The quaternionic/hypercomplex-correspondence

Vicente Cortés and Kazuyuki Hasegawa

April 15, 2019

Abstract

Given a quaternionic manifold M with a certain U(1)-symmetry, we construct a hypercomplex manifold M' of the same dimension. This construction generalizes the quaternionic Kähler/hyper-Kähler-correspondence. As an example of this construction, we obtain a compact homogeneous hypercomplex manifold which does not admit any hyper-Kähler structure. Therefore our construction is a proper generalization of the quaternionic Kähler/hyper-Kähler-correspondence.

1 Introduction

Let us recall that there exist constructions due to Andriy Haydys, called the QK/HK-correspondence and the HK/QK-correspondence, which relate quaternionic Kähler manifolds to hyper-Kähler manifolds of the same dimension [9]. These constructions have been generalized to include possibly indefinite metrics [2, 1]. In this way the supergravity c-map metric and a one-parameter deformation thereof have been described as an application of the HK/QK-correspondence with indefinite initial hyper-Kähler data. Many complete quaternionic Kähler manifolds can be obtained in this way, see for instance [6] for co-homogeneity one examples.

The main result of this paper, see Theorem 6.4, is a construction of a hypercomplex manifold from a quaternionic manifold with a U(1)-action, which we may call the *quaternionic/hypercomplex-correspondence* (Q/H-correspondence for short). This construction generalizes the QK/HK-correspondence.

In [18, 11, 17], it is shown that with every quaternionic manifold M one can associate an $\mathbb{H}^*/\{\pm 1\}$-bundle over M and a hypercomplex structure on the total space of the bundle. More precisely [17], there exists a one-parameter family of $\mathbb{H}^*/\{\pm 1\}$-bundles such that, given a quaternionic connection on M, each of the bundles is endowed with an almost hypercomplex structure. For a particular choice of the parameter, the almost hypercomplex structure is integrable and independent of the connection. Here we will

2010 Mathematics Subject Classification. Primary 53C10. Secondary 53C56, 53C26.

Keywords and phrases. quaternionic manifold, hypercomplex manifold, hypercomplex moment map, quaternionic/hypercomplex-correspondence.

adopt a different point of view. Instead of a one-parameter family of bundles, we will define a single principal $\mathbb{H}^*/\{\pm 1\}$-bundle, which we call the *Swann bundle*, endowed with a one-parameter family of almost hypercomplex structures (still depending on a quaternionic connection). Again we find that, for a particular choice of the parameter, namely $c = -4(n+1)$, the almost hypercomplex structure is always integrable and independent of the connection, see Proposition 3.3. Here $4n = \dim M$. For all other values of the parameter, we show that the almost hypercomplex structure is integrable if and only if all $I \in Q$, where Q denotes the quaternionic structure, are skew-symmetric with respect to the skew-symmetric part of the Ricci-curvature, see Theorem 3.6.

Now we briefly explain how we obtain the Q/H-correspondence. Given an infinitesimal automorphism X of a quaternionic manifold (M, Q, ∇) endowed with a quaternionic connection ∇, we show that the natural lift $\hat{X}$ of X to the Swann bundle $\hat{M}$ preserves each member of the one-parameter family of almost hypercomplex structures. The next step is to perform a hypercomplex reduction with respect to $\hat{X}$. Recall that hypercomplex reduction was introduced by Dominic Joyce in [11]. It is defined as the quotient of a level set of a moment map by the group action. The construction is based on the notion of a moment map in this context as defined in [11]. Here we define the moment map for the infinitesimal automorphism $\hat{X}$ by the equation (5.4) and analyse Joyce's conditions in Proposition 5.8. Assuming that $\hat{X}$ generates a free $\mathrm{U}(1)$-action, we can finally perform the reduction obtaining a hypercomplex manifold M'. Otherwise, we can construct the hypercomplex structure on a submanifold transversal to the foliation defined by $\hat{X}$ (on some open submanifold of $\hat{M}$).

Examples of our Q/H-correspondence include compact homogeneous hypercomplex manifolds. Indeed, starting with a homogeneous quaternionic Hopf manifold

$$(\mathbb{R}^{>0}/\langle\lambda\rangle) \times \frac{\mathrm{Sp}(n)\mathrm{U}(1)}{\mathrm{Sp}(n-1)\triangle_{\mathrm{U}(1)}},$$

we obtain a homogeneous hypercomplex Hopf manifold

$$(\mathbb{R}^{>0}/\langle\lambda\rangle) \times \frac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-1)}$$

by the Q/H-correspondence, see Example 7.8. Note that this hypercomplex manifold does not admit any hyper-Kähler structure for topological reasons. Therefore our construction is a proper generalization of the QK/HK-correspondence.

2 Preliminaries

Throughout this paper, all manifolds are assumed to be smooth and without boundary and maps are assumed to be smooth unless otherwise mentioned. The space of sections of a vector bundle $E \to M$ is denoted by $\Gamma(E)$.

We say that M is a *quaternionic manifold* with the quaternionic structure Q if Q is a subbundle of $\mathrm{End}(TM)$ of rank 3 which at every point $x \in M$ is spanned by endomorphisms I_1, I_2, $I_3 \in \mathrm{End}(T_xM)$ satisfying

$$I_1^2 = I_2^2 = I_3^2 = -\mathrm{id}, \; I_1 I_2 = -I_2 I_1 = I_3, \tag{2.1}$$

and there exists a torsion-free connection ∇ on M such that ∇ preserves Q, that is, $\nabla_X \Gamma(Q) \subset \Gamma(Q)$ for all $X \in \Gamma(TM)$. Note that we use the same letter ∇ for the connection on $\mathrm{End}(TM)$ induced by ∇ if there is no confusion. Such a torsion-free connection ∇ is called a *quaternionic connection* and the triplet (I_1, I_2, I_3) is called an *admissible frame* of Q at x. The dimension of a quaternionic manifold M is denoted by $4n$. Note that a quaternionic connection is not unique, in fact, there is the following result [7, 5].

Lemma 2.1. *Let ∇^1 and ∇^2 be quaternionic connections on (M, Q). Then there exists a 1-form ξ on M such that*

$$\nabla^2_X Y = \nabla^1_X Y + S^\xi_X Y \tag{2.2}$$

for all X, $Y \in \Gamma(TM)$, where S^ξ is defined by

$$\begin{aligned} S^\xi_X Y = \xi(X)Y + \xi(Y)X - \xi(I_1X)I_1Y - \xi(I_1Y)I_1X \\ - \xi(I_2X)I_2Y - \xi(I_2Y)I_2X - \xi(I_3X)I_3Y - \xi(I_3Y)I_3X. \end{aligned}$$

Conversely, for a given quaternionic connection ∇^1, the connection ∇^2 given by the equation above is also a quaternionic connection.

An *almost hypercomplex manifold* is defined to be a manifold M endowed with 3 almost complex structures I_1, I_2, I_3 satisfying the quaternionic relations (2.1). If I_1, I_2, I_3 are integrable, then M is called a *hypercomplex manifold*. There exists a unique torsion-free connection on a hypercomplex manifold for which the hypercomplex structures are parallel. It is called the *Obata connection* [14]. Obviously, hypercomplex manifolds are quaternionic manifolds with $Q = \langle I_1, I_2, I_3 \rangle$.

3 The canonical family of almost hypercomplex structures on the Swann bundle $\hat{M}$

In this section we will define a principal $\mathbb{R}^{>0} \times \mathrm{SO}(3)$-bundle $\hat{M} \to M$ over a quaternionic manifold (M, Q) equipped with a quaternionic connection ∇ and endow $\hat{M}$ with a one-parameter family of almost hypercomplex structures depending on the quaternionic connection ∇. Then we will study the integrability of the hypercomplex structure and its dependence (or independence) on the choice of ∇ for different values of the parameter.

3.1 The principal bundle $\hat{M} \to M$

Let S be the principal SO(3)-bundle of admissible frames (I_1, I_2, I_3) over a quaternionic manifold (M, Q). The principal action τ of $g \in \mathrm{SO}(3)$ is given by $\tau(s, g) = sg^\varepsilon$ for $s = (I_1, I_2, I_3) \in S$, where $\varepsilon = 1$ (resp. $\varepsilon = -1$) if S is considered as a right (resp. left)-principal bundle. The bundle projection of S is denoted by π_S. We take a basis (e_1, e_2, e_3) of $\mathbb{R}^3 \cong \mathrm{Im}\,\mathbb{H} \cong \mathfrak{sp}(1) \cong \mathfrak{so}(3)$ so that

$$[e_\alpha, e_\beta] = 2e_\gamma$$

for any cyclic permutation (α, β, γ). Hereafter (α, β, γ) will be always a cyclic permutation, whenever the three letters appear in an expression. A quaternionic connection induces a principal connection $\theta : TS \to \mathfrak{so}(3)$ and we denote $\theta = \sum \theta^\alpha e_\alpha$. Moreover we consider the principal $\mathbb{R}^{>0}$-bundle $S_0 := (\Lambda^{4n}(T^*M)\backslash\{0\})/\{\pm 1\}$ over M, where $\mathbb{R}^{>0} = \{a \in \mathbb{R} \mid a > 0\}$. The principal $\mathbb{R}^{>0}$-action τ_0 on S_0 is given by scalar multiplication $\tau_0(\rho, a) := \rho a^\varepsilon$ $(\varepsilon = \pm 1)$ for $\rho \in S_0$ and $a \in \mathbb{R}^{>0}$. The bundle projection of S_0 is denoted by π_{S_0}. A quaternionic connection induces also a principal connection $\theta_0 : TS_0 \to \mathbb{R} = \mathrm{Lie}\,(\mathbb{R}^{>0})$. The product $S_0 \times S$ is a principal $\mathbb{R}^{>0} \times \mathrm{SO}(3)$-bundle over $M \times M$ whose principal action is $\tau_0 \times \tau$. The $\mathbb{R}^4(\cong \mathbb{R} \oplus \mathfrak{so}(3))$-valued 1-form $(\theta_0 \circ pr_{TS_0}, \theta \circ pr_{TS}) = (\theta_0 \circ pr_{TS_0}, \theta_1 \circ pr_{TS}, \theta_2 \circ pr_{TS}, \theta_3 \circ pr_{TS})$ is a principal connection on $S_0 \times S$, where pr_{TS_0} (resp. pr_{TS}) is the projection from $T(S_0 \times S) \cong TS_0 \times TS$ onto TS_0 (resp. TS). Note that the Lie group $\mathbb{R}^{>0} \times \mathrm{SO}(3) = \mathbb{H}^*/\{\pm 1\}$.

Let $\triangle : M \to M \times M$ be the diagonal map defined by $\triangle(x) = (x, x)$ for each $x \in M$. The pullback bundle

$$\hat{M} := \triangle^*(S_0 \times S) = \{(x, (\rho, s)) \in M \times (S_0 \times S) \mid x = \pi_{S_0}(\rho) = \pi_S(s)\}$$

is a principal $\mathbb{R}^{>0} \times \mathrm{SO}(3)$-bundle over M and $\bar{\theta} := \triangle_\#^*(\theta_0 \circ pr_{TS_0}, \theta \circ pr_{TS})$ is a principal connection on $\hat{M}$, where $\triangle_\# : \hat{M} \to S_0 \times S$ is the canonical bundle map. The bundle projection of $\hat{M}$ onto M is denoted by $\hat{\pi}$. Using the bundle projections $\hat{\pi}$, π_{S_0}, π_S and the principal connections $\bar{\theta}$, θ_0, θ, we have the decomposition

$$T\hat{M} = \bar{\mathcal{V}} \oplus \bar{\mathcal{H}},\ TS_0 = \mathcal{V}_0 \oplus \mathcal{H}_0,\ TS = \mathcal{V} \oplus \mathcal{H}, \tag{3.1}$$

where $\bar{\mathcal{V}} = \mathrm{Ker}\,\hat{\pi}_*$, $\bar{\mathcal{H}} = \mathrm{Ker}\,\bar{\theta}$ and so on. It holds that $(\triangle_\#)_*(\bar{\mathcal{V}}_{(x,(\rho,s))}) = (\mathcal{V}_0)_\rho \times \mathcal{V}_s$ and $(\triangle_\#)_*(\bar{\mathcal{H}}_{(x,(\rho,s))}) \subset (\mathcal{H}_0)_\rho \times \mathcal{H}_s$ for each $(x, (\rho, s)) \in \hat{M}$. Set $\triangle_S := pr_{TS} \circ (\triangle_\#)_*$ and $\triangle_{S_0} := pr_{TS_0} \circ (\triangle_\#)_*$. The principal actions on $\hat{M}$, $S_0 \times S$, S_0 and S induce fundamental vector fields. We denote by $\widetilde{A}$ the fundamental vector field corresponding to a Lie algebra element A, irrespective of the manifold on which the vector field is defined, and set $Z_\alpha = \widetilde{e}_\alpha$ $(\alpha = 1, 2, 3)$. Note that $[Z_\alpha, Z_\beta] = 2\varepsilon Z_\gamma$.

3.2 The canonical family of almost hypercomplex structures

Let (M, Q) be a quaternionic manifold, ∇ a quaternionic connection and $\hat{\pi} : \hat{M} \to M$ the principal $\mathbb{R}^{>0} \times \mathrm{SO}(3)$-bundle with connection $\bar{\theta}$ constructed in the previous subsection. In this subsection, we define a canonical family of almost hypercomplex structures on $\hat{M}$ and consider their integrability.

Set $e_0 := 1 \in \mathbb{R}\,(\cong T_1\mathbb{R}^{>0})$ and $Z_0^c := c\,\widetilde{e}_0$ for a nonzero real number c. We denote the horizontal lifts relative to the connections $\bar{\theta}$, θ, θ_0 by $(\ \cdot\ )^{\bar{h}}$, $(\ \cdot\ )^h$, $(\ \cdot\ )^{h_0}$, respectively. An almost hypercomplex structure $(\hat{I}_1^{\bar{\theta},c}, \hat{I}_2^{\bar{\theta},c}, \hat{I}_3^{\bar{\theta},c})$ on $\hat{M}$ is defined by

$$\hat{I}_\alpha^{\bar{\theta},c} Z_0^c = -Z_\alpha, \quad \hat{I}_\alpha^{\bar{\theta},c} Z_\alpha = Z_0^c, \quad \hat{I}_\alpha^{\bar{\theta},c} Z_\beta = Z_\gamma, \quad \hat{I}_\alpha^{\bar{\theta},c} Z_\gamma = -Z_\beta$$

and

$$(\hat{I}_\alpha^{\bar{\theta},c})_{(x,(\rho,s))}(X) = (I_\alpha(\hat{\pi}_* X))^{\bar{h}}_{(x,(\rho,s))}$$

for all horizontal vector X at $(x,(\rho,s)) \in \hat{M}$, where $s=(I_1,I_2,I_3)$. Note that the triple $(\hat{I}_1^{\bar{\theta},c}, \hat{I}_2^{\bar{\theta},c}, \hat{I}_3^{\bar{\theta},c})$ depends on the connection form $\bar{\theta}$ and c.

Lemma 3.1. *For any horizontal lift $X^{\bar{h}} \in \bar{\mathcal{H}}_{(x,(\rho,s))}$ at $(x,(\rho,s)) \in \hat{M}$, we have $(\triangle_\#)_* X^{\bar{h}} = (X^{h_0}_\rho, X^h_s)$. In particular, it holds $(\triangle_\#)_*((\hat{I}_\alpha^{\bar{\theta},c})_{(x,(\rho,s))}(X^{\bar{h}})) = ((I_\alpha X)^{h_0}_\rho, (I_\alpha X)^h_s)$, where $s=(I_1,I_2,I_3)$. As a consequence, the horizontal lift $X^{\bar{h}}$ of a vector field X on M is $\triangle_\#$-related to the vector field (X^{h_0}, X^h), which is the horizontal lift of (X,X):*

$$(\triangle_\#)_* X^{\bar{h}} = (X^{h_0}, X^h) \circ \triangle_\#.$$

Proof. $(\triangle_\#)_* X^{\bar{h}}$ and (X^{h_0}, X^h) are horizontal vectors of $S_0 \times S$, since applying the connection form $(\theta_0 \circ pr_{TS_0}, \theta \circ pr_{TS})$ on both vectors gives zero. On the other hand, applying $(\pi_{S_0} \times \pi_S)_*$ on both vectors gives (X,X) because of $(\pi_{S_0} \times \pi_S) \circ \triangle_\# = \triangle \circ \hat{\pi}$. This proves $(\triangle_\#)_* X^{\bar{h}} = (X^{h_0}, X^h) \circ \triangle_\#$. Now it is easy to obtain $(\triangle_\#)_*((\hat{I}_\alpha^{\bar{\theta},c})_{(x,(\rho,s))}(X^{\bar{h}})) = ((I_\alpha X)^{h_0}_\rho, (I_\alpha X)^h_s)$ using that $(\hat{I}_\alpha^{\bar{\theta},c})_{(x,(\rho,s))}(X^{\bar{h}}) = (I_\alpha X)^{\bar{h}}$. □

Lemma 3.2. *Let ∇^1 and $\nabla^2 = \nabla^1 + S^\xi$ be quaternionic connections on (M,Q), where $\xi \in \Gamma(T^*M)$. We denote the almost hypercomplex structure defined above with respect to ∇^i $(i=1,2)$ and $c \neq 0$ by $\hat{I}_\alpha^{i,c}$ $(\alpha = 1,2,3)$. Then we have*

$$\hat{I}_\alpha^{1,c} - \hat{I}_\alpha^{2,c} = \varepsilon \left(1 + \frac{4(n+1)}{c}\right) ((\hat{\pi}^*\xi) \otimes Z_\alpha + ((\hat{\pi}^*(\xi \circ I_\alpha)) \otimes Z_0^c)$$

at each point $(x,(\rho,s)) \in \hat{M}$, where $s=(I_1,I_2,I_3)$.

Proof. We consider any point $(x,(\rho,s)) \in \hat{M}$, $s=(I_1,I_2,I_3)$, and omit the reference point in the proof. The corresponding connection forms induced by ∇^i are denoted by $\bar{\theta}^i$, $\theta^i = (\theta^i_1, \theta^i_2, \theta^i_3)$, θ^i_0 $(i=1,2)$, respectively. The tangent bundle $T\hat{M}$ is decomposed into $T\hat{M} = \bar{\mathcal{V}} \oplus \bar{\mathcal{H}}^1 = \bar{\mathcal{V}} \oplus \bar{\mathcal{H}}^2$, where $\bar{\mathcal{H}}^i = \operatorname{Ker} \bar{\theta}^i$. We express any tangent vector X of $\hat{M}$ as

$$X = Y^{\bar{h}_i} + \sum_{\delta=1}^3 a^i_\delta Z_\delta + b^i Z_0^c,$$

where $Y \in TM$. By the definition of $\hat{I}_\alpha^{i,c}$, we see

$$\hat{I}_\alpha^{i,c}(X) = (I_\alpha Y)^{\bar{h}_i} + a^i_\alpha Z_0^c + a^i_\beta Z_\gamma - a^i_\gamma Z_\beta - b^i Z_\alpha.$$

Since

$$\begin{aligned} \bar{\theta}^1(X) &= \sum_{\delta=1}^3 a^1_\delta e_\delta + cb^1 e_0 = \bar{\theta}^1(Y^{\bar{h}_2}) + \sum_{\delta=1}^3 a^2_\delta e_\delta + cb^2 e_0 \\ &= \sum_{\delta=1}^3 \theta^1_\delta(\triangle_S Y^{\bar{h}_2}) e_\delta + \theta^1_0(\triangle_{S_0} Y^{\bar{h}_2}) e_0 + \sum_{\delta=1}^3 a^2_\delta e_\delta + cb^2 e_0, \end{aligned}$$

we have $b^1 = b^2 + (1/c)\theta^1_0(\triangle_{S_0} Y^{\bar{h}_2})$ and $a^1_\delta = a^2_\delta + \theta^1_\delta(\triangle_S Y^{\bar{h}_2})$ ($\delta = 1, 2, 3$). Therefore it holds

$$\begin{aligned}
\hat{I}^{1,c}_\alpha(X) &= (I_\alpha Y)^{\bar{h}_1} + a^1_\alpha Z^c_0 + a^1_\beta Z_\gamma - a^1_\gamma Z_\beta - b^1 Z_\alpha \\
&= (I_\alpha Y)^{\bar{h}_1} + (a^2_\alpha + \theta^1_\alpha(\triangle_S Y^{\bar{h}_2}))Z^c_0 + (a^2_\beta + \theta^1_\beta(\triangle_S Y^{\bar{h}_2}))Z_\gamma \\
&\quad - (a^2_\gamma + \theta^1_\gamma(\triangle_S Y^{\bar{h}_2}))Z_\beta - (b^2 + (1/c)\theta^1_0(\triangle_{S_0} Y^{\bar{h}_2}))Z_\alpha \\
&= (I_\alpha Y)^{\bar{h}_1} - (I_\alpha Y)^{\bar{h}_2} + \hat{I}^{2,c}_\alpha(X) \\
&\quad + \theta^1_\alpha(\triangle_S Y^{\bar{h}_2})Z^c_0 + \theta^1_\beta(\triangle_S Y^{\bar{h}_2})Z_\gamma - \theta^1_\gamma(\triangle_S Y^{\bar{h}_2})Z_\beta - (1/c)\theta^1_0(\triangle_{S_0} Y^{\bar{h}_2})Z_\alpha.
\end{aligned}$$

Let $s_0 : U \to S_0$ and $s : U \to S$ be local sections defined on an open set U in M. Then $\bar{s} := (s_0, s) \circ \triangle$ is a local section of $\hat{M}$. The pull backs of θ^i, θ^i_0 to U are denoted by $\theta^{i,U}$ and $\theta^{i,U}_0$. If we define the one forms $\theta^{i,U}_\alpha$ by $\theta^{i,U} = s^*\theta^i = (1/2)\sum(\theta^{i,U}_\alpha)e_\alpha$. From Lemma 2.1 and

$$\nabla^i I_\alpha = \varepsilon(\theta^{i,U}_\gamma \otimes I_\beta - \theta^{i,U}_\beta \otimes I_\gamma) \quad (i = 1, 2)$$

one can check that

$$\begin{aligned}
\theta^{2,U}_\delta - \theta^{1,U}_\delta &= -2\varepsilon(\xi \circ I_\delta) \quad (\delta = 1, 2, 3), \\
\theta^{2,U}_0 - \theta^{1,U}_0 &= 4\varepsilon(n + 1)\xi.
\end{aligned} \tag{3.2}$$

It is easy to see that

$$\begin{aligned}
Y^{\bar{h}_1} - Y^{\bar{h}_2} &= \bar{s}_*(Y) - v_1(\bar{s}_*(Y)) - \bar{s}_*(Y) + v_2(\bar{s}_*(Y)) \\
&= -\sum_{\delta=1}^{3}(\theta^1_\delta(s_*Y) - \theta^2_\delta(s_*Y))Z_\delta - (1/c)(\theta^1_0(s_{0*}Y) - \theta^2_0(s_{0*}Y))Z^c_0 \\
&= -\frac{1}{2}\sum_{\delta=1}^{3}(\theta^{1,U}_\delta(Y) - \theta^{2,U}_\delta(Y))Z_\delta - (1/c)(\theta^{1,U}_0(Y) - \theta^{2,U}_0(Y))Z^c_0 \\
&\overset{(3.2)}{=} -\varepsilon\sum_{\delta=1}^{3}\xi(I_\delta Y)Z_\delta + \frac{4\varepsilon(n+1)}{c}\xi(Y)Z^c_0, \qquad (3.3)
\end{aligned}$$

where $v_i : T\hat{M} \to \bar{\mathcal{V}}$ is the projection with respect to $\bar{\theta}^i$ ($i = 1, 2$). Finally we obtain

$$\begin{aligned}
&\hat{I}^{1,c}_\alpha(X) - \hat{I}^{2,c}_\alpha(X) \\
&= -\varepsilon\sum_{\delta=1}^{3}\xi(I_\delta I_\alpha Y)Z_\delta + \frac{4\varepsilon(n+1)}{c}\xi(I_\alpha Y)Z^c_0 \\
&\quad + \theta^1_\alpha(\triangle_S Y^{\bar{h}_2})Z^c_0 + \theta^1_\beta(\triangle_S Y^{\bar{h}_2})Z_\gamma - \theta^1_\gamma(\triangle_S Y^{\bar{h}_2})Z_\beta - (1/c)\theta^1_0(\triangle_{S_0} Y^{\bar{h}_2})Z_\alpha \\
&\overset{(*)}{=} -\varepsilon\sum_{\delta=1}^{3}\xi(I_\delta I_\alpha Y)Z_\delta + \frac{4\varepsilon(n+1)}{c}\xi(I_\alpha Y)Z^c_0 \\
&\quad + \varepsilon\xi(I_\alpha Y)Z^c_0 + \varepsilon\xi(I_\beta Y)Z_\gamma - \varepsilon\xi(I_\gamma Y)Z_\beta + \frac{4\varepsilon(n+1)}{c}\xi(Y)Z_\alpha \\
&= \left(\varepsilon\xi(Y) + \frac{4(n+1)\varepsilon}{c}\xi(Y)\right)Z_\alpha + \left(\varepsilon\xi(I_\alpha Y) + \frac{4(n+1)\varepsilon}{c}\xi(I_\alpha Y)\right)Z^c_0,
\end{aligned}$$

where in the step $(*)$ of the calculation we have computed

$$\theta^1_\alpha(\triangle_S Y^{\bar{h}_2}) = \theta^1_\alpha(Y^{h_2}) = \theta^1_\alpha(Y^{h_2} - Y^{h_1}) \overset{(3.3)}{=} \varepsilon\xi(I_\alpha Y)$$

and similarly for the other terms. □

The following proposition is an immediate consequence of Lemma 3.2, cf. the result with [17, Proposition 3.3].

Proposition 3.3. *The almost hypercomplex structure is independent of the choice of quaternionic connection if and only if $c = -4(n+1)$.*

Next we investigate transformation properties of the structures $\hat{I}^{\bar{\theta},c}_\alpha$ $(\alpha = 1,2,3)$ under the principal action.

Lemma 3.4. *We have $L_{Z_0}\hat{I}^{\bar{\theta},c}_\alpha = L_{Z_\alpha}\hat{I}^{\bar{\theta},c}_\alpha = 0$, $L_{Z_\alpha}\hat{I}^{\bar{\theta},c}_\beta = 2\varepsilon\hat{I}^{\bar{\theta},c}_\gamma$ and $L_{Z_\alpha}\hat{I}^{\bar{\theta},c}_\gamma = -2\varepsilon\hat{I}^{\bar{\theta},c}_\beta$.*

Proof. Note first that the principal action generated by the vector fields Z_a, $a = 0, \ldots, 3$, preserves the horizontal and vertical distributions. Moreover, the central vector field Z_0 commutes with the principal action and thus preserves the three canonical almost complex structures $\hat{I}^{\bar{\theta},c}_\alpha$.

Next we observe that it is easy to check the above equations on the vertical distribution by evaluating them on $Z^c_0, \ldots, Z_3$. So it only remains to check them on the horizontal distribution. Let $\{\phi_t\}_{t\in\mathbb{R}}$ be the flow of Z_1. Since

$$\phi_t((x,(\rho,s))) = (x,(\rho,(I_1,(\cos 2\varepsilon t)I_2 + (\sin 2\varepsilon t)I_3,(-\sin 2\varepsilon t)I_2 + (\cos 2\varepsilon t)I_3)))$$

for $(x,(\rho,s)) \in \hat{M}$, where $s = (I_1, I_2, I_3)$ and the horizontal lift of any vector field or tangent vector of M is invariant under ϕ_t, we have

$$\begin{aligned}
(L_{Z_1}\hat{I}^{\bar{\theta},c}_2)_{(x,(\rho,s))}(Y^h) &= [Z_1, \hat{I}^{\bar{\theta},c}_2 Y^h]_{(x,(\rho,s))} \\
&= \frac{d}{dt}\phi^{-1}_{t*}((\hat{I}^{\bar{\theta},c}_2 Y^h)_{\phi_t((x,(\rho,s)))})\Big|_{t=0} \\
&= \frac{d}{dt}\phi^{-1}_{t*}((\cos 2\varepsilon t)(I_2Y)^h_{\phi_t((x,(\rho,s)))} + (\sin 2\varepsilon t)(I_3Y)^h_{\phi_t((x,(\rho,s)))})\Big|_{t=0} \\
&= \frac{d}{dt}\left((\cos 2\varepsilon t)(I_2Y)^h_{(x,(\rho,s))} + (\sin 2\varepsilon t)(I_3Y)^h_{(x,(\rho,s))}\right)\Big|_{t=0} \\
&= 2\varepsilon(I_3Y)^h_{(x,(\rho,s))} = 2\varepsilon(\hat{I}^{\bar{\theta},c}_3)_{(x,(\rho,s))}Y^h
\end{aligned}$$

and similarly $L_{Z_1}\hat{I}^{\bar{\theta},c}_1 = 0$, which imply $L_{Z_1}\hat{I}^{\bar{\theta},c}_3 = -2\varepsilon\hat{I}^{\bar{\theta},c}_2$. □

The Nijenhuis tensor for $\hat{I}^{\bar{\theta},c}_\alpha$ is given by

$$N^\alpha(U,V) = [U,V] + \hat{I}^{\bar{\theta},c}_\alpha[\hat{I}^{\bar{\theta},c}_\alpha U, V] + \hat{I}^{\bar{\theta},c}_\alpha[U, \hat{I}^{\bar{\theta},c}_\alpha V] - [\hat{I}^{\bar{\theta},c}_\alpha U, \hat{I}^{\bar{\theta},c}_\alpha V]$$

for U, $V \in \Gamma(T\hat{M})$. Let $\bar{\Omega}$ (resp. Ω) be the curvature form of $\bar{\theta}$ (resp. θ). Take a local section $s : U \to S$ defined an open set U of M. The pull back of $\Omega = \sum_{\alpha=1}^{3} \Omega_\alpha e_\alpha$ by s is denoted by Ω^U. Since the curvature form is horizontal, we have

$$\varepsilon\Omega|_{s(U)} = \pi_S^* \Omega^U|_{s(U)}. \tag{3.4}$$

If we define the two-forms Ω_α^U by $\Omega^U = (1/2)\sum \Omega_\alpha^U e_\alpha$ and denote by $\bar{\nabla}$ the connection on Q induced by ∇, then we have

$$R_{X,Y}^{\bar{\nabla}} I_\alpha = [R_{X,Y}^{\nabla}, I_\alpha] = \left[\frac{1}{2}\sum \Omega_\delta^U(X,Y)I_\delta, I_\alpha\right] = \Omega_\gamma^U(X,Y)I_\beta - \Omega_\beta^U(X,Y)I_\gamma,$$

which implies

$$\Omega_\alpha^U(X,Y) = -\frac{1}{2n}\mathrm{Tr} I_\alpha R_{X,Y}^{\nabla} \tag{3.5}$$

for X, $Y \in TM$. In fact, multiplying the equation $R_{X,Y}^{\nabla} \circ I_\alpha - I_\alpha \circ R_{X,Y}^{\nabla} = \Omega_\gamma^U(X,Y)I_\beta - \Omega_\beta^U(X,Y)I_\gamma$ with I_β, we obtain

$$I_\beta \circ R_{X,Y}^{\nabla} \circ I_\alpha + I_\gamma \circ R_{X,Y}^{\nabla} = -\Omega_\gamma^U(X,Y)\mathrm{id} - \Omega_\beta^U(X,Y)I_\alpha.$$

Taking the trace proves (3.5). Let Ric^∇ be the Ricci curvature of ∇ and its symmetric (resp. anti-symmetric) part is denoted by $(Ric^\nabla)^s$ (resp. $(Ric^\nabla)^a$). The Nijenhuis tensors of the canonical almost complex structures on the bundle $\hat{M}$ over the quaternionic manifold (M, Q, ∇) are computed in the next lemma.

Lemma 3.5. *If $n > 1$ or Q is anti-self-dual provided $n = 1$, we have*

$$N^\alpha(Z_0^c, Z_i) = 0 \ \text{ for } \ 1 \le i \le 3, \tag{3.6}$$

$$N^\alpha(Z_i, Z_j) = 0 \ \text{ for } \ 1 \le i, j \le 3, \tag{3.7}$$

$$N^\alpha(Z_0^c, X^{\bar{h}}) = 0, \tag{3.8}$$

$$N^\alpha(Z_i, X^{\bar{h}}) = 0 \ \text{ for } \ 1 \le i \le 3, \tag{3.9}$$

$$\begin{aligned} &\bar{\theta}(N^\alpha(X^{\bar{h}}, Y^{\bar{h}})_{(x,(\rho,s))}) \\ &\quad = \frac{4\varepsilon(n+1)+\varepsilon c}{2(n+1)}\left((Ric^\nabla)^a(X,Y) - (Ric^\nabla)^a(I_\alpha X, I_\alpha Y)\right)e_0 \\ &\quad - \frac{4\varepsilon(n+1)+\varepsilon c}{2c(n+1)}\left((Ric^\nabla)^a(X, I_\alpha Y) + (Ric^\nabla)^a(I_\alpha X, Y)\right)e_\alpha \ \text{ and} \end{aligned} \tag{3.10}$$

$$\hat{\pi}_*(N^\alpha(X^{\bar{h}}, Y^{\bar{h}})) = 0 \ \text{ for } \ X, Y \in \Gamma(TM), \tag{3.11}$$

where $(x, (\rho, s)) \in \hat{M}$ $(s = (I_1, I_2, I_3))$.

Proof. We write $\hat{I}_\alpha = \hat{I}_\alpha^{\bar{\theta},c}$ for simplicity in the proof of this lemma. It is easy to see that (3.6–3.9) hold by the definition of the almost hypercomplex structure on $\hat{M}$ and Lemma 3.4. In fact, for example, we have $N^\alpha(Z_\beta, Z_\gamma) = [Z_\beta, Z_\gamma] + [Z_\gamma, Z_\beta] = 0$ and

$N^\alpha(Z_\beta, X^{\bar h}) = \hat I_\alpha[Z_\beta, \hat I_\alpha X^{\bar h}] - [\hat I_\alpha Z_\beta, \hat I_\alpha X^{\bar h}] = \hat I_\alpha(-2\varepsilon \hat I_\gamma)(X^{\bar h}) - 2\varepsilon \hat I_\beta(X^{\bar h}) = 0$. The other equations are proved similarly. Next we show (3.10). It holds $(\theta_i \circ \hat I_\alpha)(Z_0^c) = -\delta_{i\alpha}$, $(\theta_i \circ \hat I_\alpha)(Z_\alpha) = c\delta_{i0}$, $(\theta_i \circ \hat I_\alpha)(Z_\beta) = \delta_{i\gamma}$, $(\theta_i \circ \hat I_\alpha)(Z_\gamma) = -\delta_{i\beta}$. Using this and Lemma 3.1, we have

$$\begin{aligned}
&\bar\theta(\hat I_\alpha[\hat I_\alpha X^{\bar h}, Y^{\bar h}])\\
=&(\bar\theta \circ \hat I_\alpha)(\sum_{i=1}^3 \theta_i(\triangle_S[\hat I_\alpha X^{\bar h}, Y^{\bar h}])Z_i + \frac{1}{c}\theta_0(\triangle_{S_0}[\hat I_\alpha X^{\bar h}, Y^{\bar h}])Z_0^c)\\
=&\sum_{j=0}^3(\sum_{i=1}^3 \theta_i(\triangle_S[\hat I_\alpha X^{\bar h}, Y^{\bar h}])(\theta_j \circ \hat I_\alpha)(Z_i)e_j + \frac{1}{c}\theta_0(\triangle_{S_0}[\hat I_\alpha X^{\bar h}, Y^{\bar h}])(\theta_j \circ \hat I_\alpha)(Z_0^c)e_j)\\
=&\sum_{j=0}^3(\theta_\alpha(\triangle_S[\hat I_\alpha X^{\bar h}, Y^{\bar h}])c\delta_{j0}e_j + \theta_\beta(\triangle_S[\hat I_\alpha X^{\bar h}, Y^{\bar h}])\delta_{j\gamma}e_j\\
&-\theta_\gamma(\triangle_S[\hat I_\alpha X^{\bar h}, Y^{\bar h}])\delta_{j\beta}e_j + \frac{1}{c}\theta_0(\triangle_{S_0}[\hat I_\alpha X^{\bar h}, Y^{\bar h}])(-\delta_{j\alpha})e_j)
\end{aligned}$$

As a consequence of Lemma 3.1, we have $\triangle_{S_0}[\hat I_\alpha X^{\bar h}, Y^{\bar h}] = [(I_\alpha X)^{h_0}, Y^{h_0}]|_{\triangle_\#(\hat M)}$ and $\triangle_S[\hat I_\alpha X^{\bar h}, Y^{\bar h}] = [(I_\alpha X)^h, Y^h]|_{\triangle_\#(\hat M)}$. By $\bar\Omega = \Omega + d\theta_0 = \sum_{\delta=1}^3 \Omega_\delta e_\delta + (d\theta_0)e_0$, it holds

$$\begin{aligned}
\bar\theta(\hat I_\alpha[\hat I_\alpha X^{\bar h}, Y^{\bar h}]) =& -c\Omega_\alpha((I_\alpha X)^h, Y^h)e_0 - \Omega_\beta((I_\alpha X)^h, Y^h)e_\gamma\\
&+\Omega_\gamma((I_\alpha X)^h, Y^h)e_\beta + \frac{1}{c}d\theta_0((I_\alpha X)^{h_0}, Y^{h_0})e_\alpha.
\end{aligned}$$

Defining

$$\begin{aligned}
A_\alpha(X^{\bar h}, Y^{\bar h}) =& -\Omega_\beta(\triangle_S X^{\bar h}, \triangle_S Y^{\bar h}) + \Omega_\beta(\triangle_S \hat I_\alpha X^{\bar h}, \triangle_S \hat I_\alpha Y^{\bar h})\\
&+\Omega_\gamma(\triangle_S \hat I_\alpha X^{\bar h}, \triangle_S Y^{\bar h}) + \Omega_\gamma(\triangle_S X^{\bar h}, \triangle_S \hat I_\alpha Y^{\bar h})
\end{aligned}$$

for X, $Y \in TM$, we obtain

$$\begin{aligned}
&\bar\theta(N^\alpha(X^{\bar h}, Y^{\bar h}))\\
=&(-d\theta_0(X^{h_0}, Y^{h_0}) + d\theta_0((I_\alpha X)^{h_0}, (I_\alpha Y)^{h_0}) - c\Omega_\alpha(X^h, (I_\alpha Y)^h) - c\Omega_\alpha((I_\alpha X)^h, Y^h)e_0\\
&(-\Omega_\alpha(X^h, Y^h) + \Omega_\alpha((I_\alpha X)^h, (I_\alpha Y)^h) + \frac{1}{c}d\theta_0((I_\alpha X)^{h_0}, Y^{h_0}) + \frac{1}{c}d\theta_0(X^{h_0}, (I_\alpha Y)^{h_0}))e_\alpha\\
&+A_\alpha(X^{\bar h}, Y^{\bar h})e_\beta + A_\alpha((I_\alpha X)^{\bar h}, Y^{\bar h})e_\gamma.
\end{aligned}$$

Next we show that the coefficients of e_0 and e_α can be described by the Ricci tensor of ∇ and that the other components vanish thanks to the integrability of the almost complex structure on the twistor space of M [18]. Set

$$B := \frac{1}{4(n+1)}(Ric^\nabla)^a + \frac{1}{4n}(Ric^\nabla)^s - \frac{1}{2n(n+2)}\Pi_h(Ric^\nabla)^s, \tag{3.12}$$

where $\Pi_h(Ric^\nabla)^s$ is the Q-hermitian $(0,2)$-tensor defined by

$$(\Pi_h(Ric^\nabla)^s)(X,Y) = \frac{1}{4}\left((Ric^\nabla)^s(X,Y) + \sum_{i=1}^3(Ric^\nabla)^s(I_iX, I_iY)\right)$$

for X, $Y \in TM$. By [5], we have

$$\Omega^U_\alpha(X,Y) = 2(B(X,I_\alpha Y) - B(Y,I_\alpha X)). \tag{3.13}$$

Then it holds

$$\Omega^U_\alpha(I_\alpha X,Y) + \Omega^U_\alpha(X,I_\alpha Y) = -\frac{1}{n+1}\left((Ric^\nabla)^a(X,Y) - (Ric^\nabla)^a(I_\alpha X,I_\alpha Y)\right).$$

Since $\varepsilon d\theta^U_0(X,Y) = \mathrm{Tr}R^\nabla_{X,Y} = -Ric^\nabla(X,Y) + Ric^\nabla(Y,X) = -2(Ric^\nabla)^a(X,Y)$ and $\Omega_\alpha(X^h,Y^h) = (1/2)\varepsilon\Omega^U_\alpha(X,Y)$ for all tangent vector X, Y on M, to prove (3.10), it is sufficient to check $A_\alpha = 0$. This is related to the integrability of the almost complex structure on the twistor space $\mathcal{Z}$ of the quaternionic manifold (M,Q) as we explain now. Recall that $\mathcal{Z} = \{A \in Q \mid A^2 = -\mathrm{id}\}$. We set

$$R^{\nabla(0,2)I}_{X,Y} := \frac{1}{4}(R^\nabla_{X,Y} + IR^\nabla_{IX,Y} + IR^\nabla_{X,IY} - R^\nabla_{IX,IY})$$

for X, $Y \in TM$ and $I \in \mathcal{Z}$. Then

$$[R^{\nabla(0,2)I}_{X,Y}, I] = 0 \tag{3.14}$$

for any $I \in \mathcal{Z}$ if $n > 1$. In the case of $\dim M = 4$, (3.14) holds if and only if Q is anti-self-dual. See [3] for example. By (3.5) and (3.14), we have $[R^{\nabla(0,2)I_\alpha}_{X,Y}, I_\alpha]I_\gamma = 0$ and thus

$$\begin{aligned} 0 =& 2\mathrm{Tr}[R^{\nabla(0,2)I_\alpha}_{X,Y}, I_\alpha]I_\gamma \\ =& \mathrm{Tr}(-I_\beta R^\nabla_{X,Y} + I_\beta R^\nabla_{I_\alpha X,I_\alpha Y} + I_\gamma R^\nabla_{I_\alpha X,Y} + I_\gamma R^\nabla_{X,I_\alpha Y}) \\ =& 2n(\Omega^U_\beta(X,Y) - \Omega^U_\beta(I_\alpha X,I_\alpha Y) - \Omega^U_\gamma(I_\alpha X,Y) - \Omega^U_\gamma(X,I_\alpha Y)) \\ =& -4n\varepsilon A_\alpha(X^{\bar h},Y^{\bar h}) \end{aligned}$$

for all X, $Y \in TM$. This proves that $A_\alpha = 0$.

Since ∇ is torsion-free, we have (3.11) by the similar calculation for the Nijenhuis tensor of the almost complex structure on the twistor space.

□

From Lemma 3.5 (and Proposition 3.3) we obtain the following result.

Theorem 3.6. *Let (M,Q) be a quaternionic manifold and ∇ a quaternionic connection. Let $(\hat{I}^{\bar\theta,c}_1, \hat{I}^{\bar\theta,c}_2, \hat{I}^{\bar\theta,c}_3)$ be the almost hypercomplex structure on $\hat{M}$. We assume that Q is anti-self-dual when $n = 1$. If $c = -4(n+1)$, then the almost hypercomplex structure is integrable (and independent of ∇). When $c \neq -4(n+1)$, the almost hypercomplex structure is integrable if and only if $(Ric^\nabla)^a$ is Q-hermitian, that is, it is hermitian with respect to I for all $I \in \mathcal{Z}$, where $\mathcal{Z}$ is the twistor space of (M,Q).*

We call $\hat{M}$ the *Swann bundle* of M, although the terminology "Swann bundle" is also used for the quotient space $\hat{M}/\mathbb{Z}$ with $c = -4(n+1)$ in [16]. From now on we will only consider the case that $(\hat{I}^{\bar\theta,c}_1, \hat{I}^{\bar\theta,c}_2, \hat{I}^{\bar\theta,c}_3)$ is a hypercomplex structure, i.e. integrable. We note that, for each fixed quaternionic connection, $(\hat{I}^{\bar\theta,c}_1, \hat{I}^{\bar\theta,c}_2, \hat{I}^{\bar\theta,c}_3) \neq (\hat{I}^{\bar\theta,c'}_1, \hat{I}^{\bar\theta,c'}_2, \hat{I}^{\bar\theta,c'}_3)$ if $c \neq c'$. Although it is obvious from the definition, we can also see it by considering the Obata connection. From Lemma 3.4, it follows that $\hat\nabla^c_{\widetilde{e}_0}\widetilde{e}_0 = (1/c)\hat\nabla^1_{\widetilde{e}_0}\widetilde{e}_0$, where $\hat\nabla^c$ is the Obata connection for the hypercomplex structure $(\hat{I}^{\bar\theta,c}_1, \hat{I}^{\bar\theta,c}_2, \hat{I}^{\bar\theta,c}_3)$.

4 An infinitesimal quaternionic vector field and its natural lift

A vector field X on (M, Q) is called *quaternionic* if its (local) flow φ_t satisfies

$$\varphi_{-t}^* I := \varphi_{t*} \circ I \circ \varphi_{t*}^{-1} \in Q$$

for all $I \in Q$ and for all t. For a connection ∇ and $X \in \Gamma(TM)$, we define

$$(L_X \nabla)_Y Z := L_X(\nabla_Y Z) - \nabla_{L_X Y} Z - \nabla_Y (L_X Z), \tag{4.1}$$

where Y, $Z \in \Gamma(TM)$. Note that $L_X \nabla$ is a tensor. In this paper, we study (M, Q) with a quaternionic vector field X which is also affine, that is $L_X \nabla = 0$. So we start by studying the condition $L_X \nabla = 0$. We define the Hessian H^∇ with respect to ∇ by

$$H^\nabla_{Y,Z} X = \nabla_Y \nabla_Z X - \nabla_{\nabla_Y Z} X$$

for X, Y, $Z \in \Gamma(TM)$. By similar arguments as in [4], we have the following.

Lemma 4.1. *Let ∇ be a quaternionic connection of (M, Q) and X a quaternionic vector field. Then the following conditions are equivalent each other.*
(1) $L_X \nabla = 0$,
(2) $R^\nabla_{X,Y} Z = -H^\nabla_{Y,Z} X$ *for all* Y, $Z \in TM$,
(3) $Ric^\nabla(X, Z) = \mathrm{Tr} H^\nabla_{(\,\cdot\,),Z} X$ *for all* $Z \in TM$.

Proof. Since X is a quaternionic vector field, $\varphi_{-t}^* \nabla$ is a quaternionic connection with respect to Q, where $\varphi_{-t}^* \nabla$ is the connection defined by

$$(\varphi_{-t}^* \nabla)_Y Z = \varphi_{t*} (\nabla_{\varphi_{t*}^{-1} Y} \varphi_{t*}^{-1} Z)$$

for Y, $Z \in \Gamma(TM)$. Therefore there exists a one form ξ_t such that $\varphi_t^* \nabla - \nabla = S^{\xi_t}$ by Lemma 2.1. Then we have

$$L_X \nabla = \frac{d}{dt} \varphi_t^* \nabla \bigg|_{t=0} = \frac{d}{dt} S^{\xi_t} \bigg|_{t=0} = S^{\xi_X}, \tag{4.2}$$

where $\xi_X = (d/dt)\xi_t|_{t=0}$. On the other hand, by a straightforward calculation, we have $(L_X \nabla)_Y Z = R^\nabla_{X,Y} Z + H^\nabla_{Y,Z} X$ for all Y, $Z \in TM$. Therefore, we have

$$(L_X \nabla)_Y Z = R^\nabla_{X,Y} Z + H^\nabla_{Y,Z} X = S^{\xi_X}_Z Y.$$

It follows that (1) $\Rightarrow$ (2). It is also easy to see that (2) $\Rightarrow$ (3) by taking a trace. Since $\mathrm{Tr} S^{\xi_X}_Z = 4(n+1)\xi_X(Z)$, we have

$$-Ric^\nabla(X, Z) + \mathrm{Tr} H^\nabla_{(\,\cdot\,),Z} X = \mathrm{Tr} S^{\xi_X}_Z = 4(n+1)\xi_X(Z).$$

If (3) holds, then we have (1). $\square$

We consider the normalizer and

$$N(Q) := \{A \in \mathrm{End}(TM) \mid [A, I] \in Q \text{ for all } I \in Q\}$$

and the centralizer

$$Z(Q) := \{A \in \mathrm{End}(TM) \mid [A, I] = 0 \text{ for all } I \in Q\}.$$

Then we see $N(Q) = Q + Z(Q) = Q + \mathbb{R}\cdot\mathrm{id} + Z_0(Q)$, where $Z_0(Q)$ is the subspace of $Z(Q)$ of trace-free tensors [5]. Let ∇ be a quaternionic connection and X a quaternionic vector field. Since $L_X I_\alpha = \nabla_X I_\alpha + [I_\alpha, (\nabla X)]$, ∇X is an element of $N(Q)$. We write $\nabla X = T + T_0$, where $T \in \Gamma(Q + \mathbb{R}\cdot\mathrm{id})$ and $T_0 \in \Gamma(Z_0(Q))$. Note that, by [5], we have explicitly

$$\begin{aligned}\nabla X =& -\frac{1}{4n}\sum_{\alpha=1}^{3}(\mathrm{Tr}(\nabla X)I_\alpha)I_\alpha \ (\in \Gamma(Q)) \\ &+\frac{1}{4n}(\mathrm{Tr}\nabla X)\mathrm{id} \ (\in C^\infty(M)\mathrm{id}) \\ &+\frac{1}{4}((\nabla X) - \sum_{\alpha=1}^{3} I_\alpha(\nabla X)I_\alpha) - \frac{1}{4n}(\mathrm{Tr}\nabla X)\mathrm{id} \ (\in \Gamma(Z_0(Q))).\end{aligned}$$

So it holds

$$T = \frac{1}{4n}\sum_{\alpha=0}^{3}\varepsilon_\alpha(\mathrm{Tr}(\nabla X)I_\alpha)I_\alpha, \tag{4.3}$$

where $I_0 = \mathrm{id}$ and $\varepsilon_0 = 1$, $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1$.

Proposition 4.2. *Let ∇ be a quaternionic connection and X a quaternionic vector field. Then $L_X\nabla = 0$ if and only if $2(Ric^\nabla)^a(X, \cdot\,) = d(\mathrm{Tr}(\nabla X))$.*

Proof. By the Bianchi identity, it holds

$$\begin{aligned}R^\nabla_{X,Y}Z &= -R^\nabla_{Z,X}Y - R^\nabla_{Y,Z}X \\ &= -R^\nabla_{Z,X}Y - H^\nabla_{Y,Z}X + H^\nabla_{Z,Y}X \\ &= -R^\nabla_{Z,X}Y - H^\nabla_{Y,Z}X + (\nabla_Z T)(Y) + (\nabla_Z T_0)(Y)\end{aligned}$$

for all Y, $Z \in TM$. Then we have $-Ric^\nabla(X, Z) = -\mathrm{Tr}R^\nabla_{Z,X} - \mathrm{Tr}H^\nabla_{(\cdot),Z}X + \mathrm{Tr}(\nabla_Z T)$, since T_0 and $\nabla_Z T_0$ are trace-free. Therefore, by Lemma 4.1, we see that $L_X\nabla = 0$ if and only if $2(Ric^\nabla)^a(Z, X) + \mathrm{Tr}(\nabla_Z T) = 0$. Finally, because $T \in \Gamma(Q + \mathbb{R}\cdot\mathrm{id})$, we obtain $\mathrm{Tr}(\nabla_Z T) = Z\mathrm{Tr}(\nabla X)$ by (4.3). This implies the conclusion. □

Recall that every Killing vector field is affine with respect of the Levi-Civita connection. This means that every quaternionic Killing vector field X on a quaternionic Kähler manifold (M, g, Q) is an example of an affine quaternionic vector field. This can be seen also by Proposition 4.2.

Corollary 4.3. *Let X be an quaternionic vector filed on a quaternionic manifold (M, Q). If there exists a volume element ν on M such that $L_X \nu = 0$, then there exists a quaternionic connection ∇ such that $L_X \nabla = 0$.*

Proof. We can find a quaternionic connection ∇ such that $\nabla \nu = 0$ by [5, Theorem 2.4]. Then Ric^{∇} is symmetric. Because $L_X = \nabla_X - (\nabla X)$ and $L_X \nu = 0$, we have $\mathrm{Tr}(\nabla X) = 0$. Now the conclusion follows from Proposition 4.2. □

If X is a quaternionic vector field with the flow $\{\varphi_t\}$, then X can be lifted to $\hat{X}$ on $\hat{M}$ as follows. We define $\hat{\varphi}_t : \hat{M} \to \hat{M}$ by

$$\hat{\varphi}_t((x, (\rho, s))) = (\varphi_t(x), (\varphi^*_{-t}\rho, (\varphi^*_{-t}I_1, \varphi^*_{-t}I_2, \varphi^*_{-t}I_3)))$$

for $(x, (\rho, s)) \in \hat{M}$, where $s = (I_1, I_2, I_3)$ and define

$$\hat{X}_{(x,(\rho,s))} = \frac{d}{dt}\hat{\varphi}_t((x, (\rho, s)))\bigg|_{t=0}.$$

The vector field $\hat{X}$ on $\hat{M}$ is called the *natural lift* of X. Since $\hat{X}$ is invariant by the principal $\mathbb{R}^{>0} \times \mathrm{SO}(3)$-action, we have the following.

Lemma 4.4. *Let $\hat{X}$ be the natural lift of a quaternionic vector field X. We have $[\hat{X}, \widetilde{B}] = 0$ for $B \in \mathbb{R} \oplus \mathfrak{so}(3)$.*

Existence of $\nu \in \Gamma(S_0)$ such that $L_X \nu = 0$ (see Corollary 4.3) is related to the following condition for $\hat{X}$.

Lemma 4.5. *Let X be a quaternionic vector field X on (M, Q). The following conditions are equivalent :*
(1) *there exists $\nu \in \Gamma(S_0)$ such that $L_X \nu = 0$,*
(2) *there exists a trivialization $S_0 \cong M \times \mathbb{R}^{>0}$ such that $\hat{X}_p \in T_s S \subset T_p \hat{M} \cong T_s S \oplus \mathbb{R}$ for all $p = (x, (\rho, s)) \in \hat{M}$.*

Proof. At first, assume that (1) holds. Then ν gives a trivialization $S_0 \cong M \times \mathbb{R}^{>0}$ and $\hat{M} = S \times \mathbb{R}^{>0}$. We denote the component of $\hat{M}$ tangent to the second factor by $\hat{X}^{\mathbb{R}}$. For any point $(x, (\rho, s)) \in \hat{M}$, we see that $\hat{X}^{\mathbb{R}}_{(x,(\rho,s))} = 0 \iff \hat{X}^{\mathbb{R}}_{(x,(\nu(x),s))} = 0 \iff (L_X \nu)_x = 0$ by Lemma 4.4. Conversely, we can obtain the desired section $\nu \in \Gamma(S_0)$ by $\nu(x) = \Phi^{-1}(x, 1)$ for each $x \in M$, where $\Phi : S_0 \to M \times \mathbb{R}^{>0}$ is a trivialization satisfying (2). □

If there exists $\nu \in \Gamma(S_0)$ such that $L_X \nu = 0$, we may assume that $\hat{X}$ is a tangent vector field on S by Lemma 4.5. From now on we will assume that the quaternionic vector field X generates a free $\mathrm{U}(1)$-action. Since $\mathrm{U}(1)$ is compact, there exists a volume form ν invariant under the group action. This also implies that there exists a quaternionic connection ∇ such that $L_X \nabla = 0$ by Corollary 4.3.

5 The hypercomplex moment map

In this section, we consider a hypercomplex moment map on the Swann bundle. In [11], a hypercomplex moment map is defined as follows.

Definition 5.1 ([11]). Let M be a hypercomplex manifold with hypercomplex structure I_1, I_2, I_3 and F a compact Lie group acting smoothly and freely on M preserving I_i $(i = 1, 2, 3)$. F acts on $\mathfrak{F} = \mathrm{Lie}\, F$ by the adjoint action. A vector field on M induced by $f \in \mathfrak{F}$ is denoted by X_f. If a triple $\mu = (\mu_1, \mu_2, \mu_3)$ of F-equivariant maps $\mu_i : M \to \mathfrak{F}^*$ $(i = 1, 2, 3)$ satisfies

$$d\mu_1 \circ I_1 = d\mu_2 \circ I_2 = d\mu_3 \circ I_3 \tag{5.1}$$

and

$$(d\mu_1 \circ I_1)(X_f) \text{ does not vanish on } M \text{ for any non-zero } f \in \mathfrak{F}, \tag{5.2}$$

then μ is called the *hypercomplex moment map* of F. The equations (5.1) are called the CR (Cauchy-Riemann) equations and the condition (5.2) is called the transversality condition.

A hypercomplex moment map produces another hypercomplex manifold by a quotient (Proposition 3.1 in [11]). Let (M, Q) be a quaternionic manifold with a quaternionic connection ∇ and an affine quaternionic vector field X. The following lemmas hold.

Lemma 5.2. *If X is an affine quaternionic vector field on (M, Q, ∇) and $\bar{\theta}$ is the principal $\mathbb{R}^{>0} \times \mathrm{SO}(3)$-connection on $\hat{M}$ induced by ∇, then $L_{\hat{X}}\bar{\theta} = 0$ and $L_{\hat{X}}\hat{I}_\alpha^{\bar{\theta},c} = 0$.*

Proof. The first equation follows from the fact that $\hat{\varphi}_t$ preserves the horizontal distribution, because $\hat{\varphi}_t$ is induced by a local flow φ_t of affine transformations preserving the quaternionic structure. Since the almost hypercomplex structure $(\hat{I}_1^{\bar{\theta},c}, \hat{I}_2^{\bar{\theta},c}, \hat{I}_3^{\bar{\theta},c})$ is canonically associated with the data (Q, ∇) on M, it is also invariant under $\hat{\varphi}_t$, which implies the second equation. □

From now on we assume that there exists $\nu \in \Gamma(S_0)$ such that $L_X\nu = 0$. Then we can identify $S_0 = M \times \mathbb{R}^{>0}$, $\hat{M} = S \times \mathbb{R}^{>0}$ and $\hat{X}$ is a tangent vector field on S by Lemma 4.5. In the next lemma, we identify S with the $\hat{\varphi}_t$-invariant submanifold $S \times \{1\} \subset \hat{M} = S \times \mathbb{R}^{>0}$.

Lemma 5.3. *Under the above assumption, $L_{\hat{X}}\theta = 0$. Moreover $\bar{\mathcal{H}}|_S = \mathcal{H}$ if and only if $\nabla\nu = 0$.*

Proof. The projection from $\mathbb{R} \oplus \mathfrak{so}(3)$ onto $\mathbb{R}$ (resp. $\mathfrak{so}(3)$) is denoted by $pr_{\mathbb{R}}$ (resp. $pr_{\mathfrak{so}(3)}$). The first statement follows from the previous lemma, since $pr_{\mathfrak{so}(3)}\bar{\theta}|_S = \theta$. The second statement follows from $pr_{\mathbb{R}}\bar{\theta}|_S = (\nu \circ \pi_S)^*\theta_0$, since $\nabla\nu = (\nu^*\theta_0) \otimes \nu$. □

For $1 \in \mathbb{R} \cong T_1\mathbb{R}^{>0}$, at $\rho \in S_0$, we have

$$(\widetilde{e_0})_\rho = \widetilde{1}_\rho = \frac{d}{dr}\rho\exp(\varepsilon t)\bigg|_{t=0} = \varepsilon\rho = \varepsilon r\frac{\partial}{\partial r}\bigg|_\rho ,$$

where r is the standard coordinate on $\mathbb{R}^{>0}$. Let ∇ be a quaternionic connection on (M,Q). We define 1-forms $\hat{\theta}^c_\alpha$ on $\hat{M}$ $(\alpha = 1,2,3)$ by

$$\hat{\theta}^c_\alpha|_{TS} := Ar^{\frac{2}{c}}\theta_\alpha \text{ and } \hat{\theta}^c_\alpha(Z^c_0) = 0$$

where $A \in \mathbb{R}$ is a constant. A symmetric tensor $\langle\theta,\theta\rangle$ is defined by

$$\langle\theta,\theta\rangle(Y,Z) = \sum_{i=1}^{3}\theta_i(Y)\theta_i(Z)$$

for Y and $Z \in T\hat{M}$ and we set

$$G^c_\alpha := -Ar^{\frac{2}{c}}\Omega_\alpha(\,\cdot\,,\hat{I}^{\bar{\theta},c}_\alpha\,\cdot\,) + 2\varepsilon Ar^{\frac{2}{c}}\langle\theta,\theta\rangle + \frac{2\varepsilon A}{c^2}r^{\frac{2}{c}-2}(dr\otimes dr). \tag{5.3}$$

Note that $G^c_1|_{\mathcal{V}\times\mathcal{V}} = G^c_2|_{\mathcal{V}\times\mathcal{V}} = G^c_3|_{\mathcal{V}\times\mathcal{V}}$, that is, the vertical components of G^c_α are independent of α.

Lemma 5.4. *We have $d\hat{\theta}^c_\alpha(Y,Z) = G^c_\alpha(Y,\hat{I}^{\bar{\theta},c}_\alpha Z)$ for Y, $Z \in T\hat{M}$.*

Proof. Put $f(r) = Ar^{\frac{2}{c}}$. Then

$$G^c_\alpha(Y,Z) = -f(r)\Omega_\alpha(Y,\hat{I}^{\bar{\theta},c}_\alpha Z) + 2\varepsilon f(r)\langle\theta,\theta\rangle(Y,Z) + \frac{2\varepsilon}{c^2}\frac{f(r)}{r^2}(dr\otimes dr)(Y,Z)$$

for Y, $Z \in T\hat{M}$. Since $Z^c_0 = \varepsilon cr\frac{\partial}{\partial r}$, we obtain

$$d\hat{\theta}^c_\alpha(Z^c_0,Z_\alpha) = \varepsilon crf'(r) = \varepsilon cr\cdot\frac{2A}{c}r^{\frac{2}{c}-1} = 2\varepsilon f(r),$$

$$G^c_\alpha(Z^c_0,\hat{I}^{\bar{\theta},c}_\alpha(Z_\alpha)) = G^c_\alpha(Z^c_0,Z^c_0) = c^2r^2\cdot\frac{2\varepsilon}{c^2}\frac{f(r)}{r^2} = 2\varepsilon f(r)$$

and

$$G^c_\alpha(Z_\alpha,\hat{I}^{\bar{\theta},c}_\alpha(Z^c_0)) = -G^c_\alpha(Z_\alpha,Z_\alpha) = -2\varepsilon f(r).$$

Moreover we have

$$d\hat{\theta}^c_\alpha(Z_\beta,Z_\gamma) = -\hat{\theta}^c_\alpha([Z_\beta,Z_\gamma]) = -2\varepsilon f(r)$$

and

$$G^c_\alpha(Z_\beta,\hat{I}^{\hat{\theta},c}_\alpha(Z_\gamma)) = -G^c_\alpha(Z_\beta,Z_\beta) = -2\varepsilon f(r),$$

similarly $G^c_\alpha(Z_\gamma,\hat{I}^{\bar{\theta},c}_\alpha(Z_\beta)) = 2\varepsilon f(r)$. Finally, we see

$$d\hat{\theta}^c_\alpha(Y^h,Z^h) = f(r)(d\theta^c_\alpha)(Y^h,Z^h) = f(r)\Omega_\alpha(Y^h,Z^h)$$

and

$$G^c_\alpha(Y^h,\hat{I}^{\bar{\theta},c}_\alpha Z^h) = f(r)\Omega_\alpha(Y^h,Z^h).$$

For other combinations of tangent vectors on $\hat{M}$, both tensors $d\hat{\theta}_\alpha$, G_α vanish. □

We define $\mu^c : \hat{M} \to \mathbb{R}^3$ by

$$\mu^c(x) = \hat{\theta}^c(\hat{X}_x) = (\hat{\theta}^c_1(\hat{X}_x), \hat{\theta}^c_2(\hat{X}_x), \hat{\theta}^c_3(\hat{X}_x)) \tag{5.4}$$

for $x \in \hat{M}$. We calculate some formulae which will be used later to determine sufficient conditions for μ^c to be a hypercomplex moment map.

Lemma 5.5. *If X is an affine quaternionic vector field on (M, Q, ∇), then we have*

$$d\mu^c_\alpha = -\iota_{\hat{X}} d\hat{\theta}^c_\alpha.$$

Proof. By Lemmas 4.4 and 5.3, $L_{\hat{X}}\theta_\alpha = 0$ and $[\hat{X}, Z^c_0] = 0$. It follows $L_{\hat{X}}\hat{\theta}^c_\alpha = 0$ and $d\mu^c_\alpha = d\iota_{\hat{X}}\hat{\theta}^c_\alpha = L_{\hat{X}}\hat{\theta}^c_\alpha - \iota_{\hat{X}}d\hat{\theta}^c_\alpha = -\iota_{\hat{X}}d\hat{\theta}^c_\alpha$. □

For the CR-condition for μ^c, we have

Lemma 5.6. *If X is an affine quaternionic vector field on (M, Q, ∇), then we have*

$$(d\mu^c_\alpha \circ \hat{I}^{\bar{\theta},c}_\alpha)(Z^c_0) = 0, \tag{5.5}$$

$$(d\mu^c_1 \circ \hat{I}^{\bar{\theta},c}_1)(\widetilde{B}) = (d\mu^c_2 \circ \hat{I}^{\bar{\theta},c}_2)(\widetilde{B}) = (d\mu^c_3 \circ \hat{I}^{\bar{\theta},c}_3)(\widetilde{B}) \text{ for any } B \in \mathfrak{so}(3), \tag{5.6}$$

$$(d\mu^c_\alpha \circ \hat{I}^{\bar{\theta},c}_\alpha)(Y) = -Ar^{\frac{2}{c}}\Omega_\alpha(\hat{X}, \hat{I}^{\bar{\theta},c}_\alpha Y), \tag{5.7}$$

for all horizontal vector Y.

Proof. By Lemmas 5.4 and 5.5, we have $d\mu^c_\alpha \circ \hat{I}^{\bar{\theta},c}_\alpha = G^c_\alpha(\hat{X}, \cdot)$. Then it is easy to see $(d\mu^c_\alpha \circ \hat{I}^{\bar{\theta},c}_\alpha)(Z^c_0) = G^c_\alpha(\hat{X}, Z^c_0) = 0$. Since $G^c_1 = G^c_2 = G^c_3$ on $\bar{\mathcal{V}} \times T\hat{M}$, we obtain (5.6). Finally for horizontal vector Y we have $(d\mu^c_\alpha \circ \hat{I}^{\bar{\theta},c}_\alpha)(Y) = G^c_\alpha(\hat{X}, Y) = -Ar^{\frac{2}{c}}\Omega_\alpha(\hat{X}, \hat{I}^{\bar{\theta},c}_\alpha Y)$. □

For the transversality condition for μ^c, we state the next lemma, which follows from the equation $d\mu^c_\alpha \circ \hat{I}^{\bar{\theta},c}_\alpha = G^c_\alpha(\hat{X}, \cdot)$.

Lemma 5.7. *We have*

$$(d\mu^c_\alpha \circ \hat{I}^{\bar{\theta},c}_\alpha)(\hat{X}) = G^c_\alpha(\hat{X}, \hat{X}) = -Ar^{\frac{2}{c}}\Omega_\alpha(\hat{X}, \hat{I}^{\bar{\theta},c}_\alpha \hat{X}) + 2\varepsilon Ar^{\frac{2}{c}}\langle\theta, \theta\rangle(\hat{X}, \hat{X}).$$

By Lemma 5.6, μ^c satisfies CR equations (5.1) if and only if

$$\Omega_1(\hat{X}, \hat{I}^{\bar{\theta},c}_1 Y) = \Omega_2(\hat{X}, \hat{I}^{\bar{\theta},c}_2 Y) = \Omega_3(\hat{X}, \hat{I}^{\bar{\theta},c}_3 Y)$$

holds for all horizontal vector Y. On the other hand, from the equations (3.12) and (3.13), Ω_α satisfies

$$\begin{aligned} 2\varepsilon\Omega_\alpha(Y^h, \hat{I}^{\bar{\theta},c}_\alpha Z^h) &= \Omega^U_\alpha(Y, I_\alpha Z) \\ &= \frac{1}{2(n+1)}\big((Ric^\nabla)^a(I_\alpha Y, I_\alpha Z) - (Ric^\nabla)^a(Y, Z)\big) \\ &\quad -\frac{1}{2n}\big((Ric^\nabla)^s(I_\alpha Y, I_\alpha Z) + (Ric^\nabla)^s(Y, Z)\big) \\ &\quad +\frac{2}{n(n+2)}(\Pi_h(Ric^\nabla)^s)(Y, Z) \end{aligned} \tag{5.8}$$

for tangent vectors Y and Z on M. In particular, if Ric^{∇} is Q-hermitian, then $\Omega_\alpha(\,\cdot\,, \hat{I}^{\bar{\theta},c}_\alpha\,\cdot\,)$ does not depend on α. We see that μ^c satisfies the CR equations (5.1) if $Ric^{\nabla}(X,Y) = Ric^{\nabla}(IX, IY)$ for all $Y \in TM$ and $I \in \mathcal{Z}$. Moreover if the vector field X on M is affine quaternionic and μ^c satisfies the CR equations, then

$$\Omega_\alpha(\hat{X}, \hat{I}^{\bar{\theta},c}_\alpha \hat{X}) (= \Omega_\beta(\hat{X}, \hat{I}^{\bar{\theta},c}_\beta \hat{X}) = \Omega_\gamma(\hat{X}, \hat{I}^{\bar{\theta},c}_\gamma \hat{X})) = -\frac{\varepsilon}{2(n+2)}(Ric^{\nabla})(X,X)\circ\hat{\pi}.$$

The following statements can be obtained for the CR equations and the transversality condition.

Proposition 5.8. *Let M be a quaternionic manifold with a quaternionic connection ∇ and X an affine quaternionic vector field on (M, Q, ∇). Assume that there exists $\nu \in \Gamma(S_0)$ such that $L_X\nu = 0$. If*

$$Ric^{\nabla}(X,Y) = Ric^{\nabla}(IX, IY)$$

for all $Y \in TM$, $I \in \mathcal{Z}$ and

$$(Ric^{\nabla})(X,X)\circ\hat{\pi} + 4(n+2)\langle\theta,\theta\rangle(\hat{X},\hat{X})$$

does not vanish on $\hat{M}$, then the map $\mu^c = Ar^{\frac{2}{c}}\theta_\alpha : \hat{M} \to \mathbb{R}^3$ $(A \neq 0)$ satisfies the CR equations and the transversality condition for any $c \neq 0$.

Also we have

Corollary 5.9. *Let M be a quaternionic manifold with a quaternionic connection ∇ and X an affine quaternionic vector field on (M, Q, ∇). Assume that there exists $\nu \in \Gamma(S_0)$ such that $L_X\nu = 0$. If Ric^{∇} is Q-hermitian and*

$$(Ric^{\nabla})(X,X)\circ\hat{\pi} + 4(n+2)\langle\theta,\theta\rangle(\hat{X},\hat{X})$$

does not vanish on $\hat{M}$, then the map $\mu^c = Ar^{\frac{2}{c}}\theta_\alpha : \hat{M} \to \mathbb{R}^3$ $(A \neq 0)$ satisfies the CR equations and the transversality condition for any $c \neq 0$.

6 The proof of the main result

In this section, we give the proof of our main result. Using the hypercomplex quotient in [11], we can obtain a hypercomplex manifold M' with certain properties. To show it, the following lemmas are needed.

Lemma 6.1. *We have $L_{\widetilde{B}}\theta = -\varepsilon[B,\theta]$. Moreover $L_{Z_\alpha}\theta_\alpha = 0$, $L_{Z_\alpha}\theta_\beta = 2\varepsilon\theta_\gamma$ and $L_{Z_\alpha}\theta_\gamma = -2\varepsilon\theta_\beta$.*

Proof. We see that $(L_{\widetilde{B}}\theta)(\widetilde{C}) = -\theta([\widetilde{B},\widetilde{C}]) = -\varepsilon[B,C] = -\varepsilon[B,\theta(\widetilde{C})]$ and $(L_{\widetilde{B}}\theta)(Y^h) = -\theta([\widetilde{B}, Y^h]) = 0 (= -[B, \theta(Y^h)])$. For the latter statements, we compute

$$\sum(L_{Z_\alpha}\theta_i)e_i = L_{Z_\alpha}\theta = -\varepsilon[e_\alpha, \sum\theta_i e_i] = -\varepsilon[e_\alpha,\theta_\beta e_\beta] - \varepsilon[e_\alpha,\theta_\gamma e_\gamma] = -2\varepsilon\theta_\beta e_\gamma + 2\varepsilon\theta_\gamma e_\beta.$$

□

By Lemma 6.1, we have

Lemma 6.2. *It holds that* $L_{Z_\alpha}\hat{\theta}^c_\alpha = 0$, $L_{Z_\alpha}\hat{\theta}^c_\beta = 2\varepsilon\hat{\theta}^c_\gamma$ *and* $L_{Z_\alpha}\hat{\theta}^c_\gamma = -2\varepsilon\hat{\theta}^c_\beta$.

From Lemma 5.3, it holds

Lemma 6.3. *If* $L_X\nabla = 0$*, then* $L_{\hat{X}}d\hat{\theta}^c_\alpha = 0$ *for* $\alpha = 1, 2, 3$.

We have the main theorems in this paper.

Theorem 6.4. *Let* (M, Q) *be a quaternionic manifold. We assume that* Q *is anti-self-dual when* $n = 1$*. Moreover assume that* $\mathrm{U}(1)$ *acts freely on* M *preserving* Q*. We denote by* X *the vector field generating the* $\mathrm{U}(1)$*-action. If* Q *admits a quaternionic connection* ∇ *such that* $L_X\nabla = 0$*,*

$$Ric^\nabla(X, Y) = Ric^\nabla(IX, IY) \tag{6.1}$$

for all $Y \in TM$*,* $I \in \mathcal{Z}$ *and*

$$(Ric^\nabla)(X, X) \circ \hat{\pi} + 4(n + 2)\langle\theta, \theta\rangle(\hat{X}, \hat{X}) \tag{6.2}$$

does not vanish on $\hat{M}$*, then the natural lift* $\hat{X}$ *generates a free* $\mathrm{U}(1)$*-action with the moment map* μ^c *defined by (5.4), where* $c = -4(n+1)$*. Then the corresponding hypercomplex quotient is a hypercomplex manifold* $(M', H = (I'_1, I'_2, I'_3))$ *with an* I'_1*-holomorphic vector field* Z *such that* $L_Z I'_2 = 2\varepsilon I'_3$*,* $L_Z I'_3 = -2\varepsilon I'_2$*. Moreover the exact* 2*-forms* $d\hat{\theta}^c_\alpha$ *on* $\hat{M}$ *induce closed* 2*-forms* Θ'_α *on* M' *which satisfy* $L_Z\Theta'_1 = 0$*,* $L_Z\Theta'_2 = 2\varepsilon\Theta'_3$*,* $L_Z\Theta'_3 = -2\varepsilon\Theta'_2$*.*

Proof. Choose $c = -4(n + 1)$. Then $\hat{M}$ is a hypercomplex manifold by Theorem 3.6. We can choose a $\mathrm{U}(1)$-invariant volume form ν on M. Then the condition (2) in Lemma 4.5 holds, so $\hat{X}$ is tangent to S, which means that the results in the previous section can be applied. Since Proposition 5.8 and the second statement of Lemma 5.2 hold, $M' = P/\mathrm{U}(1)$ is a hypercomplex manifold with the induced hypercomplex structure I'_1, I'_2, I'_3 by [11, Proposition 3.1], where P is the level set $(\mu^c)^{-1}((1, 0, 0))$. Based on the proof of [11, Proposition 3.1], take $V = \{v \in TP \mid (d\mu^c_\alpha \circ \hat{I}^{\hat{\theta},c}_\alpha)(v) = 0\}$. Then we see that $TP = V \oplus \langle\hat{X}\rangle$. In particular, $\pi_P^* TM' \cong V$, where $\pi_P : P \to M'$ is the quotient map. The vector field $\hat{I}^{\hat{\theta},c}_1 Z^c_0 = Z_1$ is tangent to P, since

$$Z_1\mu^c_\alpha|_P = (2\varepsilon\delta_{2\alpha}\mu^c_3 - 2\varepsilon\delta_{3\alpha}\mu^c_2)|_P$$

by Lemma 6.2. By Lemma 4.4, Z_1 is a projectable vector field, that is, $Z := \pi_{P*}(Z_1)$ is a vector field on M'. The vector field Z satisfies $(L_Z I'_\alpha)(U) = \pi_{P*}((L_{Z_1}\hat{I}^{\hat{\theta},c}_\alpha)(U_P))$, where $U_P \in \Gamma(V)$ is any projectable vector field and $U = \pi_{P*}(U_P)$ is its projection. In fact, this can be obtained from $I'_\alpha \circ \pi_{P*} \circ pr_V = \pi_{P*} \circ pr_V \circ \hat{I}^{\hat{\theta},c}_\alpha$, where $pr_V : T\hat{M}|_P \to V$ is the projection with respect to the $\hat{I}^{\hat{\theta},c}_\alpha$-invariant decomposition

$$(T\hat{M})|_P = V \oplus \langle\hat{X}, \hat{I}^{\hat{\theta},c}_1\hat{X}, \hat{I}^{\hat{\theta},c}_2\hat{X}, \hat{I}^{\hat{\theta},c}_3\hat{X}\rangle.$$

Therefore, by Lemmas 3.4 and 4.4, we see that Z is a I'_1-holomorphic vector field such that $L_Z I'_2 = 2\varepsilon I'_3$ and $L_Z I'_3 = -2\varepsilon I'_2$. Finally, by Lemma 6.3, we can define 2-forms Θ'_1, Θ'_2, Θ'_3 on M' by $\Theta'_\alpha(U, W) = (d\hat{\theta}^c_\alpha)(U_P, W_P)$ for $U = \pi_{P*}(U_P)$ and $W = \pi_{P*}(W_P)$. It is clear that these forms are closed. Finally we see that these forms satisfy the desired conditions by Lemma 6.2. □

Remark 6.5. In Theorem 6.4, the same conclusion can be obtained under the assumption that the action induced by $\hat{X}$ is free instead of the assumption that the action induced by X is free.

In the case that Ric^{∇} is Q-hermitian, we have the following theorem.

Theorem 6.6. *Let (M, Q) be a quaternionic manifold. We assume that Q is anti-self-dual when $n = 1$. Moreover assume that* U(1) *acts freely on M preserving Q. We denote by X the vector field generating the* U(1)*-action. If Q admits a quaternionic connection ∇ such that $L_X \nabla = 0$, Ric^{∇} is Q-hermitian and*

$$(Ric^{\nabla})(X, X) \circ \hat{\pi} + 4(n+2)\langle\theta, \theta\rangle(\hat{X}, \hat{X}) \tag{6.3}$$

does not vanish on $\hat{M}$, then there exists a 1*-parameter family $\{(M'^c, H^c = (I_1'^c, I_2'^c, I_3'^c))\}_{c\neq 0}$ of hypercomplex manifolds with an $I_1'^c$-holomorphic vector field Z^c on M'^c such that $L_{Z^c} I_2'^c = 2\varepsilon I_3'^c$, $L_{Z^c} I_3'^c = -2\varepsilon I_2'^c$. Moreover the exact* 2*-forms $d\hat{\theta}^c_\alpha$ on $\hat{M}$ give a* 1*-parameter family $\{(\Theta_1'^c, \Theta_2'^c, \Theta_3'^c)\}_{c\neq 0}$ of triplets of closed* 2*-forms on M'^c such that $L_{Z^c}\Theta_1'^c = 0$, $L_{Z^c}\Theta_2'^c = 2\varepsilon\Theta_3'^c$, $L_{Z^c}\Theta_3'^c = -2\varepsilon\Theta_2'^c$ and*

$$\Theta_1'^c(\,\cdot\,, I_1'^c\,\cdot\,) = \Theta_2'^c(\,\cdot\,, I_2'^c\,\cdot\,) = \Theta_3'^c(\,\cdot\,, I_3'^c\,\cdot\,). \tag{6.4}$$

Proof. Since Ric^{∇} is Q-hermitian, the almost hypercomplex structures $(\hat{I}_1^{\hat{\theta},c}, \hat{I}_2^{\hat{\theta},c}, \hat{I}_3^{\hat{\theta},c})$ on $\hat{M}$ are integrable for all $c \neq 0$ by Theorem 3.6. Following the same procedure as in the proof of Theorem 6.4, we obtain the claims with exception of the equation (6.4). The latter equation follows from Q-hermitian assumption for Ric^{∇} using Lemma 5.4 and (5.8). □

The assumption (6.2) is formulated in terms of objects on the Swann bundle $\hat{M}$. We have the following corollary under assumptions formulated directly on M.

Corollary 6.7. *Let (M, Q) be a quaternionic manifold. We assume that Q is anti-self-dual when $n = 1$. Moreover assume that* U(1) *acts freely on M preserving Q. We denote by X the vector field generating the* U(1)*-action. If Q admits a quaternionic connection ∇ such that $L_X\nabla = 0$, $(Ric^{\nabla})^s(X, X) > 0$ and (6.1) is satisfied (resp. Ric^{∇} is Q-hermitian), then we have the same conclusion as Theorem 6.4 (resp. Theorem 6.6).*

We call the correspondence from (M, Q, X) to (M', H, Z) or to $\{(M'^c, H^c, Z^c)\}_{c\neq 0}$ described in Theorems 6.4 and 6.6 the *Quaternionic/Hypercomplex-correspondence* (*Q/H-correspondence* for short).

A relation with Swann's twist construction. Now we explain how M' considered just as a smooth manifold can be related to M by Swann's twist construction. Consider the Lie subgroup $\mathrm{U}(1)_{Z_1} := \{g \in \mathrm{SO}(3) \mid Ad_g e_1 = e_1\}$ of SO(3), which can be identified with U(1). Notice this group is different from the group $\langle\hat{X}\rangle \cong \mathrm{U}(1)$ generated by $\hat{X}$. Then $P = (\mu^c)^{-1}((1,0,0)) = (\mu^c)^{-1}(e_1)$ is a principal $\mathrm{U}(1)_{Z_1}$-bundle over $\hat{\pi}(P)$ with a connection $\iota_P^*(\theta_1)$, where $\iota_P : P \to \hat{M}$ is the inclusion map from P. In fact, the calculation

$$\mu^c(pg) = \hat{\theta}(\hat{X}_{pg}) = \hat{\theta}(R_{g*}\hat{X}_p) = Ad_{g^{-1}}\hat{\theta}(\hat{X}_p) = Ad_{g^{-1}}e_1 = e_1$$

for $p \in P$ and $g \in \mathrm{U}(1)_{Z_1}$ shows that P is invariant under $\mathrm{U}(1)_{Z_1}$. In particular, $P \cap \hat{\pi}^{-1}(x)$ is a union of circles ($\mathrm{U}(1)_{Z_1}$-orbits). Since the functions $\theta_\alpha(\hat{X})|_{\hat{\pi}^{-1}(x)}$ on $\hat{\pi}^{-1}(x) \cong \mathbb{H}^*/\{\pm 1\}$ are linear in the natural coordinates on $\mathbb{H} \cong \mathbb{R}^4$ and $\hat{\theta}_\alpha = Ar^{\frac{2}{c}}\theta_\alpha(\hat{X})$, we see that the above intersection $P \cap \hat{\pi}^{-1}(x)$ is a single circle. Recall [20] that Swann's twist construction produces a new manifold M' from a manifold M with the following twist data: a vector field ξ, a two form F and a function a on M. More precisely, ξ generates a U(1)-action, F is an invariant closed 2-form which is the curvature form of a connection form on a principal U(1)-bundle, and a is non-vanishing and satisfies $da = -\iota_\xi F$. It was shown in [13] that the HK/QK-correspondence can be described using the twist construction and a so-called elementary deformation of the metric.

In the setting of the Q/H-correspondence, let $s : U \to P$, $U \subset \hat{\pi}(P)$ be a local section. Then we define a two form F and a function a on $\hat{\pi}(P)$ by

$$F := s^*(d(\iota_P^*\theta_1)) = s^*(d\theta_1) = s^*\Omega_1,$$
$$a := s^*(\theta_1(\hat{X}) \circ \iota_P)) = \theta_1(\hat{X}) \circ s, = s^*(\theta_1(\hat{X})).$$

Note that both F and a are independent of the choice of s. Then we have

Proposition 6.8. *As a smooth manifold, M' obtained by the Q/H-correspondence is a twist of $\hat{\pi}(P)$ in the sense of* [20] *with the twist data $(\xi = X, F, a)$ as above.*

Proof. Since $L_X \nabla = 0$, we have

$$\begin{aligned} da = s^*(d\iota_{\hat{X}}\theta_1) &= -s^*(d\theta_1(\hat{X},\ \cdot\ )) = -\Omega_1(\hat{X}, s_*(\ \cdot\ )) \\ &= -\Omega_1(X^h, s_*(\ \cdot\ )) = -\Omega_1(s_*(X), s_*(\ \cdot\ )) = -F(X,\ \cdot\ ). \end{aligned}$$

Also we obtain $L_X F = (\iota_X d + d\iota_X)F = -dda = 0$. ☐

Note that the complex structures I'_α are not $\mathcal{H}$-related to I_α in the sense of [20], because the invariant subbundle $V \subset T\hat{M}$ does not coincides with $\mathcal{H}$ in general.

7 Examples

In this section, we give examples.

QK/HK-correspondence: When M is a possibly indefinite quaternionic Kähler manifold with non zero scalar curvature, we can take the Levi-Civita connection ∇ as a quaternionic connection and if there exists a non-zero quaternionic Killing vector field X on M, then we can take X as the affine quaternionic vector field in the Q/H-correspondence. The tensor field (5.3) gives a (pseudo-)hyper-Kähler metric on $\hat{M}$ and (6.4) gives a (pseudo-)hyper-Kähler metric on M' if $\hat{X}$ is time-like or space-like (see [1]). Therefore our Q/H-correspondence is a generalization of the QK/HK-correspondence. The following example is well-known (see [10, 8, 19] for example).

Example 7.1 (The cotangent bundle $T^*\mathbb{C}P^n$ as a hyper-Kähler manifold)**.** Consider the quaternionic (right-)projective space $M = \mathbb{H}P^n$ with the standard quaternionic structure. We can choose the Levi-Civita connection ∇ of the standard quaternionic Kähler metric on M as a quaternionic connection. Then we see the Swann bundle $\hat{M} = (\mathbb{H}^{n+1}\backslash\{0\})/\{\pm 1\} \to M$ as a hypercomplex manifold with the hypercomplex structure $(\hat{I}_1^{\bar{\theta},c}, \hat{I}_2^{\bar{\theta},c}, \hat{I}_3^{\bar{\theta},c})$, where $\bar{\theta}$ is the principal connection associated with the Levi-Civita connection. Let X be the vector field on M which generates the U(1)-action on M by quaternionic affine transformations defined by $e^{i\theta} \cdot [z_0, \dots, z_n] := [e^{i\theta} z_0, \dots, e^{i\theta} z_n]$ for $e^{i\theta} \in \mathrm{U}(1)$ and $[z_0, \dots, z_n] \in M$. It holds that the Ricci tensor Ric^{∇} is Q-hermitian and $Ric^{\nabla}(X, X) > 0$. Then we can apply Corollary 6.7. Note that the action induces the well-known hyper-Kähler moment map on $\hat{M}$ when $c = 1$. The hyper-Kähler metric $G^1_\alpha = G^1_\beta = G^1_\gamma$ is given by a constant multiple of the standard Euclidean flat metric on $\mathbb{H}^{n+1}\backslash\{0\}$. Applying the QK/HK-correspondence (which amounts to taking the hyper-Kähler quotient of $\hat{M}$ with respect to the vector field $\hat{X}$) to this example yields Calabi's hyper-Kähler structure on $T^*\mathbb{C}P^n$.

Hypercomplex manifold with the Obata connection: Let $(M, (I_1, I_2, I_3))$ be a hypercomplex manifold and ∇ its Obata connection on M. We recall the Obata connection is a canonical torsion-free connection preserving the hypercomplex structure [14]. In particular, it is a quaternionic connection with respect to the quaternionic structure $Q = \langle I_1, I_2, I_3 \rangle$. Assume that a vector field X with the flow $\{\varphi_t\}_{t\in\mathbb{R}}$ on M is given, which generates a free action of $\mathrm{U}(1) = \mathbb{R}/2\pi\mathbb{Z}$ on M such that

$$L_X I_1 = 0, \quad L_X I_2 = 2\varepsilon I_3 \quad \text{and} \quad L_X I_3 = -2\varepsilon I_2. \tag{7.1}$$

Then it holds

$$\varphi^*_{-t} I_1 = I_1,\ \varphi^*_{-t} I_2 = (\cos(2\varepsilon t)) I_2 + (\sin(2\varepsilon t)) I_3,\ \varphi^*_{-t} I_3 = (-\sin(2\varepsilon t)) I_2 + (\cos(2\varepsilon t)) I_3.$$

This shows that X is a quaternionic vector field for the quaternionic structure $Q = \langle I_1, I_2, I_3 \rangle$. Since $(\varphi^*_{-t}\nabla)$ is the Obata connection for the hypercomplex structure $(\varphi^*_{-t} I_1, \varphi^*_{-t} I_2, \varphi^*_{-t} I_3)$, $(\varphi^*_{-t}\nabla)$ is again a quaternionic connection for Q. By the explicit expression of the Obata connection in [5], we have

$$\frac{d}{dt}(\varphi^*_{-t}\nabla) = 0,$$

and hence $\varphi^*_{-t}\nabla = \nabla$ for all t. It follows that $L_X \nabla = 0$. Because the Ricci curvature of the Obata connection is skew symmetric and Q-hermitian by Corollary 1.6 in [5], we can apply the Q/H-correspondence to (M, Q, ∇) obtaining a hypercomplex manifold M'. The manifolds M and M' are related as follows.

Proposition 7.2. *M is a double covering space of M'.*

Proof. The hypercomplex structure is a global section $s : M \to S$

$$x \mapsto s(x) = (I_1(x), I_2(x), I_3(x)),$$

and defines a global trivialization of the principal SO(3)-bundle S. Take a U(1)-invariant volume form. Since

$$\begin{aligned}\hat{M} &= \{(x, s(x)g, r) \mid x \in M,\ g \in \mathrm{SO}(3),\ r > 0\} \\ &\cong M \times \mathrm{SO}(3) \times \mathbb{R}^{>0} \supset M \times \mathrm{SO}(3) \times \{1\} \cong M \times \mathrm{SO}(3) = S\end{aligned}$$

and $(I_1, \varphi_{-t}^* I_2, \varphi_{-t}^* I_3) = (I_1, I_2, I_3) g_{\varepsilon t} = (I_1, I_2, I_3) g_t^{\varepsilon}$, where

$$g_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2t) & -\sin(2t) \\ 0 & \sin(2t) & \cos(2t) \end{pmatrix}, \tag{7.2}$$

we can write $\hat{\varphi}_t(x, (I_1, I_2, I_3), r) = (\varphi_t(x), (I_1, I_2, I_3) g_{\varepsilon t}, r)$ and hence $\hat{X}_{s(x)} = X^h_{s(x)} + (\widetilde{e}_1)_{s(x)}$. Therefore we see that $\hat{X}_{s(x)g^{\varepsilon}} = X^h_x + (\widetilde{Ad_{g^{-\varepsilon}} e_1})_{s(x)g^{\varepsilon}}$, where $g \in \mathrm{SO}(3)$. Then the moment map $\mu^c : \hat{M} \to \mathfrak{so}(3) (= \mathbb{R}^3)$ on $\hat{M}$ is given by

$$\mu^c(p) = A r^{\frac{2}{c}} \theta(\hat{X}_p) = A r^{\frac{2}{c}} g^{-\varepsilon} e_1 g^{\varepsilon}$$

at any point $p = (x, s(x)g^{\varepsilon}, r) \in \hat{M}$. The level set $P := (\mu^c)^{-1}(e_1) = (\mu^c)^{-1}((1,0,0))$ is given by

$$\{(x, s(x)g^{\varepsilon}, r) \in \hat{M} \mid x \in M,\ A r^{\frac{2}{c}} g^{-\varepsilon} e_1 g^{\varepsilon} = e_1\}.$$

Hence we have

$$P = \{(x, s(x)g^{\varepsilon}, A^{-\frac{c}{2}}) \in \hat{M} \mid x \in M,\ g \in \mathrm{U}(1)\} \cong M \times \mathrm{U}(1). \tag{7.3}$$

We obtain a hypercomplex manifold $M' = P/\langle \hat{X} \rangle$, where $\langle \hat{X} \rangle \cong \mathrm{U}(1)$. Define a map $k : M \to M'$ by $k(x) = \pi_P((x, s(x), A^{-\frac{c}{2}}))$ for each $x \in M$, where $\pi_P : P \to M' = P/\langle \hat{X} \rangle$ is the quotient map. Since $k^{-1}(y) = \pi_P^{-1}(y) \cap (s(M) \times \{A^{-\frac{c}{2}}\})$ consists of exactly two points for each $y \in M'$ by (7.2), k is a double covering map. By (5.7) in Lemma 5.6, it holds $V_p = \{v \in T_p P \mid (d\mu^c_{\alpha} \circ \hat{I}^{\hat{\theta},c}_{\alpha})(v) = 0\} = \mathcal{H}_p$. It follows that $\pi_P^*(TM') \cong \mathcal{H}|_P$, where $\mathcal{H}$ is the horizontal subbundle with respect to the Obata connection. Since $s_*(Y) = Y^h$ for $Y \in TM$, we have

$$k_*(I_{\alpha} Y) = \pi_{P*}(s_* I_{\alpha} Y) = \pi_{P*}((I_{\alpha} Y)^h) = I'_{\alpha}(\pi_{P*}(Y^h)) = I'_{\alpha}(\pi_{P*} s_* Y) = I'_{\alpha}(k_* Y).$$

Therefore $k : M \to M'$ is a double covering map satisfying $k_* \circ I_{\alpha} = I'_{\alpha} \circ k_*$. □

Note that M' is obtained by the twist data $(X, F = 0, a = 1)$.

Example 7.3. For the Swann bundle $\hat{M}$ of a quaternionic manifold (M, Q), we see that $Z_1 = -\hat{I}_1^{\bar{\theta},c} Z_0^c$ satisfies the conditions required above by Lemma 3.4. So $\hat{M}$ is a double covering space of $(\hat{M})'$.

Quaternionic Hopf manifold: Consider $\mathbb{H}^n \cong \mathbb{R}^{4n}$ as a right-vector space over the quaternions. Set $\tilde{M} := \mathbb{H}^n \backslash \{0\}$. The standard hypercomplex structure $\tilde{H} = (\tilde{I}_1, \tilde{I}_2, \tilde{I}_3)$ on $\tilde{M}$ is defined by $\tilde{I}_1 = R_i, \tilde{I}_2 = R_j, \tilde{I}_3 = -R_k$, where R_q is the right-multiplication by

$q \in \mathbb{H}$. The hypercomplex structure $\tilde{H}$ gives a global section $s : \tilde{M} \to S(\cong \tilde{M} \times \mathrm{SO}(3))$ as in the previous example. The corresponding quaternionic structure is denoted by $\tilde{Q} = \langle \tilde{I}_1, \tilde{I}_2, \tilde{I}_3 \rangle$. Let $\tilde{g}$ be the standard flat hyper-Kähler metric on $\tilde{M}$, $A \in \mathrm{Sp}(n)\mathrm{Sp}(1)$ and $\lambda > 1$. Then $\gamma := \lambda A$ generates a group $\Gamma = \langle \gamma \rangle$ of homotheties which acts freely and properly discontinuously on the simply connected manifold $(\tilde{M}, \tilde{g})$. We can identify $\tilde{M}$ with $\mathbb{R} \times S^{4n-1}$ by means of the diffeomorphism $v \mapsto (t, v/\|v\|)$, where $t = \log \|v\| / \log \lambda$. Under this identification, γ corresponds to the transformation

$$T_A : \mathbb{R} \times S^{4n-1} \to \mathbb{R} \times S^{4n-1}, \ \ (t, v) \mapsto (t+1, Av). \tag{7.4}$$

The quotient $\tilde{M}/\Gamma \cong (\mathbb{R} \times S^{4n-1})/\langle T_A \rangle$ is diffeomorphic to $S^1 \times S^{4n-1}$ and inherits a quaternionic structure Q and a quaternionic connection ∇, both invariant under the centralizer $G^Q := Z_{\mathrm{GL}(n,\mathbb{H})\mathrm{Sp}(1)}(\{\gamma\})$ of γ in $\mathrm{GL}(n, \mathbb{H})\mathrm{Sp}(1)$. (In particular, if $A \in \mathrm{Sp}(n)$, then $\tilde{M}/\Gamma$ inherits a hypercomplex structure H and its Obata connection ∇, both invariant under the centralizer $G^H := Z_{\mathrm{GL}(n,\mathbb{H})}(\{\gamma\})$ of γ in $\mathrm{GL}(n, \mathbb{H})$.) In fact, the quaternionic structure $\tilde{Q}$ on $\tilde{M}$ is $\mathrm{GL}(n, \mathbb{H})\mathrm{Sp}(1)$-invariant and induces therefore an almost quaternionic structure Q on $\tilde{M}/\Gamma$, since $\Gamma \subset \mathrm{GL}(n, \mathbb{H})\mathrm{Sp}(1)$. Moreover, the Levi-Civita connection connection $\tilde{\nabla}$ on $(\tilde{M}, \tilde{g})$, which coincides with the Obata connection with respect to $\tilde{H}$, is invariant under all homotheties of $\tilde{M}$. Since Γ acts by homotheties, we see that $\tilde{\nabla}$ induces a torsion-free connection ∇ on $\tilde{M}/\Gamma$, which preserves Q. This means that Q is a quaternionic structure on $\tilde{M}/\Gamma$. The group G^Q acts on $\tilde{M}/\Gamma$ preserving the data (Q, ∇). If $A \in \mathrm{Sp}(n)$, then Γ preserves the hypercomplex structure $\tilde{H}$ on $\tilde{M}$ and thus induces a hypercomplex structure H and $\tilde{\nabla}$ induces the Obata connection ∇ on $(\tilde{M}/\Gamma, H)$. The centralizer G^H of $\gamma = A$ in $\mathrm{GL}(n, \mathbb{H})$ acts on $\tilde{M}/\Gamma$ preserving (H, ∇). We say that $(\tilde{M}/\Gamma, Q)$ (resp. $(\tilde{M}/\Gamma, H)$) is a *quaternionic* (*resp. hypercomplex*) *Hopf manifold*. Note that the hypercomplex Hopf manifolds are sometimes called quaternionic Hopf manifolds (see [15] for example).

Now taking $A = R_q$ for some unit quaternion $q \neq \pm 1$, we have a quaternionic Hopf manifold $M = \tilde{M}/\Gamma$. Then we see $G^Q = \mathrm{GL}(n, \mathbb{H})\mathrm{U}(1) = \mathbb{R}^{>0} \times \mathrm{SL}(n, \mathbb{H})\mathrm{U}(1)$, where $\mathrm{U}(1)$ denotes the centralizer of q in $\mathrm{Sp}(1)$. Up to an automorphism of $\mathrm{Sp}(1)$, we can assume that

$$\mathrm{U}(1) = \{e^{i\theta} \mid \theta \in \mathbb{R}\}.$$

We take a subgroup $\mathbb{R}^{>0} \times \mathrm{Sp}(n)\mathrm{U}(1)$ of G^Q, which acts on M transitively. The isotropy subgroup is given by $\langle \lambda \rangle \times \mathrm{Sp}(n-1)\triangle_{\mathrm{U}(1)}$, where $\triangle_{U(1)}$ is a diagonally embedded subgroup of $\mathrm{Sp}(n)\mathrm{U}(1) \subset \mathrm{Sp}(n)\mathrm{Sp}(1)$ which is isomorphic to $\mathrm{U}(1)$. This has an expression as

$$\mathrm{Sp}(n-1)\triangle_{\mathrm{U}(1)} = \left\{ \left[\left(\begin{array}{c|ccc} e^{i\theta} & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A & \\ 0 & & & \end{array} \right), \ e^{i\theta} \right] \ \middle| \ A \in \mathrm{Sp}(n-1), e^{i\theta} \in \mathrm{U}(1) \right\}.$$

As described above, we obtain an invariant quaternionic structure on the homogeneous space

$$M = (\mathbb{R}^{>0}/\langle \lambda \rangle) \times \frac{\mathrm{Sp}(n)\mathrm{U}(1)}{\mathrm{Sp}(n-1)\triangle_{\mathrm{U}(1)}}.$$

Remark 7.4. In particular, for $n = 1$, this yields a left invariant quaternionic structure on $\mathrm{U}(1) \times \mathrm{Sp}(1)$. For $n = 2$, we obtain an invariant quaternionic structure on the homogeneous space

$$\mathrm{U}(1) \times \frac{\mathrm{Sp}(2)\mathrm{U}(1)}{\mathrm{Sp}(1)\triangle_{\mathrm{U}(1)}} = \frac{T^2 \cdot \mathrm{Sp}(2)}{\mathrm{U}(2)}.$$

Note that the homogeneous quaternionic space $T^2 \cdot \mathrm{Sp}(2)/\mathrm{U}(2)$ has a finite covering of the form $(T^2 \times G)/\mathrm{U}(2)$, where G is a compact semisimple Lie group, namely $\mathrm{Sp}(2)$. This presentation is of the form $(T^k \times G)/\mathrm{U}(2)$ as considered in [12].

Consider the $\mathrm{U}(1)$-action on $\tilde{M}$ defined by the right-multiplication by elements of $\mathrm{U}(1)(\subset \mathbb{R}^{>0} \times \mathrm{Sp}(n)\mathrm{U}(1) \subset G^Q) : z \mapsto z \cdot e^{\varepsilon i t}$ $(z \in \tilde{M})$. Then the corresponding vector field $\tilde{X}$ satisfies $\tilde{X}_z = \varepsilon z i = \varepsilon \tilde{I}_1 z$ for $z \in \tilde{M}$. Moreover we see that the relations (7.1) in the previous example hold, that is, $L_{\tilde{X}}\tilde{I}_1 = 0$, $L_{\tilde{X}}\tilde{I}_2 = 2\varepsilon\tilde{I}_3$, $L_{\tilde{X}}\tilde{I}_3 = -2\varepsilon\tilde{I}_2$. The $\mathrm{U}(1)$-action preserving the quaternionic structure induces one on M and $\tilde{X}$ induces the vector field X on M generating the latter $\mathrm{U}(1)$-action on M. Considering the hypercomplex moment map on the Swann bundle $\hat{\tilde{M}}$ (resp. $\hat{M}$) of $\tilde{M}$ (resp. M) and the level set $\tilde{P} \subset \hat{\tilde{M}}$ (resp. $P \subset \hat{M}$) of the corresponding moment map, we can obtain a hypercomplex manifold $\tilde{M}'$ (resp. M'). In fact, since $Ric^{\tilde{\nabla}} = 0$ (resp. $Ric^{\nabla} = 0$) and $\hat{\tilde{X}}$ (resp. $\hat{X}$) is not horizontal, the Q/H-correspondence can be applied to $\tilde{M}$ (resp. M), cf (6.2).

Now we consider $\tilde{M}_+ := \tilde{M}/\{\pm 1\}$ and $M_+ := M/\{\pm 1\}$. The quotient maps by the action of the group $\{\pm 1\} \cong \mathbb{Z}_2$ on the manifolds are denoted by $\tilde{\pi}_+ : \tilde{M} \to \tilde{M}_+$ and $\pi_+ : M \to M_+$, respectively. The induced objects on $\tilde{M}_+$ and M_+ are denoted by the same letter. We obtain a hypercomplex isomorphism between $\tilde{M}'$ and $\tilde{M}_+$ as follows. Define $\tilde{f} : \tilde{M}' \to \tilde{M}_+$ by $\tilde{f}(x) = \tilde{\pi}_+(\hat{\tilde{\pi}}(u))$ for any $x \in \tilde{M}'$, where $\hat{\tilde{\pi}} : \hat{\tilde{M}} \to \tilde{M}$ is the bundle projection and $u \in \pi_{\tilde{P}}^{-1}(x) \cap (s(\tilde{M}) \times \{A^{-\frac{c}{2}}\})$. Since $\pi_{\tilde{P}}^{-1}(x) \cap (s(\tilde{M}) \times \{A^{-\frac{c}{2}}\})$ consists of exactly two points of the form $\{(\pm p, \tilde{H}, A^{-\frac{c}{2}})\}$ as we observed in the proof of Proposition 7.2, $\tilde{f}$ is well-defined. It is easy to see

$$\tilde{f} \circ \pi_{\tilde{P}} = \tilde{\pi}_+ \circ \hat{\tilde{\pi}} \tag{7.5}$$

on $s(\tilde{M}) \times \{A^{-\frac{c}{2}}\}$ by the definition of $\tilde{f}$. Furthermore we have

$$\tilde{f} \circ \tilde{k} = \tilde{\pi}_+, \tag{7.6}$$

where $\tilde{k} : \tilde{M} \to \tilde{M}'$ is the double covering as in Proposition 7.2 for $\tilde{M}$. In fact, from (7.5), it follows that $\tilde{f}(\tilde{k}(x)) = \tilde{f}(\pi_{\tilde{P}}(s(x), A^{-\frac{c}{2}})) = \tilde{\pi}_+(\hat{\tilde{\pi}}(s(x), A^{-\frac{c}{2}})) = \tilde{\pi}_+(x)$ for all $x \in \tilde{M}$.

Lemma 7.5. *The map $\tilde{f} : \tilde{M}' \to \tilde{M}_+$ is an isomorphism of hypercomplex manifolds.*

Proof. To prove that $\tilde{f}$ is injective, let $x_1, x_2 \in \tilde{M}'$ such that $\tilde{f}(x_1) = \tilde{f}(x_2)$. There exists $y_a \in \tilde{M}$ such that $x_a = \pi_{\tilde{P}}(s(y_a), A^{-\frac{c}{2}})$ $(a = 1, 2)$. Since $\tilde{f}(x_1) = \tilde{f}(x_2)$ and (7.5), we have $\hat{\tilde{\pi}}(s(y_1), A^{-\frac{c}{2}}) = \pm\hat{\tilde{\pi}}(s(y_2), A^{-\frac{c}{2}})$, that is, $y_1 = \pm y_2$, or equivalently $y_1 = \varphi_0(y_2)$ or $y_1 = \varphi_\pi(y_2)$. Therefore, we see that $(s(y_1), A^{-\frac{c}{2}}) = (s(\varphi_\delta(y_2)), A^{-\frac{c}{2}})$, where $\delta = 0$ or π. Then we have $x_1 = \pi_{\tilde{P}}(s(y_1), A^{-\frac{c}{2}}) = \pi_{\tilde{P}}(s(y_2), A^{-\frac{c}{2}}) = x_2$, which means $\tilde{f}$ is injective.

To show that $\tilde{f}$ is surjective, let $z \in \tilde{M}_+$ and choose $y \in \tilde{M}$ such that $z = \tilde{\pi}_+(y)$. By (7.6), we obtain $z = \tilde{\pi}_+(y) = \tilde{f}(\tilde{k}(y))$. Hence $\tilde{f}$ is surjective. The lift of $v \in T\tilde{M}'$ to $\mathcal{H}$ is denoted by $v^{h'}$. By

$$(I'_\alpha)_x(v) = \pi_{\tilde{P}*u}((\hat{I}_\alpha^{\bar{\theta},c})_u(v^{h'})) = \pi_{\tilde{P}*u}(((I_\alpha)_{\tilde{f}(x)}(\hat{\tilde{\pi}}_{*u}(v^{h'})))^h),$$

then

$$\begin{aligned}\tilde{f}_{*x}((I'_\alpha)_x(v)) &\overset{(7.5)}{=} \tilde{\pi}_{+*}(\hat{\tilde{\pi}}_{*u}(((I_\alpha)_{\tilde{f}(x)}((\hat{\tilde{\pi}}_{*u})(v^{h'})))^h)) \\ &= (I_\alpha)_{\tilde{f}(x)}(\tilde{\pi}_{+*}(\hat{\tilde{\pi}}_{*u}(v^{h'}))) \overset{(7.5)}{=} (I_\alpha)_{\tilde{f}(x)}(\tilde{f}_{*x}(v))\end{aligned}$$

at each point $x \in \tilde{M}'$, where $u \in \pi_{\tilde{P}}^{-1}(x) \cap (s(\tilde{M}) \times \{A^{-\frac{c}{2}}\})$. This shows that the hypercomplex manifolds M' and M are isomorphic. □

Set $F := \tilde{f} \circ \pi_{\tilde{P}} : \tilde{P} \to \tilde{M}_+$. Hereafter we will denote the equivalence class with respect to the action of a group K by $[\,\cdot\,]_K$.

Lemma 7.6. *We have $F(\gamma \cdot y) = \lambda \cdot F(y)$ for all $y \in \tilde{P}$.*

Proof. For any point $y = (p, \tilde{H}g, A^{-\frac{c}{2}})$ ($p \in \tilde{M}$ and $g \in \mathrm{U}(1)$) of $\tilde{P}$, we have $\gamma \cdot y = (\lambda p i, \tilde{H}g\rho(i), A^{-\frac{c}{2}})$ by (7.3), where $\rho : \mathrm{Sp}(1) \to \mathrm{SO}(3)$ is the standard double covering. Therefore we obtain

$$\gamma \cdot [y]_{\langle\hat{\hat{X}}\rangle} = [(\lambda p i, \tilde{H}g\rho(i), A^{-\frac{c}{2}})]_{\langle\hat{\hat{X}}\rangle} = [(\pm\lambda p\hat{g}^{-1}, \tilde{H}, A^{-\frac{c}{2}})]_{\langle\hat{\hat{X}}\rangle},$$

where $\hat{g} \in \mathrm{Sp}(1)$ such that $\rho(\hat{g}) = g$. Then it holds

$$F(\gamma \cdot y) = \tilde{f}(\gamma \cdot [y]_{\langle\hat{\hat{X}}\rangle}) = \tilde{f}([(\pm\lambda p\hat{g}^{-1}, \tilde{H}, A^{-\frac{c}{2}})]_{\langle\hat{\hat{X}}\rangle}) = \tilde{\pi}_+(\pm\lambda p\hat{g}^{-1}) = \lambda\tilde{f}([y]_{\langle\hat{\hat{X}}\rangle}) = \lambda F(y)$$

from (7.5). □

Note that M' has an induced $\{\pm 1\}$-action, since the lifted action of $\{\pm 1\}$ to the Swann bundle $\hat{\tilde{M}}$ commutes with Γ and $\mathrm{U}(1)$. Let $\pi'_+ : M' \to M'_+$ be the quotient map of the action by $\{\pm 1\}$ on M'. We can define a map

$$\Phi : M'_+(= \pi'_+(M')) \to \tilde{M}_+/\langle\lambda\rangle$$

as follows. Take any $x \in M'_+$. Then there exists $y \in \tilde{P}$ such that $x = \pi'_+(\pi_P([y]_\Gamma))$ and we set $\Phi(x) := [F(y)]_{\langle\lambda\rangle}$. We shall show that $[F(y)]_{\langle\lambda\rangle}$ is independent of the choice of y. If $(x =)\pi'_+(\pi_P([y_1]_\Gamma)) = \pi'_+(\pi_P([y_2]_\Gamma))$, there exist $\delta \in \{\pm 1\}$, $g \in \mathrm{U}(1)$ and $l \in \mathbb{Z}$ such that $y_1 = \delta \cdot g \cdot \gamma^l \cdot y_2$. By Lemma 7.6 and the definitions of $\tilde{f}$ and $\pi_{\tilde{P}}$, we see

$$F(y_1) = F(\delta \cdot g \cdot \gamma^l \cdot y_2) = \lambda^l F(y_2),$$

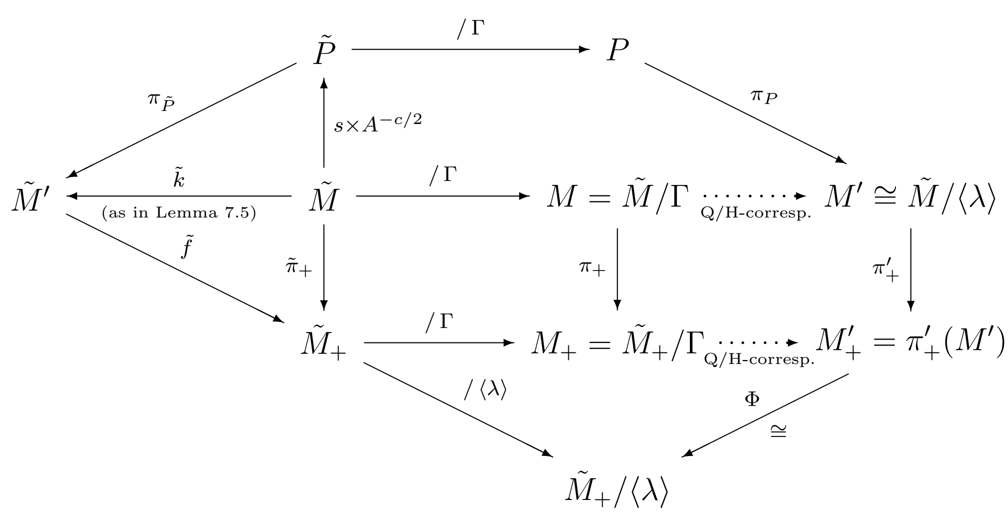
which implies $[F(y_1)]_{\langle\lambda\rangle} = [F(y_2)]_{\langle\lambda\rangle}$. Moreover we have

Lemma 7.7. *The map $\Phi : M'_+ \to \tilde{M}_+/\langle\lambda\rangle$ is an isomorphism.*

Proof. To prove that Φ is injective, let $x_1,\, x_2 \in M'_+ = \pi'_+(M')$ such that $\Phi(x_1) = \Phi(x_2)$. There exists $y_1,\, y_2 \in \tilde{P}$ such that $x_a = \pi'_+(\pi_P([y_a]_\Gamma))$ $(a = 1, 2)$. Since $[F(y_1)]_{\langle\lambda\rangle} = [F(y_2)]_{\langle\lambda\rangle}$, there exists $l \in \mathbb{Z}$ such that $F(y_1) = \lambda^l \cdot F(y_2) = F(\gamma^l \cdot y_2)$. Then we have $\tilde{f}(\pi_{\tilde{P}}(y_1)) = \tilde{f}(\pi_{\tilde{P}}(\gamma^l \cdot y_2))$, so there exists $g \in \mathrm{U}(1)$ such that $y_1 = g \cdot \gamma^l \cdot y_2$. Therefore we obtain

$$x_1 = \pi'_+(\pi_P([y_1]_\Gamma)) = \pi'_+(\pi_P([g \cdot \gamma^l \cdot y_2]_\Gamma)) = \pi'_+(\pi_P([y_2]_\Gamma)) = x_2.$$

So Φ is injective. Next we shall show that Φ is surjective. Take any $z \in \tilde{M}_+/\langle\lambda\rangle$. There exists $y \in \tilde{P}$ such that $z = [F(y)]_{\langle\lambda\rangle}$. Setting $x = \pi'_+(\pi_P([y]_\Gamma))$, we have $\Phi(x) = z$, which means Φ is surjective. Since the hypercomplex structures are invariant under actions of all groups $\mathrm{U}(1) = \langle\hat{X}\rangle = \langle\hat{\hat{X}}\rangle$, Γ, $\langle\lambda\rangle$ and $\{\pm 1\}$ in the argument, Φ is a hypercomplex isomorphism. $\square$



Therefore, by Lemma 7.7, the hypercomplex manifold M'_+ obtained from M_+ by the Q/H-correspondence is identified with $\tilde{M}_+/\langle\lambda\rangle$. Since M' is the double covering space of M'_+, we have

$$M' \cong \tilde{M}/\langle\lambda\rangle.$$

The centralizer G^H of λ is $\mathrm{GL}(n, \mathbb{H}) = \mathbb{R}^{>0} \times \mathrm{SL}(n, \mathbb{H})$ and take a subgroup $\mathbb{R}^{>0} \times \mathrm{Sp}(n)$ of G^H. As we explained, $\tilde{M}/\langle\lambda\rangle$ can be expressed by the homogeneous space

$$\tilde{M}/\langle\lambda\rangle = (\mathbb{R}^{>0}/\langle\lambda\rangle) \times \frac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-1)}.$$

Finally, we summarize the discussion as follows.

Example 7.8. The hypercomplex manifold

$$M' = (\mathbb{R}^{>0}/\langle\lambda\rangle) \times \frac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-1)}$$

is obtained by the Q/H-correspondence from the quaternionic manifold

$$M = (\mathbb{R}^{>0}/\langle\lambda\rangle) \times \frac{\mathrm{Sp}(n)\mathrm{U}(1)}{\mathrm{Sp}(n-1)\triangle_{\mathrm{U}(1)}}.$$

(Note that we are considering the invariant quaternionic (resp. hypercomplex) structure on M (resp. M') described above.)

We remark that M' does not admit any hyper-Kähler structure for topological reasons, since M' is diffeomorphic to $S^1 \times S^{4n-1}$. Therefore our Q/H-correspondence yields examples which can not appear in the QK/HK-correspondence.

Acknowledgments. Research by the first author is partially funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy – EXC 2121 Quantum Universe – 390833306. This paper was prepared during the second author's stay at Universität Hamburg on his sabbatical leave (April 2018 - March 2019). He would like to thank the Department of Mathematics of Universität Hamburg for the hospitality. His research is partially supported by JSPS KAKENHI Grant Number 18K03272.

References

[1] D. Alekseevsky, V. Cortés, M. Dyckmanns and T. Mohaupt, *Quaternionic Kähler metrics associated with special Kähler manifolds*, J. Geom. Phys. **92** (2015), 271-287.

[2] D. Alekseevsky, V. Cortés, and T. Mohaupt, *Conification of Kähler and hyper-Kähler manifolds*, Comm. Math. Phys. **324** (2013), 637-655.

[3] D. Alekseevsky and M. Graev, *G-structures of twistor type and their twistor spaces*, J. Geom. Phys. **10** (1993), 203-229.

[4] D. Alekseevsky and S. Marchiafava, *Quaternionic Transformations and the First Eigenvalues of Laplacian on a Quaternionic Kähler Manifold*, ESI (The Erwin Schrödinger International Institute for Mathematical Physics) **150** (1994).

[5] D. Alekseevsky and S. Marchiafava, *Quaternionic structures on a manifold and subordinated structures*, Ann. Mat. Pura Appl. (4) **171** (1996), 205-273.

[6] V. Cortés, M. Dyckmanns, M. Jüngling and D. Lindemann, *A class of cubic hypersurfaces and quaternionic Kähler manifolds of co-homogeneity one*, preprint (arXiv:1701.7882).

[7] S. Fujimura, *Q-connections and their changes on an almost quaternion manifold*, Hokkaido Math. J. **5** (1976), 239-248.

[8] K. Galicki and B. H. Lawson, *Quaternionic reduction and quaternionic orbifolds*, Math. Ann. **282** (1988), 1-21.

[9] A. Haydys, *Hyper-Kähler and quaternionic Kähler manifolds with S^1-symmetries*, J. Geom. Phys. **58** (2008), 293-306.

[10] N. Hitchin, *Metrics on moduli spaces*, Contemp. Math. 58 (1986), 157-178 (Lefschetz Centennial conference. Proceedings on algebraic geometry).

[11] D. Joyce, *The hypercomplex quotient and the quaternionic quotient*, Math. Ann. **290** (1991), 323-340.

[12] D. Joyce, *Compact hypercomplex and quaternionic manifolds*, J. Differential Geometry, **35** (1992), 743-761.

[13] O. Macia and A. Swann, *Twist geometry of the c-map*, Commun. Math. Phys. **336** (2015), 1329-1357.

[14] M. Obata, *Affine connections on manifolds with almost complex, quaternion or Hermitian structure*, Jap. J. Math., **26** (1956), 43-79.

[15] L. Ornea and P. Piccinni, *Locally conformal Kähler manifold structures in quaternionic geometry*, Trans. Amer. Math. Soc., **349** (1997), 641-355.

[16] H. Pedersen, *Hypercomplex Geometry*, Proceedings of the Second Meeting on Quaternionic Structures in Mathematics and Physics (2001), 313-320.

[17] H. Pedersen, Y. Poon and A. Swann, *Hypercomplex structures associated to quaternionic manifolds*, Differential Geom. Appl. **9** (1998), 273-292.

[18] S. Salamon, *Differential geometry of quaternionic manifolds*, Ann. Scient. Éc. Norm. Sup. **19** (1986), 31-55.

[19] A. Swann, *Hyper-Kähler and quaternionic Kähler*, Math. Ann. **289** (1991), 421-450.

[20] A. Swann, *Twisting hermitian and hypercomplex geometries*, Duke Math. J. **155** (2010), 403-431.

Vicente Cortés
Department of Mathematics
and Center for Mathematical Physics
University of Hamburg
Bundesstraße 55,
D-20146 Hamburg, Germany.
email:vicente.cortes@uni-hamburg.de

Kazuyuki Hasegawa
Faculty of teacher education
Institute of human and social sciences
Kanazawa university
Kakuma-machi, Kanazawa,
Ishikawa, 920-1192, Japan.
e-mail:kazuhase@staff.kanazawa-u.ac.jp