

The Polarized Two-Loop Massive Pure Singlet Wilson Coefficient for Deep-Inelastic Scattering

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Abstract

We calculate the polarized massive two-loop pure singlet Wilson coefficient contributing to the structure functions $g_1(x, Q^2)$ analytically in the whole kinematic region. The Wilson coefficient contains Kummer-elliptic integrals. We derive the representation in the asymptotic region $Q^2 \gg m^2$, retaining power corrections, and in the threshold region. The massless Wilson coefficient is recalculated. The corresponding twist-2 corrections to the structure function $g_2(x, Q^2)$ are obtained by the Wandzura-Wilczek relation. Numerical results are presented.

1 Introduction

Analytic expressions in perturbative Quantum Chromodynamics, as for partonic sub-system scattering cross sections and Wilson coefficients, provide an excellent basis for both numerical studies, fits to precision data and provide analytic insight into the structure of Feynman integrals. Analytic results also allow to derive important limiting cases and make it easier to incorporate resummations in specific kinematic regions. With a growing number of scales and loops for the respective processes it becomes more difficult to obtain analytic results, although there has been significant progress in analytic integration methods recently; for a survey cf. [1]. The precise knowledge of these corrections is of importance to measure the polarized parton densities in high energy collisions and to determine, related to it, the strong coupling constant $\alpha_s(M_Z)$ and the heavy quark masses, cf. [2].

The first two-loop QCD heavy flavor corrections to the polarized structure function $g_1(x, Q^2)$ have been calculated in [3] in the asymptotic region $Q^2 \gg m^2$, where Q^2 denotes the virtuality of the exchanged photon and m the mass of the heavy quark. The asymptotic two-loop QCD corrections have been recalculated in [4,5]. In [3] the region of low values of Q^2 has been modeled by an ansatz. The leading threshold resummation for the gluonic contributions has been studied in [6]. The complete two-loop polarized heavy flavor Wilson coefficient in the non-singlet case has been calculated analytically in the tagged flavor case in [3] and for the complete contribution to the structure function $g_1(x, Q^2)$ in [7], also completing former work on the polarized Bjorken sum rule in [8]. Numerical results for the polarized two-loop heavy flavor case have been given in [9] recently. Finally, in the non-singlet case the asymptotic contributions have been calculated to three-loop order analytically in [10,11].

In the present paper we follow Ref. [12] in the unpolarized case and calculate the polarized pure singlet two-loop heavy flavor corrections for the structure function $g_1(x, Q^2)$ in the whole kinematic range analytically. We also compute the corresponding massless contributions, which have first been calculated in [13] and later in [14]. Since the calculations are carried out using dimensional regularization in $D = 4 + \varepsilon$ dimensions one may work in the Larin-scheme [15]¹ and perform a finite renormalization to the $\overline{\text{M}}$ -scheme [17,18] afterwards. We derive both the result in the asymptotic range $Q^2 \gg m^2$, see also Refs. [3,5], and in the threshold region. Numerical results are presented. Various technical aspects of the present calculation can be found in Ref. [12] already.

The paper is organized as follows. In Section 2 we summarize basic relations for the polarized deep-inelastic scattering cross section. In Section 3 the result for the massless pure singlet Wilson coefficient $C_{g_1}^{(2),\text{PS}}$ is presented. The recalculation of the massless Wilson coefficient is necessary, since in Ref. [13] different schemes have been used in part. The corresponding massive Wilson coefficient is calculated in Section 4. The corresponding results for the twist-2 contributions to the structure function $g_2(x, Q^2)$ can be obtained by using the Wandzura-Wilczek relation [19], as has been shown for the massless quarkonic [20,21] and gluonic [22] cases, for diffractive scattering [23], non-forward scattering [24], and the target mass corrections [25,26]. Limiting cases are studied in Section 5 and numerical results are presented in Section 6. The conclusions are given in Section 7. Some Mellin convolutions appearing due to renormalization are listed in the Appendix.

2 The Deep-inelastic Scattering Cross Section

The scattering cross sections for deep-inelastic charged lepton scattering of polarized nucleons

¹For other γ_5 schemes see Refs. [16].

are obtained polarizing the incoming lepton longitudinally and the target nucleon either longitudinally or transversally, resulting into the spin 4-vectors S_L and S_T ,

$$S_L = (0, 0, 0; M) \quad (1)$$

$$S_T = M(0, \cos(\beta), \sin(\beta); 0) \quad (2)$$

in the nucleon rest frame. One has $S_L \cdot p = S_T \cdot p = 0$, with p the nucleon 4-momentum. The scattering cross sections are given by, cf. e.g. [25, 27],

$$\frac{d^2\sigma(\lambda, \pm S_L)}{dxdy} = \pm 2\pi S \left[-2\lambda y \left(2 - y - \frac{2xyM^2}{S} \right) xg_1(x, Q^2) + 8\lambda \frac{yxM^2}{S} xg_2(x, Q^2) \right] \quad (3)$$

$$\begin{aligned} \frac{d^3\sigma(\lambda, \pm S_T)}{dxdyd\phi} &= \pm S \frac{\alpha^2}{Q^4} 2\sqrt{\frac{M^2}{S}} \sqrt{xy \left[1 - y - \frac{xyM^2}{S} \right]} \cos(\beta - \phi) \\ &\times \left[-2\lambda y xg_1(x, Q^2) - 4\lambda xg_2(x, Q^2) \right] \end{aligned} \quad (4)$$

for pure virtual photon exchange. Here S denotes the cms energy of the process, M is the nucleon mass, λ the degree of lepton polarization, $\alpha = e^2/(4\pi)$ is the fine structure constant, $Q^2 = -q^2$ denotes the photon virtuality and $x = Q^2/(Sy)$, $y = l \cdot q/p \cdot q$ are the Bjorken variables with l the incoming charged lepton and proton momenta, $S = (p + l)^2$ and ϕ is the azimuthal angle of the final state lepton, which can be integrated over in the case of longitudinal polarization.

In the following we will present a series of relations in Mellin- N space for convenience. The respective quantities in momentum-fraction z -space are related to those in Mellin-space by the transformation

$$\mathbf{M}[f(z)](N) = \int_0^1 dz z^{N-1} f(z). \quad (5)$$

The structure function $g_1(N, Q^2)$ is given in the twist-two approximation using the factorization theorems [28] by

$$\begin{aligned} g_1(N, Q^2) &= \frac{1}{2} \left[\frac{1}{N_F} \sum_{k=0}^{N_F} e_k^2 \left\{ \Sigma(N, \mu_F^2) C_q^{\text{PS}} \left(N, \frac{Q^2}{\mu_F^2} \right) + G(N, \mu_F^2) C_g^{\text{S}} \left(N, \frac{Q^2}{\mu_F^2} \right) \right\} \right. \\ &\quad \left. + \Delta(N, \mu_F^2) C_q^{\text{NS}} \left(N, \frac{Q^2}{\mu_F^2} \right) \right]. \end{aligned} \quad (6)$$

Here

$$\Sigma(N) = \sum_{k=1}^{N_F} [q(N) + \bar{q}(N)] \quad (7)$$

denotes the singlet distribution, $G(N)$ the gluon distribution, $\Delta(N)$ the flavor non-singlet distribution

$$\Delta(N) = \sum_{i=1}^{N_F} \left[e_i^2 - \frac{1}{N_F} \sum_{k=1}^{N_F} e_k^2 \right] [q_i(N) + \bar{q}_i(N)]. \quad (8)$$

e_k labels the electric charge of the k th light quark.

The Mellin transform of the structure function g_2 is related to that of g_1 by

$$g_2(N, Q^2) = -\frac{N-1}{N}g_1(N, Q^2) \quad (9)$$

or

$$g_2(x, Q^2) = -g_1(x, Q^2) + \int_0^1 \frac{dy}{y} g_1(y, Q^2). \quad (10)$$

Note that in the massive pure singlet case the support of both structure functions is limited by $0 < x < 1/(1 + 4m^2/Q^2)$ due to the production of two heavy quarks.

The different steps in the renormalization and factorization of the polarized massless Wilson coefficients have been described in [13, 29] and for the massive Wilson coefficients in [5] using the Larin–scheme. In the present case the finite renormalization moving to the $\overline{\text{M}}$ –scheme only affects the massless or massive Wilson coefficient by adding the term $-z_{\text{PS}}^{(2)}$, Eq. (32).

3 The Massless Wilson Coefficient

The Feynman diagrams contributing to the polarized massless two–loop Wilson coefficient are shown in Figure 1.

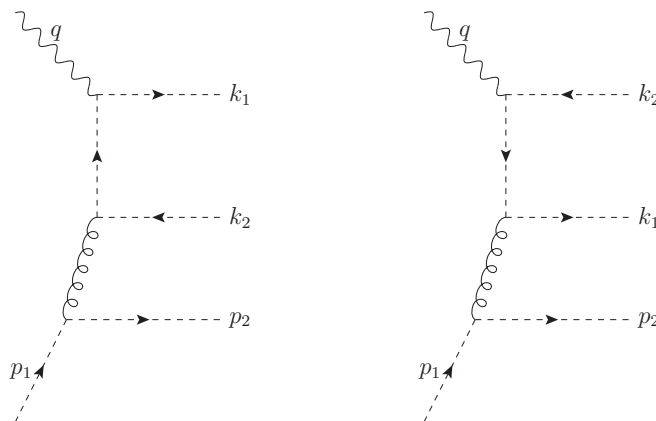


Figure 1: The Feynman diagrams for the massless and massive two–loop pure singlet Wilson coefficient.

forming the amplitude squared $G_{\mu\nu}$, where the indices refer to the coupling of the virtual photon. Here all quark lines are massless. The massless resp. massive Wilson coefficients are obtained following Ref. [13], Eqs. (3.7–3.18). The corresponding phase space integrals have been given in Ref. [12]. We apply the Larin–scheme [15] in which the contraction of the free indices of the two appearing Levi–Civita tensors have to be performed in D dimensions.

The unrenormalized two–loop massless pure singlet Wilson coefficient reads in Mellin– N space

$$\hat{C}_{g_1}^{(2),\text{PS}} = \hat{a}_s^2 S_\varepsilon^2 \left(\frac{Q^2}{\mu^2} \right)^\varepsilon \left\{ \frac{1}{\varepsilon^2} \frac{1}{2} P_{qg}^{(0)} P_{gq}^{(0)} + \frac{1}{\varepsilon} \left[\frac{1}{2} P_{qq}^{(1),\text{PS}} + P_{gq}^{(0)} \bar{c}_{g_1.g}^{(1)} \right] + \bar{c}_{g_1.q}^{(2),\text{PS}} + P_{qg}^{(0)} a_{g_1.g}^{(1)} \right\}, \quad (11)$$

where $\hat{a}_s = \hat{g}_s^2/(4\pi)^2$ denotes the unrenormalized strong coupling constant, $S_\varepsilon = \exp[\frac{\varepsilon}{2}(\gamma_E - \ln(4\pi))]$ the spherical factor and γ_E the Euler–Mascheroni constant, $\bar{c}_i^{(k)}$ and $a_{g_1,g}^{(1)}$ are the expansion coefficients of the one-loop Wilson coefficient with

$$\hat{C}_{g_1,g}^{(1)} = \hat{a}_s S_\varepsilon \left(\frac{Q^2}{\mu^2} \right)^{\varepsilon/2} \left[\frac{1}{\varepsilon} P_{qg} + \bar{c}_{g_1,g}^{(1)} + \varepsilon a_{g_1,g}^{(1)} \right], \quad (12)$$

given by the Feynman diagrams in Figure 2,

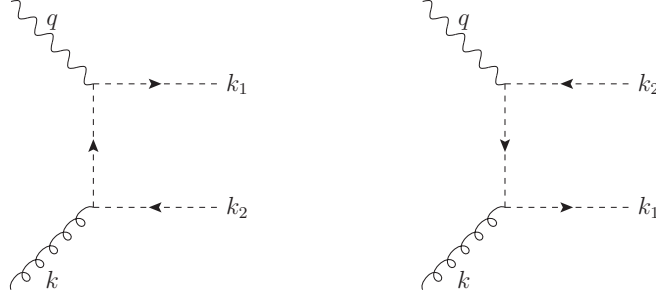


Figure 2: The one-loop Feynman diagrams.

where all quark lines are massless. One obtains

$$\bar{c}_{g_1,g}^{(1)} = 4T_F N_F \left[-(2z - 1)[H_1 + H_0] + 3 - 4z \right] \quad (13)$$

$$a_{g_1,g}^{(1)} = T_F N_F \left[-12 + 16z + 3(1 - 2z)\zeta_2 - (6 - 8z)(H_0 + H_1) - (1 - 2z)(H_0 + H_1)^2 \right]. \quad (14)$$

The contributing splitting functions [30–34] are

$$P_{qg}(z) = 8T_F N_F [z^2 - (1 - z)^2] \quad (15)$$

$$P_{gq}(z) = 4C_F \frac{1 - (1 - z)^2}{z} \quad (16)$$

$$P_{qq}^{(1),\text{PS}}(z) = 16C_F T_F N_F \left[1 - z - (1 - 3z)H_0 - (1 + z)H_0^2 \right], \quad (17)$$

where $C_F = (N_c^2 - 1)/2N_c$, $T_F = 1/2$ and $N_c = 3$ for the gauge group $SU(3)_c$ of Quantum Chromodynamics. In the following we will use the convention

$$\hat{f}(N_F) = f(N_F + 1) - f(N_F). \quad (18)$$

The harmonic polylogarithms are given by [35]

$$H_{b,\bar{a}}(z) = \int_0^z dy f_b(y) H_{\bar{a}}(y), \quad H_\emptyset = 1, \quad b, a_i \in \{-1, 0, 1\}, \quad (19)$$

and the letters f_c read

$$f_0(z) = \frac{1}{z}, \quad f_1(z) = \frac{1}{1 - z}, \quad f_{-1}(z) = \frac{1}{1 + z}. \quad (20)$$

We use the shorthand notation $H_{\bar{a}}(z) \equiv H_{\bar{a}}$ in case of the argument z . The harmonic polylogarithms are dual to the harmonic sums [36, 37] by the Mellin transformation (5).

In the Larin–scheme we obtain

$$\begin{aligned}
\hat{C}_{g_1}^{(2),\text{PS,L}} &= -C_F T_F N_F \left\{ -\frac{1}{\varepsilon^2} [80(1-z) + 32(1+z)H_0] + \frac{1}{\varepsilon} [184(1-z) - 32(1+z)\zeta_2 \right. \\
&\quad + 40(3-z)H_0 + 24(1+z)H_0^2 + 80(1-z)H_1 + 32(1+z)H_{0,1}] - \frac{1432}{3}(1-z) \\
&\quad - \frac{4}{3}(233 - 43z)H_0 + \frac{32(1+z)^3}{3z}H_{-1}H_0 - \frac{2}{3}(129 - 15z + 8z^2)H_0^2 \\
&\quad - \frac{28}{3}(1+z)H_0^3 - (1-z)[184 + 80H_0]H_1 - 40(1-z)H_1^2 - (1+z)[40 + 32H_0]H_{0,1} \\
&\quad - \frac{32(1+z)^3}{3z}H_{0,-1} + 16(1+z)[H_{0,0,1} - 2H_{0,1,1}] \\
&\quad \left. + \left[\frac{4}{3}(129 - 45z + 8z^2) + 56(1+z)H_0 \right] \zeta_2 + 16(1+z)\zeta_3, \right. \tag{21}
\end{aligned}$$

performing the phase space integrations as has been outlined in [12, 29].

At $O(a_s^2)$ neither the renormalization of the heavy quark mass nor the coupling constant contributes in case of the massive or massless pure singlet Wilson coefficient. The poles in ε in Eq. (21) are due to collinear singularities only, which have to be factorized. One may proceed as follows. The unfactorized quarkonic Wilson coefficients for the structure function g_1 , $\hat{C}_{1,q}^{\text{NS,S}}$ in Mellin–space are given by

$$\hat{C}_{1,q}^{\text{NS}} = \Gamma_{qq}^{\text{NS}} C_q^{\text{NS}} \tag{22}$$

$$\hat{C}_{1,q}^{\text{S}} = \Gamma_{qq}^{\text{S}} C_q^{\text{S}} + \Gamma_{gq}^{\text{S}} C_g^{\text{S}}. \tag{23}$$

The pure singlet contribution is obtained by

$$\begin{aligned}
\hat{C}_{1,q}^{\text{PS}} &= \hat{C}_{1,q}^{\text{S}} - \hat{C}_{1,q}^{\text{NS}} \\
&= \Gamma_{qq}^{\text{S}} C_q^{\text{S}} - \Gamma_{qq}^{\text{NS}} C_q^{\text{NS}} + \Gamma_{gq}^{\text{S}} C_g^{\text{S}} \tag{24}
\end{aligned}$$

$$= [\Gamma_{qq}^{\text{NS}} + \Gamma_{qq}^{\text{S}}] [C_q^{\text{NS}} + C_q^{\text{PS}}] - \Gamma_{qq}^{\text{NS}} C_q^{\text{NS}} + \Gamma_{gq}^{\text{S}} C_g^{\text{S}} \tag{25}$$

with

$$\Gamma_{gq}^{(0)} = \hat{a}_s S_\varepsilon \left(\frac{\mu_F^2}{\mu^2} \right)^{\varepsilon/2} \frac{1}{\varepsilon} P_{gq}^{(0)}, \tag{26}$$

$$\Gamma_{qq}^{(1),\text{PS}} = \hat{a}_s^2 S_\varepsilon^2 \left(\frac{\mu_F^2}{\mu^2} \right)^\varepsilon \left[\frac{1}{\varepsilon^2} P_{qq}^{(0)} P_{gq}^{(0)} + \frac{1}{\varepsilon} P_{qq}^{(1),\text{PS}} \right]. \tag{27}$$

and

$$\hat{C}_{1,q}^{\text{PS}} = a_s^2 \left\{ \frac{1}{\varepsilon^2} \frac{1}{2} P_{qq}^{(0)} P_{gq}^{(0)} + \frac{1}{\varepsilon} \left[\frac{1}{2} P_{qq}^{(1),\text{PS}} + P_{gq}^{(0)} C_g^{(1)} \right] + C_q^{(2),\text{PS}} \right\}. \tag{28}$$

The factorized massless pure singlet two–loop Wilson coefficient $C_{g_1}^{(2),\text{PS,L}}$ is given by

$$C_{g_1}^{(2),\text{PS,L}} \left(z, \frac{Q^2}{\mu^2} \right) = a_s^2 \left\{ \frac{1}{8} P_{qq}^{(0)} P_{gq}^{(0)} L_M^2 + \frac{1}{2} [P_{qq}^{(1),\text{PS}} + P_{gq}^{(0)} \bar{c}_g^{(1)}] L_M + \bar{c}_q^{(2),\text{PS}} \right\}, \tag{29}$$

where

$$L_M = \ln \left(\frac{Q^2}{\mu} \right). \quad (30)$$

Here we set $\mu_F = \mu$, and work with a single scale only for the factorization and renormalization scale; $a_s = g_s^2/(4\pi)$ is the running coupling constant and the spherical factor S_ε , as usually, is set to one at the end of the calculation. Note that the splitting function $P_{qq}^{(1),\text{PS}}$ is correctly obtained, cf. [33, 34], despite working in the Larin–scheme, cf. also [18].

The massless Wilson coefficient in the $\overline{\text{M}}$ -scheme is obtained by the following finite renormalization

$$C_{g_1}^{(2),\text{PS},\text{M}} = C_{g_1}^{(2),\text{PS},\text{L}} - z_{\text{PS}}^{(2)}, \quad (31)$$

with [17]

$$z_{\text{PS}}^{(2)} = C_F T_F N_F \left[16(1-z) + 8(3-z)H_0 + 4(2+z)H_0^2 \right], \quad (32)$$

cf. [5, 18]. $C_{g_1}^{(2),\text{PS},\text{M}}$ in z -space is given by

$$\begin{aligned} C_{g_1}^{(2),\text{PS},\text{M}} \left(z, \frac{Q^2}{\mu^2} \right) &= a_s^2 C_F T_F N_F \left\{ [20(1-z) + 8(1+z)H_0] L_M^2 - [(1-z)(88 + 40H_1) \right. \\ &\quad + 16(1+z)(H_0^2 + H_{0,1} - \zeta_2) + 32(2-z)H_0] L_M + \frac{760}{3}(1-z) \\ &\quad + \frac{4}{3}(119 - 13z)H_0 - \frac{32(1+z)^3}{3z} H_{-1}H_0 + \frac{2}{3}[75 - 15z + 8z^2]H_0^2 \\ &\quad + \frac{20}{3}(1+z)H_0^3 + (1-z)(88 + 40H_0)H_1 + 20(1-z)H_1^2 + [8(3+z) \\ &\quad + 16(1+z)H_0]H_{0,1} + \frac{32(1+z)^3}{3z} H_{0,-1} + 16(1+z)H_{0,1,1} \\ &\quad \left. - 32 \left[\frac{1}{3}(9 - 3z + z^2) + (1+z)H_0 \right] \zeta_2 - 16(1+z)\zeta_3 \right\}. \quad (33) \end{aligned}$$

4 The Massive Wilson Coefficient

The kinematic domain for the massive Wilson coefficient is given by

$$0 < z < \frac{Q^2}{4m^2 + Q^2}. \quad (34)$$

The unrenormalized two-loop massive pure singlet Wilson coefficient reads in Mellin– N space

$$\hat{H}_{g_1}^{(2),\text{PS},\text{L}} = \hat{a}_s^2 S_\varepsilon^2 \left(\frac{Q^2}{\mu^2} \right)^\varepsilon \left\{ \frac{1}{\varepsilon} P_{gq}^{(0)} h_{g_1,g}^{(1)} + h_{g_1,q}^{(2),\text{PS},\text{L}} + P_{gq}^{(0)} \bar{b}_{g_1,g}^{(1)} \right\}. \quad (35)$$

The contributing Feynman diagrams are shown in Figure 1, where now the outgoing quark lines with momenta k_1 and k_2 are taken massive, see also Figure 2. Here $\bar{h}_{g_1,g}^{(1)}$ [38–40] and $b_{g_1,g}^{(1)}$ are the expansion coefficients of the one-loop Wilson coefficient

$$\hat{H}_{g_1,g}^{(1)} = \hat{a}_s S_\varepsilon \left(\frac{Q^2}{\mu^2} \right)^{\varepsilon/2} [h_{g_1,g}^{(1)} + \varepsilon \bar{b}_{g_1,g}^{(1)}] \quad (36)$$

given by the diagrams in Figure 2 now with massive quark lines. The expansion coefficients are given by

$$h_{g_1,g}^{(1)} = 4T_F \left[(3-4z)\beta - (1-2z)H_0 \left(\frac{1+\beta}{1-\beta} \right) \right] \quad (37)$$

$$\begin{aligned} \bar{b}_{g_1,g}^{(1)} &= T_F \left\{ -4(3-4z)\beta + (1-2z)H_0^2 \left(\frac{1-\beta}{1+\beta} \right) - 2 \left[(3-4z)\beta + (1-2z)H_0 \left(\frac{1-\beta}{1+\beta} \right) \right] \right. \\ &\quad \left. \times [H_0 + H_1 - 2\ln(\beta)] + 4(1-2z)H_{0,1} \left(\frac{2\beta}{1+\beta} \right) \right\}. \end{aligned} \quad (38)$$

Here β denotes the velocity of the produced heavy quarks,

$$\beta = \sqrt{1 - \frac{4m^2}{Q^2} \frac{z}{1-z}}. \quad (39)$$

Since the two heavy quarks do not induce collinear divergences the mass factorization in the massive case reads

$$\hat{H}_{g_1}^{(2),\text{PS,L}} = H_{g_1}^{(2),\text{PS}} + \Gamma_{gq} \otimes H_{g_1,g}^{(1),\text{PS}}. \quad (40)$$

Here the Mellin convolution of two functions in z -space is given by

$$A(z) \otimes B(z) = \int_0^1 dz_1 \int_0^1 dz_2 \delta(z - z_1 z_2) A(z_1) B(z_2). \quad (41)$$

We find

$$\begin{aligned} H_{g_1}^{(2),\text{PS,L}} &= \hat{a}_s^2 S_\varepsilon^2 \left\{ \left(\frac{Q^2}{\mu^2} \right)^\varepsilon \left[\frac{1}{\varepsilon} P_{gq}^{(0)} h_{g_1}^{(1)} + h_{g_1}^{(2),\text{PS}} + P_{gq}^{(0)} \bar{b}_{g_1}^{(1)} \right] \right. \\ &\quad \left. - \left(\frac{\mu_F^2}{\mu^2} \right)^{\varepsilon/2} \left(\frac{Q^2}{\mu^2} \right)^{\varepsilon/2} \left[\frac{1}{\varepsilon} P_{gq}^{(0)} h_{g_1}^{(1)} + P_{gq}^{(0)} \bar{b}_{g_1}^{(1)} \right] \right\}. \end{aligned} \quad (42)$$

Identifying the renormalization and factorization scale, $\mu = \mu_F$, we finally obtain

$$H_{g_1}^{(2),\text{PS}} = a_s^2 \left[\frac{1}{2} P_{gq}^{(0)} h_{g_1}^{(1)} L_M + h_{g_1}^{(2),\text{PS}} \right] + O(\varepsilon). \quad (43)$$

Note that in the pure singlet case neither the heavy quark mass nor the coupling constant is renormalized at two-loop order.

The massive pure singlet Wilson coefficient is obtained as a four-fold integral over two angular and two energy variables, cf. [12] for details of the calculation. These integrals are systematically turned into iterative integrals. This process leads to a set of letters, through which these integrals are defined, see also [41]. We use the code [42] in `Mathematica`, which also uses the routine `DSolveRational` of the package `HolonomicFunctions` [43]; see [44, 45] for the general theory underlying [42]. We also refer to [46] for the simpler case when no singularities are present at the endpoints of integration, which, however, does not apply here.

The result can be expressed in terms of iterative integrals of the following twelve partly square-root valued letters,

$$f_{w_1}(t) = \frac{1}{1-kt}, \quad (44)$$

$$f_{w_2}(t) = \frac{1}{1+kt}, \quad (45)$$

$$f_{w_3}(t) = \frac{1}{\beta+t}, \quad (46)$$

$$f_{w_4}(t) = \frac{1}{\beta-t}, \quad (47)$$

$$f_{w_5}(t) = \frac{1}{k-z-(1-z)kt}, \quad (48)$$

$$f_{w_6}(t) = \frac{1}{k+z-(1-z)kt}, \quad (49)$$

$$f_{w_7}(t) = \frac{1}{k-z+(1-z)kt}, \quad (50)$$

$$f_{w_8}(t) = \frac{1}{k+z+(1-z)kt}, \quad (51)$$

$$f_{w_9}(t) = \frac{t}{k^2(1-t^2(1-z^2))-z^2}, \quad (52)$$

$$f_{w_{10}}(t) = \frac{1}{t\sqrt{1-t^2}\sqrt{1-k^2t^2}}, \quad (53)$$

$$f_{w_{11}}(t) = \frac{t}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}, \quad (54)$$

$$f_{w_{12}}(t) = \frac{t}{\sqrt{1-t^2}\sqrt{1-k^2t^2}(k^2(1-t^2(1-z^2))-z^2)}, \quad (55)$$

given before in Ref. [12], and the letters spanning the harmonic polylogarithms. These iterative integrals have maximally weight $w = 3$ and belong to the Kummer–elliptic integrals, [12], in general. The variable k is defined by

$$k = \frac{\sqrt{z}}{\sqrt{1-(1-z)\beta^2}}. \quad (56)$$

One obtains for the following analytic result of the massive polarized two–loop Wilson coefficient

$$\begin{aligned} H_{g_1}^{(2),\text{PS,L}} = & C_F T_F \left\{ -\frac{16(1-z)P_1}{3k^2} \left\{ H_{w_5,0} - H_{w_6,0} + H_{w_7,0} - H_{w_8,0} - [H_{w_5} - H_{w_6} + H_{w_7} - H_{w_8}] \right. \right. \\ & \times H_0 \left. \right\} - \frac{8P_2}{3k^2} H_{w_2,-1} - \frac{8P_3}{3k^2} H_1 H_{w_1} + \frac{4(1-z)P_4}{3k^2 z} \left\{ H_{w_6,1} - H_{w_8,-1} - H_{w_6} H_1 \right. \\ & \left. + H_{w_8} H_{-1} \right\} - \frac{4(1-z)P_5}{3k^2 z} \left\{ H_{w_6,-1} - H_{w_8,1} + H_{w_8} H_1 - H_{w_6} H_{-1} \right\} \\ & + \frac{4(1-z)P_6}{3k^2 z} \left\{ H_{w_5,-1} - H_{w_7,1} + H_{w_7} H_1 - H_{w_5} H_{-1} \right\} - \frac{4(1-z)P_7}{3k^2 z} \left\{ H_{w_5,1} \right. \\ & \left. - H_{w_7,-1} - H_{w_5} H_1 + H_{w_7} H_{-1} \right\} + \frac{16P_8}{3(1-k\beta)} H_{w_1} - \frac{16P_9}{3(1+k\beta)} H_{w_2} \\ & + \frac{8(1-z)P_{10}}{3(k(2-z)-z)(1-k\beta)} H_{w_5} + \frac{8(1-z)P_{11}}{3(k(2-z)+z)(1+k\beta)} H_{w_6} \\ & - \frac{8(1-z)P_{12}}{3(k(2-z)-z)(1+k\beta)} H_{w_7} - \frac{8(1-z)P_{13}}{3(k(2-z)+z)(1-k\beta)} H_{w_8} \end{aligned}$$

$$\begin{aligned}
& + \frac{32P_{14}}{3k^2(k^2(2-z)^2 - z^2)} H_1 - \frac{32P_{15}}{3k^2(k^2(2-z)^2 - z^2)} H_{-1} \\
& + \frac{1216}{3} (1-z)\beta + 8(1-z)(1-2z) [H_1 + H_{-1} - 2\beta] (\ln(z) + \ln(1-z)) \\
& + 16(1+2z) \left\{ 2(H_{w_1, w_4} + H_{w_2, w_4} + H_{w_3, w_1} + H_{w_3, w_2}) \right. \\
& + k(H_{w_1}^2 - H_{w_2}^2) + [-2\ln(k^2 - z) + 6\ln(k) - \ln(1 - k^2) + \ln(k^2 - z^2) - 2H_{w_3}] \\
& \times (H_{w_1} + H_{w_2}) + k(1-z) [H_{w_5, w_1} + H_{w_6, w_2} - H_{w_7, w_2} - H_{w_8, w_1}] \\
& \left. - k(1-z) [H_{w_5} - H_{w_8}] H_{w_1} - k(1-z) [H_{w_6} - H_{w_7}] H_{w_2} \right\} \\
& + 16(1-z)(7-2z)\beta \ln(k^2 - z^2) + 8 \left(7 - \left(2 - \frac{1}{k^2} \right) z \right) \left\{ 2H_1 H_0 - 6\ln(k) H_1 \right. \\
& + \ln(1 - k^2) H_1 + 2\ln(k^2 - z) H_1 + 2H_{w_3} H_1 - 6\ln(k) H_{-1} + \ln(1 - k^2) H_{-1} \\
& + 2\ln(k^2 - z) H_{-1} + 2H_{-1} H_{w_3} - 2H_{0,1} - 2H_{1, w_4} - 2H_{w_3, 1} - 2H_{w_3, -1} + 2H_{-1, 0} \\
& \left. - 2H_{-1, w_4} \right\} + \frac{32}{3} \left(-3k^2 + \frac{z^2}{k^2} - 3(1+z^2) \right) [H_{w_1} + H_{w_2}] H_0 + 8\ln(k^2 - z^2) \\
& \times \left(-8 + \frac{(-1+5k^2)z}{k^2} - 2z^2 \right) [H_1 + H_{-1}] + \frac{8}{3} \left(6k^2 - \frac{3z}{k} - \frac{2z^2}{k^2} + 6(2+2z \right. \\
& + z^2) - 3k(8-5z+2z^2) \right) [H_{w_1} H_{-1} - H_{w_2} H_1] + \frac{8}{3} \left(6k^2 + \frac{3z}{k} - \frac{2z^2}{k^2} + 6(2+2z \right. \\
& + z^2) + 3k(8-5z+2z^2) \right) H_{w_2} H_{-1} + \frac{4}{3} \left(87 + \frac{4}{z} - \frac{9(-1+2k^2)z}{k^2} \right) [H_1^2 - H_{-1}^2] \\
& - \frac{16}{3} k(1-z)(3+2z+3z^2) \left\{ \frac{1}{\sqrt{z}} [H_{w_{10}, w_5} - H_{w_{10}, w_6} + H_{w_{10}, w_7} - H_{w_{10}, w_8}] \right. \\
& \left. - \frac{k}{\sqrt{z}} \left(H_{w_5, w_{11}} + H_{w_6, w_{11}} + H_{w_7, w_{11}} + H_{w_8, w_{11}} - [H_{w_5} + H_{w_6} + H_{w_7} + H_{w_8}] H_{w_{11}} \right) \right. \\
& \left. - 2k(1-k^2)\sqrt{z} \left(H_{w_5, w_{12}} + H_{w_6, w_{12}} + H_{w_7, w_{12}} + H_{w_8, w_{12}} - [H_{w_5} + H_{w_6} + H_{w_7} \right. \right. \\
& \left. \left. + H_{w_8}] H_{w_{12}} \right) - \frac{2}{\sqrt{z}(1-z)} [H_{w_{10}, w_1} + H_{w_{10}, w_2}] - \frac{2(1-k^2)\sqrt{z}}{1-z} [H_{w_{12}, 1} \right. \\
& \left. + H_{w_{12}, -1}] \right\} - \frac{384(k^2 - z)}{k^2\beta} H_{w_3} - \frac{8}{3} \left(39 - \frac{4}{z} + \frac{9(1-2k^2)z}{k^2} \right) H_{-1} H_1 \\
& + 32 \left(k^2 - (2-z)z - \frac{z^2}{3k^2} \right) H_{w_1, 0} + \frac{8}{3} \left(6k^2 + \frac{3z}{k} - \frac{2z^2}{k^2} - 3kz(1-2z) \right. \\
& - 6(1+4z-z^2) \left. \right) H_{w_1, 1} + \frac{8}{3} \left(6 - 6k^2 + 24z + \frac{3z}{k} - 6z^2 + \frac{2z^2}{k^2} \right. \\
& \left. - 3kz(1-2z) \right) H_{w_1, -1} + 32 \left(k^2 - (2-z)z - \frac{z^2}{3k^2} \right) H_{w_2, 0} + \frac{8}{3} \left(6k^2 - \frac{3z}{k} \right. \\
& \left. + 3k(1-2z)z - \frac{2z^2}{k^2} - 6(1+4z-z^2) \right) H_{w_2, 1} + \left(208 - \frac{64}{3z} + \frac{48(1-2k^2)z}{k^2} \right) \\
& \times H_{-1, 1} - \frac{64k^2(1-z^2)(1+3z^2)}{3z} \left\{ H_{w_9, 1} + H_{w_9, -1} - k(1-z) [H_{w_9, w_5} + H_{w_9, w_6} \right.
\end{aligned}$$

$$\begin{aligned}
& + \mathbb{H}_{w_9, w_7} + \mathbb{H}_{w_9, w_8} \Big\} + 8(1+z) \Big\{ -4\mathbb{H}_{0,1,1} - 4\mathbb{H}_{0,-1,1} + 20\mathbb{H}_{1,1,1} + 4\mathbb{H}_{1,1,w_4} \\
& + 4\mathbb{H}_{1,-1,w_4} - 4\mathbb{H}_{w_3,1,1} + 4\mathbb{H}_{w_3,1,-1} - 4\mathbb{H}_{w_3,-1,1} + 4\mathbb{H}_{w_3,-1,-1} + 4\mathbb{H}_{-1,1,0} \\
& + 16\mathbb{H}_{-1,1,1} - 4\mathbb{H}_{-1,1,w_4} + 4\mathbb{H}_{-1,-1,0} + 16\mathbb{H}_{-1,-1,1} - 4\mathbb{H}_{-1,-1,w_4} + 20\mathbb{H}_{-1,-1,-1} \\
& + \left(\ln(1-k^2) - \ln(k^2-z^2) + 2\ln(k^2-z) - 6\ln(k) \right) [\mathbb{H}_{-1}^2 - \mathbb{H}_1^2 - 2\mathbb{H}_1\mathbb{H}_{-1} \\
& + 4\mathbb{H}_{-1,1}] + [(10\mathbb{H}_{-1} - 4\mathbb{H}_{w_3})\mathbb{H}_1 - 4\mathbb{H}_0\mathbb{H}_1 + 2\mathbb{H}_1^2 - 2\mathbb{H}_{1,1} - 4\mathbb{H}_{w_3,1} - 4\mathbb{H}_{w_3,-1} \\
& - 12\mathbb{H}_{-1,1} - 10\mathbb{H}_{-1,-1}]\mathbb{H}_{-1} + [4\mathbb{H}_{1,1} + 8\mathbb{H}_{-1,1} + 4\mathbb{H}_{-1,-1} - 4\mathbb{H}_1^2]\mathbb{H}_{w_3} + [4\mathbb{H}_0,1 \\
& + 4\mathbb{H}_{0,-1} - 10\mathbb{H}_{1,1} + 4\mathbb{H}_{w_3,1} + 4\mathbb{H}_{w_3,-1} - 4\mathbb{H}_{-1,1} - 10\mathbb{H}_{-1,-1}]\mathbb{H}_1 \\
& + [-4\mathbb{H}_1^2 + 4\mathbb{H}_{1,1} + 4\mathbb{H}_{-1,1}]\mathbb{H}_0 \Big\} + 32k(1+z) \Big\{ \mathbb{H}_{w_1,1,0} + \mathbb{H}_{w_1,1,1} - \mathbb{H}_{w_1,1,w_4} \\
& - \mathbb{H}_{w_1,1,-1} + \mathbb{H}_{w_1,-1,0} + \mathbb{H}_{w_1,-1,1} - \mathbb{H}_{w_1,-1,w_4} - \mathbb{H}_{w_1,-1,-1} - \mathbb{H}_{w_2,1,0} - \mathbb{H}_{w_2,1,1} \\
& + \mathbb{H}_{w_2,1,w_4} + \mathbb{H}_{w_2,1,-1} - \mathbb{H}_{w_2,-1,0} - \mathbb{H}_{w_2,-1,1} + \mathbb{H}_{w_2,-1,w_4} + \mathbb{H}_{w_2,-1,-1} + \mathbb{H}_{w_3,1,w_1} \\
& - \mathbb{H}_{w_3,1,w_2} + \mathbb{H}_{w_3,-1,w_1} - \mathbb{H}_{w_3,-1,w_2} + \frac{1}{2} [\ln(k^2-z^2) - 2\ln(k^2-z) + 6\ln(k) \\
& - \ln(1-k^2)] [\mathbb{H}_{w_2,-1} - \mathbb{H}_{w_1,-1} + \mathbb{H}_{w_2,1} - \mathbb{H}_{w_1,1}] + \frac{1}{2} \left(-\mathbb{H}_{w_1,1} - \mathbb{H}_{w_1,-1} + \mathbb{H}_{w_2,1} \right. \\
& \left. + \mathbb{H}_{w_2,-1} + [\mathbb{H}_{w_1} - \mathbb{H}_{w_2}]\mathbb{H}_1 \right) \mathbb{H}_{-1} + [\mathbb{H}_{w_1,1} + \mathbb{H}_{w_1,-1} - \mathbb{H}_{w_2,1} - \mathbb{H}_{w_2,-1}]\mathbb{H}_{w_3} \\
& - \frac{1}{2} \left(\mathbb{H}_{1,1} - \mathbb{H}_1^2 + 2\mathbb{H}_{w_3,1} + 2\mathbb{H}_{w_3,-1} + 2\mathbb{H}_{-1,1} + \mathbb{H}_{-1,-1} + k\mathbb{H}_{w_1,1} + k\mathbb{H}_{w_1,-1} \right. \\
& \left. - k\mathbb{H}_{w_2,1} - k\mathbb{H}_{w_2,-1} \right) [\mathbb{H}_{w_1} - \mathbb{H}_{w_2}] + \frac{1}{2} [\mathbb{H}_{w_1,1} + \mathbb{H}_{w_1,-1} - \mathbb{H}_{w_2,1} - \mathbb{H}_{w_2,-1}]\mathbb{H}_1 \Big\} \\
& - 16k(1-z^2) \Big\{ -\mathbb{H}_{1,w_4,w_5} - \mathbb{H}_{1,w_4,w_6} - \mathbb{H}_{1,w_4,w_7} - \mathbb{H}_{1,w_4,w_8} + \mathbb{H}_{w_5,1,1} - \mathbb{H}_{w_5,1,-1} \\
& + \mathbb{H}_{w_5,w_3,1} - \mathbb{H}_{w_5,w_3,-1} + \mathbb{H}_{w_6,1,1} - \mathbb{H}_{w_6,1,-1} + \mathbb{H}_{w_6,w_3,1} - \mathbb{H}_{w_6,w_3,-1} + \mathbb{H}_{w_7,w_3,1} \\
& - \mathbb{H}_{w_7,w_3,-1} - \mathbb{H}_{w_7,-1,1} + \mathbb{H}_{w_7,-1,-1} + \mathbb{H}_{w_8,w_3,1} - \mathbb{H}_{w_8,w_3,-1} - \mathbb{H}_{w_8,-1,1} + \mathbb{H}_{w_8,-1,-1} \\
& + \mathbb{H}_{-1,w_4,w_5} + \mathbb{H}_{-1,w_4,w_6} + \mathbb{H}_{-1,w_4,w_7} + \mathbb{H}_{-1,w_4,w_8} + k[\mathbb{H}_{w_1,w_4,w_5} + \mathbb{H}_{w_1,w_4,w_6} \\
& + \mathbb{H}_{w_1,w_4,w_7} + \mathbb{H}_{w_1,w_4,w_8} - \mathbb{H}_{w_2,w_4,w_5} - \mathbb{H}_{w_2,w_4,w_6} - \mathbb{H}_{w_2,w_4,w_7} - \mathbb{H}_{w_2,w_4,w_8} - \mathbb{H}_{w_5,1,w_1} \\
& + \mathbb{H}_{w_5,1,w_2} - \mathbb{H}_{w_5,w_3,w_1} + \mathbb{H}_{w_5,w_3,w_2} - \mathbb{H}_{w_6,1,w_1} + \mathbb{H}_{w_6,1,w_2} - \mathbb{H}_{w_6,w_3,w_1} + \mathbb{H}_{w_6,w_3,w_2} \\
& - \mathbb{H}_{w_7,w_3,w_1} + \mathbb{H}_{w_7,w_3,w_2} + \mathbb{H}_{w_7,-1,w_1} - \mathbb{H}_{w_7,-1,w_2} - \mathbb{H}_{w_8,w_3,w_1} + \mathbb{H}_{w_8,w_3,w_2} \\
& + \mathbb{H}_{w_8,-1,w_1} - \mathbb{H}_{w_8,-1,w_2}] + \left(-\mathbb{H}_{1,1} - \mathbb{H}_{w_3,1} + \mathbb{H}_{w_3,-1} - \mathbb{H}_{-1,1} + \mathbb{H}_1^2 - k\mathbb{H}_{w_1,1} \right. \\
& \left. + k\mathbb{H}_{w_2,1} + k\mathbb{H}_{w_3,w_1} - k\mathbb{H}_{w_3,w_2} \right) [\mathbb{H}_{w_5} + \mathbb{H}_{w_6}] + \left(-\mathbb{H}_{w_3,1} + \mathbb{H}_{w_3,-1} + \mathbb{H}_{-1,1} \right. \\
& \left. + \mathbb{H}_{-1,-1} - \mathbb{H}_1\mathbb{H}_{-1} + k\mathbb{H}_{w_1,-1} - k\mathbb{H}_{w_2,-1} + k\mathbb{H}_{w_3,w_1} - k\mathbb{H}_{w_3,w_2} \right) [\mathbb{H}_{w_7} + \mathbb{H}_{w_8}] \\
& + \left((\mathbb{H}_{w_5} + \mathbb{H}_{w_6} + \mathbb{H}_{w_7} + \mathbb{H}_{w_8})\mathbb{H}_{w_3} - \mathbb{H}_{w_5,1} - \mathbb{H}_{w_5,w_3} - \mathbb{H}_{w_6,1} - \mathbb{H}_{w_6,w_3} - \mathbb{H}_{w_7,w_3} \right. \\
& \left. + \mathbb{H}_{w_7,-1} - \mathbb{H}_{w_8,w_3} + \mathbb{H}_{w_8,-1} \right) \\
& \times [\mathbb{H}_1 - \mathbb{H}_{-1} - k(\mathbb{H}_{w_1} - \mathbb{H}_{w_2})] \Big\} + 576(1-z)\beta \ln(k) - 96(1-z)\beta \ln(1-k^2)
\end{aligned}$$

$$\begin{aligned}
& \left. -192(1-z)\beta H_0 - 192(1-z)\beta \ln(k^2 - z) \right\} \\
& + \frac{1}{2} P_{gq}^{(0)} \otimes h_{g_1}^{(1)} L_M - P_{gq}^{(0)} \otimes \bar{b}_{g_1}^{(1)}. \tag{57}
\end{aligned}$$

The remaining convolutions appearing in Eq. (57) are given in Appendix A. Here and in the Appendix the argument of the iterative integrals $H_{\bar{a}}$ is β .

The polynomials P_i in Eq. (57) read

$$P_1 = 3k^4 + 3k^2(z^2 + 1) - z^2, \tag{58}$$

$$P_2 = 6k^4 + 3k^3z(2z - 1) + 6k^2(z^2 - 4z - 1) + 3kz - 2z^2, \tag{59}$$

$$P_3 = 6k^4 + 3k^3(2z^2 - 5z + 8) + 6k^2(z^2 + 2z + 2) + 3kz - 2z^2, \tag{60}$$

$$P_4 = 6k^4z + k^3(-16z^3 + 33z^2 - 24z + 8) + 6k^2z(z^2 + 2z + 2) - 3kz^2 - 2z^3, \tag{61}$$

$$P_5 = 6k^4z + k^3(-4z^3 + 3z^2 + 24z + 8) + 6k^2z(z^2 + 2z + 2) + 3kz^2 - 2z^3, \tag{62}$$

$$P_6 = 6k^4z + k^3(4z^3 - 3z^2 - 24z - 8) + 6k^2z(z^2 + 2z + 2) - 3kz^2 - 2z^3, \tag{63}$$

$$P_7 = 6k^4z + k^3(16z^3 - 33z^2 + 24z - 8) + 6k^2z(z^2 + 2z + 2) + 3kz^2 - 2z^3, \tag{64}$$

$$P_8 = 3\beta k^3(z - 3) + 3k^2(z - 4) + 4\beta k(3z^2 - 13z + 3) - 12z^2 + 46z + 9, \tag{65}$$

$$P_9 = 3\beta k^3(z - 3) - 3k^2(z - 4) + 4\beta k(3z^2 - 13z + 3) + 12z^2 - 46z - 9, \tag{66}$$

$$\begin{aligned}
P_{10} = & 3\beta k^4(1 - z)^2 + k^3(-35\beta + (3 - 16\beta)z^2 + 3(17\beta - 9)z + 39) + k^2(-52\beta + 12\beta z^3 \\
& + (22 - 81\beta)z^2 + (121\beta - 72)z + 35) + k(12(\beta - 1)z^3 + (75 - 38\beta)z^2 \\
& + (26\beta - 88)z + 10) + z(-12z^2 + 32z - 5), \tag{67}
\end{aligned}$$

$$\begin{aligned}
P_{11} = & 3\beta k^4(1 - z)^2 + k^3(35\beta + (16\beta - 3)z^2 + (27 - 51\beta)z - 39) + k^2(-52\beta + 12\beta z^3 \\
& + (22 - 81\beta)z^2 + (121\beta - 72)z + 35) + k(-12(\beta - 1)z^3 + (38\beta - 75)z^2 \\
& + (88 - 26\beta)z - 10) + z(-12z^2 + 32z - 5), \tag{68}
\end{aligned}$$

$$\begin{aligned}
P_{12} = & 3\beta k^4(1 - z)^2 - k^3(35\beta + (16\beta + 3)z^2 - 3(17\beta + 9)z + 39) + k^2(-52\beta + 12\beta z^3 \\
& - (81\beta + 22)z^2 + (121\beta + 72)z - 35) + k(12(\beta + 1)z^3 - (38\beta + 75)z^2 \\
& + (26\beta + 88)z - 10) + z(12z^2 - 32z + 5), \tag{69}
\end{aligned}$$

$$\begin{aligned}
P_{13} = & 3\beta k^4(1 - z)^2 + k^3(35\beta + (16\beta + 3)z^2 - 3(17\beta + 9)z + 39) + k^2(-52\beta + 12\beta z^3 \\
& - (81\beta + 22)z^2 + (121\beta + 72)z - 35) + k(-12(\beta + 1)z^3 + (38\beta + 75)z^2 \\
& - 2(13\beta + 44)z + 10) + z(12z^2 - 32z + 5), \tag{70}
\end{aligned}$$

$$\begin{aligned}
P_{14} = & k^4(-3(36\beta + 1) + (27\beta - 10)z^3 + (37 - 135\beta)z^2 + (216\beta - 34)z) \\
& + k^2z((3 - 27\beta)z^2 + (27\beta + 28)z - 28) + 7z^3, \tag{71}
\end{aligned}$$

$$\begin{aligned}
P_{15} = & k^4(-108\beta + (27\beta + 10)z^3 - (135\beta + 37)z^2 + (216\beta + 34)z + 3) \\
& + k^2z(-3(9\beta + 1)z^2 + (27\beta - 28)z + 28) - 7z^3. \tag{72}
\end{aligned}$$

We finally perform the transformation to the \mathbf{M} -scheme for the massive two-loop pure singlet Wilson coefficient. It is given by

$$H_{g_1}^{(2),\text{PS},\mathbf{M}} = H_{g_1}^{(2),\text{PS},\mathbf{L}} - \frac{1}{N_F} z_{\text{PS}}^{(2)}. \tag{73}$$

5 The Asymptotic and Threshold Expansions

The complete expressions calculated in Section 4 allow now to perform the asymptotic expansion for $Q^2 \gg m^2$ and the threshold expansion for $\beta \ll 1$.

In the asymptotic limit $Q^2 \gg m^2$ the first expansion coefficients of the polarized massive pure singlet Wilson coefficient read setting $\mu^2 = Q^2$

$$\begin{aligned}
H_{g_1}^{(2),\text{PS},\text{M}} = & C_{FTF} \left\{ - (20(1-z) + 8(1+z)H_0) \ln^2 \left(\frac{m^2}{Q^2} \right) - (8(1-z) - 8(1-3z)H_0 \right. \\
& - 8(1+z)H_0^2) \ln \left(\frac{m^2}{Q^2} \right) + \frac{592}{3}(1-z) + \left(\frac{256}{3}(2-z) - \frac{32(1+z)^3}{3z} H_{-1} \right) H_0 \\
& + \frac{8}{3}(21+2z^2)H_0^2 + \frac{16}{3}(1+z)H_0^3 + \left(88(1-z) + 80(1-z)H_0 \right) H_1 + 20(1-z)H_1^2 \\
& - \left(16(1-3z) - 32(1+z)H_0 \right) H_{0,1} + \frac{32(1+z)^3}{3z} H_{0,-1} - 32(1+z)H_{0,0,1} \\
& + 16(1+z)H_{0,1,1} - \left(\frac{32}{3}(9-3z+z^2) + 32(1+z)H_0 \right) \zeta_2 + 16(1+z)\zeta_3 \\
& - \left[16(1-z) + 8(3-z)H_0 + 4(2+z)H_0^2 \right] \\
& + \frac{m^2}{Q^2} \left[\left(16(1-z)(1-3z) - 32zH_0 \right) \ln \left(\frac{m^2}{Q^2} \right) + 8(18-12z-7z^2) \right. \\
& \left. + 16(6+z+6z^2)H_0 + 16zH_0^2 + 16(3-7z+3z^2)H_1 \right] \\
& + \left(\frac{m^2}{Q^2} \right)^2 \left[-4(1-z)(3+4z) \ln^2 \left(\frac{m^2}{Q^2} \right) + \left(\frac{4P_{18}}{1-z} - 16(1-z)(5+4z)H_0 \right. \right. \\
& - 8(1-z)(5+4z)H_1) \ln \left(\frac{m^2}{Q^2} \right) + \frac{2P_{19}}{3(1-z)^2} + \left(\frac{16P_{16}}{1-z} - 64(1-z^2)H_{-1} \right) H_0 \\
& + \left(\frac{4P_{17}}{1-z} - 32(1-z)H_0 \right) H_1 - 4(1-z)(7+4z)H_1^2 - 16(1-z)(3+4z)H_{0,1} \\
& \left. \left. + 64(1-z^2)H_{0,-1} + 16(1-z)\zeta_2 \right] + O(\kappa^3 \ln^2(\kappa)) \right\}, \tag{74}
\end{aligned}$$

with the polynomials

$$P_{16} = 3z^4 + z^3 - 11z^2 + 13z - 7, \tag{75}$$

$$P_{17} = 6z^4 + 2z^3 - 63z^2 + 84z - 32, \tag{76}$$

$$P_{18} = 6z^4 + 2z^3 - 57z^2 + 76z - 28, \tag{77}$$

$$P_{19} = 15z^5 - 27z^4 + 393z^3 - 1079z^2 + 1069z - 339. \tag{78}$$

In this expansion the Kummer–elliptic integrals turn into harmonic polylogarithms. The leading term, which is free of power corrections of $O((m^2/Q^2)^k)$, $k \in \mathbb{N}$, $k \geq 1$, can be predicted using the representation of the massive Wilson coefficient by massive operator matrix elements (OMEs), cf. [3, 5, 47, 48], and massless Wilson coefficients,

$$H_{g_1}^{(2),\text{PS}} \left(z, \frac{Q^2}{m^2} \right) = A_{Qq}^{(2),\text{PS}}(N_F + 1) + \hat{C}_{g_1}^{(2),\text{PS}}(N_F + 1). \tag{79}$$

Here the massless Wilson coefficient $\hat{C}_{g_1}^{(2),\text{PS}}(N_F + 1)$ is the one given in Section 3 normalized by

$N_F + 1$. The massive two-loop operator matrix element $A_{Qq}^{(2),\text{PS}}$ in Mellin space reads

$$A_{Qq}^{(2),\text{PS}} = -\frac{1}{8}\hat{P}_{qq}^{(0)}P_{qq}^{(0)}\ln^2\left(\frac{m^2}{\mu^2}\right) - \frac{1}{2}\hat{P}_{qq}^{(1),\text{PS}}\ln\left(\frac{m^2}{\mu^2}\right) + \frac{1}{8}\hat{P}_{qq}^{(0)}P_{qq}^{(0)}\zeta_2 + a_{Qq}^{(2),\text{PS}}, \quad (80)$$

cf. [3, 5, 47, 48]; for its renormalization see Ref. [49]. The constant part of the unrenormalized polarized OME $a_{Qq}^{(2),\text{PS}}$ is given by [3, 5]

$$a_{Qq}^{(2),\text{PS}}(z) = C_F T_F \left\{ \begin{aligned} & -72(1-z) - 12(1+5z)H_0 - 2(1-3z)H_0^2 - \frac{4}{3}(1+z)H_0^3 + 40(1-z)H_0H_1 \\ & - (40(1-z) - 16(1+z)H_0)H_{0,1} - 32(1+z)H_{0,0,1} - (20(1-z) - 8(1+z)H_0)\zeta_2 \\ & + 32(1+z)\zeta_3 \end{aligned} \right\} \quad (81)$$

in z -space. The calculation of $A_{Qq}^{(2),\text{PS}}$ is performed in the Larin-scheme. One has to apply the tensor decomposition method, however, to obtain the correct result. These aspects are discussed in Ref. [5] in detail.

The threshold expansion of the Wilson coefficients for $\beta \ll 1$ is given by

$$H_{g_1}^{(1)}\left(z, \frac{Q^2}{m^2}\right) = 4T_F\beta \left\{ 1 - \frac{2}{3}(1-2z)\beta^2 - \frac{2}{5}(1-2z)\beta^4 - \frac{2}{7}(1-2z)\beta^6 - \frac{2}{9}(1-2z)\beta^8 + O(\beta^{10}) \right\}, \quad (82)$$

$$H_{g_1}^{(2),\text{PS,L}}\left(z, \frac{Q^2}{m^2}\right) = C_F T_F (1-z)\beta^3 \left\{ \begin{aligned} & -\frac{256}{9} + \frac{16}{3}[\ln(1-z) - \ln(z) + 4\ln(2\beta)] \\ & + \beta^2 \left(-\frac{32}{75}(41+20z) + \frac{16}{5}[\ln(1-z) - \ln(z) + 4\ln(2\beta)] \right) \\ & - \beta^4 \left(\frac{16(2723+20504z-12352z^2)}{11025} - \frac{16}{105}(7+16z-8z^2)[\ln(1-z) \right. \\ & \left. - \ln(z) + 4\ln(2\beta)] \right) + \beta^6 \left(\frac{16(47203-909904z+950864z^2-345728z^3)}{297675} \right. \\ & \left. + \frac{16}{945}(1+272z-232z^2+64z^3)[\ln(1-z) - \ln(z) + 4\ln(2\beta)] \right) \\ & \left. + O(\beta^8) \right\}. \end{aligned} \right\} \quad (83)$$

6 Numerical Results

Let us now illustrate the analytic results numerically. In Figure 3 the two-loop heavy flavor Wilson coefficient $H_{g_1}^{(2),\text{PS,M}}$ is shown as a function of z for different values of $Q^2 \in [10, 10^4] \text{ GeV}^2$, setting the charm quark mass to $m_c = 1.59 \text{ GeV}$, cf. [11].

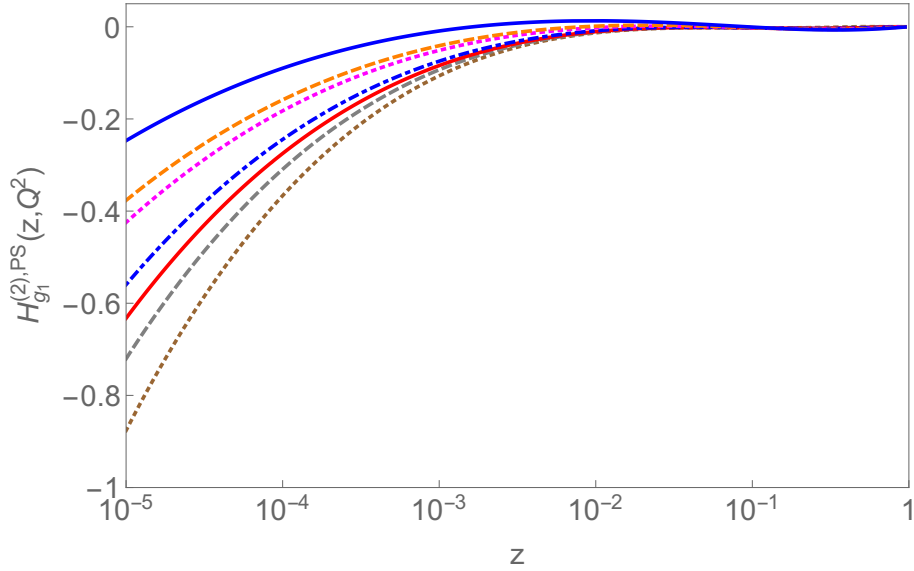


Figure 3: The Wilson coefficient $H_{g_1}^{(2),PS}$ as a function of z for different values of Q^2 and the scale choice $\mu^2 = \mu_F^2 = Q^2$. Upper full line (Blue): $Q^2 = 10^4$ GeV²; upper dashed line (Orange): $Q^2 = 10^3$ GeV²; upper dotted line (Magenta): $Q^2 = 500$ GeV²; dash-dotted line (Blue): $Q^2 = 100$ GeV²; lower full line (Red): $Q^2 = 50$ GeV²; lower dashed line (Gray): $Q^2 = 25$ GeV²; lower dotted line (Brown): $Q^2 = 10$ GeV².

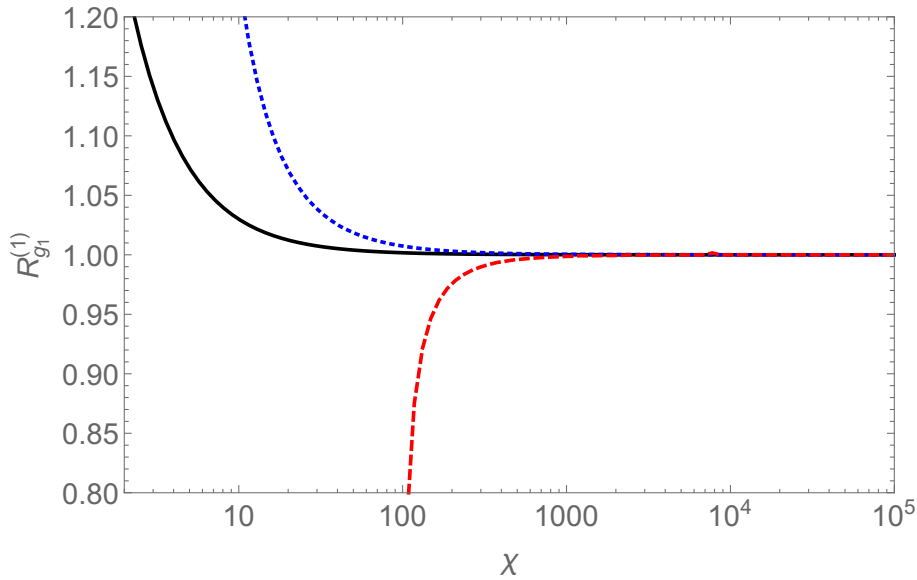


Figure 4: The ratio $R_{g_1}^{(1)}$, Eq. (84), as a function of $\chi = Q^2/m^2$. Solid line: $z = 10^{-4}$; dotted line: $z = 10^{-2}$; dashed line: $z = 1/2$.

For large values of Q^2 these results approach the asymptotic result for $H_{g_1}^{(2),PS}$. In the small x region this Wilson coefficient is negative.

Next we study the ratios

$$R_{g_1}^{(1)} = \frac{H_{g_1}^{(2),PS}}{\tilde{H}_{g_1}^{(2),PS}}(\mu = \mu_F = m), \quad (84)$$

comparing the full (57) and the asymptotic results, \tilde{H} , (74) for the leading term in Figure 4.

For $H_{g_1}^{(2),\text{PS}}$ the asymptotic expansion agrees with the full calculation up to $Q^2/m^2 \equiv \chi = 10$ to about 2% for $z = 10^{-4}$, $\chi = 40$ for $z = 10^{-2}$ and $\chi = 200$ for $z = 1/2$. However, the Wilson coefficients are very small already for the last value. Similar to the ratio of the full and asymptotic Wilson coefficient we define the ratio

$$R_{g_1} = \frac{g_1^{(2),\text{PS}}}{\tilde{g}_1^{(2),\text{PS}}}, \quad (85)$$

where $\tilde{g}_{1,q}^{(2),\text{PS}}$ is the structure function obtained by using the expansion of the respective Wilson coefficient up to the desired level. The corresponding results are depicted in Figure 5. We use the parameterization of the parton distributions Ref. [50] at NLO with the corresponding values of $\alpha_s(Q^2)$ at NNLO [51] to compare to previous non-singlet results in [10]. Demanding an agreement within $\pm 2\%$ for g_1^{PS} in the range $z \in [10^{-4}, 10^{-2}, 1/2]$ leads to values $Q_0^2/m^2 \in [5, 5, 13]$ of the $O((m^2/Q^2)^2)$ improved result, $Q_0^2/m^2 \in [10, 12, 30]$ of the $O(m^2/Q^2)$ improved result, and $Q_0^2/m^2 \in [12, 100, 170]$ for the asymptotic result.

In Figures 6 we show the complete results for the two-loop pure singlet contributions to xg_1 and xg_2 as a function of x for a series of Q^2 -values. Both functions show an oscillatory behaviour, which is enlarged for xg_2 due to the Wandzura–Wilczek relation.

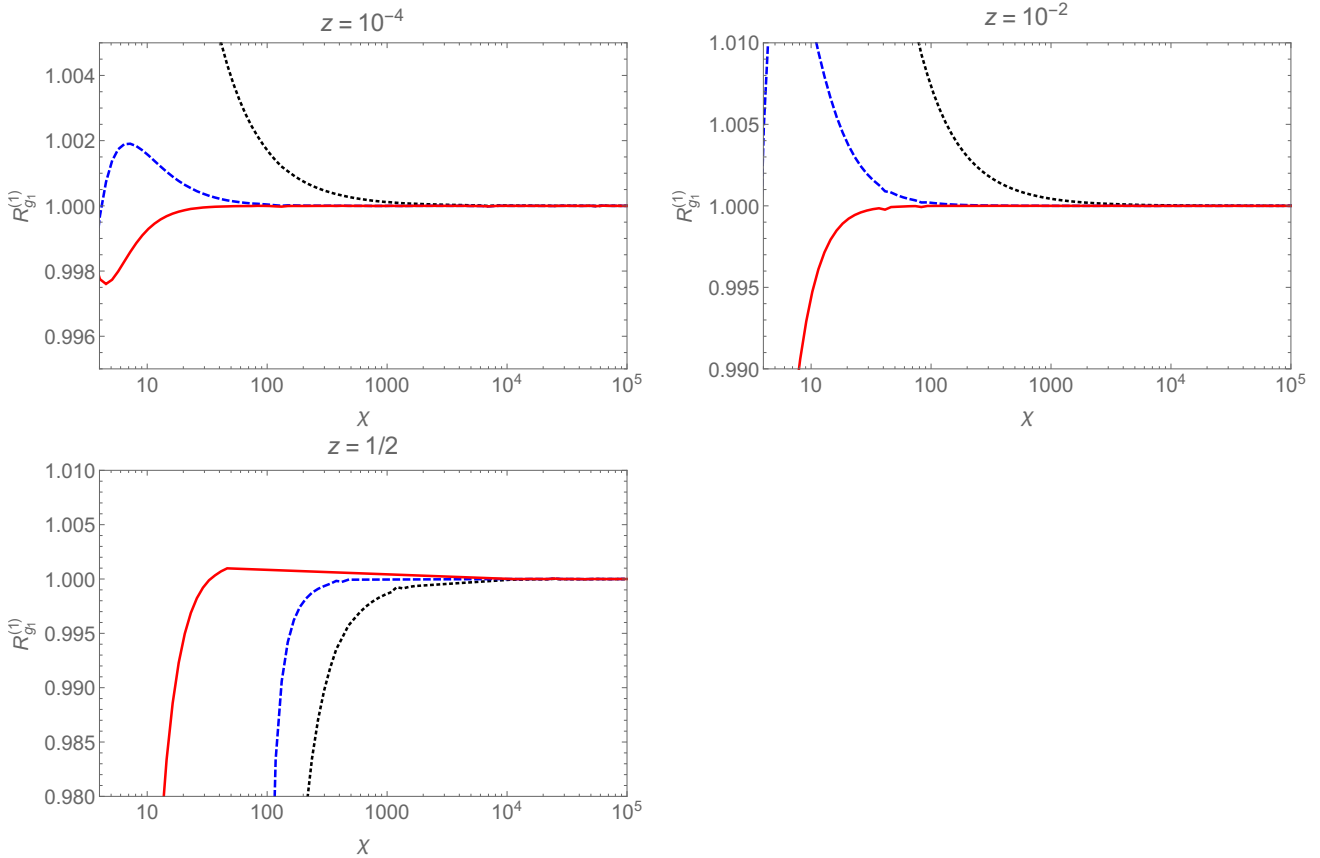


Figure 5: The ratio $R_{g_1}^{(1)}$, Eq. (84), as a function of $\chi = Q^2/m^2$ for different values of z gradually improved with κ suppressed terms. Dotted lines: asymptotic result; dashed lines: $O(m^2/Q^2)$ improved; solid lines : $O((m^2/Q^2)^2)$ improved.

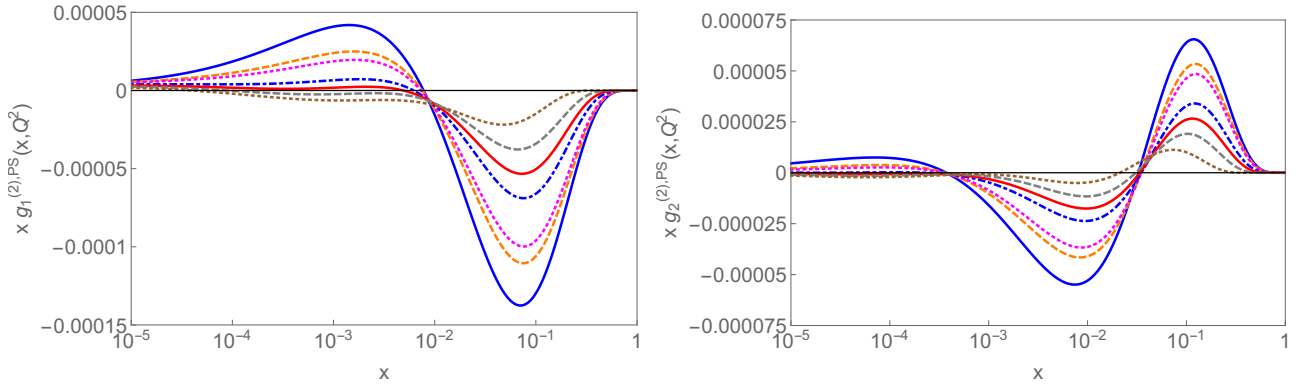


Figure 6: The pure singlet contributions $xg_1^{(2),\text{PS}}$ and $xg_2^{(2),\text{PS}}$ for different values of Q^2 and the scale choice $\mu^2 = \mu_F^2 = Q^2$. Full line (Blue): $Q^2 = 10^4$ GeV²; dashed line (Orange): $Q^2 = 10^3$ GeV²; dotted line (Magenta): $Q^2 = 500$ GeV²; dash-dotted line (Blue): $Q^2 = 100$ GeV²; full line (Red): $Q^2 = 50$ GeV²; dashed line (Gray): $Q^2 = 25$ GeV²; dotted line (Brown): $Q^2 = 10$ GeV², using the parameterization of the parton distribution [50].

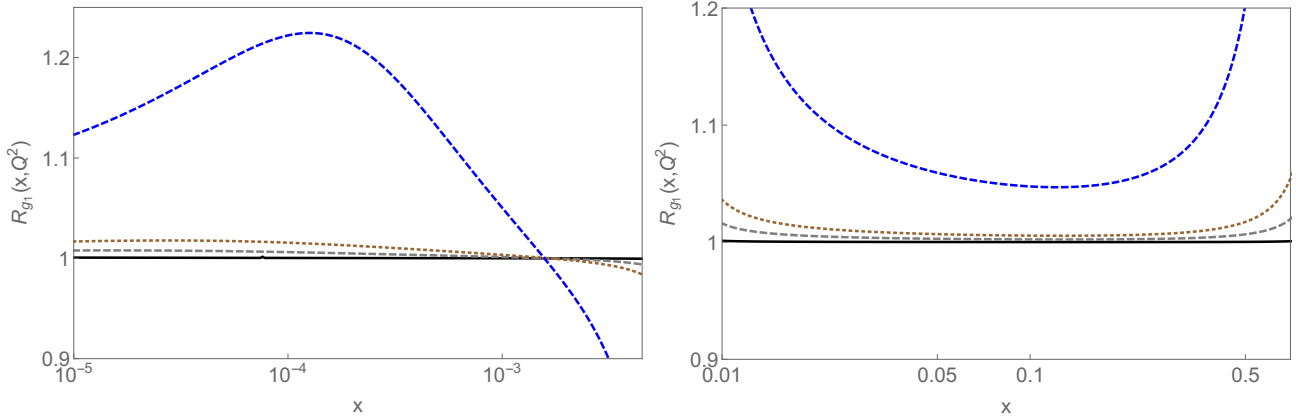


Figure 7: The ratio of the structure function $g_1^{(2),\text{PS}}$ in the full calculation over the asymptotic approximation for different values of Q^2 and the scale choice $\mu^2 = \mu_F^2 = Q^2$. Full line (Black): $Q^2 = 10^4$ GeV²; dashed line (Gray): $Q^2 = 10^3$ GeV²; dotted line (Brown): $Q^2 = 500$ GeV²; lower dashed line (Blue): $Q^2 = 100$ GeV², using the parameterization of the parton distribution [50]

In Figure 7 we illustrate the ratios Eq. (85) as a function of x for different values of Q^2 for g_1^{PS} comparing the asymptotic result to the full result. For a better visibility and to avoid to depict zero transitions in the denominator we separate the small x and large x part into two plots. The corrections behave widely flat in x for larger values of Q^2 and develop some profile for $Q^2 < 100$ GeV².

In Figure 8 we depict the ratio of the full result over the $O((m^2/Q^2)^2)$ improved asymptotic results for g_1^{PS} as a function of x for a series of Q^2 -values, again separating the small x and the large x ranges because of zero transitions for this ratio. For $Q^2 \gtrsim 100$ GeV² the ratios are rather flat and are close to one. The line for $Q^2 = 100$ GeV² for $x > 0.5$ deviates from one by more than 5%. Larger deviations are found for $Q^2 = 50$ GeV², where the 5% margin is only met for $x < 3 \cdot 10^{-3}$. We limited the expansion to terms of $\sim O((m^2/Q^2)^2)$ in the present paper, but

higher order terms can be given straightforwardly. The expanded expressions do also allow direct Mellin transforms and provide a suitable analytic basis for Mellin–space programmes.

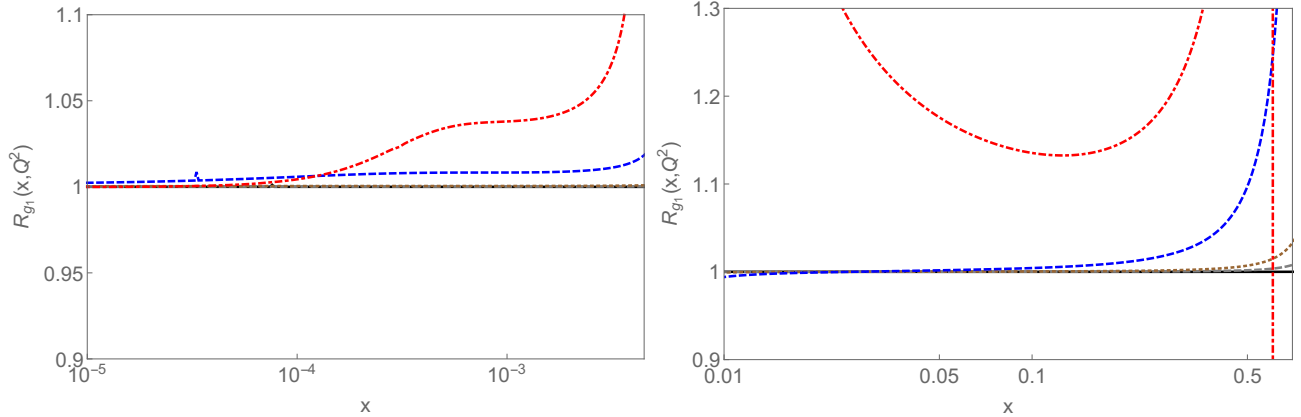


Figure 8: The ratio of the pure singlet structure function $g_1^{(2),\text{PS}}$ in the full calculation over the $O((m^2/Q^2)^2)$ improved approximation for different values of Q^2 and the scale choice $\mu^2 = \mu_F^2 = Q^2$. Full lines (Black): $Q^2 = 10^4$ GeV²; dashed lines (Gray): $Q^2 = 10^3$ GeV²; dotted lines (Brown): $Q^2 = 500$ GeV²; dashed lines (Blue): $Q^2 = 100$ GeV²; dash-dotted lines (Red): $Q^2 = 50$ GeV², using the parameterization of the parton distribution [50].

7 Conclusions

We have calculated the massless and massive polarized two–loop pure singlet Wilson coefficients for deep–inelastic scattering in analytic form. The calculation has been performed in the Larin–scheme, with a final finite renormalization to the $\overline{\text{M}}$ –scheme, which has been introduced in Refs. [17, 18]. The massless Wilson coefficient can be expressed by the harmonic polylogarithms in z –space and harmonic sums in Mellin– N space. In the massive case the polarized two–loop pure singlet Wilson coefficient is also given by iterative integrals, however, of a more general kind, the Kummer–elliptic integrals, here based on an alphabet of 12 letters, cf. [12]. From the expansion of the massive Wilson coefficient in the region $Q^2 \gg m^2$ one obtains the asymptotic result, which can be given in terms of a massive OME and the massless Wilson coefficient, cf. [5]. In the region of lower values of Q^2 and larger values of x , the power corrections to the massive two–loop Wilson coefficient are essential. From the available analytic result one can construct the series in m^2/Q^2 analytically. Since the deep–inelastic process is usually considered only for virtualities $Q^2 \gtrsim 5$ GeV², this series gives the proper numerical representation in case of the charm–quark corrections retaining a relatively small number of terms. The latter representation has the advantage that it can be transformed into Mellin space directly, since the expansion coefficients are given in terms of harmonic polylogarithms in z –space.

A Contributing Expressions due to Renormalization

In the following we list the Mellin–convolutions, which occurred in Eq. (57). These are convolutions with leading order splitting functions, referring to the parameter $\kappa = m^2/Q^2$.

$$P_{gq}^{(0)} \otimes h_{g_1}^{(1)} = C_F T_F \left\{ -192(1-z)\beta + 32(1+2z)\sqrt{1+4\kappa} \ln \left(\frac{\sqrt{1+4\kappa} - \beta}{\sqrt{1+4\kappa} + \beta} \right) \right.$$

$$\begin{aligned}
& + \left[-64(1+z) \ln(1 + \sqrt{1+4\kappa}) + 64(1+z) \ln(\beta + \sqrt{1+4\kappa}) \right. \\
& - 32(1+z) \ln\left(\frac{\sqrt{1+4\kappa}-1}{\sqrt{1+4\kappa}+1}\right) + 32(1+z) \ln\left(\frac{\sqrt{1+4\kappa}-\beta}{\sqrt{1+4\kappa}+\beta}\right) \\
& \left. - 16(7-z(1-4\kappa)) + 32(1+z) \ln(2) - 32(1+z) \ln(1+\beta) \right] \ln\left(\frac{1-\beta}{1+\beta}\right) \\
& - 32(1+z) \text{Li}_2\left(\frac{1-\beta}{2}\right) + 32(1+z) \text{Li}_2\left(\frac{1+\beta}{2}\right) - 16(1+z) \ln^2\left(\frac{1-\beta}{1+\beta}\right) \\
& - 32(1+z) \text{Li}_2\left(\frac{1+\beta}{1-\sqrt{1+4\kappa}}\right) + 32(1+z) \text{Li}_2\left(\frac{\beta-1}{\sqrt{1+4\kappa}-1}\right) \\
& \left. + 32(1+z) \text{Li}_2\left(\frac{1-\beta}{1+\sqrt{1+4\kappa}}\right) - 32(1+z) \text{Li}_2\left(\frac{1+\beta}{1+\sqrt{1+4\kappa}}\right) \right\}, \quad (86)
\end{aligned}$$

$$\begin{aligned}
P_{gq}^{(0)} \otimes \bar{b}_{g_1}^{(1)} & = C_F T_F \left\{ 208(1-z)\beta + \frac{16(1-k^2)}{k} \ln^2(1-k) - \frac{4}{k^2} (2k^2z - 7k^2 - z) \left\{ 4H_1H_0 \right. \right. \\
& + 2\ln(1-k)[H_1 + H_{-1}] - 4\ln(k)[H_1 + H_{-1}] + 2\ln(1+k)[H_1 + H_{-1}] + H_1^2 \\
& \left. - 2H_1H_{-1} - H_{-1}^2 - 4H_{0,1} + 4H_{-1,0} + 4H_{-1,1} \right\} - \frac{8}{k^2} (4k^2 + z + 7k^2z - 12k^2\beta \\
& + 12k^2z\beta)H_{-1} - \frac{8}{k^2} (4k^2 + z + 7k^2z + 12k^2\beta - 12k^2z\beta)H_1 \\
& + 32(1+2z) \left\{ (1 + \ln(k)) [H_{w_1} + H_{w_2}] - H_{w_1,0} - H_{w_2,0} - \frac{1}{2} [H_{w_1,1} - H_{w_1,-1} \right. \\
& \left. + H_{w_2,1} - H_{w_2,-1}] \right\} - 96(1-z)\beta [\ln(1-k^2) - 2\ln(k) + 2H_0] \\
& - 16 \left((k^2 + 2z) \ln(1-k) + (2 - k^2 + 2z) \ln(1+k) \right) H_{w_1} \\
& - 16 \left((2 - k^2 + 2z) \ln(1-k) + (k^2 + 2z) \ln(1+k) \right) H_{w_2} \\
& + 8(1+z) \left[H_1H_{-1}^2 - 4H_{0,1,1} + 4H_{-1,0,1} + 8H_{-1,1,0} + 8H_{-1,1,1} + 4H_{-1,-1,0} \right. \\
& + 8H_{-1,-1,1} - \frac{1}{3} [H_1^3 + H_{-1}^3] + [H_1^2 - 4H_{-1,1}]H_{-1} + [4H_{0,1} - 4H_{-1,0} - 4H_{-1,1} \\
& \left. - 2H_1H_0]H_1 + [H_{-1}^2 - H_1^2 - 2H_1H_{-1} + 4H_{-1,1}] (\ln(1-k^2) - 2\ln(k)) \right. \\
& \left. + 2(\zeta_2 - \ln^2(2)) [\ln(1-k^2) - \ln(1-z)] \right] + 32 [\ln(1-k^2) - \ln(1-z)] \ln(2) \\
& + 16k(1+z) \left[2H_{w_1,1,0} + H_{w_1,1,1} - H_{w_1,1,-1} + 2H_{w_1,-1,0} + H_{w_1,-1,1} - H_{w_1,-1,-1} \right. \\
& - 2H_{w_2,1,0} - H_{w_2,1,1} + H_{w_2,1,-1} - 2H_{w_2,-1,0} - H_{w_2,-1,1} + H_{w_2,-1,-1} \\
& \left. + (\ln(1-k^2) - 2\ln(k)) [H_{w_1,1} + H_{w_1,-1} - H_{w_2,1} - H_{w_2,-1}] \right. \\
& \left. + (\zeta_2 - \ln^2(2)) [H_{w_1} - H_{w_2}] \right] + 32k \ln(2) [H_{w_1} - H_{w_2}] \\
& \left. - \frac{16(1-k^2)}{k} \left(\ln^2(1+k) + \ln(1-z) [\ln(1-k) - \ln(1+k)] \right) \right\}. \quad (87)
\end{aligned}$$

Here, $\text{Li}_2(x)$, denotes the dilogarithm [52],

$$\text{Li}_2(x) = - \int_0^x \frac{dz}{z} \ln(1-z). \quad (88)$$

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