarithms, cf. Section 5.1, cannot be given. Yet, one may use q-series in the numerical representation expanding to a certain power. This, however, is equivalent to the numerical representation given in Section 4, where no further analytic continuation is necessary.

6 The ρ-parameter

Finally we would like to present numerical results on the ρ-parameter with a finite quark mass ratio, given in Ref. [44]. The ρ-parameter is defined by

$$\rho = 1 + \frac{\Pi_T^Z(0)}{M_Z^2} - \frac{\Pi_T^W(0)}{M_W^2} \equiv 1 + \Delta\rho, \tag{74}$$

with $\Pi_T^k(0)$ the respective transversal self energies at zero momentum and M_k the masses of the Z and W bosons. Here the correction is given by

$$\Delta\rho = \frac{3G_F m_t^2}{8\pi^2\sqrt{2}} \left(\delta^{(0)} + \frac{\alpha_s}{\pi}\delta^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \delta^{(2)} + \mathscr{O}(\alpha_s^3) \right), \tag{75}$$

where G_F is the Fermi-constant, m_t denotes the heavy fermion mass, and $x = m_b^2/m_t^2$ the ratio of the masses of the light and the heavy partner squared.

The radiative corrections allow to set limits on heavy fermions in case of doublet mass splitting, which was important to determine the precise mass region of the top-quark [127]. Radiative corrections were calculated in Refs. [67, 127–133]. In Ref. [43] we calculated the analytic form of the yet missing master integrals. They can now be evaluated numerically starting from a complete analytic representation. We insert our results into the representation given in [67].

The expression for the $\delta^{(2)}$, Eq. (75), in terms of the master integrals in the $\overline{\text{MS}}$ scheme, is given by Eq. (76), where we only show the contributions due to the iterative non-iterative integrals.

$$\delta^{(2)}(x) = \cdots + C_F \left(C_F - \frac{C_A}{2} \right) \left[\frac{11 - x^2}{12(1-x^2)^2} f_{8a}(x) + \frac{9 - x^2}{3(1-x^2)^2} f_{9a}(x) \right.$$
$$\left. + \frac{1}{12} f_{10a}(x) + \frac{5 - 39x^2}{36(1-x^2)^2} f_{8b}(x) + \frac{1 - 9x^2}{9(1-x^2)^2} f_{9b}(x) + \frac{x^2}{12} f_{10b}(x) \right]$$

$$+\frac{C_F T_F}{9(1-x^2)^3}\Bigg[(5x^4-28x^2-9)f_{8a}(x)+\frac{1-3x^2}{3x^2}(9x^4+9x^2-2)f_{8b}(x)$$
$$+(9-x^2)(x^4-6x^2-3)f_{9a}(x)+\frac{1-9x^2}{3x^2}(3x^4+6x^2-1)f_{9b}(x)\Bigg]. \quad (76)$$

The different functions $f_i(a)$ are given in Ref. [43]. The behaviour of the correction term $\delta^{(2)}(x)$ is shown in Figure 2. The color factor signals that it stems from the non-planar part of the problem. In the limit of $m_t \to \infty$ the numerical value $\delta^{(2)}(0) = -3.969$ is obtained in agreement with [129]. In the limit of zero mass splitting the correction vanishes.

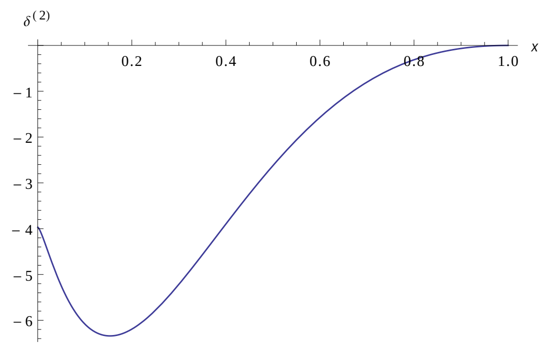

Fig. 2 *The two-mass contributions to $\delta^{(2)}$ as a function of x.*

7 Conclusions

In the analytic calculation of zero- and single-scale Feynman diagrams in the most simple cases iterative integral and indefinite nested sum representations are sufficient. Here either the system of differential or difference equations factorizes to first order [29]. All these cases can be solved algorithmically, cf. [23], in whatsoever basis. The function spaces, which represent the solutions for the cases having been studied so far, are completely known and the associated numerical implementations are widely available.

At present an important target of research are the cases in which the level of non-factorization is of second or higher order. Also in these cases the general structure of the formal solutions is known. In case of the differential equations they are given by the variation of constant, over the solutions of the homogeneous equations. Here the latter ones have no iterative solutions. They can be written as (multiple) Mellin-Barnes [134] integrals [135] and by this cast into a multiple integral representation in which the next integration variable cannot be completely transformed into the integral boundaries. Therefore, these integrals are of non-iterative character. In summary, one obtains iterative integrals over these non-iterative integrals as the main structure [41, 43].

From the mathematical point of view one would like to understand the non-iterative integrals emerging on the different levels of non-factorization in more detail. In the 2nd order case the corresponding differential equations have ${}_2F_1$-solutions with specific rational parameters and rational functions in x as argument. This is generally due to the fact that the corresponding differential equations have more than three singularities. There is a decision algorithm, cf. [43, 76, 77], whether or not the ${}_2F_1$-solutions can be mapped on complete elliptic integrals or not. Furthermore, one may investigate using the criteria given in [79, 80] whether representations in terms of complete elliptic integrals of the first kind are sufficient in special cases. In the elliptic case one may consider representation in terms of modular forms, which are in general meromorphic. A sub-class of only holomorphic modular forms, cf. e.g. [54, 61], also exists in a series of interesting cases. Finally, complete elliptic integrals of the first and second kind with argument x or $(1-x)$ do not form a 2nd order problem, if considered in N space, where they have a representation in hypergeometric terms.

The level of non-factorization for single-scale Feynman integrals at second order is widely understood and throughly tied up with ${}_2F_1$-solutions. Their properties allow to derive also analytic solutions. Corresponding series expansions in the complex plane allow for numerical implementations since their convergence regions do sufficiently overlap.

Much less is known in case of third and higher order non-factorization. Cases of this kind will emerge in future calculations. Here one is not advised to apply the pure *integral* approach of differential equations [23, 27]. To recognize the nature of the integrals contributing here it is useful to apply the dispersive approach to the corresponding integrals first [136]. Even multiple cuts may be necessary to unravel the emerging structures. In this way, once again, *non-iterative* integrals are obtained. This has been the easiest approach to solve the sun-rise graph also, cf. [48]. This method will be of use to unravel further levels and to establish the links needed to known mathematical structures or at least to guide the way to work out the corresponding mathematics, if it is not know yet.

Again the analytic calculation of Feynman integrals shows the rich mathematical structures behind these quantities and leads to an intense cooperation between theoretical physics, different branches of mathematics and computer algebra. During the last 30 years an enormous development has been taking place, but much more is going to come.

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