Paracontrolled distribution approach to stochastic Volterra equations

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Abstract

Based on the notion of paracontrolled distributions, we provide existence and uniqueness results for rough Volterra equations of convolution type with potentially singular kernels and driven by the newly introduced class of convolutional rough paths. The existence of such rough paths above a wide class of stochastic processes including the fractional Brownian motion is shown. As applications we consider various types of rough and stochastic (partial) differential equations such as rough differential equations with delay, stochastic Volterra equations driven by Gaussian processes and moving average equations driven by Lévy processes.

Key words: fractional Brownian motion, Gaussian process, Itô-Lyons map, paradifferential calculus, rough differential equation, stochastic Volterra equation.
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1 Introduction

Stochastic Volterra equations serve as mathematical models for numerous random phenomena appearing in various areas such as biology, physics and mathematical finance. In the present work we consider Volterra equations of convolution type, which in their simplest form are given by

$$u(t) = u_0 + \int_{-\infty}^{t} \varphi(t-s)\sigma(u(s))\,\mathrm{d}\vartheta(s), \quad t \in \mathbb{R}, \tag{1.1}$$

where $\vartheta\colon \mathbb{R} \to \mathbb{R}^m$ is a (random) input signal, e.g., an m-dimensional (fractional) Brownian motion, $u_0 \in \mathbb{R}^n$, $\varphi\colon \mathbb{R} \to \mathbb{R}$ is the so-called kernel and $\sigma\colon \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ is a vector field. Since the pioneering works of Berger and Mizel [7, 8], stochastic Volterra equations have been studied in different settings and generality by a vast number of authors, see e.g. [38, 36, 12, 44].

This wide class of equations covers many stochastic differential and integral equations as special cases such as ordinary stochastic differential equations, classical stochastic Volterra integral equations, stochastic equations involving fractional derivatives (noting that singular kernels correspond to Fourier multipliers) and moving average equations driven by Lévy processes. Recently, Volterra equations attracted additional attention from the mathematical finance community because stochastic Volterra equations with singular kernels φ constitute very suitable models for the unpredictable and rough behaviour of volatility in financial markets, cf. [1, 28, 17].

Rough path theory initiated by Lyons [31] provides an innovative approach to the theory of stochastic differential equations leading to many novel insights. One of the fundamental results of rough path theory is the continuity of the solution map $\vartheta \mapsto u$, known as the Itô-Lyons map, for controlled differential equations driven by rough paths. This continuity statement had significant impact over the past decades and found many applications, see [32, 21].

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The main goal of this article is to develop a pathwise approach to and a solution theory for Volterra equations driven by rough paths, which allow for regular as well as singular kernels. In particular, we prove the local Lipschitz continuity of the Itô-Lyons map for Volterra equations generalising (in some directions) the above mentioned fundamental result. Many implications of the rough path theory seem thus to be feasible for Volterra equations.

For this purpose we first establish the existence of a unique solution to the Volterra equation (1.1) driven by singals ϑ with sufficient regularity, based on Littlewood-Paley theory and Bony's paraproduct. In order to extend the existence and uniqueness results to a rough path setting, we rely on the notion of paracontrolled distributions, which was introduced by Gubinelli et al. [24]. The paracontrolled distribution approach is particularly suitable for the pathwise analysis of Volterra equations of convolution type because of the following two key observations: Firstly, the convolution operator appearing in (1.1) fits nicely together with the underlying Fourier and Littlewood-Paley analysis since the convolution operator is, for instance, a local operation in the Fourier domain. The second advantage of paracontrolled distributions is that the driving rough path ϑ (or the underlying model using the language of regularity structures [26]) can be chosen adapted to the specific equation which turns out to be essential for the solution theory involving singular kernels.

Volterra equations driven by rough paths have so far only been studied by Deya and Tindel [14, 15]. They have demonstrated that classical rough path theory can be utilised to handle Volterra equations driven by rough paths. The approach in [14, 15] requires a deep and heavy analysis leading to strong regularity assumptions on the kernel φ, namely $\varphi \in C^3$, and thus excluding singular kernels. This is mainly caused by relying on the classical space of (geometric) rough paths, which have been designed to treat ordinary rough differential equations.

Using the flexibility of the paracontrolled distribution approach, we introduce the notion of convolutional rough paths by including the convolution kernel φ in the definition of the so-called resonant term. The later notion can be seen as the analogue to geometric rough paths in the paracontrolled distribution stetting. We prove that the Itô-Lyons map has a locally Lipschitz continuous extension from the space of smooth paths to the space of convolutional rough paths. Hence, the Volterra equation (1.1) driven by a level-2 convolutional rough path possesses a unique solution. This ansatz leads to rather weak regularity assumptions on the kernel φ requiring less than Lipschitz continuity and thus allowing especially for singular kernels.

In addition to the above mentioned modelling, there is a particular interest in singular kernels, e.g. [12, 13, 43], because of their links to stochastic differential equations with fractional derivatives [30] and to a large class of semilinear stochastic partial differential equations [44]. The here developed paracontrolled distribution approach to Volterra equations can thus also be viewed as a step towards these applications. However, exploiting these directions more comprehensively would require extensions based on higher order paracontrolled calculus, see [4], or to infinite dimensional spaces, cf. [33], which is beyond the scope of the present article.

While it is necessary for singular kernels to be included in the definition of the rough path, in the case of regular kernels, say φ is at least Lipschitz continuous, the existence of the convolutional rough path can be reduced to the existence of a generic rough path, i.e., independent of the specific kernel. Moreover, considering the regularity of the driving signal in Besov spaces, our analysis builds on [37] and interestingly the continuity results hold for some Volterra equations driven by convolutional rough paths with jumps, contributing to the recent extension of rough path theory to càdlàg paths, cf. [11, 22].

In order to apply the pathwise solution theory for Volterra equations driven by convolutional rough paths to stochastic Volterra equtaions, we construct convolutional rough paths for a large class of stochastic processes satisfying a hypercontractivity property. Examples include many Gaussian processes such as fractional Brownian motion with Hurst index $H > 1/3$. As a consequence, we obtain unique solutions to stochastic Volterra equations driven by Gaussian processes, extending most literature which focuses on driving signals given by semi-martinagles. Another advantage of the pathwise approach is that it can immediately deal with stochastic Volterra equations with anticipating coefficients, cf. the seminal work of [36].

Plan of the paper: In Section 2 the functional analytic foundation is provided. Section 3 establishes the existence and uniqueness results for Volterra equations. The connection to the classical rough path theory and the probabilistic construction of the resonant term for suitable stochastic processes can be found in Section 4. Applications of the pathwise results to various types of stochastic Volterra equations are presented in Section 5. Appendix A collects several auxiliary lemmas concerning Besov spaces.

1.1 Setting up the Volterra equation

In the rest of the paper, we study the following class of Volterra equations of convolution type

$$u(t) = u_0(t) + \big(\varphi_1 * (\sigma_1(u)\xi_1)\big)(t) + \big(\varphi_2 * (\sigma_2(u)\xi_2)\big)(t), \quad t \in \mathbb{R}, \tag{1.2}$$

where

- the convolution operator $*$ is defined by

  $$(f * g)(y) := \int_{\mathbb{R}} f(y - x) g(x) \, \mathrm{d}x, \quad y \in \mathbb{R},$$

  with the usual generalization for distributions f and g,
- $u_0 \colon \mathbb{R} \to \mathbb{R}^n$ is the initial condition,
- $\varphi_j \colon \mathbb{R} \to \mathbb{R}$ are the kernels (or kernel functions) for $j = 1, 2$,
- $\sigma_j \colon \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ are vector fields for $j = 1, 2$,
- $\xi_1 \colon \mathbb{R} \to \mathbb{R}^m$ and $\xi_2 \colon \mathbb{R} \to \mathbb{R}^m$ is a possibly rough and smoother signal, respectively.

Comparing (1.1) and (1.2), the signal ξ_1 corresponds to the (distributional) derivative of ϑ and the integral boundaries $(-\infty, t]$ are included via kernel functions of the form $\varphi_1 = \mathbb{1}_{[0,\infty)}\varphi$. Throughout the paper we refer to $\varphi_1 * (\sigma_1(u)\xi_1)$ as the *rough term* and to $\varphi_2 * (\sigma_2(u)\xi_2)$ as the *drift term*, making for simplicity the assumption that also φ_1, ξ_1 are less regular than φ_2, ξ_2, respectively. Let us remark that distinguishing between a rough and a drift term allows for sharper regularity conditions on the respective vector fields σ_1 and σ_2, cf. the notion of (p, q)-rough paths [29].

2 Bony's paraproduct and Besov spaces

Let us briefly set up the functional analytic framework. We begin by recalling the notion of Besov spaces in terms of the Littlewood-Paley decomposition. For a more general introduction we refer to Bahouri et al. [2], Sawano [41] and Triebel [42].

For the sake of clarification let us mention that $L^p(\mathbb{R}^d, \mathbb{R}^{m\times n})$ denotes the space of Lebesgue p-integrable functions with norm $\|\cdot\|_{L^p}$ for $p \in [1, \infty)$ and $L^\infty(\mathbb{R}^d, \mathbb{R}^{m\times n})$ denotes the space of bounded functions with corresponding norm $\|\cdot\|_\infty$. The space of Schwartz functions on $\mathbb{R}^d$ is denoted by $\mathcal{S}(\mathbb{R}^d) := \mathcal{S}(\mathbb{R}^d, \mathbb{R}^{m\times n})$ and its dual by $\mathcal{S}'(\mathbb{R}^d) := \mathcal{S}'(\mathbb{R}^d, \mathbb{R}^{m\times n})$, which is the space of tempered distributions.

For a function $f \in L^1(\mathbb{R}^d, \mathbb{R}^{m\times n})$ the Fourier transform and its inverse are defined by

$$\mathcal{F}f(z) := \int_{\mathbb{R}^d} e^{-i\langle z, x\rangle} f(x) \, \mathrm{d}x \quad \text{and} \quad \mathcal{F}^{-1}f(z) := (2\pi)^{-d}\mathcal{F}f(-z).$$

If $f \in \mathcal{S}'(\mathbb{R}^d)$, then the usual generalization of the Fourier transform is considered.

The Littlewood-Paley theory is based on a localization in the frequency domain by a *dyadic partition of unity* (χ, ρ), that is, χ and ρ are non-negative infinitely differentiable radial functions on $\mathbb{R}^d$ such that $\operatorname{supp}\chi \subseteq \mathcal{B}$ and $\operatorname{supp}\rho \subseteq \mathcal{A}$ for a ball $\mathcal{B} \subseteq \mathbb{R}^d$ and an annulus $\mathcal{A} \subseteq \mathbb{R}^d$,

$\chi(z)+\sum_{j\geq 0}\rho(2^{-j}z)=1$ for all $z\in\mathbb{R}^d$, $\operatorname{supp}(\chi)\cap\operatorname{supp}(\rho(2^{-j}\cdot))=\emptyset$ for $j\geq 1$, and $\operatorname{supp}(\rho(2^{-i}\cdot))\cap\operatorname{supp}(\rho(2^{-j}\cdot))=\emptyset$ for $|i-j|>1$. We set throughout

$$\rho_{-1}:=\chi \quad \text{and} \quad \rho_j:=\rho(2^{-j}\cdot) \quad \text{for } j\geqslant 0.$$

Given a dyadic partition of unity (χ,ρ), the *Littlewood-Paley blocks* are defined by

$$\Delta_{-1}f:=\mathcal{F}^{-1}(\rho_{-1}\mathcal{F}f) \quad \text{and} \quad \Delta_j f:=\mathcal{F}^{-1}(\rho_j\mathcal{F}f) \quad \text{for } j\geq 0.$$

Note that $\Delta_j f$ is a smooth function for every $j\geq -1$ and for every $f\in\mathcal{S}'(\mathbb{R}^d)$ one has $f=\sum_{j\geq -1}\Delta_j f$. For $\alpha\in\mathbb{R}$ and $p,q\in[1,\infty]$ the *Besov space* $\mathcal{B}^\alpha_{p,q}(\mathbb{R}^d,\mathbb{R}^{m\times n})$ is given by

$$\begin{aligned}\mathcal{B}^\alpha_{p,q}(\mathbb{R}^d,\mathbb{R}^{m\times n}) &:= \big\{f\in\mathcal{S}'(\mathbb{R}^d,\mathbb{R}^{m\times n}) \;:\; \|f\|_{\alpha,p,q}<\infty\big\}\\ \text{with} \quad \|f\|_{\alpha,p,q} &:= \Big\|\big(2^{j\alpha}\|\Delta_j f\|_{L^p}\big)_{j\geqslant -1}\Big\|_{\ell^q}.\end{aligned}$$

Although the norm $\|\cdot\|_{\alpha,p,q}$ depends on the dyadic partition (χ,ρ), different dyadic partitions of unity lead to equivalent norms (see [2, Corollary 2.70]). Whenever the dimension of the image space is clear from the context, we write $\mathcal{B}^\alpha_{p,q}(\mathbb{R}^d):=\mathcal{B}^\alpha_{p,q}(\mathbb{R}^d,\mathbb{R}^{m\times n})$ and $\mathcal{B}^\alpha_{p,q}:=\mathcal{B}^\alpha_{p,q}(\mathbb{R},\mathbb{R}^{m\times n})$ and analogous abbreviations for $L^p(\mathbb{R}^d,\mathbb{R}^{m\times n})$. The special case of Hölder-Zygmund spaces is denoted by $\mathcal{C}^\alpha:=\mathcal{B}^\alpha_{\infty,\infty}$ with corresponding norms $\|\cdot\|_{\mathcal{C}^\alpha}:=\|\cdot\|_{\alpha,\infty,\infty}$ for $\alpha>0$. In the following we will frequently apply embedding results for Besov spaces, which can be found for example in [42, Proposition 2.5.7 and Theorem 2.7.1].

Let us fix the notation $A_\vartheta\lesssim B_\vartheta$, for a generic parameter ϑ, meaning that $A_\vartheta\leqslant CB_\vartheta$ for some constant $C>0$ independent of ϑ. We write $A_\vartheta\sim B_\vartheta$ if $A_\vartheta\lesssim B_\vartheta$ and $B_\vartheta\lesssim A_\vartheta$. For integers $j_\vartheta,k_\vartheta\in\mathbb{Z}$ we write $j_\vartheta\lesssim k_\vartheta$ if there is some $N\in\mathbb{N}$ such that $j_\vartheta\leqslant k_\vartheta+N$, and $j_\vartheta\sim k_\vartheta$ if $j_\vartheta\lesssim k_\vartheta$ and $k_\vartheta\lesssim j_\vartheta$.

Given $f\in\mathcal{B}^\alpha_{p_1,q_1}(\mathbb{R}^d)$ and $g\in\mathcal{B}^\beta_{p_2,q_2}(\mathbb{R}^d)$, we can formally decompose the product fg in terms of Littlewood-Paley blocks as

$$fg=\sum_{j\geq -1}\sum_{i\geq -1}\Delta_i f\Delta_j g = T_f g + T_g f + \pi(f,g) \tag{2.1}$$

where

$$T_f g := \sum_{j\geq -1}\Big(\sum_{i\leq j-2}\Delta_i f\Big)\Delta_j g \quad \text{and} \quad \pi(f,g):=\sum_{|i-j|\leq 1}\Delta_i f\Delta_j g.$$

This decomposition was originally introduced by Bony [9] and $\pi(f,g)$ is usually called *resonant term*. The following paraproduct estimates verify the importance of *Bony's decomposition*. For the proof of this lemma, we refer to [2, Theorem 2.82 and 2.85] and [37, Lemma 2.1].

Lemma 2.1 (Bony's paraproduct estimates)**.** *Let $\alpha,\beta\in\mathbb{R}$ and $p_1,p_2,q_1,q_2\in[1,\infty]$ and suppose that*

$$\frac{1}{p}:=\frac{1}{p_1}+\frac{1}{p_2}\leqslant 1 \quad \textit{and} \quad \frac{1}{q}:=\frac{1}{q_1}+\frac{1}{q_2}\leqslant 1.$$

(i) If $(f,g)\in L^{p_1}(\mathbb{R}^d)\times B^\beta_{p_2,q}(\mathbb{R}^d)$, then $\|T_f g\|_{\beta,p,q}\lesssim\|f\|_{L^{p_1}}\|g\|_{\beta,p_2,q}$.

(ii) If $\alpha<0$ and $(f,g)\in B^\alpha_{p_1,q_1}(\mathbb{R}^d)\times B^\beta_{p_2,q_2}(\mathbb{R}^d)$, then $\|T_f g\|_{\alpha+\beta,p,q}\lesssim\|f\|_{\alpha,p_1,q_1}\|g\|_{\beta,p_2,q_2}$.

(iii) If $\alpha+\beta>0$ and $(f,g)\in B^\alpha_{p_1,q_1}(\mathbb{R}^d)\times B^\beta_{p_2,q_2}(\mathbb{R}^d)$, then $\|\pi(f,g)\|_{\alpha+\beta,p,q}\lesssim\|f\|_{\alpha,p_1,q_1}\|g\|_{\beta,p_2,q_2}$.

In order to analyse the smoothing property of the convolution operator $*$ appearing in the Volterra equation (1.2), we provide the following Young inequality and its proof since the authors are not aware of a reference for this result in the stated generality.

Lemma 2.2 (Generalized Young's inequality)**.** *Let $\alpha, \beta \in \mathbb{R}$, $d \in \mathbb{N}$ and $p_1, p_2, q_1, q_2 \in [1, \infty]$ satisfying*

$$0 \leqslant \frac{1}{p} := \frac{1}{p_1} + \frac{1}{p_2} - 1 \leqslant 1 \quad \text{and} \quad 0 \leqslant \frac{1}{q} := \frac{1}{q_1} + \frac{1}{q_2} \leqslant 1.$$

Then, for any $f \in \mathcal{B}^{\alpha}_{p_1,q_1}(\mathbb{R}^d)$ and $g \in \mathcal{B}^{\beta}_{p_2,q_2}(\mathbb{R}^d)$ we have $f * g \in \mathcal{B}^{\alpha+\beta}_{p,q}(\mathbb{R}^d)$ with

$$\|f * g\|_{\alpha+\beta,p,q} \lesssim \|f\|_{\alpha,p_1,q_1} \|g\|_{\beta,p_2,q_2}.$$

Proof. The Littlewood-Paley blocks of the convolution satisfy

$$\Delta_j(f * g) = \mathcal{F}^{-1}\big[\rho_j \mathcal{F} f \mathcal{F} g\big] = \mathcal{F}^{-1}[\rho_j^{1/2} \mathcal{F} f] * \mathcal{F}^{-1}[\rho_j^{1/2} \mathcal{F} g], \quad j \geqslant -1.$$

Using Young's inequality for L^p-spaces, we bound

$$2^{j(\alpha+\beta)} \|\Delta_j(f * g)\|_{L^p} \leqslant \big(2^{j\alpha} \|\mathcal{F}^{-1}[\rho_j^{1/2} \mathcal{F} f]\|_{L^{p_1}}\big)\big(2^{j\beta} \|\mathcal{F}^{-1}[\rho_j^{1/2} \mathcal{F} g]\|_{L^{p_2}}\big).$$

Hence, by the Cauchy-Schwarz inequality it suffices to show

$$\big\|\big(2^{j\alpha} \|\mathcal{F}^{-1}[\rho_j^{1/2} \mathcal{F} f]\|_{L^{p_1}}\big)_{j \geqslant -1}\big\|_{\ell^{q_1}} \lesssim \|f\|_{\alpha,p_1,q_1} \tag{2.2}$$

(and consequently the analogous estimate holds true for g). To verify (2.2), we decompose $f = \sum_j \Delta_j f$. Due to the compact support of ρ_j and the classical Young inequality, we obtain

$$\begin{aligned} 2^{j\alpha} \|\mathcal{F}^{-1}[\rho_j^{1/2} \mathcal{F} f]\|_{L^{p_1}} &\leqslant 2^{j\alpha} \sum_{j'} \big\|\mathcal{F}^{-1}\big[\rho_j^{1/2} \mathcal{F}[\Delta_{j'} f]\big]\big\|_{L^{p_1}} \\ &\leqslant 2^{j\alpha} \sum_{|j-j'| \leqslant 1} \|\mathcal{F}^{-1}[\rho_j^{1/2}]\|_{L^1} \|\Delta_j f\|_{L^{p_1}} \\ &\lesssim \sum_{j'} \big(2^{-(j'-j)\alpha} \mathbb{1}_{[-1,1]}(j' - j)\big)\big(2^{j'\alpha} \|\Delta_{j'} f\|_{L^{p_1}}\big). \end{aligned}$$

Again by Young's inequality (applied to ℓ^{q_1}) we conclude

$$\big\|\big(2^{j\alpha} \|\mathcal{F}^{-1}[\rho_j^{1/2} \mathcal{F} f]\|_{L^{p_1}}\big)_{j \geqslant -1}\big\|_{\ell^{q_1}} \leqslant \big\|\big(2^{-j\alpha} \mathbb{1}_{[-1,1]}(j)\big)_{j \geqslant -1}\big\|_{\ell^1} \|f\|_{\alpha,p_1,q_1} \leqslant (2^{|\alpha|} + 2) \|f\|_{\alpha,p_1,q_1}.$$

□

In order to quantify the regularity of the vector fields appearing in the Volterra equation (1.2) we follow the convention by Stein (cf. [19, Definition 3.1]): For operator-valued functions $F \colon \mathbb{R}^m \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ we write $F \in C^k$ for $k \in \mathbb{N}$, if F is bounded, continuous and k-times differentiable with bounded and continuous derivatives. The first and second derivative are denoted by F' and F'', respectively, and higher derivatives by $F^{(k)}$. The space C^k is equipped with the norms

$$\|F\|_\infty := \sup_{x \in \mathbb{R}^m} \|F(x)\| \quad \text{and} \quad \|F\|_{C^k} := \|F\|_\infty + \sum_{j=1}^{k} \|F^{(k)}\|_\infty,$$

where $\|\cdot\|$ denotes the corresponding operator norms.

3 Existence and uniqueness results for Volterra equations

Let us briefly recall the Volterra equation (1.2), which was given by

$$u(t) = u_0(t) + \big(\varphi_1 * (\sigma_1(u)\xi_1)\big)(t) + \big(\varphi_2 * (\sigma_2(u)\xi_2)\big)(t), \quad t \in \mathbb{R},$$

for an initial condition $u_0 \colon \mathbb{R} \to \mathbb{R}^n$, vector fields $\sigma_j \colon \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$, kernel functions $\varphi_j \colon \mathbb{R} \to \mathbb{R}$ and driving signals $\xi_j \colon \mathbb{R} \to \mathbb{R}^m$, for $j = 1, 2$. While the convolution is always well-defined for any function or distribution in a Besov space (cf. Lemma 2.2), the product $\sigma_j(u)\xi_j$ requires sufficient Besov regularity of the involved functions (cf. Lemma 2.1). This statement will be made precise in the next subsection.

3.1 Regular driving signals

To analyse the product $\sigma_j(u)\xi_j$ more carefully, we suppose that the driving signals satisfy $\xi_j \in \mathcal{B}^{\beta_j-1}_{p,\infty}$ with $\beta_j > 0$ and $p \geq 2$, for $j = 1, 2$. We further assume that the corresponding solution u of the Volterra equation (1.2) fulfills $u \in \mathcal{B}^{\alpha}_{p,\infty}$ for some regularity $\alpha \geqslant \frac{1}{p}$. In this case Bony's decomposition (2.1), the paraproduct estimates (Lemma 2.1) and the Besov embedding $B^{\alpha+\beta_j-1}_{p/2,\infty} \subseteq B^{\beta_j-1}_{p,\infty}$ applied to the problematic product yields

$$\sigma_j(u)\xi_j = \underbrace{T_{\sigma_j(u)}\xi_j}_{\in\mathcal{B}^{\beta_j-1}_{p,\infty}} + \underbrace{\pi(\sigma_j(u),\xi_j)}_{\in\mathcal{B}^{\alpha+\beta_j-1}_{p/2,\infty}} + \underbrace{T_{\xi_j}\sigma_j(u)}_{\mathcal{B}^{\alpha+\beta_j-1}_{p/2,\infty}} \in \mathcal{B}^{\beta_j-1}_{p,\infty} \quad \text{if } \alpha+\beta_j > 1,\, p \geqslant 2. \tag{3.1}$$

Notice that the Young type condition $\alpha + \beta_j > 1$ is crucial for the regularity estimate of the resonant term $\pi(\sigma_j(u), \xi_j)$. If $\varphi_j \in \mathcal{B}^{\gamma_j}_{1,\infty}$ for some $\gamma_j \geqslant 0$, then Young's inequality (Lemma 2.2) combined with (3.1) yields

$$\varphi_j * (\sigma_j(u)\xi_j) \in \mathcal{B}^{\beta_j+\gamma_j-1}_{p,\infty}.$$

In view of the Volterra equation (1.2) we obtain the relationship

$$\alpha = \min\{\beta_1 + \gamma_1 - 1, \beta_2 + \gamma_2 - 1\}.$$

In the following we associate the "rougher" signal with the first convolution term and thus assume $\beta_1 \leqslant \beta_2$ and $\gamma_1 \leqslant \gamma_2$. The Young type condition $\alpha + \beta_j > 1$ is then equivalent to $2\beta_1 + \gamma_1 > 2$.

Applying a fixed point argument, we first prove the existence of a unique solution to the Volterra equation (1.2) in this Young setting. Afterwards we will relax the regularity assumptions allowing for a more irregular driving signal ξ_1 in (1.2), see Subsection 3.2.

Proposition 3.1. *Let $p \geqslant 2$, $0 < \beta_1 \leqslant \beta_2$ and $0 < \gamma_1 \leqslant \gamma_2$ such that*

$$\alpha := \beta_1 + \gamma_1 - 1 \in (1/p, 1] \quad \textit{and} \quad 2\beta_1 + \gamma_1 > 2.$$

Suppose $u_0 \in \mathcal{B}^{\alpha}_{p,\infty}$, $\xi_j \in \mathcal{B}^{\beta_j-1}_{p,\infty}$, $\varphi_j \in \mathcal{B}^{\gamma_j}_{1,\infty}$ and $\sigma_j \in C^2$ with $\sigma_j(0) = 0$, for $j = 1, 2$. If $\max_{j=1,2} \|\sigma_j\|_{C^2}$ is sufficiently small depending on $\max_{j=1,2} \|\varphi_j\|_{\gamma_j,1,\infty}$, $\max_{j=1,2} \|\xi_j\|_{\beta_j-1,p,\infty}$ and $\|u_0\|_{\alpha,p,\infty}$, then the Volterra equation (1.2) has a unique solution $u \in \mathcal{B}^{\alpha}_{p,\infty}$.

Let us remark that the assumption $\alpha > \frac{1}{p}$ in Proposition 3.1 is only used for the embedding $\mathcal{B}^{\alpha}_{p,\infty} \subseteq L^\infty$. If we separately control the norms $\|\cdot\|_{\alpha,p.\infty}$ and $\|\cdot\|_\infty$ of the solution u, we may allow for $u \in \mathcal{B}^{1/p}_{p,\infty}$. This implies that the solution u of the Volterra equation (1.2) may have jumps but these jumps can only come from the initial condition u_0. This observation leads to the next proposition.

Proposition 3.2. *Let $p \geqslant 2$, $0 < \beta_1 \leqslant \beta_2$, $0 < \gamma_1 \leqslant \gamma_2$ such that*

$$\beta_1 + \gamma_1 - 1 \in (1/p, 1] \quad \textit{and} \quad \beta_1 + 1/p > 1.$$

Suppose $u_0 \in \mathcal{B}^{1/p}_{p,\infty} \cap L^\infty$, $\xi_j \in \mathcal{B}^{\beta_j-1}_{p,\infty}$, $\varphi_j \in \mathcal{B}^{\gamma_j}_{1,\infty}$ and $\sigma_j \in C^2$ with $\sigma_j(0) = 0$, for $j = 1, 2$. If $\max_{j=1,2} \|\sigma_j\|_{C^2}$ is sufficiently small depending on $\|u_0\|_{\frac{1}{p},p,\infty} + \|u_0\|_\infty$, $\max_{j=1,2} \|\varphi_j\|_{\gamma_j,1,\infty}$ and $\max_{j=1,2} \|\xi_j\|_{\beta_j-1,p,\infty}$, then the Volterra equation (1.2) has a unique solution $u \in \mathcal{B}^{1/p}_{p,\infty} \cap L^\infty$.

Let us remark that Proposition 3.1 is not a corollary of Proposition 3.2. However, since the corresponding proofs work analogously, we present here only the proof of Proposition 3.2 in order to avoid redundance.

Proof of Proposition 3.2. We study the solution map

$$\Phi\colon \mathcal{B}^{\frac{1}{p}}_{p,\infty} \cap L^\infty \to \mathcal{B}^{\frac{1}{p}}_{p,\infty} \cap L^\infty, \quad v \mapsto u := u_0 + \varphi_1 * \big(\sigma_1(v)\xi_1\big) + \varphi_2 * \big(\sigma_2(v)\xi_2\big).$$

If Φ is a well-defined map and a contraction, then the assertion follows from Banach's fixed point theorem.

Step 1: The map Φ is well-defined. Indeed, by Young's inequality (Lemma 2.2), the Besov embeddings $B^{1/p+\beta_j-1}_{p/2,\infty} \subseteq B^{\beta_j-1}_{p,\infty}$, $B^{\gamma_j+\beta_j-1}_{p,\infty} \subseteq B^{1/p}_{p,\infty}$ and $B^{\beta_j+\gamma_j-1}_{p,\infty} \subseteq L^\infty$ for $j=1,2$ and Bony's decomposition we have

$$
\begin{aligned}
&\|\Phi(v)\|_{\frac{1}{p},p,\infty} + \|\Phi(v)\|_\infty \\
&\quad\lesssim \|u_0\|_{\frac{1}{p},p,\infty} + \|u_0\|_\infty + \sum_{j=1,2} \|\varphi_j\|_{\gamma_j,1,\infty}\|\sigma_j(v)\xi_j\|_{\beta_j-1,p,\infty} \\
&\quad\lesssim \|u_0\|_{\frac{1}{p},p,\infty} + \|u_0\|_\infty + \sum_{j=1,2} \big(\|T_{\sigma_j(v)}\xi_j\|_{\beta_j-1,p,\infty} + \|\pi(\sigma_j(v),\xi_j)\|_{\frac{1}{p}+\beta_j-1,p/2,\infty} \\
&\qquad\qquad\qquad + \|T_{\xi_j}\sigma_j(v)\|_{\frac{1}{p}+\beta_j-1,p/2,\infty}\big)\|\varphi_j\|_{\gamma_j,1,\infty}.
\end{aligned}
$$

The paraproduct estimates (Lemma 2.1) and Lemma A.3 yield

$$
\begin{aligned}
&\|\Phi(v)\|_{\frac{1}{p},p,\infty} + \|\Phi(v)\|_\infty \\
&\quad\lesssim \|u_0\|_{\frac{1}{p},p,\infty} + \|u_0\|_\infty + \sum_{j=1,2} \|\varphi_j\|_{\gamma_j,1,\infty}\big(\|\sigma_j(v)\|_{\frac{1}{p},p,\infty} + \|\sigma_j(v)\|_\infty\big)\|\xi_j\|_{\beta_j-1,p,\infty} \\
&\quad\lesssim \|u_0\|_{\frac{1}{p},p,\infty} + \|u_0\|_\infty + \big(\|v\|_{\frac{1}{p},p,\infty} + \|v\|_\infty\big)\sum_{j=1,2} \|\varphi_j\|_{\gamma_j,1,\infty}\|\sigma_j\|_{C^1}\|\xi_j\|_{\beta_j-1,p,\infty}.
\end{aligned}
\tag{3.2}
$$

Hence, $\Phi(v) \in \mathcal{B}^{\frac{1}{p}}_{p,\infty} \cap L^\infty$ for every $v \in \mathcal{B}^{\frac{1}{p}}_{p,\infty} \cap L^\infty$.

Step 2: Invariance of Φ. We now verify that Φ maps the ball

$$
\mathcal{B}_K := \Big\{ v \in \mathcal{B}^{\frac{1}{p}}_{p,\infty} \cap L^\infty \;:\; \|v\|_{\frac{1}{p},p,\infty} + \|v\|_\infty \le 2K^2 \Big\} \subseteq \mathcal{B}^{\frac{1}{p}}_{p,\infty} \cap L^\infty
$$

into itself for some suitable constant $K \in \mathbb{R}$. Due to (3.2), there exists some $K \geqslant 1$ such that $\|u_0\|_{\frac{1}{p},p,\infty} + \|u_0\|_\infty \leqslant K$ and

$$
\begin{aligned}
&\|\Phi(v)\|_{\frac{1}{p},p,\infty} + \|\Phi(v)\|_\infty \\
&\quad\leqslant K\bigg(\|u_0\|_{\frac{1}{p},p,\infty} + \|u_0\|_\infty + \big(\|v\|_{\frac{1}{p},p,\infty} + \|v\|_\infty\big)\sum_{j=1,2} \|\varphi_j\|_{\gamma_j,1,\infty}\|\sigma_j\|_{C^1}\|\xi_j\|_{\beta_j-1,p,\infty}\bigg).
\end{aligned}
$$

If $\max_{j=1,2}\|\sigma_j\|_{C^1}$ is sufficiently small such that

$$
\max_{j=1,2} \|\varphi_j\|_{\gamma_j,1,\infty}\|\sigma_j\|_{C^1}\|\xi_j\|_{\beta_j-1,p,\infty} \le \frac{1}{4K},
$$

then for any $v \in \mathcal{B}_K$ we obtain $\|\Phi(v)\|_{\frac{1}{p},p,\infty} + \|\Phi(v)\|_\infty \le K^2 + K^2 \le 2K^2$.

Step 3: Φ is a contraction. To deduce the Lipschitz continuity of Φ on $\mathcal{B}_K$, let $v_1, v_2 \in \mathcal{B}_K$. By Young's inequality (Lemma 2.2) and the auxiliary Lemmas A.2 and A.3 we deduce

$$
\begin{aligned}
&\|\Phi(v_1)-\Phi(v_2)\|_{\frac{1}{p},p,\infty} + \|\Phi(v_1)-\Phi(v_2)\|_\infty \\
&\quad\lesssim \sum_{j=1,2} \|\varphi_j\|_{\gamma_j,1,\infty}\big(\|\sigma_j(v_1)-\sigma_j(v_2)\|_{\frac{1}{p},p,\infty} + \|\sigma_j(v_1)-\sigma_j(v_2)\|_\infty\big)\|\xi_j\|_{\beta_j-1,p,\infty} \\
&\quad\lesssim \Big(\sum_{j=1,2} \|\sigma_j\|_{C^2}\|\varphi_j\|_{\gamma_j,1,\infty}\|\xi_j\|_{\beta_j-1,p,\infty}\Big)\big(1 + \|v_1\|_{\frac{1}{p},p,\infty} + \|v_1\|_\infty + \|v_2\|_{\frac{1}{p},p,\infty} + \|v_2\|_\infty\big) \\
&\qquad\times \big(\|v_1-v_2\|_{\frac{1}{p},p,\infty} + \|v_1-v_2\|_\infty\big) \\
&\quad\lesssim \max_{j=1,2}\|\sigma_j\|_{C^2}\Big(\sum_{j=1,2} \|\varphi_j\|_{\gamma_j,1,\infty}\|\xi_j\|_{\beta_j-1,p,\infty}\Big)(1+4K^2)\big(\|v_1-v_2\|_{\frac{1}{p},p,\infty} + \|v_1-v_2\|_\infty\big).
\end{aligned}
$$

In conclusion, $\Phi\colon \mathcal{B}_K \to \mathcal{B}_K$ is Lipschitz continuous and it is a contraction for sufficiently small $\max_{j=1,2}\|\sigma_j\|_{C^2}$ depending on $\|u_0\|_{\frac{1}{p},p,\infty}+\|u_0\|_\infty$, $\max_{j=1,2}\|\varphi_j\|_{\gamma_j,1,\infty}$ and $\max_{j=1,2}\|\xi_j\|_{\beta_j-1,p,\infty}$. □

Remark 3.3. One can bypass the flatness condition on the vector fields σ_1 and σ_2, that is, $\max_{j=1,2}\|\sigma_j\|_{C^2}$ was assumed to be sufficiently small, by assuming that the kernel functions φ_1,φ_2 as well as the driving signals ξ_1,ξ_2 are supported on the positive real line, cf. Subsection 3.4.

3.2 Rough driving signals

The regularity assumptions on the driving signals proposed in Proposition 3.1, for obtaining a unique solution to the Volterra equation (1.2), are usually too strong for applications in probability theory. Namely, we have imposed the smoothness condition $\alpha+\beta_1>1$, which means $\beta_1>\frac{2-\gamma_1}{2}$. For instance, for ordinary differential equations we have $\gamma_1=1$ and thus $\beta_1>\frac{1}{2}$ excluding stochastic differential equations driven by the Brownian motion. In the sequel we will generalise this condition for the first convolution term in (1.2) to $2\alpha+\beta_1>1$ being equivalent to $\beta_1>\frac{3-2\gamma_1}{3}$. In the case $\gamma_1=1$ we then require $\beta_1>\frac{1}{3}$ which is in line with the classical rough path theory with one iterated integral. This paves the way for a wide range of applications of our results to, e.g., fractional Brownian motion, martingales and Lévy processes, see Section 5.

As discussed before, under the weaker regularity condition $\beta_1>\frac{3-2\gamma_1}{3}$ one main difficulty is to give a rigorous meaning to the product $\sigma_1(u)\xi_1$, cf. (3.1). To overcome this issue, we adapt the paracontrolled approach introduced by Gubinelli et al. [24]. In order to profit from the smoothing effect of the convolution with φ_1, we choose a paracontrolled ansatz that reflects the convolution structure of equation (1.2).

Abbreviating the regular terms by $u_{0,2}:=u_0+\varphi_2*(\sigma_2(u)\xi_2)$ and using Bony's decomposition (2.1), we may write

$$u=u_{0,2}+\varphi_1*\big(T_{\sigma_1(u)}\xi_1+\pi(\sigma_1(u),\xi_1)+T_{\xi_1}\sigma_1(u)\big).$$

Since the term $\varphi_1*T_{\sigma_1(u)}\xi_1$ is the least regular one, we choose the ansatz:

$$u=u_{0,2}+\varphi_1*T_{\sigma_1(u)}\xi_1+u^*$$

with remainder

$$u^*:=\varphi_1*\big(\pi(\sigma_1(u),\xi_1)+T_{\xi_1}\sigma_1(u)\big), \tag{3.3}$$

which is of regularity $\alpha+\beta_1-1+\gamma_1=2\alpha$ assuming everything is well-defined. However, this is a priori not true due to the resonant term $\pi(\sigma_1(u),\xi_1)$. To analyze this term, we use a linearization of $\sigma_1(u)$ (see [37, Proposition 4.1]) and again the ansatz for u to decompose the critical term $\pi(\sigma_1(u),\xi_1)$ into

$$\begin{aligned}\pi(\sigma_1(u),\xi_1)&=\sigma_1'(u)\pi(u,\xi_1)+\Pi_{\sigma_1}(u,\xi_1)\\&=\sigma_1'(u)\big(\pi(u_{0,2},\xi_1)+\pi(\varphi_1*T_{\sigma_1(u)}\xi_1,\xi_1)+\pi(u^*,\xi_1)\big)+\Pi_{\sigma_1}(u,\xi_1),\end{aligned} \tag{3.4}$$

where

$$\Pi_{\sigma_1}(u,\xi_1):=\pi(\sigma_1(u),\xi_1)-\sigma_1'(u)\pi(u,\xi_1)\in\mathcal{B}^{2\alpha+\beta_1-1}_{p/3,\infty}. \tag{3.5}$$

At this point the resonant term $\pi(\varphi_1*T_{\sigma_1(u)}\xi_1,\xi_1)$ is not yet well-defined. In order to continue our analysis, we need to compare

$$\pi(\varphi_1*T_{\sigma_1(u)}\xi_1,\xi_1)\quad\text{and}\quad\pi(\varphi_1*\xi_1,\xi_1),$$

which is indeed possible thanks to the following lemma.

Lemma 3.4. *Suppose there exists a constant $r \in \mathbb{R}$ such that for some $\gamma \geqslant 0$*

$$\varphi \in \mathcal{B}^{\gamma}_{1,\infty} \quad \text{and} \quad (\cdot - r)\varphi \in \mathcal{B}^{\gamma+1}_{1,\infty}.$$

If $f \in \mathcal{B}^{\alpha}_{p_1,\infty}$ and $g \in \mathcal{B}^{\beta}_{p_2,\infty}$ with $\alpha \in (0,1)$, $\beta \in \mathbb{R}$ and $p_1, p_2 \in [1,\infty]$ such that $\frac{1}{p} := \frac{1}{p_1} + \frac{1}{p_2} \leqslant 1$, then

$$R_\varphi(f,g) := \varphi * T_f g - T_{f(\cdot - r)}(\varphi * g) \in \mathcal{B}^{\alpha+\beta+\gamma}_{p,\infty}$$

with

$$\big\|R_\varphi(f,g)\big\|_{\alpha+\beta+\gamma,p,\infty} \lesssim \big(\|\varphi\|_{\gamma,1,\infty} + \|(\cdot + r)\varphi\|_{\gamma+1,1,\infty}\big)\|f\|_{\alpha,p_1,\infty}\|g\|_{\beta,p_2,\infty}.$$

Applying Lemma 3.4, we can write the "undefined" resonant term $\pi(\varphi_1 * T_{\sigma_1(u)}\xi_1, \xi_1)$ in (3.4) as

$$\begin{aligned} &\pi(\varphi_1 * T_{\sigma_1(u)}\xi_1, \xi_1) \\ &\quad = \pi(T_{\sigma_1(u(\cdot - r_1))}(\varphi_1 * \xi_1), \xi_1) + \pi(R_{\varphi_1}(\sigma_1(u), \xi_1), \xi_1) \\ &\quad = \sigma_1(u(\cdot - r_1))\pi(\varphi_1 * \xi_1, \xi_1) + \Gamma(\sigma_1(u(\cdot - r_1)), \varphi_1 * \xi_1, \xi_1) + \pi(R_{\varphi_1}(\sigma_1(u), \xi_1), \xi_1), \end{aligned} \tag{3.6}$$

for some $r_1 \in \mathbb{R}$ and where we used the commutator

$$\Gamma(f,g,h) := \pi(T_f g, h) - f\pi(g,h), \quad f,g,h \in \mathcal{S}'(\mathbb{R}),$$

satisfying

$$\|\Gamma(f,g,h)\|_{a+b+c,p/3,q} \lesssim \|f\|_{a,p,\infty}\|g\|_{b,p,\infty}\|h\|_{c,p,\infty}, \tag{3.7}$$

for $p \geqslant 3$, $a \in (0,1)$ and $b, c \in \mathbb{R}$ with $a+b+c > 0$ and $b + c < 0$, see the so-called commutator lemma [37, Lemma 4.4]. We thus have reduced the critical term $\pi(\sigma_1(u), \xi_1)$ to the resonant term $\pi(\varphi_1 * \xi_1, \xi_1)$. The latter one does not depend on the particular equation (1.2) in the sense that it neither depends on u nor on σ_1, but only on the signal ξ_1 and the convolution kernel φ_1.

Proof of Lemma 3.4. We may assume that $f \in \mathcal{B}^{\alpha+1/p_1+1}_{p_1,\infty}$ being a dense subset of $\mathcal{B}^{\alpha}_{p_1,\infty}$ such that the result will follow by continuity. We will use the notation $S_j f := \sum_{k<j-1} \Delta_j f$ for $f \in \mathcal{S}'(\mathbb{R})$. Noting that

$$\Delta_j(\varphi * g) = \mathcal{F}^{-1}\rho_j * \varphi * g = \varphi * (\Delta_j g), \tag{3.8}$$

and since $\sum_j \Delta_j(\varphi * g)$ converges if $\sum_j \Delta_j g$ converges by Lemma 2.2, we have

$$\varphi * T_f g(x) - T_{f(\cdot - r)}(\varphi * g)(x) = \sum_{j \geq -1} R_j(x)$$

with

$$\begin{aligned} R_j(x) :=& \varphi * \big(S_{j-1} f \Delta_j g\big)(x) - S_{j-1} f(x-r)(\varphi * \Delta_j g)(x) \\ =& \int_{\mathbb{R}} \varphi(x-z)\big(S_{j-1}f(z) - S_{j-1}f(x-r)\big)\Delta_j g(z)\,\mathrm{d}z \\ =& \int_0^1 \int_{\mathbb{R}} (z-x+r)\varphi(x-z) S_{j-1}f'(x-r+t(z-x+r))\Delta_j g(z)\,\mathrm{d}z\,\mathrm{d}t \end{aligned} \tag{3.9}$$

where we apply Fubini's theorem in the last line using that f' is bounded. Since $(y-r)\varphi(y) \in L^1(\mathbb{R}) \subseteq \mathcal{B}^1_{1,\infty}$, the Fourier transform of R_j is well-defined and we have

$$\begin{aligned} \mathcal{F}R_j(\xi) &= \int_0^1 \int_{\mathbb{R}^2} e^{i\xi(x-z)}(z-x+r)\varphi(x-z)e^{i\xi z} S_{j-1}f'(x-r+t(z-x+r))\Delta_j g(z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^1 \int_{\mathbb{R}^2} e^{i\xi x}(x-r)\varphi(x)e^{i\xi z} S_{j-1}f'(x+z-r-tx+tr))\Delta_j g(z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

$$= -\int_0^1 \int_{\mathbb{R}} e^{i\xi x}(x-r)\varphi(x)\mathcal{F}\big[S_{j-1}\big(f'(\cdot+(1-t)(x-r))\big)\Delta_j g\big](\xi)\,\mathrm{d}x\,\mathrm{d}t.$$

Since $\mathcal{F}\big[\Delta_j g S_{j-1}\big(f'(\cdot+(1-t)(x-r))\big)\big]$ is supported on an annulus with radius of order 2^j (uniformly in t and x), we conclude that $\Delta_k R_j = 0$ if $|k-j| \gtrsim 1$. Consequently,

$$\|\varphi * T_f g(x) - T_{f(\cdot - r)}(\varphi * g)(x)\|_{\alpha+\beta+\gamma,p,\infty} \lesssim \sup_{k\geq -1} 2^{k(\alpha+\beta+\gamma)} \sum_{j\sim k} \|\Delta_k R_j\|_{L^p}. \tag{3.10}$$

Let us introduce $\varphi_k := \mathcal{F}^{-1}\rho_k * \varphi$ and recall the operator $[\Delta_k, f]g := \Delta_k(fg) - f\Delta_k g$. We have

$$\begin{aligned}\Delta_k R_j &= \varphi_k * \big(S_{j-1} f \Delta_j g\big) - \Delta_k\big(S_{j-1} f(\cdot - r)(\varphi * \Delta_j g)\big)\\ &= \varphi_k * \big(S_{j-1} f \Delta_j g\big) - S_{j-1} f(\cdot - r)(\varphi_k * \Delta_j g) - [\Delta_k, S_{j-1} f(\cdot - r)](\varphi * \Delta_j g).\end{aligned} \tag{3.11}$$

For the third term in the above display [37, Lemma 4.3], (3.8) and Lemma 2.2 yield

$$\begin{aligned}\sum_{j\sim k} \big\|[\Delta_k, S_{j-1} f(\cdot - r)](\varphi * \Delta_j g)\big\|_{L^p} &\leq \sum_{j\sim k} 2^{-k\alpha}\|S_{j-1} f\|_{\alpha,p_1,\infty}\|\Delta_j(\varphi * g)\|_{L^{p_2}}\\ &\lesssim \sum_{j\sim k} 2^{-k\alpha}\|f\|_{\alpha,p_1,\infty} 2^{-j(\beta+\gamma)}\|\varphi * g\|_{\beta+\gamma,p_2,\infty}\\ &\lesssim 2^{-k(\alpha+\beta+\gamma)}\|\varphi\|_{\gamma,1,\infty}\|f\|_{\alpha,p_1,\infty}\|g\|_{\beta,p_2,\infty}.\end{aligned} \tag{3.12}$$

Exactly as in (3.9) the first two terms in (3.11) can be written as

$$\begin{aligned}&\varphi_k * \big(S_{j-1} f \Delta_j g\big)(x) - S_{j-1} f(x-r)(\varphi_k * \Delta_j g)(x)\\ &\quad= \int_0^1 \int_{\mathbb{R}} (z-x+r)\varphi_k(x-z)\Delta_j g(z) S_{j-1} f'(x+t(z-x)+(t-1)r)\,\mathrm{d}z\,\mathrm{d}t.\end{aligned}$$

Abbreviating $\widetilde{\varphi}_k(x) = (x-r)\varphi_k(x)$, Hölder's inequality gives

$$\begin{aligned}&\big|\varphi_k * \big(S_{j-1} f \Delta_j g\big)(x) - S_{j-1} f(x-r)(\varphi_k * \Delta_j g)(x)\big|\\ &\quad\leqslant \int_0^1 \int_{\mathbb{R}} |\widetilde{\varphi}_k(x-z)|^{1-1/p}|\widetilde{\varphi}_k(x-z)|^{1/p}\big|S_{j-1} f'\big(x+t(z-x)+(t-1)r\big)\Delta_j g(z)\big|\,\mathrm{d}z\,\mathrm{d}t\\ &\quad\leqslant \|\widetilde{\varphi}_k\|_{L^1}^{1-1/p}\int_0^1 \Big(\int_{\mathbb{R}} |\widetilde{\varphi}_k(x-z)|\big|S_{j-1} f'\big(x+t(z-x)+(t-1)r\big)\Delta_j g(z)\big|^p\,\mathrm{d}z\Big)^{1/p}\mathrm{d}t.\end{aligned}$$

Using that $\alpha - 1 < 0$ and [2, Proposition 2.79], we obtain by

$$\begin{aligned}&\big\|\varphi_k * \big(S_{j-1} f \Delta_j g\big) - S_{j-1} f(\varphi_k * \Delta_j g)\big\|_{L^p}\\ &\quad\lesssim \|\widetilde{\varphi}_k\|_{L^1}^{1-1/p}\Big(\int_0^1 \int_{\mathbb{R}^2} |\widetilde{\varphi}_k(x)|\big|\Delta_j g(z) S_{j-1} f'(x+z-tx+(t-1)r)\big|^p\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t\Big)^{1/p}\\ &\quad\leqslant \|\widetilde{\varphi}_k\|_{L^1}^{1-1/p}\Big(\int_0^1 \int_{\mathbb{R}} |\widetilde{\varphi}_k(x)|\|\Delta_j g\|_{L^{p_2}}^p\|S_{j-1} f'\|_{L^{p_1}}^p\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/p}\\ &\quad\lesssim 2^{-j(\alpha+\beta+\gamma)}\|g\|_{\beta,p_2,\infty}\|f'\|_{\alpha-1,p_1,\infty} 2^{j(\gamma+1)}\|\widetilde{\varphi}_k(x)\|_{L^1}.\end{aligned} \tag{3.13}$$

Since $\|f'\|_{\alpha-1,p_1,\infty} \lesssim \|f\|_{\alpha,p_1,\infty}$, it suffices to show $\|\widetilde{\varphi}_k\|_{L^1} = \|(x-r)\varphi_k(x)\|_{L^1} \lesssim 2^{-k(\gamma+1)}$. Note that

$$\begin{aligned}(x-r)\varphi_k(x) &= \int_{\mathbb{R}} (x-z+z-r)\mathcal{F}^{-1}\rho_k(x-z)\varphi(z)\,\mathrm{d}z\\ &= \int_{\mathbb{R}} (x-z)\mathcal{F}^{-1}\rho_k(x-z)\varphi(z)\,\mathrm{d}z + \int_{\mathbb{R}} \mathcal{F}^{-1}\rho_k(x-z)(z-r)\varphi(z)\,\mathrm{d}z\\ &= \big((y\mathcal{F}^{-1}\rho_k(y)) * \varphi\big)(x) + \big(\mathcal{F}^{-1}\rho_k * ((y-r)\varphi(y))\big)(x)\end{aligned}$$

$$
\begin{aligned}
&= -i\big(\mathcal{F}^{-1}[\rho_k'] * \varphi\big)(x) + \Delta_k\big((y-r)\varphi(y)\big)(x)\\
&= -i\sum_{j\sim k} \mathcal{F}^{-1}[\rho_k'] * \Delta_j\varphi(x) + \Delta_k\big((y-r)\varphi(y)\big)(x),
\end{aligned}
$$

which by Young's inequality implies

$$
\begin{aligned}
\|(x-r)\varphi_k(x)\|_{L^1} &\leqslant \sum_{j\sim k} \|\mathcal{F}^{-1}[\rho_k']\|_{L^1}\|\Delta_j\varphi\|_{L^1} + \|\Delta_k\big((y-r)\varphi(y)\big)\|_{L^1}\\
&= \sum_{j\sim k} \|\mathcal{F}^{-1}[\rho'](2^k\cdot)\|_{L^1}\|\Delta_j\varphi\|_{L^1} + \|\Delta_k\big((y-r)\varphi(y)\big)\|_{L^1} \qquad (3.14)\\
&= 2^{-k(\gamma+1)}\|\mathcal{F}^{-1}[\rho']\|_{L^1}\|\varphi\|_{\gamma,1,\infty} + 2^{-k(\gamma+1)}\|(\cdot-r)\varphi\|_{\gamma+1,1,\infty}.
\end{aligned}
$$

Finally, we combine the estimates (3.12), (3.13) and (3.14) to get

$$
\sum_{j\sim k} \|\Delta_k R_j\|_{L^p} \lesssim 2^{-k(\alpha+\beta+\gamma)}\big(\|\varphi\|_{\gamma,1,\infty} + \|(\cdot-r)\varphi\|_{\gamma+1,1,\infty}\big)\|f\|_{\alpha,p_1,\infty}\|g\|_{\beta,p_2,\infty}.
$$

In view of (3.10) we have proven the asserted bound for $\big\|R_\varphi(f,g)\big\|_{\alpha+\beta+\gamma,p,\infty}$ and in particular $R_\varphi(f,g)\in\mathcal{B}^{\alpha+\beta+\gamma}_{p,\infty}$. □

Remark 3.5. Lemma 3.4 can be seen as a counterpart to the integration by parts formula as used in the context of classical rough differential equations of the form $Du = F(u)\xi$ with the differential operator D and a signal $\xi\in\mathcal{B}^{\beta-1}_{p,\infty}$, see for example [24, 37]. Defining the integration operator $I := D^{-1}$ to be the inverse of D and denoting by ϑ the solution of $D\vartheta = \xi$, one has

$$
T_{F(u)}\vartheta = I(DT_{F(u)}\vartheta) = IT_{F(u)}\xi + IT_{DF(u)}\vartheta,
$$

where the second term is of regularity 2α. Heuristically speaking, for Volterra equations we replace the integration operator $I\colon f\mapsto I(f)$ by the convolution operator $f\mapsto\varphi * f$ and set $\vartheta := \varphi * \xi$.

The resonant term $\pi(\varphi_1 * \xi_1, \xi_1)$ appearing in (3.6) turns out to be the necessary "additional information" one needs to postulate in order to give a meaning to the Volterra equation (1.2) with rough driving signals ξ_1. It corresponds to the iterated integrals in rough path theory (cf. [31, 32, 21]) or the models in Hairer's theory of regularity structures (cf. [26, 27]). For the construction of $\pi(\varphi_1 * \xi_1, \xi_1)$ for certain stochastic processes we refer to Section 4.

In the present context we introduce the notion of convolutional rough paths.

Definition 3.6. Let $\beta,\gamma>0$, $p\in[2,\infty]$ and set $\alpha := \beta+\gamma-1$. The space of smooth functions $\xi\colon\mathbb{R}\to\mathbb{R}^n$ with compact support is denoted by $\mathcal{C}_c^\infty$. Given a function $\varphi\in\mathcal{B}^\gamma_{1,\infty}$, the closure of the set

$$
\big\{\big(\xi,\pi(\varphi*\xi,\xi)\big)\ :\ \xi\in\mathcal{C}_c^\infty\big\}\subseteq\mathcal{B}^{\beta-1}_{p,\infty}\times\mathcal{B}^{\alpha+\beta-1}_{p/2,\infty}
$$

with respect to the norm $\|\xi\|_{\beta-1,p,\infty} + \|\pi(\varphi*\xi,\xi)\|_{\alpha+\beta-1,p/2,\infty}$ is denoted by $\mathcal{B}^{\beta,\gamma}_p(\varphi)$ and $(\xi,\mu)\in\mathcal{B}^{\beta,\gamma}_p(\varphi)$ is called *convolutional rough path*.

Assuming $\pi(\varphi_1*\xi_1,\xi_1)$ is well-defined, by the previous analysis we know that u^* from (3.3) is also well-defined. Hence, Bony's decomposition and Lemma 3.4 allow to rewrite the rough term $\varphi_1 * (\sigma_1(u)\xi_1)$ as

$$
\begin{aligned}
\varphi_1 * (\sigma_1(u)\xi_1) &= \varphi_1 * \big(T_{\sigma_1(u)}\xi_1 + \pi(\sigma_1(u),\xi_1) + T_{\xi_1}\sigma_1(u)\big)\\
&= T_{\sigma_1(u(\cdot-r_1))}(\varphi_1*\xi_1) + \underbrace{u^* + R_{\varphi_1}(\sigma_1(u),\xi_1)}_{\in\mathcal{B}^{2\alpha}_{p/2,\infty}}.
\end{aligned}
$$

For the more regular drift term $\varphi_2 * (\sigma_2(u)\xi_2)$ we observe (using similar calculations as in the Young setting and Lemma 3.4) a control structure with respect to $\varphi_2 * \xi_2$:

$$
\varphi_2 * (\sigma_2(u)\xi_2) = \varphi_2 * T_{\sigma_2(u)}\xi_2 + \varphi_2 * \big(\underbrace{\pi(\sigma_2(u),\xi_2) + T_{\xi_2}(\sigma_2(u))}_{\in\mathcal{B}^{\alpha+\beta_2-1}_{p/2,\infty}}\big)
$$

$$= T_{\sigma_2(u(\cdot - r_2))}(\varphi_2 * \xi_2) + \underbrace{\varphi_2 * \big(\pi(\sigma_2(u), \xi_2) + T_{\xi_2}(\sigma_2(u))\big) + R_{\varphi_2}(\sigma_2(u), \xi_2)}_{\in \mathcal{B}^{2\alpha}_{p/2,\infty}},$$

for some $r_2 \in \mathbb{R}$. Therefore, the ansatz for a solution u to the Volterra equation (1.2) leads to the following "paracontrolled" structure:

Definition 3.7. Let $p \geqslant 1$ and $\alpha > 1/p$. A function $v \in \mathcal{B}^{\alpha}_{p,\infty}$ is called *paracontrolled* by $w_1, w_2 \in \mathcal{B}^{\alpha}_{p,\infty}$ if there are $v^{(1)}, v^{(2)} \in \mathcal{B}^{\alpha}_{p,\infty}$ such that $v^{\#} := v - T_{v^{(1)}} w_1 - T_{v^{(2)}} w_2 \in \mathcal{B}^{2\alpha}_{p/2,\infty}$. The space of all such triples $(v, v^{(1)}, v^{(2)}) \in (\mathcal{B}^{\alpha}_{p,\infty})^3$ where v paracontrolled by $w_1, w_2 \in \mathcal{B}^{\alpha}_{p,\infty}$ is denoted by $\mathcal{D}^{\alpha}_p(w_1, w_2)$ equipped with the norm

$$\|v^{(1)}\|_{\alpha,p,\infty} + \|v^{(2)}\|_{\alpha,p,\infty} + \|v - T_{v^{(1)}} w_1 - T_{v^{(2)}} w_2\|_{2\alpha,p/2,\infty}.$$

Remark 3.8. Note that for any $v^{(1)}, v^{(2)}, w_1, w_2 \in \mathcal{B}^{\alpha}_{p,\infty}$ and $v^{\#} \in \mathcal{B}^{2\alpha}_{p,\infty}$ the function $v := T_{v^{(1)}} w_1 - T_{v^{(2)}} w_2 + v^{\#}$ is paracontrolled by w_1, w_2 and, in particular, v is an element of $\mathcal{B}^{\alpha}_{p,\infty}$. Indeed, Lemma 2.1 and the embeddings $\mathcal{B}^{2\alpha}_{p/2,\infty} \subseteq \mathcal{B}^{\alpha}_{p,\infty} \subseteq L^{\infty}$ imply

$$\|v\|_{\alpha,p,\infty} \lesssim \sum_{j=1,2} \|T_{v^{(j)}} w_j\|_{\alpha,p,\infty} + \|v^{\#}\|_{\alpha,p,\infty} \lesssim \sum_{j=1,2} \|v^{(j)}\|_{\alpha,p,\infty} \|w_j\|_{\alpha,p,\infty} + \|v^{\#}\|_{2\alpha,p/2,\infty}.$$

It is natural to require the same paracontrolled structure for the initial condition u_0 as for the solution for u. In other words, u_0 is assumed to be of the form

$$u_0 = T_{u_0^{(1)}}(\varphi_1 * \xi_1) + T_{u_0^{(2)}}(\varphi_2 * \xi_2) + u_0^{\#} \quad \text{for some } u_0^{(1)}, u_0^{(2)} \in \mathcal{B}^{\alpha}_{p,\infty},\ u_0^{\#} \in \mathcal{B}^{2\alpha}_{p/2,\infty}.$$

Remark 3.9. A similar requirement for initial conditions u_0 appears in the context of delay differential equations driven by rough paths where u_0 is usually a path and not only a constant, cf. Neuenkirch et al. [35, Theorem 1.1]. Hence, in order to ensure that the rough path integral is well-defined, Neuenkirch et al. [35] suppose the initial condition to be a controlled path in the sense of Gubinelli [23].

To sum up, the ansatz reads as

$$u = T_{u_0^{(1)} + \sigma_1(u(\cdot - r_1))}(\varphi_1 * \xi_1) + T_{u_0^{(2)} + \sigma_2(u(\cdot - r_2))}(\varphi_2 * \xi_2) + u^{\#} \tag{3.15}$$

with

$$u^{\#} := u_0^{\#} + \sum_{j=1,2} \big(\varphi_j * \big(\pi(\sigma_j(u), \xi_j) + T_{\xi_j}\sigma_j(u)\big) + R_{\varphi_j}(\sigma_j(u), \xi_j)\big) \in \mathcal{B}^{2\alpha}_{p/2,\infty}. \tag{3.16}$$

Note that, imposing the Young type condition $\alpha + \beta_2 > 1$ on the regularity of the drift term $\varphi_2 * (\sigma_2(u)\xi_2)$ ensures especially that the cross terms $\pi(\varphi_1 * \xi_1, \xi_2)$ and $\pi(\varphi_2 * \xi_2, \xi_1)$ are well-defined.

Postulating the paracontrolled structure for the initial condition, we show in the following that the *Itô-Lyons map* $\widehat{S} := \widehat{S}_{\varphi_1,\varphi_2}$ given by

$$\begin{aligned} \widehat{S} \colon (\mathcal{B}^{\alpha}_{p,\infty})^2 \times \mathcal{B}^{2\alpha}_{p/2,\infty} \times \mathcal{B}^{\beta_1,\gamma_1}_p(\varphi_1) \times \mathcal{B}^{\beta_2}_{p,\infty} &\to \mathcal{B}^{\alpha}_{p,\infty}, \\ \big(u_0^{(1)}, u_0^{(2)}, u_0^{\#}, (\xi_1, \mu), \xi_2\big) &\mapsto u, \end{aligned} \tag{3.17}$$

where u is the solution to the Volterra equation (1.2) given the initial condition $u_0 := T_{u_0^{(1)}}(\varphi_1 * \xi_1) + T_{u_0^{(2)}}(\varphi_2 * \xi_2) + u_0^{\#}$ and the inputs (ξ_1, μ), ξ_2, has indeed a unique locally Lipschitz continuous extension from smooth driving signals $(\xi_1, \pi(\varphi_1 * \xi_1, \xi_1))$ to the space of convolutional rough paths (ξ_1, μ). For fixed signals $((\xi_1, \mu), \xi_2)$ the ansatz from above and the proof of the following theorem reveals that the Itô-Lyons maps, more precisely, $\mathcal{D}^{\alpha}_p(\varphi_1 * \xi_1, \varphi_2 * \xi_2)$ into $\mathcal{D}^{\alpha}_p(\varphi_1 * \xi_1, \varphi_2 * \xi_2)$.

Theorem 3.10. *Let $p \in [3,\infty]$, $0 < \beta_1 \leqslant \beta_2 \leqslant 1$ and $0 < \gamma_1 \leqslant \gamma_2$ satisfy $\alpha := \beta_1 + \gamma_1 - 1 \in (\frac{1}{3}, 1)$,* $2\alpha + \beta_1 > 1$ and $\alpha + \beta_2 > 1$. *For*

(i) $\sigma_1 \in C^3$ *and* $\sigma_2 \in C^2$ *with* $\sigma_1(0) = \sigma_2(0) = 0$,

(ii) $\varphi_j \in \mathcal{B}^{\gamma_j}_{1,\infty}$ *such that there exists* $r_j \in \mathbb{R}$ *with* $\|(\cdot - r_j)\varphi_j\|_{\gamma_j+1,1,\infty} < \infty$, *for* $j = 1, 2$,

(iii) $(\xi_1, \mu) \in \mathcal{B}^{\beta_1-1,\gamma_1}_p(\varphi_1)$ *and* $\xi_2 \in \mathcal{B}^{\beta_2-1}_{p,\infty}$,

(iv) $(u_0^{(1)}, u_0^{(2)}, u_0^{\#}) \in (\mathcal{B}^{\alpha}_{p,\infty})^2 \times \mathcal{B}^{2\alpha}_{p/2,\infty}$,

the Volterra equation (1.2) with initial condition $u_0 = T_{u_0^{(1)}}(\varphi_1 * \xi_1) + T_{u_0^{(2)}}(\varphi_2 * \xi_2) + u_0^{\#}$ *has a unique solution if* $\Delta := \|\sigma_1\|_{C^3}\|\varphi_1\|_{\gamma_1,1,\infty} + \|\sigma_2\|_{C^2}\|\varphi_2\|_{\gamma_2,1,\infty}$ *is sufficiently small depending on* $((u_0^{(1)}, u_0^{(2)}, u_0^{\#}), (\xi_1, \mu), \xi_2)$ *and* φ_1, φ_2. *Moreover, the Itô-Lyons map* $\widehat{S}$ *from (3.17) is locally Lipschitz continuous around* $((u_0^{(1)}, u_0^{(2)}, u_0^{\#}), (\xi_1, \mu), \xi_2)$.

Remark 3.11. Theorem 3.10 provides the local Lipschitz continuity of the Itô-Lyons map $\widehat{S}$ on the rough path space $\mathcal{B}^{\beta_1,\gamma_1}_p(\varphi_1)$, which contains (convolutional) geometric rough paths with jumps. Indeed, considering $\gamma_1 > 1$ and $p = 3$, the parameter assumptions in Theorem 3.10 only require

$$\beta_1 > 1 - \frac{2}{3}\gamma_1 \quad \text{and} \quad \beta_1 > \frac{1}{3} + 1 - \gamma_1.$$

Hence, we can choose $\beta_1 < 1/p$, which implies that $\mathcal{B}^{\beta_1,\gamma_1}_p(\varphi_1)$ contains discontinuous paths, and Theorem 3.10 is still applicable.

The existence of a continuous extension of the Itô-Lyons map from the space of smooth paths to a space of geometric rough paths containing discontinuous paths seems to be due to the smoothing property of the kernel function $\varphi_1 \in \mathcal{B}^{\gamma_1}_{1,\infty}$ for $\gamma_1 > 1$, cf. [37, Remark 5.11]. Note that even if the driving rough path may possess jumps, the solution of the Volterra equation is still a continuous functions as we require $\alpha > 1/3$. In order to obtain a continuous extension of the Itô-Lyons map acting on smooth paths to discontinuous rough paths in the case of classical rough differential equations (corresponding to $\gamma_1 = 1$) requires to consider the rough paths enhanced with an additional information given by the so-called path functionals, see the work of Chevyrev and Friz [11].

3.3 Proof of Theorem 3.10

Most objects appearing the paracontrolled approach to the Volterra equation (1.2) come only with local Lipschitz estimates of their Besov norms. Therefore, as a first step towards a proof of Theorem 3.10 we provide the a priori bounds for solutions of the Volterra equation (1.2).

Proposition 3.12. *Let $p \in [3,\infty]$, $0 < \beta_1 \leqslant \beta_2 \leqslant 1$ and $0 < \gamma_1 \leqslant \gamma_2$ satisfy $\alpha := \beta_1 + \gamma_1 - 1 \in (\frac{1}{3}, 1)$,* $2\alpha + \beta_1 > 1$ and $\alpha + \beta_2 > 1$. *Suppose that*

(i) $\sigma_1 \in C^2$ *and* $\sigma_2 \in C^1$ *with* $\sigma_1(0) = \sigma_2(0) = 0$,

(ii) $\varphi_j \in \mathcal{B}^{\gamma_j}_{1,\infty}$ *such that there exists* $r_j \in \mathbb{R}$ *with* $\|(\cdot - r_j)\varphi_j\|_{\gamma_j+1,1,\infty} < \infty$, *for* $j = 1, 2$,

(iii) $\xi_1 \in \mathcal{C}^\infty_c$ *and* $\xi_2 \in \mathcal{B}^{\beta_2-1}_{p,\infty}$,

(iv) $(u_0^{(1)}, u_0^{(2)}, u_0^{\#}) \in (\mathcal{B}^{\alpha}_{p,\infty})^2 \times \mathcal{B}^{2\alpha}_{p/2,\infty}$.

Let $u_0 := u_0^{\#} + T_{u_0^{(1)}}(\varphi_1 * \xi_1) + T_{u_0^{(2)}}(\varphi_2 * \xi_2)$ *be the paracontrolled initial condition. Setting*

$$\Delta := \|\sigma_1\|_{C^2}\|\varphi_1\|_{\gamma_1,1,\infty} + \|\sigma_2\|_{C^1}\|\varphi_2\|_{\gamma_2,1,\infty}, \qquad C_\sigma := \|\sigma_1\|_{C^1} + \|\sigma_2\|_{C^1} + 1,$$

$$C_\varphi := \sum_{j=1,2} \big(\|\varphi_j\|_{\gamma_j,1,\infty} + \|(\cdot - r_j)\varphi_j\|_{\gamma_j+1,1,\infty} \big) + 1, \quad C_{u_0} := \|u_0^{(1)}\|_{\alpha,p,\infty} + \|u_0^{(2)}\|_{\alpha,p,\infty} + 1$$

and

$$C_\xi := \|\pi(\varphi_1 * \xi_1, \xi_1)\|_{\alpha+\beta_1-1,p/2,\infty} + \sum_{j=1,2} \Big(1 + \sum_{k=1,2} \|\varphi_k\|_{\gamma_k,1,\infty} \|\xi_k\|_{\beta_k-1,p,\infty} \Big) \|\xi_j\|_{\beta_j-1,p,\infty},$$

there is a constant $c > 0$ depending only on α and p such that, if $\Delta C_\sigma C_\varphi C_\xi C_{u_0} \leqslant c$, then

$$\begin{aligned} \|u\|_{\alpha,p,\infty} &\leqslant 2\|u_0\|_{\alpha,p,\infty} + \|u_0^\#\|_{2\alpha,p/2,\infty} + 1 \\ &\lesssim \sum_{j=1,2} \|\varphi_j\|_{\gamma_j,1,\infty} \|\xi_j\|_{\beta_j-1,p,\infty} \|u_0^{(j)}\|_{\alpha,p,\infty} + \|u_0^\#\|_{2\alpha,p/2,\infty} + 1. \end{aligned}$$

Proof. Using

$$u = u_0 + \sum_{j=1,2} \varphi_j * \big(T_{\sigma_j(u)} \xi_j + \pi(\sigma_j(u), \xi_j) + T_{\xi_j} \sigma_j(u) \big),$$

$\alpha \leqslant \beta_j + \gamma_j - 1$, the generalized Young inequality (Lemma 2.2) and Besov embeddings (as $\alpha > 1/p$), we have

$$\begin{aligned} \|u\|_{\alpha,p,\infty} \lesssim \|u_0\|_{\alpha,p,\infty} + \sum_{j=1,2} \|\varphi_j\|_{\gamma_j,1,\infty} \big(&\|T_{\sigma_j(u)} \xi_j\|_{\beta_j-1,p,\infty} \\ &+ \|\pi(\sigma_j(u), \xi_j)\|_{\alpha+\beta_j-1,p/2,\infty} + \|T_{\xi_j} \sigma_j(u)\|_{\beta_j-1,p,\infty} \big). \end{aligned}$$

By the paraproduct estimates (Lemma 2.1) and Lemma A.3 we obtain for $j = 1, 2$

$$\|T_{\sigma_j(u)} \xi_j\|_{\beta_j-1,p,\infty} \lesssim \|\sigma_j(u)\|_\infty \|\xi_j\|_{\beta_j-1,p,\infty} \lesssim \|\sigma_j\|_\infty \|\xi_j\|_{\beta_j-1,p,\infty}$$

and

$$\|T_{\xi_j} \sigma_j(u)\|_{\beta_j-1,p,\infty} \lesssim \|\xi_j\|_{\beta_j-1,p,\infty} \|\sigma_j(u)\|_{\alpha,p,\infty} \lesssim \|\xi_j\|_{\beta_j-1,p,\infty} \|\sigma_j\|_{C^1} \|u\|_{\alpha,p,\infty}. \tag{3.18}$$

We now need to bound the resonant terms $\|\pi(\sigma_j(u), \xi_j)\|_{\alpha+\beta_j-1,p/2,\infty}$ for $j = 1, 2$. For $j = 2$ we apply again the paraproduct estimates (Lemma 2.1) and Lemma A.3 to get

$$\|\pi(\sigma_2(u), \xi_2)\|_{\alpha+\beta_2-1,p/2,\infty} \lesssim \|\sigma_2(u)\|_{\alpha,p,\infty} \|\xi_2\|_{\beta_2-1,p,\infty} \lesssim \|\sigma_2\|_{C^1} \|u\|_{\alpha,p,\infty} \|\xi_2\|_{\beta_2-1,p,\infty}$$

using the assumption $\alpha + \beta_2 - 1 > 0$.

For $j = 1$, in order to avoid a quadratic bound of $\Pi_\sigma(u, \xi)$ (cf. (3.5) and [37, Proposition 4.1]), we apply the linearization from Lemma A.4, which provides a function $S_{\sigma_1}(u) \in \mathcal{B}^{2\alpha}_{p/2,\infty}$ such that

$$\pi(\sigma_1(u), \xi_1) = \pi(T_{\sigma_1'(u)} u, \xi_1) + \pi(S_{\sigma_1}(u), \xi_1).$$

Writing the ansatz (3.15) as

$$u = \sum_{k=1,2} T_{\widetilde{u}_k} (\varphi_k * \xi_k) + u^\# \quad \text{with} \quad \widetilde{u}_k := u_0^{(k)} + \sigma_k(u(\cdot - r_k)), \quad k = 1, 2, \tag{3.19}$$

and in combination with the commutator estimate (3.7), we find that

$$\begin{aligned} \pi(\sigma_1(u), \xi_1) &= \sum_{k=1,2} \pi\big(T_{\sigma_1'(u)} \big(T_{\widetilde{u}_k} (\varphi_k * \xi_k) \big), \xi_1 \big) + \pi(T_{\sigma_1'(u)} u^\#, \xi_1) + \pi(S_{\sigma_1}(u), \xi_j) \\ &= \sum_{k=1,2} \Big(\sigma_1'(u) \pi(T_{\widetilde{u}_k} (\varphi_k * \xi_k), \xi_1) + \Gamma(\sigma_1'(u), T_{\widetilde{u}_k} (\varphi_k * \xi_k), \xi_1) \Big) \\ &\qquad + \pi(T_{\sigma_1'(u)} u^\#, \xi_1) + \pi(S_{\sigma_1}(u), \xi_1) \end{aligned}$$

$$= \sum_{k=1,2} \Big(\sigma_1'(u)\widetilde{u}_k \pi(\varphi_k * \xi_k, \xi_1) + \sigma_1'(u)\Gamma(\widetilde{u}_k, \varphi_k * \xi_k, \xi_1)$$
$$+ \Gamma(\sigma_1'(u), T_{\widetilde{u}_k}(\varphi_k * \xi_k), \xi_1) \Big) + \pi(T_{\sigma_1'(u)}u^{\#}, \xi_1) + \pi(S_{\sigma_1}(u), \xi_1).$$

In the following we estimate these five terms, with $k = 1, 2$, frequently using Besov embeddings ($\alpha > 1/p$), the paraproduct estimates (Lemma 2.1) and the auxiliary Besov estimates (Lemma A.1, A.2 and A.3).

Defining $F(x, y) := \sigma_1'(x)\sigma_k(y)$ and owing to $2\alpha + \beta_1 > 1$, we have

$$\begin{aligned}
&\|\sigma_1'(u)\widetilde{u}_k \pi(\varphi_k * \xi_k, \xi_1)\|_{\alpha+\beta_1-1,p/2,\infty} \\
&\quad \lesssim \|\sigma_1'(u)\widetilde{u}_k\|_{\alpha,p,\infty} \|\pi(\varphi_k * \xi_k, \xi_1)\|_{\alpha+\beta_1-1,p/2,\infty} \\
&\quad \lesssim \big(\|\sigma_1'(u)\|_{\alpha,p,\infty}\|u_0^{(k)}\|_{\alpha,p,\infty} + \|\sigma_1'(u)\sigma_k(u(\cdot - r_k))\|_{\alpha,p,\infty}\big) \|\pi(\varphi_k * \xi_k, \xi_1)\|_{\alpha+\beta_1-1,p/2,\infty} \\
&\quad \lesssim \big(\|\sigma_1'\|_{C^1}\|u\|_{\alpha,p,\infty}\|u_0^{(k)}\|_{\alpha,p,\infty} + \|F(u, u(\cdot - r_k))\|_{\alpha,p,\infty}\big) \|\pi(\varphi_k * \xi_k, \xi_1)\|_{\alpha+\beta_1-1,p/2,\infty} \\
&\quad \lesssim \big(\|\sigma_1'\|_{C^1}\|u\|_{\alpha,p,\infty}\|u_0^{(k)}\|_{\alpha,p,\infty} + \|\sigma_k\|_{C^1}\|\sigma_1\|_{C^2}(\|u\|_{\alpha,p,\infty} + \|u(\cdot - r_k))\|_{\alpha,p,\infty})\big) \\
&\qquad \times \|\pi(\varphi_k * \xi_k, \xi_1)\|_{\alpha+\beta_1-1,p/2,\infty} \\
&\quad \lesssim \|\sigma_1\|_{C^2}\big(\|u_0^{(k)}\|_{\alpha,p,\infty} + \|\sigma_k\|_{C^1}\big)\|\pi(\varphi_k * \xi_k, \xi_1)\|_{\alpha+\beta_1-1,p/2,\infty}\|u\|_{\alpha,p,\infty}.
\end{aligned}$$

Applying the commutator estimate (3.7) and Young's inequality (Lemma 2.2), we obtain

$$\begin{aligned}
&\|\sigma_1'(u)\Gamma(\widetilde{u}_k, \varphi_k * \xi_k, \xi_1)\|_{\alpha+\beta_1-1,p/2,\infty} \\
&\quad \lesssim \|\sigma_1'(u)\|_\infty \|\Gamma(\widetilde{u}_k, \varphi_k * \xi_k, \xi_1)\|_{2\alpha+\beta_1-1,p/3,\infty} \\
&\quad \lesssim \|\sigma_1'\|_\infty \|u_0^{(k)} + \sigma_k(u(\cdot - r_k))\|_{\alpha,p,\infty}\|\varphi_k * \xi_k\|_{\alpha,p,\infty}\|\xi_1\|_{\beta_1-1,p,\infty} \\
&\quad \lesssim \|\sigma_1\|_{C^1}\big(\|u_0^{(k)}\|_{\alpha,p,\infty} + \|\sigma_k\|_{C^1}\|u\|_{\alpha,p,\infty}\big)\|\varphi_k\|_{\gamma_k,1,\infty}\|\xi_k\|_{\beta_k-1,p,\infty}\|\xi_1\|_{\beta_1-1,p,\infty}
\end{aligned}$$

and similarly

$$\begin{aligned}
&\|\Gamma(\sigma_1'(u), T_{\widetilde{u}_k}(\varphi_k * \xi_k), \xi_1)\|_{\alpha+\beta_1-1,p/2,\infty} \\
&\quad \lesssim \|\sigma_1'(u)\|_{\alpha,p,\infty}\|T_{\widetilde{u}_k}(\varphi_k * \xi_k)\|_{\alpha,p,\infty}\|\xi_1\|_{\beta_1-1,p,\infty} \\
&\quad \lesssim \|\sigma_1\|_{C^2}\|u\|_{\alpha,p,\infty}\big(\|u_0^{(k)}\|_\infty + \|\sigma_k(u(\cdot - r_k))\|_\infty\big)\|\varphi_k * \xi_k\|_{\alpha,p,\infty}\|\xi_1\|_{\beta_1-1,p,\infty} \\
&\quad \lesssim \|\sigma_1\|_{C^2}\big(\|u_0^{(k)}\|_{\alpha,p,\infty} + \|\sigma_k\|_\infty\big)\|\varphi_k\|_{\gamma_k,1,\infty}\|u\|_{\alpha,p,\infty}\|\xi_k\|_{\beta_k-1,p,\infty}\|\xi_1\|_{\beta_1-1,p,\infty}.
\end{aligned}$$

From Bony's estimates (Lemma 2.1) we deduce that

$$\begin{aligned}
&\|\pi(T_{\sigma_1'(u)}u^{\#}, \xi_1)\|_{\alpha+\beta_1-1,p/2,\infty} \\
&\quad \lesssim \|T_{\sigma_1'(u)}u^{\#}\|_{2\alpha,p/2,\infty}\|\xi_1\|_{\beta_1-1,p,\infty} \lesssim \|\sigma_1'\|_\infty\|u^{\#}\|_{2\alpha,p/2,\infty}\|\xi_1\|_{\beta_1-1,p,\infty}.
\end{aligned}$$

Finally, Lemma A.4 shows

$$\begin{aligned}
&\|\pi(S_{\sigma_1}(u), \xi_1)\|_{\alpha+\beta_1-1,p/2,\infty} \\
&\quad \lesssim \|S_{\sigma_1}(u)\|_{2\alpha,p/2,\infty}\|\xi_1\|_{\beta_1-1,p,\infty} \\
&\quad \lesssim \|\sigma_1\|_{C^2}\|\xi_1\|_{\beta_1-1,p,\infty}\Big(1 + \sum_{k=1,2}\|\widetilde{u}_k\|_\infty\|\varphi_k * \xi_k\|_{\alpha,p,\infty}\Big)\big(\|u\|_{\alpha,p,\infty} + \|u^{\#}\|_{2\alpha,p/2,\infty}\big) \\
&\quad \lesssim \|\sigma_1\|_{C^2}\|\xi_1\|_{\beta_1-1,p,\infty}\Big(1 + \sum_{k=1,2}\big(\|u_0^{(k)}\|_\infty + \|\sigma_k(u(\cdot - r_k))\|_\infty\big)\|\varphi_k\|_{\gamma_k,1,\infty}\|\xi_k\|_{\beta_k-1,p,\infty}\Big) \\
&\qquad \times \big(\|u\|_{\alpha,p,\infty} + \|u^{\#}\|_{2\alpha,p/2,\infty}\big) \\
&\quad \lesssim \|\sigma_1\|_{C^2}\|\xi_1\|_{\beta_1-1,p,\infty}\Big(1 + \sum_{k=1,2}\big(\|u_0^{(k)}\|_\infty + \|\sigma_k\|_\infty\big)\|\varphi_k\|_{\gamma_k,1,\infty}\|\xi_k\|_{\beta_k-1,p,\infty}\Big)
\end{aligned}$$

$$\times \big(\|u\|_{\alpha,p,\infty} + \|u^{\#}\|_{2\alpha,p/2,\infty}\big).$$

Summarizing, we have

$$\begin{aligned}
&\|\pi(\sigma_1(u),\xi_1)\|_{\alpha+\beta_1-1,p/2,\infty} \\
&\quad\lesssim \|\sigma_1\|_{C^2}(\|\sigma_1\|_{C^1} + \|\sigma_2\|_{C^1} + 1) \\
&\qquad \times \Big(\|\xi_1\|_{\beta_1-1,p,\infty} + \sum_{k=1,2}\big(\|\varphi_k\|_{\gamma_k,1,\infty}\|\xi_k\|_{\beta_k-1,p,\infty}\|\xi_1\|_{\beta_1-1,p,\infty} + \|\pi(\varphi_k * \xi_k,\xi_1)\|_{\alpha+\beta_1-1,p/2,\infty}\big)\Big) \\
&\qquad \times \big(\|u_0^{(1)}\|_{\alpha,p,\infty} + \|u_0^{(2)}\|_{\alpha,p,\infty} + 1\big)\big(\|u\|_{\alpha,p,\infty} + \|u^{\#}\|_{2\alpha,p/2,\infty} + 1\big).
\end{aligned}$$

Since $\gamma_2+\beta_2+\beta_1-2 \geqslant \alpha+\beta_2-1 > 0$, we can estimate the resonant term for $k=2$ by

$$\begin{aligned}
\|\pi(\varphi_2 * \xi_2,\xi_1)\|_{\alpha+\beta_1-1,p/2,\infty} &\lesssim \|\pi(\varphi_2 * \xi_2,\xi_1)\|_{\beta_2+\beta_1+\gamma_2-2,p/2,\infty} \\
&\lesssim \|\varphi_2 * \xi_2\|_{\beta_2+\gamma_2-1,p,\infty}\|\xi_1\|_{\beta_1-1,p,\infty} \\
&\lesssim \|\varphi_2\|_{\gamma_2,1,\infty}\|\xi_2\|_{\beta_2-1,p,\infty}\|\xi_1\|_{\beta_1-1,p,\infty},
\end{aligned}$$

where we used Bony's estimates (Lemma 2.1) and Young's inequality (Lemma 2.2).

With the definitions from Proposition 3.12 we thus obtain

$$\begin{aligned}
&\|\varphi_1 * \pi(\sigma_1(u),\xi_1)\|_{2\alpha,p/2,\infty} + \|\varphi_2 * \pi(\sigma_2(u),\xi_2)\|_{2\alpha,p/2,\infty} \\
&\quad\lesssim \Delta C_\sigma C_\xi C_{u_0}\big(\|u\|_{\alpha,p,\infty} + \|u^{\#}\|_{2\alpha,p/2,\infty} + 1\big)
\end{aligned} \tag{3.20}$$

and

$$\|u\|_{\alpha,p,\infty} \lesssim \|u_0\|_{\alpha,p,\infty} + \Delta C_\sigma C_\xi C_{u_0}\big(\|u\|_{\alpha,p,\infty} + \|u^{\#}\|_{2\alpha,p/2,\infty} + 1\big). \tag{3.21}$$

Moreover, by the formula for $u^{\#}$ as given in (3.16), by the estimates (3.18), (3.20) and Lemma 3.4 we see

$$\begin{aligned}
\|u^{\#}\|_{2\alpha,p,\infty} &\leqslant \|u_0^{\#}\|_{2\alpha,p/2,\infty} + \sum_{j=1,2}\big(\|\varphi_j * \pi(\sigma_j(u),\xi_j)\|_{2\alpha,p/2,\infty} \\
&\qquad + \|\varphi_j * (T_{\xi_j}\sigma_j(u))\|_{2\alpha,p/2,\infty} + \|R_{\varphi_j}(\sigma_j(u),\xi_j)\|_{2\alpha,p/2,\infty}\big) \\
&\leqslant \|u_0^{\#}\|_{2\alpha,p/2,\infty} + C\Delta C_\sigma C_\varphi C_\xi C_{u_0}\big(\|u\|_{\alpha,p,\infty} + \|u^{\#}\|_{2\alpha,p/2,\infty} + 1\big)
\end{aligned}$$

for a constant $C>0$. Assuming $C\Delta C_\sigma C_\varphi C_\xi C_{u_0} \leqslant 1/2$, one gets

$$\|u^{\#}\|_{2\alpha,p/2,\infty} \leqslant \|u\|_{\alpha,p,\infty} + 2\|u_0^{\#}\|_{2\alpha,p/2,\infty} + 1.$$

Combining this with (3.21), we have for another constant $C'>0$

$$\|u\|_{\alpha,p,\infty} \leqslant \|u_0\|_{\alpha,p,\infty} + C'\Delta C_\sigma C_\xi C_{u_0}\big(\|u\|_{\alpha,p,\infty} + \|u_0^{\#}\|_{2\alpha,p/2,\infty} + 1\big).$$

Therefore, $\|u\|_{\alpha,p,\infty} \leqslant 2\|u_0\|_{\alpha,p,\infty} + \|u_0^{\#}\|_{2\alpha,p/2,\infty} + 1$ provided $C'\Delta C_\sigma C_\xi C_{u_0} \leqslant 1/2$. □

Finally, we can establish the existence of a unique local Lipschitz continuous extension of the Itô-Lyons map $\widehat{S}$ from (3.17) and thus conclude the existence of unique solution of the Volterra equation (1.2) for the rough setting by approximating the convolutional rough paths with smooth functions. As the estimates work analogously to the proof of Proposition 3.12, we present only the key estimates without giving to many details.

Proof of Theorem 3.10. For $i=1,2$ let $(\xi_1^i,\xi_2^i) \in \mathcal{C}_c^\infty \times \mathcal{B}_{p,\infty}^{\beta_2-1}$ be two signals and $(u_0^{(1),i}, u_0^{(2),i}, u^{\#,i}) \in (\mathcal{B}_{p,\infty}^{\alpha})^2 \times \mathcal{B}_{p/2,\infty}^{2\alpha}$ be two initial conditions. Let $M \geq 1$ be a constant such that

$$C_\varphi, C_{\xi^i}, C_{u_0^i}, \|u_0^{\#,i}\|_{2\alpha,p/2,\infty} \leq M, \quad \text{for} \quad i=1,2,$$

using the definitions from Proposition 3.12. Assuming that

$$L_\sigma := \big(\|\sigma_1\|_{C^3} + \|\sigma_2\|_{C^2}\big)\big(1 + \|\sigma_1\|_{C^3} + \|\sigma_2\|_{C^2}\big)$$

is sufficiently small depending on M, Proposition 3.1 implies the existence of corresponding unique solutions u^1, u^2 to the Volterra equation (1.2) and additionally Proposition 3.12 leads to the bound

$$\|u^i\|_{\alpha,p,\infty} \lesssim M^2, \quad i = 1, 2.$$

Based on the ansatz for u^1, u^2 (see (3.19)) and Young's inequality (Lemma 2.2), we observe

$$\begin{aligned} &\|u^1 - u^2\|_{\alpha,p,\infty} \\ &\lesssim \|u_0^1 - u_0^2\|_{\alpha,p,\infty} + \sum_{j=1,2} \|\varphi_j\|_{\gamma_j,1,\infty}\big(\|T_{\sigma_j(u^1)}\xi_j^1 - T_{\sigma_j(u^2)}\xi_j^2\|_{\beta_j-1,p,\infty} \\ &\qquad + \|\pi(\sigma_j(u^1),\xi_j^1) - \pi(\sigma_j(u^2),\xi_j^2)\|_{\alpha+\beta_j-1,p/2,\infty} + \|T_{\xi_j^1}\sigma_j(u^1) - T_{\xi_j^2}\sigma_j(u^2)\|_{\beta_j-1,p,\infty}\big). \end{aligned} \tag{3.22}$$

By the paraproduct estimates (Lemma 2.1) and Lemma A.3 we obtain

$$\begin{aligned} \|T_{\sigma_j(u^1)}\xi_j^1 - T_{\sigma_j(u^2)}\xi_j^2\|_{\beta_j-1,p,\infty} \lesssim \|\sigma_j\|_\infty \|\xi_j^1 - \xi_j^2\|_{\beta_j-1,p,\infty} + \|\sigma_j\|_{C^2}\|\xi^2\|_{\beta_j-1,p,\infty} \\ \times \big(1 + \|u^1\|_{\alpha,p,\infty} + \|u^2\|_{\alpha,p,\infty}\big)\|u^1 - u^2\|_{\alpha,p,\infty} \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} &\|T_{\xi_j^1}\sigma_j(u^1) - T_{\xi_j^2}\sigma(u^2)\|_{\beta_j-1,p,\infty} \\ &\lesssim \|\sigma_j\|_{C^2}\|\xi_j^1\|_{\beta_j-1,p,\infty}\big(1 + \|u^1\|_{\alpha,p,\infty} + \|u^2\|_{\alpha,p,\infty}\big)\|u^1 - u^2\|_{\alpha,p,\infty} \\ &\quad + \|\sigma_j\|_{C^1}\|u^2\|_{\alpha,p,\infty}\|\xi_j^1 - \xi_j^2\|_{\beta_j-1,p,\infty}. \end{aligned} \tag{3.24}$$

It remains to show the local Lipschitz continuity of $\pi(\sigma_j(u^i),\xi_j^i)$. For $j = 2$, due to $\alpha+\beta_2-1 > 0$, we directly apply the paraproduct estimates (Lemma 2.1) and Lemma A.3 to get

$$\begin{aligned} &\|\pi(\sigma_2(u^1),\xi_2^1) - \pi(\sigma_2(u^2),\xi_2^2)\|_{\alpha+\beta_2-1,p/2,\infty} \\ &\lesssim \|\sigma_2\|_{C^2}\|\xi_2^1\|_{\beta_2-1,p,\infty}\big(1 + \|u^1\|_{\alpha,p,\infty} + \|u^2\|_{\alpha,p,\infty}\big)\|u^1 - u^2\|_{\alpha,p,\infty} \\ &\quad + \|\sigma_2\|_{C^1}\|u^2\|_{\alpha,p,\infty}\|\xi_2^1 - \xi_2^2\|_{\beta_2-1,p,\infty}. \end{aligned}$$

For $j = 1$ we linearise $\pi(\sigma_1(u^i),\xi_1^i)$ more carefully using Lemma A.4, the ansatz and the commutator estimate (3.7). Rewriting the ansatz (3.15) as

$$u^i = \sum_{k=1,2} T_{\widetilde{u}_k^i}(\varphi_k * \xi_k^i) + u^{\#,i}$$

with

$$\begin{aligned} u^{\#,i} &:= u_0^{\#,i} + \sum_{j=1,2}\big(\varphi_j * \big(\pi(\sigma_j(u^i),\xi_j^i) + T_{\xi_j^i}\sigma_j(u^i)\big) + R_{\varphi_j}(\sigma_j(u^i),\xi_j^i)\big), \\ \widetilde{u}_k^i &:= u_0^{(k),i} + \sigma_k(u^i(\cdot - r_k)), \quad k = 1, 2, \end{aligned}$$

we find as in the proof of Proposition 3.12 that

$$\begin{aligned} \pi(\sigma_1(u^i),\xi_1^i) &= \sum_{k=1,2}\Bigg(\sigma_1'(u^i)\widetilde{u}_k^i\pi(\varphi_k * \xi_k^i,\xi_1^i) + \sigma_1'(u^i)\Gamma(\widetilde{u}_k^i,\varphi_k * \xi_k^i,\xi_1^i) \\ &\qquad + \Gamma(\sigma_1'(u^i),T_{\widetilde{u}_k^i}(\varphi_k * \xi_k^i),\xi_1^i)\Bigg) + \pi(T_{\sigma_1'(u^i)}u^{\#,i},\xi_1^i) + \pi(S_{\sigma_1}(u^i),\xi_1^i) \\ &=: \sum_{k=1,2}\big(D_1^{k,i} + D_2^{k,i} + D_3^{k,i}\big) + D_4^i + D_5^i. \end{aligned}$$

We estimate the differences of these five terms, with $k = 1, 2$, using again Besov embeddings ($\alpha > 1/p$), the paraproduct estimates (Lemma 2.1) and the auxiliary Besov estimates (Lemma A.1, A.2 and A.3). In order to abbreviate theses estimates, let us introduce

$$\begin{aligned}
\widetilde{C}_u &:= \Big(1 + \sum_{i,k=1}^{2} \big(\|u^i\|_{\alpha,p,\infty} + \|u^{\#,i}\|_{2\alpha,p/2,\infty} + \|u_0^{(k),i}\|_{\alpha,p,\infty}\big)\Big)^2, \\
\widetilde{C}_\xi &:= 1 + \|\pi(\varphi_1 * \xi_1^1, \xi_1^1)\|_{\alpha+\beta_1-1,p/2,\infty} + \sum_{i,k=1}^{2} \|\xi_k^i\|_{\beta_k-1,p,\infty}, \\
\widetilde{C}_\varphi &:= \|\varphi_1\|_{\gamma_1,1,\infty} + \|\varphi_2\|_{\gamma_2,1,\infty}.
\end{aligned}$$

For the first term we have

$$\begin{aligned}
&\|D_1^{k,1} - D_1^{k,2}\|_{\alpha+\beta_1-1,p/2,\infty} \\
&\quad \lesssim L_\sigma \widetilde{C}_\xi \widetilde{C}_u \big(\|u_0^{(k),1} - u_0^{(k),2}\|_{\alpha,p,\infty} + \|u^1 - u^2\|_{\alpha,p,\infty} \\
&\qquad\qquad + \|\pi(\varphi_k * \xi_k^1, \xi_1^1) - \pi(\varphi_k * \xi_k^2, \xi_1^2)\|_{\alpha+\beta_1-1,p/2,\infty}\big).
\end{aligned}$$

Applying the commutator estimate (3.7) and Young's inequality (Lemma 2.2), we obtain

$$\begin{aligned}
&\|D_2^{k,1} - D_2^{k,2}\|_{\alpha+\beta_1-1,p/2,\infty} \\
&\quad \lesssim L_\sigma \widetilde{C}_\varphi \widetilde{C}_\xi^2 \widetilde{C}_u \big(\|u_0^{(k),1} - u_0^{(k),2}\|_{\alpha,p,\infty} + \|u^1 - u^2\|_{\alpha,p,\infty} \\
&\qquad\qquad + \|\xi_k^1 - \xi_k^2\|_{\beta_k-1,p,\infty} + \|\xi_1^1 - \xi_1^2\|_{\beta_1-1,p,\infty}\big).
\end{aligned}$$

The commutator estimate and Young's inequality moreover yield

$$\begin{aligned}
&\|D_3^{k,1} - D_3^{k,2}\|_{\alpha+\beta_1-1,p/2,\infty} \\
&\quad \lesssim L_\sigma \widetilde{C}_\varphi \widetilde{C}_\xi^2 \widetilde{C}_u \big(\|u^1 - u^2\|_{\alpha,p,\infty} + \|u_0^{(k),1} - u_0^{(k),2}\|_{\alpha,p,\infty} \\
&\qquad\qquad + \|\xi_k^1 - \xi_k^2\|_{\beta_k-1,p,\infty} + \|\xi_1^1 - \xi_1^2\|_{\beta_1-1,p,\infty}\big).
\end{aligned}$$

Applying Lemma 3.4, we deduce that

$$\begin{aligned}
&\|D_4^1 - D_4^2\|_{\alpha+\beta_1-1,p/2,\infty} \\
&\quad \lesssim L_\sigma \widetilde{C}_\xi \widetilde{C}_u \big(\|u^1 - u^2\|_{\alpha,p,\infty} + \|u^{\#,1} - u^{\#,2}\|_{2\alpha,p/2,\infty} + \|\xi_1^1 - \xi_1^2\|_{\beta_1-1,p,\infty}\big).
\end{aligned}$$

Finally, [37, Lemna 4.2] leads to

$$\|D_5^1 - D_5^2\|_{\alpha+\beta_1-1,p/2,\infty} \lesssim L_\sigma \widetilde{C}_\xi^2 \widetilde{C}_u \big(\|u^1 - u^2\|_{\alpha,p,\infty} + \|\xi_1^1 - \xi_1^2\|_{\beta_1-1,p,\infty}\big).$$

Relying additionally on the estimate

$$\begin{aligned}
&\|\pi(\varphi_2 * \xi_2^1, \xi_1^1) - \pi(\varphi_2 * \xi_2^2, \xi_1^2)\|_{\alpha+\beta_1-1,p/2,\infty} \\
&\quad \le \|\varphi_2\|_{\gamma_2-1,1,\infty} \|\xi_1^1\|_{\beta_1-1,p,\infty} \|\xi_2^1 - \xi_2^2\|_{\beta_2-1,p,\infty} + \|\varphi_2\|_{\gamma_2,1,\infty} \|\xi_2^1\|_{\beta_2-1,p,\infty} \|\xi_1^1 - \xi_1^2\|_{\beta_1-1,p/2,\infty},
\end{aligned}$$

we conclude that there exist a constant $C(M)$ such that

$$\begin{aligned}
&\|\pi(\sigma_1(u^1), \xi_1^1) - \pi(\sigma_1(u^2), \xi_1^2)\|_{2\alpha,p/2,\infty} \\
&\quad \lesssim L_\sigma C(M) \bigg(\|u^1 - u^2\|_{\alpha,p,\infty} + \|\pi(\varphi_1 * \xi_1^1, \xi_1^1) - \pi(\varphi_1 * \xi_1^2, \xi_1^2)\|_{\alpha+\beta_1-1,p/2,\infty} \\
&\qquad + \sum_{j=1,2} \big(\|\xi_j^1 - \xi_j^2\|_{\beta_1-1,,p,\infty} + \|u_0^{(j),1} - u_0^{(j),2}\|_{\alpha,p,\infty}\big) + \|u^{\#,1} - u^{\#,2}\|_{2\alpha,p/2,\infty}\bigg).
\end{aligned}$$

The last term can be further estimated by

$$
\begin{aligned}
&\|u^{\#,1}-u^{\#,2}\|_{2\alpha,p/2,\infty}\\
&\quad\le \|u_0^{\#,1}-u_0^{\#,2}\|_{2\alpha,p/2,\infty}+\sum_{j=1,2}\|\varphi_j\|_{\gamma_j,1,\infty}\big(\|\pi(\sigma_j(u^1),\xi_j^1)-\pi(\sigma_j(u^2),\xi_j^2)\|_{2\alpha,p/2,\infty}\\
&\qquad+\|T_{\xi_j^1}\sigma_j(u^1)-T_{\xi_j^2}\sigma_j(u^2)\|_{2\alpha,p/2,\infty}+\|R_{\varphi_j}(\sigma_j(u^1),\xi_j^1)-R_{\varphi_j}(\sigma_j(u^2),\xi_j^2)\|_{2\alpha,p/2,\infty}\big)\\
&\quad\lesssim \|u_0^{\#,1}-u_0^{\#,2}\|_{2\alpha,p/2,\infty}+\widetilde{C}_\varphi L_\sigma C(M)\bigg(\|u^1-u^2\|_{\alpha,p,\infty}+\|u^{\#,1}-u^{\#,2}\|_{2\alpha,p/2,\infty}\\
&\qquad+\sum_{j=1,2}\big(\|\xi_j^1-\xi_j^2\|_{\beta_1-1,,p,\infty}+\|u_0^{(j),1}-u_0^{(j),2}\|_{\alpha,p,\infty}\big)\\
&\qquad+\|\pi(\varphi_1*\xi_1^1,\xi_1^1)-\pi(\varphi_1*\xi_1^2,\xi_1^2)\|_{\alpha+\beta_1-1,p/2,\infty}\bigg),
\end{aligned}
$$

where we used that $R_\varphi(\cdot,\cdot)$ from Lemma 3.4 is a bounded linear operator by its definition. For L_σ small enough the last inequality in combination with (3.22) (3.23) and (3.24) implies

$$
\begin{aligned}
\|u^1-u^2\|_{\alpha,p,\infty}\lesssim \|u_0^{\#,1}-u_0^{\#,2}\|_{2\alpha,p/2,\infty}+\widehat{C}\bigg(&\|\pi(\varphi_1*\xi_1^1,\xi_1^1)-\pi(\varphi_1*\xi_1^2,\xi_1^2)\|_{\alpha+\beta_1-1,p/2,\infty}\\
&+\sum_{j=1,2}\big(\|\xi_j^1-\xi_j^2\|_{\beta_1-1,,p,\infty}+\|u_0^{(j),1}-u_0^{(j),2}\|_{\alpha,p,\infty}\bigg),
\end{aligned}
$$

for some constant $\widehat{C}:=C(L_\sigma,M)>0$. This Lipschitz estimate allows to extend the Itô-Lyons map (3.17) from smooth driving signals ξ_1 with compact support to the space of convolutional rough paths. □

3.4 Solutions for general vector fields

In Theorem 3.10 we assumed that

$$\Delta=\|\sigma_1\|_{C^3}\|\varphi_1\|_{\gamma_1,1,\infty}+\|\sigma_2\|_{C^2}\|\varphi_2\|_{\gamma_2,1,\infty}$$

is sufficiently small, which can be interpreted as a flatness condition on the vector fields σ_1,σ_2. In this subsection we discuss how the existence and uniqueness results can be extended to general vector fields σ_1,σ_2 applying a scaling argument in the spirit of Gubinelli et al. [24] to a localized version of (1.2). Interestingly, Δ is small if the (localised) kernels φ_1,φ_2 are supported on a sufficiently small domain and if $\gamma_1,\gamma_2<1$, cf. Remark 3.16.

Theorem 3.13. *Let $p\in[3,\infty]$, $0<\beta_1\leqslant\beta_2\leqslant 1$ and $0<\gamma_1\leqslant\gamma_2$ satisfy $\alpha:=\beta_1+\gamma_1-1\in(\frac{1}{3},1)$,* $\alpha+\beta_1<1<2\alpha+\beta_1$ and $\alpha+\beta_2>1$. *If $\gamma_j>1$ for $j=1,2$, let also $\beta_j>1/p$ be fullfilled. Suppose that*

(i) $\sigma_1\in C^3$ and $\sigma_2\in C^2$ with $\sigma_1(0)=\sigma_2(0)=0$,

(ii) $\varphi_j\in\mathcal{B}^{\gamma_j}_{1,\infty}$ such that there exists $r_j\in\mathbb{R}$ with $\|(\cdot-r_j)\varphi_j\|_{\gamma_j+1,1,\infty}<\infty$, for $j=1,2$,

(iii) $(\xi_1,\mu)\in\mathcal{B}_p^{\beta_1-1,\gamma_1}(\varphi_1)$ and $\xi_2\in\mathcal{B}^{\beta_2-1}_{p,\infty}$,

(iv) $u_0\in\mathcal{B}^{2\alpha}_{p/2,\infty}$.

Additionally, we impose the structural assumption on the kernel φ_1:

(v) There is some $\psi\in\mathcal{B}^{s+\delta}_{1,\infty}$ for $\delta>(2-2\beta_1)\vee 1$ and $s\in[0,1)$ such that $\varphi_1(x)=(x-r_1)^{-s}\mathbb{1}_{(r_1,\infty)}(x)\psi(x)$.

Let χ be a C^∞ function with $\operatorname{supp}\chi \subseteq [-2,2]$ *and* $\chi(x) = 1$ *for* $x \in [-1,1]$*. Then there is some* $\lambda \in (0,1)$ *depending on* $(u_0, (\xi_1, \mu), \xi_2)$*,* φ_1, φ_2 *and* σ_1, σ_2*, such that the localized Volterra equation*

$$u(t) = u_0^{loc,\lambda}(t) + \big(\varphi_1^{loc,\lambda} * (\sigma_1(u)\xi_1)\big)(t) + \big(\varphi_2^{loc,\lambda} * (\sigma_2(u)\xi_2)\big)(t), \quad t \in \mathbb{R}, \tag{3.25}$$

with kernels $\varphi_j^{loc,\lambda} := \chi(\lambda^{-1}\cdot)\varphi_j$*,* $j = 1,2$*, and initial condition* $u_0^{loc,\lambda} := \chi(\lambda^{-1}\cdot)u_0$ *has a unique solution in the space* $\mathcal{B}^{\alpha-\varepsilon}_{p,\infty}$ *for any* $\varepsilon > 0$*.*

Proof. Let us introduce the dilation operator $\Lambda_\lambda f := f(\lambda\cdot)$ for any $f \in \mathcal{S}'$. For $\delta, \lambda > 0$, we first observe that

$$\begin{aligned} u &= u_0^{loc,\lambda} + \sum_{j=1,2} \varphi_j^{loc,\lambda} * (\sigma_j(u)\xi_j) \\ &= u_0^{loc,\lambda} + \sum_{j=1,2} \int_{\mathbb{R}} \frac{\lambda}{\delta} \varphi_j^{loc,\lambda}(\cdot - \lambda s)\delta\sigma_j\big(u(\lambda s)\big)\Lambda_\lambda \xi_j(s)\,\mathrm{d}s. \end{aligned}$$

Therefore, u solves (3.25) if and only if $\widetilde{u} := \Lambda_\lambda u$ solves

$$\widetilde{u} = \Lambda_\lambda u_0^{loc,\lambda} + \sum_{j=1,2} \int_{\mathbb{R}} \frac{\lambda}{\delta} \Lambda_\lambda \varphi_j^{loc,\lambda}(\cdot - s)\delta\sigma_j\big(\widetilde{u}(s)\big)\Lambda_\lambda \xi_j(s)\,\mathrm{d}s. \tag{3.26}$$

Applying the dilation estimate from [37, Lem. 2.3] we have

$$\big\|\Lambda_\lambda \xi_j\big\|_{\beta_j - 1,p,\infty} \lesssim (1 + \lambda^{\beta_j - 1}|\log\lambda|)\lambda^{-1/p}\|\xi_j\|_{\beta_j-1,p,\infty}. \tag{3.27}$$

The auxiliary Lemma A.5 yields

$$\begin{aligned} &\|\Lambda_\lambda \varphi_j^{loc,\lambda}\|_{\gamma_j,1,\infty} = \|\chi\Lambda_\lambda\varphi_j\|_{\gamma_j,1,\infty} \lesssim \lambda^{(\gamma'\wedge 1)-1}|\log\lambda|\|\varphi_j\|_{\gamma_j,1,\infty} \qquad \text{for any } \gamma' < \gamma_j, \\ &\|\Lambda_\lambda u_0^{loc,\lambda}\|_{2\alpha,p/2,\infty} = \|\chi\Lambda_\lambda u_0\|_{2\alpha,p/2,\infty} \lesssim \lambda^{\alpha - 1/p}|\log\lambda|\|u_0\|_{2\alpha,p/2,\infty}. \end{aligned} \tag{3.28}$$

We now may choose δ such that the norms of the scaled noise and kernels remain bounded while $\|\delta\sigma_j\|_{C^3} \to 0$ for $\delta \to 0$. Due to the assumptions on the parameters, we have $\frac{1}{p} < \beta_j + (\gamma_j \wedge 1) - 1$ such that there is some $0 < \tau < (\beta_j + (\gamma_j \wedge 1) - 1 - 1/p)/2$ and we can choose $\delta = \lambda^{\beta_j + (1\wedge\gamma_j) - 1 - 1/p - 2\tau}$. Setting $\widetilde{u}_0^{loc} := \Lambda_\lambda u_0^{loc}$ and

$$\widetilde{\xi}_j := \lambda^{1+1/p-\beta_j+\tau}\Lambda_\lambda\xi_j, \quad \widetilde{\varphi}_j^{loc} := \lambda^{1-(1\wedge\gamma_j)+\tau}\Lambda_\lambda\varphi_j^{loc}, \quad \widetilde{\sigma}_j := \delta\sigma_j,$$

we obtain from (3.26) the dilated representation

$$\widetilde{u} := \widetilde{u}_0^{loc} + \sum_{j=1,2} \big(\widetilde{\varphi}_j^{loc} * (\widetilde{\sigma}_j(\widetilde{u})\widetilde{\xi}_j\big). \tag{3.29}$$

Owing to (3.27) and (3.28), we have uniformly in $\lambda > 0$

$$\|\widetilde{\xi}_j\|_{\beta_j-1,p,\infty} \lesssim \|\xi_j\|_{\beta_j-1,p,\infty}, \qquad \|\widetilde{\varphi}_j^{loc}\|_{\gamma_j,1,\infty} \lesssim \|\varphi_j\|_{\gamma_j,1,\infty}, \qquad \|\widetilde{u}_0^{loc}\|_{2\alpha,p/2,\infty} \lesssim \|u_0\|_{2\alpha,p/2,\infty}.$$

We may now choose λ and thus δ sufficiently small such that Theorem 3.10 applies to (3.29) when γ_j and α are replaced by $\widetilde{\gamma}_j := \gamma_j - \varepsilon$ and $\widetilde{\alpha} := \alpha - \varepsilon = \beta_1 + \widetilde{\gamma}_1 - 1$, respectively, for some sufficiently small $\varepsilon > 0$. Since $\|\widetilde{\varphi}_j^{loc}\|_{\widetilde{\gamma}_j,1,\infty} \lesssim \|\widetilde{\varphi}_j^{loc}\|_{\gamma_j,1,\infty} \lesssim \|\varphi_j\|_{\gamma_j,1,\infty}$, it only remains to verify bounds for $\|(\cdot - r_j)\widetilde{\varphi}_j^{loc}\|_{\widetilde{\gamma}_j+1,1,\infty}$ and $\|\pi(\widetilde{\varphi}_1^{loc} * \widetilde{\xi}_1, \widetilde{\xi}_1)\|_{\widetilde{\alpha}+\beta_1-1,p/2,\infty}$ uniformly in λ. Setting $r_j = 0$ without loss of generality, we obtain from Lemma A.5 for $\gamma' = 1 \wedge \gamma_j - \tau/2$

$$\begin{aligned} \|x\widetilde{\varphi}_j^{loc}(x)\|_{\widetilde{\gamma}_j+1,1,\infty} &= \lambda^{-(1\wedge\gamma_j)+\tau}\|\chi(x)\chi(x/2)\Lambda_\lambda(x\varphi_j(x))\|_{\widetilde{\gamma}_j+1,1,\infty} \\ &\lesssim \lambda^{\gamma' - (1\wedge\gamma_j)+\tau}|\log\lambda|\big(\|x\varphi_j(x)\|_{\gamma_j+1,1,\infty} + \|\varphi_j\|_{\gamma_j,1,\infty}\big). \end{aligned}$$

Moreover, we have due to [37, Lem. 2.3], Lemma A.6 and $\alpha+\beta_1<1$:

$$\begin{aligned}
&\|\pi(\widetilde{\varphi}_1^{loc}*\widetilde{\xi}_1,\widetilde{\xi}_1)\|_{\widetilde{\alpha}+\beta_1-1,p/2,\infty}\\
&\quad=\lambda^{2+2/p-2\beta_1-(\gamma_1\wedge 1)+3\tau}\|\pi(\Lambda_\lambda(\varphi_1^{loc,\lambda}*\xi_1),\Lambda_\lambda\xi_1)\|_{\widetilde{\alpha}+\beta_1-1,p/2,\infty}\\
&\quad\le\lambda^{2+2/p-2\beta_1-(\gamma_1\wedge 1)+3\tau}\big(\|\Lambda_\lambda(\pi(\varphi_1^{loc,\lambda}*\xi_1),\xi_1))\|_{\widetilde{\alpha}+\beta_1-1,p/2,\infty}\\
&\qquad+\|\pi(\Lambda_\lambda(\varphi_1^{loc,\lambda}*\xi_1),\Lambda_\lambda\xi_1)-\Lambda_\lambda(\pi(\varphi_1^{loc,\lambda}*\xi_1,\xi_1))\|_{\widetilde{\alpha}+\beta_1-1,p/2,\infty}\big)\\
&\quad\lesssim\lambda^{\widetilde{\alpha}+1-\beta_1-(\gamma_1\wedge 1)+3\tau}|\log\lambda|\|\pi(\varphi_1^{loc,\lambda}*\xi_1,\xi_1)\|_{\widetilde{\alpha}+\beta_1-1,p/2,\infty}\\
&\qquad+\lambda^{\widetilde{\alpha}+1-\beta_1-(\gamma_1\wedge 1)+3\tau}\|\varphi_1^{loc,\lambda}*\xi_1\|_{\widetilde{\alpha},p,\infty}\|\xi_1\|_{\beta_1-1,p,\infty}\\
&\qquad+\lambda^{2+2/p-2\beta_1-(\gamma_1\wedge 1)+3\tau}\|\Lambda_\lambda(\varphi_1^{loc,\lambda}*\xi_1)\|_{\widetilde{\alpha},p,\infty}\|\Lambda_\lambda\xi_1\|_{\beta_1-1,p,\infty}.
\end{aligned}\tag{3.30}$$

The last two terms in (3.30) can be bounded by Young's inequality

$$\lambda^{\widetilde{\alpha}+1-\beta_1-(\gamma_1\wedge 1)+3\tau}\|\varphi_1^{loc,\lambda}*\xi_1\|_{\widetilde{\alpha},p,\infty}\|\xi_1\|_{\beta_1-1,p,\infty}\lesssim\lambda^{3\tau}\|\varphi_1^{loc,\lambda}\|_{\widetilde{\gamma},p,\infty}\|\xi_1\|^2_{\beta_1-1,p,\infty}$$

and, in combination with [37, Lem. 2.3] and Lemma (A.5) for $\varepsilon<\tau$,

$$\begin{aligned}
&\lambda^{2+2/p-2\beta_1-(\gamma_1\wedge 1)+3\tau}\|\Lambda_\lambda(\varphi_1^{loc,\lambda}*\xi_1)\|_{\widetilde{\alpha},p,\infty}\|\Lambda_\lambda\xi_1\|_{\beta_1-1,p,\infty}\\
&\quad\lesssim\lambda^{3+2/p-2\beta_1-(\gamma_1\wedge 1)+3\tau}\|\Lambda_\lambda\varphi_1^{loc,\lambda}\|_{\widetilde{\gamma},1,\infty}\|\Lambda_\lambda\xi_1\|^2_{\beta_1-1,p,\infty}\\
&\quad\leqslant\lambda^{\tau}|\log\lambda|^3\|\varphi_1^{loc,\lambda}\|_{\widetilde{\gamma},1,\infty}\|\xi_1\|^2_{\beta_1-1,p,\infty}.
\end{aligned}$$

Choosing $q,q'\in[1,\infty)$ such that $\frac{1}{q'}+\frac{1}{q}=1$ and $\gamma_1>\frac{1}{q}>\widetilde{\gamma}_1$, we observe

$$\begin{aligned}
&\|\varphi_1^{loc,\lambda}\|_{\widetilde{\gamma},1,\infty}\\
&\quad\lesssim\|T_{\varphi_1}\chi(\lambda^{-1}\cdot)\|_{\widetilde{\gamma}_1,1,\infty}+\|T_{\chi(\lambda^{-1}\cdot)}\varphi_1\|_{\widetilde{\gamma}_1,1,\infty}+\|\pi(\varphi_1,\chi(\lambda^{-1}\cdot))\|_{\widetilde{\gamma}_1,1,\infty}\\
&\quad\lesssim\|\varphi_1\|_{L^{q'}}\|\chi(\lambda^{-1}\cdot)\|_{\widetilde{\gamma}_1,q,\infty}+\|\chi(\lambda^{-1}\cdot)\|_{L^\infty}\|\varphi_1\|_{\widetilde{\gamma}_1,1,\infty}+\|\varphi_1\|_{\widetilde{\gamma}_1,1,\infty}\|\chi(\lambda^{-1}\cdot)\|_{\beta_1-1,\infty,\infty}\\
&\quad\lesssim\|\varphi_1\|_{\gamma_1,1,\infty}(1+\lambda^{-\widetilde{\gamma}_1}|\log\lambda^{-1}|)\lambda^{\frac{1}{q}}\|\chi\|_{\widetilde{\gamma}_1,q,\infty}+\|\chi\|_{L^\infty}\|\varphi_1\|_{\gamma_1,1,\infty}\\
&\qquad+\|\varphi_1\|_{\gamma_1,1,\infty}(1+\lambda^{1-\beta_1}|\log\lambda^{-1}|)\|\chi\|_{\beta_1-1,\infty,\infty},
\end{aligned}\tag{3.31}$$

where we applied Bony's decomposition, [37, Lem. 2.3] and Besov embeddings. Hence,

$$\|\varphi_1^{loc,\lambda}\|_{\widetilde{\gamma},1,\infty}\lesssim\|\varphi_1\|_{\gamma_1,1,\infty}$$

and we can estimate (3.30) by

$$\begin{aligned}
&\|\pi(\widetilde{\varphi}_1^{loc}*\widetilde{\xi}_1,\widetilde{\xi}_1)\|_{\widetilde{\alpha}+\beta_1-1,p/2,\infty}\\
&\quad\lesssim\lambda^{\widetilde{\alpha}+1-\beta_1-(\gamma_1\wedge 1)+3\tau}|\log\lambda|\|\pi(\varphi_1^{loc,\lambda}*\xi_1,\xi_1)\|_{\widetilde{\alpha}+\beta_1-1,p/2,\infty}+\|\varphi_1\|_{\widetilde{\gamma},p,\infty}\|\xi_1\|^2_{\beta_1-1,p,\infty}\\
&\quad\lesssim\|\pi(\varphi_1*\xi_1,\xi_1)\|_{\alpha+\beta_1-1,p/2,\infty}+\big\|\pi\big(((1-\Lambda_{1/\lambda}\chi)\varphi_1)*\xi_1,\xi_1\big)\big\|_{\widetilde{\alpha}+\beta_1-1,p/2,\infty}\\
&\qquad+\|\varphi_1\|_{\widetilde{\gamma},p,\infty}\|\xi_1\|^2_{\beta_1-1,p,\infty}.
\end{aligned}\tag{3.32}$$

It remains to estimate the term $\pi\big(((1-\Lambda_{1/\lambda}\chi)\varphi_1)*\xi_1,\xi_1\big)$ since the other terms can be seen to be uniformly bounded in $\lambda\in(0,1]$ keeping in mind (3.31). We use that the potential irregularity of φ_1 at the origin is smoothed out. Setting $\varepsilon':=(1-\alpha-\beta_1)+\varepsilon$ such that $\varepsilon-2(\beta_1-1)=\gamma+\varepsilon'$, we can bound

$$\begin{aligned}
\big\|\pi\big(((1-\Lambda_{1/\lambda}\chi)\varphi_1)*\xi_1,\xi_1\big)\big\|_{\widetilde{\alpha}+\beta_1-1,p/2,\infty}&\lesssim\big\|\pi\big(((1-\Lambda_{1/\lambda}\chi)\varphi_1)*\xi_1,\xi_1\big)\big\|_{\varepsilon,p/2,\infty}\\
&\lesssim\|((1-\Lambda_{1/\lambda}\chi)\varphi_1)*\xi_1\|_{\varepsilon-\beta_1+1,p,\infty}\|\xi_1\|_{\beta_1-1,p,\infty}\\
&\lesssim\|((1-\Lambda_{1/\lambda}\chi)\varphi_1)\|_{\gamma+\varepsilon',1,\infty}\|\xi_1\|^2_{\beta_1-1,p,\infty}.
\end{aligned}$$

We will now use the kernel assumption $\varphi_1(x) = x^{-s}\psi(x)\mathbb{1}_{[0,\infty)}(x)$. According to [42, Corollary 2.9.3] and the proof of [42, Theorem 2.9.1], the extension operator

$$S_0\colon \{f \in \mathcal{B}^\delta_{p,\infty}(\mathbb{R}_+) : f(0) = 0\} \to \mathcal{B}^\delta_{p,\infty}(\mathbb{R}), \quad f \mapsto \widetilde{f}(x) := \begin{cases} f(x), & x \geqslant 0 \\ 0, & x < 0 \end{cases},$$

is bounded and linear if $\frac{1}{p} < \delta < \frac{1}{p} + 1$. In particular, for any function $f \in \mathcal{B}^\delta_{p,\infty}$ with $f(0) = 0$ we conclude with restriction $f|_{\mathbb{R}_+}$ to $\mathbb{R}_+$ that

$$\begin{aligned} \|f\mathbb{1}_{[0,\infty)}\|_{\delta,p,q} &\lesssim \|f|_{\mathbb{R}_+}\|_{\mathcal{B}^\delta_{p,\infty}(\mathbb{R}_+)} \\ &= \inf\big\{\|g\|_{\delta,p,\infty} : g \in \mathcal{B}^\delta_{p,\infty}, g(x) = f(x)\,\forall x \geqslant 0\big\} \leqslant \|f\|_{\delta,p,\infty}. \end{aligned} \tag{3.33}$$

Since χ is constant one in a neighbourhood of the origin, we may apply (3.33) to $f(x) = (1 - \chi(\lambda^{-1}x))x^{-s}\psi(x)$ and any $\delta \in ((\gamma + \varepsilon') \vee 1, 2)$. Together with Lemma A.5 we obtain for $\varepsilon'' \in (s, 1)$

$$\begin{aligned} \|(1 - \chi(\lambda^{-1}\cdot))\varphi\|_{\gamma+\varepsilon',1,\infty} &\lesssim \Big\|\frac{1 - \chi(x/\lambda)}{x^s}\psi\Big\|_{\delta,1,\infty} = \lambda^{-s}\Big\|\frac{1 - \chi(\lambda^{-1}x)}{(x/\lambda)^s}\psi\Big\|_{\delta,1,\infty} \\ &\lesssim \lambda^{\varepsilon''-s}|\log\lambda|\|x^{-s}(1 - \chi(x))\|_{\delta,1,\infty}\|\psi\|_{\delta,1/(1-\varepsilon''),\infty} \\ &\lesssim \|x^{-s}(1 - \chi(x))\|_{\delta,1,\infty}\|\psi\|_{\delta+\varepsilon'',1,\infty}. \end{aligned}$$

In combination with (3.32), we observe a uniform bound for $\|\pi(\widetilde{\varphi}^{loc}_1 * \widetilde{\xi}_1, \widetilde{\xi}_1)\|_{\widetilde{\alpha}+\beta_1-1,p/2,\infty}$ which concludes the proof. ☐

Remark 3.14. Note that under the support assumptions $\operatorname{supp}\varphi_i \subseteq [0,\infty)$ and $\operatorname{supp}\xi_i \subseteq [0,\infty)$ for $i = 1, 2$ the solution of the localized equation (3.25) coincide with the solution of the original Volterra equation (1.2) on a small time horizon, provided the initial condition is, e.g., a constant or has sufficiently small support. Based on this observation, one can iteratively solve the Volterra equation (1.2) in order to obtain a global solution using a classical pasting argument. In case of Volterra equations this procedure will require carefully chosen support conditions on the kernel functions and the noise terms. In the special case of classical rough differential equations (which corresponds to $\varphi_1 = \varphi_2 = \mathbb{1}_{[0,\infty)}$, see Subsection 5.1) such procedure was carried out in, e.g., [24] and [37].

Remark 3.15. The assumption (v) on the kernel φ_1 is fairly flexible and covers many typical applications. For $s = 0$ we may replace $\mathbb{1}_{(0,\infty)}$ by $\mathbb{1}_{[0,\infty)}$ and we obtain a class of regular kernels $\varphi_1 = \mathbb{1}_{[0,\infty)}\psi$ for some $\psi \in \mathcal{B}^\delta_{1,\infty}$, $\delta \in (1 \vee (\gamma_1 + 1 - \alpha - \beta_1), \gamma_1 + \alpha)$, (setting $r_1 = 0$ for simplicity). In this case the singularity at 0 is not more severe than a jump such that we recover many features of ordinary rough differential equations, especially $\gamma_1 = 1$. The condition $\psi \in \mathcal{B}^\delta_{1,\infty}$ is quite weak and includes, for instance, the kernels studied in [15] where $\psi \in C^3$. On the one hand δ has to be larger than γ such that ψ is more regular than φ_1 itself and on the other hand $\delta > 1$ ensures that ψ is continuous. For $s > 0$ and $\psi(0) \neq 0$ the kernel is singular. Note that the degree of the singularity is constrained by the regularity assumption $\varphi_1 \in \mathcal{B}^{\gamma_1}_{1,\infty}$ implying $s \leqslant 1 - \gamma_1$. For example, if ξ_1 is white noise, then $\alpha > 1/3$ implies $\gamma > 5/6$ such that we require $s \in [0, 1/6)$. For further examples we refer to Section 5.

Remark 3.16. More generally, for singular kernels φ_1 which do not satisfy assumption (v), a uniform bound (in λ) of the localised resonant term $\|\pi(\varphi^{loc,\lambda}_1 * \xi_1, \xi_1)\|_{\alpha+\beta_1-1-\varepsilon,p/2,\infty}$ from (3.30) could be directly assumed. Indeed, we will see in the stochastic construction below (see the proof of Theorem 4.6) that this resonant term is typically of order $\|\varphi^{loc,\lambda}_1\|_{\gamma,1,\infty}$, which can be bounded by Lemma A.5 as

$$\begin{aligned} \|\varphi^{loc,\lambda}_1\|_{\gamma-\varepsilon,1,\infty} &= \|\chi(\lambda^{-1}\cdot)\varphi_1\|_{\gamma-\varepsilon,1,\infty} \\ &\lesssim \lambda^\varepsilon|\log\lambda|\|\chi\|_{\gamma-\varepsilon,1,\infty}\|\varphi_1\|_{\gamma-\varepsilon,1/(1-\varepsilon),\infty} \lesssim \lambda^{\varepsilon/2}\|\chi\|_{\gamma,1,\infty}\|\varphi_1\|_{\gamma,1,\infty}. \end{aligned}$$

Note that the last estimate is arbitrary small for sufficiently small λ, Theorem 3.10 can then be directly applied to the localised equation (3.25) without an additional scaling argument if $\gamma_1, \gamma_2 < 1$.

4 The resonant term

In order to apply the existence and unique results provided in Section 3 to stochastic Volterra equations, it is often necessary to construct the resonant term $\pi(\varphi * \xi,\xi)$ for the driving stochastic processes. In the case of regular kernels $\varphi \in \mathcal{B}^1_{1,\infty}$, the existence of the resonant term $\pi(\varphi * \xi,\xi)$ is equivalent to the existence of the classical rough path, see Subsection 4.1. However, for singular kernels $\varphi \in \mathcal{B}^\delta_{1,\infty}$ with $\delta < 1$ this equivalence does not hold anymore and it is necessary to include the kernel φ in the definition of the "rough path", see Example 4.3. Therefore, we provide a probabilistic construction of convolutional rough paths for a wide class of Gaussian processes in Subsection 4.2.

4.1 Relation to rough path theory

For a regular kernel $\varphi = \mathbb{1}_{[0,\infty)}\psi$ and a rough signal ξ the resonant term $\pi(\varphi*\xi,\xi)$ can be reduced to the resonant term $\pi(\mathbb{1}_{[0,\infty)} * \xi, \xi) = \pi(\int_{-\infty}^t \mathrm{d}\xi(s), \xi)$ between ξ and its anti-derivative. The latter corresponds to the classical rough path integral, cf. [24]. Considering the Volterra equation on some bounded time interval, we may use $\pi((\mathbb{1}_{[0,\infty)}\chi) * \xi, \xi)$ instead of $\pi(\mathbb{1}_{[0,\infty)} * \xi, \xi)$ where χ is some smooth compactly supported function being constant one in a neighbourhood of the origin. Note that χ only ensures integrability of the kernel, while the characteristic regularity properties of $\mathbb{1}_{[0,\infty)}$ are preserved. In particular, the (weak) derivative of $(\mathbb{1}_{[0,\infty)}\chi) * \xi$ is ξ up to some additional smooth remainder.

Lemma 4.1. *Let $\xi \in \mathcal{B}^{\beta-1}_{p,\infty}$ for $\beta > 0$, $p \in [2,\infty]$ and $(\xi^n)_n \subseteq \mathcal{S}$ be such that $\xi^n \to \xi$ in $\mathcal{B}^{\beta-1}_{p,\infty}$ as $n \to \infty$. Suppose that $\chi \in C^\infty$ is a smooth compactly supported function with $\chi(0) = 1$ and $\varphi := \psi \mathbb{1}_{[0,\infty)} \in \mathcal{B}^1_{1,\infty}$ for some $\psi \in \mathcal{B}^\delta_{1,\infty}$ with $\delta \in (1 \vee 2(1-\beta), 2)$ and $\psi(0) \neq 0$. Then, $\pi(\varphi * \xi, \xi) := \lim_{n\to\infty} \pi(\varphi * \xi^n, \xi^n)$ exists in $\mathcal{B}^{2\beta-1}_{p/2,\infty}$ if and only if $\pi\big((\mathbb{1}_{[0,\infty)}\chi) * \xi, \xi\big) := \lim_{n\to\infty} \pi\big((\mathbb{1}_{[0,\infty)}\chi) * \xi^n, \xi^n\big)$ exists in $\mathcal{B}^{2\beta-1}_{p/2,\infty}$. In this case, one has*

$$\pi(\varphi * \xi, \xi) - \varphi(0)\pi\big((\mathbb{1}_{[0,\infty)}\chi) * \xi, \xi\big) \in \mathcal{B}^{\delta-2(1-\beta)}_{p/2,\infty}.$$

Proof. Let $(\xi^n)_n \subseteq \mathcal{S}$ be such that $\xi^n \to \xi$ in $\mathcal{B}^{\beta-1}_{p,\infty}$ and $\pi(\varphi * \xi, \xi) := \lim_{n\to\infty} \pi(\varphi * \xi^n, \xi^n)$ in $\mathcal{B}^{2\beta-1}_{p/2,\infty}$. We first observe that

$$\begin{aligned}
&\pi\big((\mathbb{1}_{[0,\infty)}\chi) * \xi^n, \xi^n\big) \\
&\quad = \psi(0)^{-1}\pi(\varphi * \xi^n, \xi^n) - \big(\psi(0)^{-1}\pi(\varphi * \xi^n, \xi^n) - \pi\big((\mathbb{1}_{[0,\infty)}\chi) * \xi^n, \xi^n\big)\big).
\end{aligned}$$

Since the first term converges by assumption, it is sufficient to consider the other two. Setting $\varepsilon := \delta - 2(1-\beta) > 0$, Bony's paraproduct estimates and the generalised Young inequality yield

$$\begin{aligned}
&\big\|\pi(\varphi * \xi^n, \xi^n) - \psi(0)\pi\big((\mathbb{1}_{[0,\infty)}\chi) * \xi^n, \xi^n\big)\big\|_{\varepsilon,p/2,\infty} \\
&\quad = \big\|\pi\big(\big((\psi - \psi(0)\chi)\mathbb{1}_{[0,\infty)}\big) * \xi^n, \xi^n\big)\big\|_{\varepsilon,p/2,\infty} \\
&\quad \lesssim \big\|\big((\psi - \psi(0)\chi)\mathbb{1}_{[0,\infty)}\big) * \xi^n\big\|_{\varepsilon-\beta_1+1,p,\infty}\|\xi^n\|_{\beta-1,p,\infty} \\
&\quad \lesssim \big\|\big((\psi - \psi(0)\chi)\mathbb{1}_{[0,\infty)}\big)\big\|_{\delta,1,\infty}\|\xi^n\|^2_{\beta-1,p,\infty}.
\end{aligned}$$

Applying the estimate (3.33) for the regularity $1 < \delta < 2$, we obtain

$$\begin{aligned}
\big\|\big((\psi - \psi(0)\chi)\mathbb{1}_{[0,\infty)}\big)\big\|_{\delta,1,\infty} &\lesssim \|\psi - \psi(0)\chi\|_{\delta,1,\infty} \\
&\leqslant \|\psi\|_{\delta,1,\infty} + |\psi(0)|\|\chi\|_{\delta,1,\infty} \lesssim \big(1 + \|\chi\|_{\delta,1,\infty}\big)\|\psi\|_{\delta,1,\infty}.
\end{aligned}$$

As $\xi^n \to \xi$ in $\mathcal{B}^{\beta-1}_{p,\infty}$ and $\mathcal{B}^{\delta-2(1-\beta)}_{p/2,\infty} \subseteq \mathcal{B}^{2\beta-1}_{p/2,\infty}$, this implies one direction of the assertion. The converse direction follows analogously. $\square$

Remark 4.2. For $\alpha + \beta_1 < 1$ the condition $\delta > 2(1-\beta) = 1 - \alpha - \beta + \gamma > \gamma$ is in line with the regular case in Theorem 3.13. Lemma 4.1 especially implies that for regular kernels the results, developed in Section 3 for convolutional rough paths, can be applied to all stochastic processes which can be enhanced to rough paths such as semi-martingales and various Gaussian processes, cf. Friz and Victoir [19].

While in the regular case the additional information can be reduced to $\pi((\mathbb{1}_{[0,\infty)}\chi) * \xi, \xi)$, the following example illustrates that for singular Volterra equations it is indeed necessary to include the kernel into the resonant term, i.e., it is not sufficient to take only this "classical" resonant term into account.

Example 4.3. Consider the following 2-dimensional Volterra equation

$$\begin{aligned} u^1 &= \varphi * \xi^1, \\ u^2 &= \varphi * (u^1 \xi^2) = \varphi * \big((\varphi * \xi^1)\xi^2\big), \end{aligned}$$

with some singular kernel $\varphi \in \mathcal{B}^{\gamma}_{1,\infty}$ for $\gamma \in (0,1)$ and $(\xi^1, \xi^2) \in \mathcal{B}^{\beta-1}_{p,\infty}$. We notice that $\pi((\mathbb{1}_{[0,\infty)}\chi) * \xi^1, \xi^2) \in \mathcal{B}^{2\beta-1}_{p/2,\infty}$ is well-defined if $2\beta > 1$, but $\pi((\varphi * \xi^1), \xi^2) \in \mathcal{B}^{2\beta-2+\gamma}_{p/2,\infty}$ is not well-defined if $2\beta < 2 - \gamma$. Hence, for $\gamma < 1$ and $1/2 < \beta < 1 - \gamma/2$ the product $(\varphi * \xi^1)\xi^2$ is not well-defined while the resonant term $\pi((\mathbb{1}_{[0,\infty)}\chi) * \xi^1, \xi^2)$ gives no additional information.

In order to make the example more explicit, we set $\xi^i = \mathrm{d}\vartheta^i$, $i = 1, 2$, with $\vartheta^i = B^i_H \widetilde{\chi}$ for fractional Brownian motions B^i_H with Hurst index $H \in (1/2, 2/3)$ and a compactly supported function $\widetilde{\chi} \in C^\infty$. Moreover, we choose the kernel $\varphi(s) = s^{r-1}\mathbb{1}_{(0,\infty)}(s)\widetilde{\chi}(s)$ for $r \in (4/3 - H, 2 - 2H)$, which is associated to the fractional integration operator of order r, cf. Section 5.3. We then have for any arbitrarily small $\varepsilon > 0$ that $(\vartheta^1, \vartheta^2) \in \mathcal{B}^{\beta}_{\infty,\infty}$ and $\varphi \in \mathcal{B}^{\gamma}_{1,\infty}$ with $\beta = H - \varepsilon$ and $\gamma = r - \varepsilon$. By the choice of r and H, we indeed have $1/2 < \beta < 1 - \gamma/2$, but also $\alpha := \beta + \gamma - 1 > 1/3$ and $2\alpha + \beta > 1$ such that Theorem 3.10 is applicable.

4.2 Stochastic construction of the resonant term

While Lemma 4.1 allows for the construction of the resonant term $\pi(\varphi * \xi, \xi)$ for a regular kernels φ and a large class of noise processes ξ via rough path theory, the aim of this section is to directly construct $\pi(\varphi * \xi, \xi)$. This is particularly interesting for singular kernels, but also gives some deeper understanding on the interplay between the analytical object $\pi(\varphi * \xi, \xi)$ and the stochastic behaviour of ξ. We investigate a class of stochastic processes admitting a series expansion

$$\xi_t = \sum_{n \geqslant 1} a_n(t)\zeta_n, \quad t \in \mathbb{R}, \tag{4.1}$$

for coefficient processes $(a_n)_{n\geqslant 1}$ and random variables ζ_n, which are all defined on a joint probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with corresponding expectation operator $\mathbb{E}$. We will impose the following assumptions:

(A) Let $(\zeta_n)_{n\geqslant 1}$ be a sequence of random variables satisfying $\mathbb{E}[\zeta_n \zeta_m] = \mathbb{1}_{\{n=m\}}$ and the following *hypercontractivity* property: For every $r \geqslant 1$ there is a constant $C_r > 0$ such that for every polynomial $P\colon \mathbb{R}^n \to \mathbb{R}$ of degree 2 we have

$$\mathbb{E}\big[|P(\zeta_1, \ldots, \zeta_n)|^r\big] \leqslant C_r \mathbb{E}\big[|P(\zeta_1, \ldots, \zeta_n)|^2\big]^{r/2}.$$

(B) Let $a_n \in \mathcal{B}^{\beta-1}_{p,1}$, $n \geqslant 1$, for some $p \geqslant 2$, $\beta \in (0,1)$ such that $\sum_{n\geqslant 1} \|a_n\|^2_{\beta-1,p,1} < \infty$.

An important class of processes satisfying these assumptions are centred Gaussian processes ξ whose covariance operator can be represented as an L^2-inner product, i.e., $\mathbb{E}[\xi_s \xi_t] = \langle f_s, f_t \rangle$ for a class of functions $(f_t)_{t\in\mathbb{R}}$. If we expand $f_t = \sum_n a_n(t)\psi_n$, $a_n(t) = \langle f_t, \psi_n \rangle$, with respect to some orthonormal basis (ψ_n), we may obtain the representation (4.1) with i.i.d. standard normal (ζ_n). Indeed, the distribution of the finite dimensional distributions of the random series then coincides with the original process by construction, such that in general only tightness has additionally to be verified.

Example 4.4.

(i) Let $(B_t)_{t\in[0,1]}$ be a Brownian motion. Its well-known Karhunen-Loève expansion is given by

$$B_t = \sqrt{2}\sum_{n=1}^{\infty} \frac{\sin\big((n-1/2)\pi t\big)}{(n-1/2)\pi}\zeta_n, \qquad t \in [0,1],$$

for i.i.d. $\zeta_n \sim \mathcal{N}(0,1)$. Using a periodic version of Brownian motion, we may consider this series for all $t \in \mathbb{R}$. Let $\xi = (\mathrm{d}B)\chi$ be the distributional derivative multiplied with a localising function $\chi \in L^p$. Then ξ admits the representation with (4.1) with $a_n(t) = \sqrt{2}\cos((n-1/2)\pi t)\chi(t)$. Since $\|a_n\|_{\beta-1,p,1}$ is of the order $n^{\beta-1}$, Assumption (B) is satisfied for all $\beta < 1/2$.

(ii) Dzhaparidze and van Zanten [16] have proved the following series expansion for the fractional Brownian motion $(X_t)_{t\in[0,1]}$ with Hurst index $H \in (0,1)$:

$$X_t = \sum_{n=1}^{\infty} \frac{\sin(x_n t)}{x_n}\sigma_n\zeta_n + \sum_{n=1}^{\infty} \frac{1-\cos(y_n t)}{y_n}\tau_n\eta_n, \qquad t \in [0,1],$$

where $(\zeta_n)_{n\geqslant 1}$ and $(\eta_n)_{n\geqslant 1}$ are independent, standard normal random variables, $x_1 < x_2 < \dots$ are the positive, real zeros of the Bessel function J_{-H} of the first kind of order $-H$ and $y_1 < y_2 < \dots$ are the positive zeros of J_{1-H}. Moreover, $\sigma_n^2 = c_H x_n^{-2H} J_{1-H}^{-2}(x_n)$ and $\tau_n^2 = c_H y_n^{-2H} J_{-H}^{-2}(y_n)$ with some explicit constant $c_H > 0$ given in [16]. X_t can be decomposed into a two-dimensional process with coordinates given by the first and the second sum, respectively. As noise process ξ, we again consider the localised derivative leading to (4.1) with $a_n(t) = (a_n^{(1)}(t), a_n^{(2)}(t)) = (\sigma_n\cos(x_n t), \tau_n \sin(y_n))^\top\chi(t)$. Noting the asymptotic expressions $\sigma_n^2 \sim \tau_n^2 \sim n^{1-2H}$ and $x_n \sim y_n \sim n$ for $n \to \infty$, cf. [16], we obtain $\|a_n^{(1)}\|_{\beta-1,p,p} \sim \sigma_n x_n^{\beta-1} \sim n^{-1/2-H+\beta}$ and $\|a_n^{(2)}\|_{\beta-1,p,p} \sim n^{-1/2-H+\beta}$. We conclude that Assumption (B) is fulfilled for $\beta < H$.

Based on the assumption (A) and (B), we first verify the Besov regularity of ξ.

Lemma 4.5. *Let $(\zeta_n)_{n\geqslant1}$ and $(a_n)_{n\geqslant1}$ fulfil Assumptions (A) and (B), respectively. Then, there is a monotone integer valued sequence $(m_n) \uparrow \infty$ such that the approximating sequence $\xi^n = \sum_{k=1}^{m_n} a_k(s)\zeta_k$ is almost surely a Cauchy sequence with respect to $\|\cdot\|_{\beta-1,p,\infty}$. In particular, the almost sure and L^p-limit*

$$\xi_t := \lim_{n\to\infty} \xi_t^n = \sum_{n\geqslant1} a_n(t)\zeta_n, \quad t \in \mathbb{R},$$

is $\mathcal{B}^{\beta-1}_{p,\infty}$-regular.

Proof. We set $m_0 = 1$ and

$$m_n := \inf\Big\{K \geqslant m_{n-1} : \sum_{k=K+1}^{\infty} \|a_k\|^2_{\beta-1,p,p} \leqslant n^{-6}\Big\}, \qquad n \geqslant 1. \tag{4.2}$$

It is sufficient to show

$$\sum_{n\geqslant1} \mathbb{P}\Big(\|\xi^{n+1} - \xi^n\|_{\beta-1,p,\infty} > b_n\Big) < \infty \tag{4.3}$$

for some sequence $(b_n) \in \ell^1$. Then, the Borel-Cantelli Lemma yields that for almost every $\omega \in \Omega$ there is some $n(\omega) \geqslant 1$ such that $\|\xi^{m+1} - \xi^m\|_{\beta-1,p,\infty} \leqslant b_m$ for all $m \geqslant n(\omega)$. Since b_m is summable, $(\xi^n)_{n\geqslant1}$ is almost surely a Cauchy sequence converging to $\xi \in \mathcal{B}^{\beta-1}_{p,\infty}$. Moreover, it suffices to consider $p < \infty$ due to the embedding $\mathcal{B}^{\beta-1}_{p,\infty} \subseteq \mathcal{B}^{\beta-1-1/p}_{\infty,\infty}$, which is sufficient if p is chosen large enough.

We now verify (4.3). By definition we have

$$\begin{aligned}\|\xi^{n+1}-\xi^n\|_{\beta-1,p,\infty} &= \Big\|\sum_{k=m_n+1}^{m_{n+1}} \zeta_k a_k\Big\|_{\beta-1,p,\infty}\\ &= \sup_{j\geqslant -1}\Big(2^{(\beta-1)j}\Big\|\sum_{k=m_n+1}^{m_{n+1}}\zeta_k(\Delta_j a_k)\Big\|_{L^p}\Big)\\ &= \sup_{j\geqslant -1}\Big(2^{(\beta-1)jp}\int_{\mathbb{R}}\Big|\sum_{k=m_n+1}^{m_{n+1}}\zeta_k(\Delta_j a_k)(x)\Big|^p \mathrm{d}x\Big)^{1/p}.\end{aligned}$$

Hence, using an union bound for the supremum and Markov's inequality we have

$$\mathbb{P}\Big(\|\xi^{n+1}-\xi^n\|_{\beta-1,p,\infty} > b_n\Big) \leq b_n^{-p}\sum_{j\in\mathbb{N}} 2^{(\beta-1)jp}\int_{\mathbb{R}}\mathbb{E}\Big[\Big|\sum_{k=m_n+1}^{m_{n+1}}\zeta_k(\Delta_j a_k)(x)\Big|^p\Big]\mathrm{d}x. \tag{4.4}$$

Using the hypercontractivity and $\mathbb{E}[\zeta_n\zeta_m]=\mathbb{1}_{\{n=m\}}$, we obtain the upper bound

$$\begin{aligned}&b_n^{-p}\sum_{j\in\mathbb{N}} 2^{(\beta-1)jp}\int_{\mathbb{R}}\mathbb{E}\Big[\Big(\sum_{k=m_n+1}^{m_{n+1}}\zeta_k(\Delta_j a_k)(x)\Big)^2\Big]^{\frac{p}{2}}\mathrm{d}x\\ &\quad= b_n^{-p}\sum_{j\in\mathbb{N}} 2^{(\beta-1)jp}\int_{\mathbb{R}}\Big(\sum_{k=m_n+1}^{m_{n+1}}(\Delta_j a_k)^2(x)\Big)^{\frac{p}{2}}\mathrm{d}x.\end{aligned}$$

We now use Hölder's inequality to obtain for any sequence $(c_k)\in\ell^1$

$$\begin{aligned}\mathbb{P}\Big(\|\xi^{n+1}-\xi^n\|_{\beta-1,p,\infty} > b_n\Big) &\lesssim b_n^{-p}\sum_{j\in\mathbb{N}} 2^{(\beta-1)jp}\Big(\sum_{k=m_n+1}^{m_{n+1}} c_k\Big)^{p/2-1}\sum_{k=m_n+1}^{m_{n+1}} c_k^{-(p/2-1)}\|\Delta_j a_k\|_{L^p}^p\\ &= b_n^{-p}\Big(\sum_{k=m_n+1}^{m_{n+1}} c_k\Big)^{p/2-1}\sum_{k=2^n+1}^{2^{n+1}} c_k^{-(p/2-1)}\|a_k\|_{\beta-1,p,p}^p\\ &\leqslant b_n^{-p}\Big(\sum_{k=m_n+1}^{m_{n+1}} c_k\Big)^{p/2}\Big(\sup_k c_k^{-1/2}\|a_k\|_{\beta-1,p,p}\Big)^p.\end{aligned}$$

Choosing $c_k := \|a_k\|_{\beta-1,p,1}^2 \geqslant \|a_k\|_{\beta-1,p,p}^2$, it remains to note that $d_n := (\sum_{k=m_n+1}^{m_{n+1}} c_k)^{1/2} \leqslant n^{-3}$ by the choice of m_n, such that we may choose $b_n = n^{-3/2}$. □

Young's inequality (Lemma 2.2) yields automatically $\varphi*\xi\in\mathcal{B}^{\gamma+\beta-1}_{p,\infty}$ for $\varphi\in\mathcal{B}^{\gamma}_{1,\infty}$. With these preperations we can verify the existence of a limit $\lim_{n\to\infty}\pi(\varphi*\xi^n,\xi^n) =: \pi(\varphi*\xi,\xi)\in\mathcal{B}^{2\beta+\gamma-2}_{p/2,\infty}$.

Theorem 4.6. *Let $(\zeta_n)_{n\geqslant 1}$ and $(a_n)_{n\geqslant 1}$ fulfil Assumptions (A) and (B), $p\geqslant 4$ and $\gamma>0$. Further, suppose that (ζ_n) are independent, and $\varphi\in\mathcal{B}^{\gamma}_{1,\infty}$. Set $\xi^n := \sum_{k=1}^{m_n} a_k\zeta_k$ for a sufficiently fast growing integer valued sequence $(m_n)\uparrow\infty$. Then $(\pi(\varphi*\xi^n,\xi^n))_{n\geqslant 1}$ is almost surely a Cauchy sequence with respect to $\|\cdot\|_{2\beta+\gamma-2,p/2,\infty}$ with almost sure and $L^{p/2}$-limit*

$$\pi(\varphi*\xi,\xi) := \lim_{n\to\infty}\pi(\varphi*\xi^n,\xi^n)\in\mathcal{B}^{2\beta+\gamma-2}_{p/2,\infty}.$$

Proof. Let $(m_n)_{n\geqslant 0}$ be as in (4.2). As in the Lemma 4.5 thanks to the Borel-Cantelli Lemma it suffices to prove for some sequence $(b_n)\in\ell^1$ and finite $p\in[1,\infty)$:

$$\sum_{n\geqslant 1}\mathbb{P}\Big(\|\pi(\varphi*\xi^{n+1},\xi^{n+1})-\pi(\varphi*\xi^n,\xi^n)\|_{2\beta+\gamma-2,p/2,\infty} > b_n\Big) < \infty.$$

Defining $\Delta_k^\varphi f := \Delta_k(\varphi * f) = \mathcal{F}^{-1}[\rho_j \mathcal{F}\varphi] * f$ for distributions f, we have

$$\begin{aligned}
&\pi(\varphi * \xi^{n+1}, \xi^{n+1}) - \pi(\varphi * \xi^n, \xi^n) \\
&= \sum_{j\geqslant 1} \sum_{k=j-1}^{j+1} \Delta_j\big(\xi^{n+1} - \xi^n\big)\Delta_k^\varphi \xi^{n+1} + \sum_{j\geqslant 1} \sum_{k=j-1}^{j+1} \Delta_j \xi^n \Delta_k^\varphi\big(\xi^{n+1} - \xi^n\big) \\
&=: T_{n,1} + T_{n,2}.
\end{aligned}$$

Since both terms can be estimated analogously, we focus on $T_{n,1}$, for which we have

$$\begin{aligned}
T_{n,1} &= \sum_{j\geqslant 1} \sum_{k=j-1}^{j+1} \Delta_j\Big(\sum_{m=m_n+1}^{m_{n+1}} \zeta_m a_m\Big) \cdot \Delta_k^\varphi\Big(\sum_{m=1}^{m_{n+1}} \zeta_k a_m\Big) \\
&= \sum_{j\geqslant 1} \sum_{k=j-1}^{j+1} \sum_{m=m_n+1}^{m_{n+1}} \sum_{m'=1}^{m_{n+1}} \zeta_m \zeta_{m'} (\Delta_j a_m)(\Delta_k^\varphi a_{m'}).
\end{aligned}$$

Hence, we get

$$\begin{aligned}
&\|T_{n,1}\|_{2\beta+\gamma-2,p/2,\infty} \\
&\quad= \sup_j \big(2^{(2\beta+\gamma-2)j} \|\Delta_j T_1\|_{L^{p/2}}\big) \\
&\quad\leqslant \sup_j \Big(2^{(2\beta+\gamma-2)j} \sum_{j'\sim j} \Big\| \sum_{k=j'-1}^{j'+1} \sum_{m=m_n+1}^{m_{n+1}} \sum_{m'=1}^{m_{n+1}} \zeta_m \zeta_{m'} (\Delta_j a_m)(\Delta_k^\varphi a_{m'}) \Big\|_{L^{p/2}}\Big) \\
&\quad\lesssim \sup_j \Bigg(2^{(2\beta+\gamma-2)j} \Big\| \sum_{k=j-1}^{j+1} \sum_{m=m_n+1}^{m_{n+1}} \sum_{m'=1}^{m_{n+1}} \zeta_m \zeta_{m'} (\Delta_j a_m)(\Delta_k^\varphi a_{m'}) \Big\|_{L^{p/2}}\Bigg).
\end{aligned}$$

As above, Markov's inequality and the hypercontractivity yield

$$\begin{aligned}
&\mathbb{P}\big(\|T_{n,1}\|_{2\beta+\gamma-2,p/2,\infty} > b_n\big) \\
&\quad\lesssim b_n^{-p/2} \sum_j 2^{(2\beta+\gamma-2)jp/2} \int_{\mathbb{R}} \mathbb{E}\Big[\Big| \sum_{k=j-1}^{j+1} \sum_{m=m_n+1}^{m_{n+1}} \sum_{m'=1}^{m_{n+1}} \zeta_m \zeta_{m'} (\Delta_j a_m)(x)(\Delta_k^\varphi a_{m'})(x)\Big|^{p/2}\Big] \,\mathrm{d}x \\
&\quad\lesssim b_n^{-p/2} \sum_j 2^{(2\beta+\gamma-2)jp/2} \int_{\mathbb{R}} \mathbb{E}\Big[\Big| \sum_{k=j-1}^{j+1} \sum_{m=m_n+1}^{m_{n+1}} \sum_{m'=1}^{m_{n+1}} \zeta_m \zeta_{m'} (\Delta_j a_m)(x)(\Delta_k^\varphi a_{m'})(x)\Big|^2\Big]^{\frac{p}{4}} \,\mathrm{d}x \\
&\quad\lesssim b_n^{-p/2} \sum_j 2^{(2\beta+\gamma-2)jp/2} \int_{\mathbb{R}} \Big(\sum_{k_1,k_2=j-1}^{j+2} \sum_{m_1,m_2=m_n+1}^{m_{n+1}} \sum_{m_1',m_2'=1}^{m_{n+1}} \mathbb{E}\big[\zeta_{m_1}\zeta_{m_1'}\zeta_{m_2}\zeta_{m_2'}\big] \\
&\qquad \times (\Delta_j a_{m_1})(x)(\Delta_{k_1}^\varphi a_{m_1'})(x)(\Delta_j a_{m_2})(x)(\Delta_{k_2}^\varphi a_{m_2'}(x)\Big)^{\frac{p}{4}} \,\mathrm{d}x.
\end{aligned}$$

In the previous sum it suffices the consider the terms where $\{m_1 = m_2, m_1' = m_2'\}$, $\{m_1 = m_1', m_2 = m_2'\}$ (being equivalent to $\{m_1 = m_2', m_2 = m_1'\}$) and $\{m_1 = m_2 = m_1' = m_2'\}$, because in all other cases $\mathbb{E}\big[\zeta_{m_1}\zeta_{m_1'}\zeta_{m_2}\zeta_{m_2'}\big]$ is zero by independence of the (ζ_m). Since all partial sums can be bounded similarly, we consider only $\{m_1 = m_2, m_1' = m_2'\}$ for brevity. This partial sum is given by

$$\begin{aligned}
S_n :=&\, b_n^{-p/2} \sum_j \Bigg(2^{(2\beta+\gamma-2)jp/2} \\
&\sum_{j-1\leqslant k_1,k_2\leqslant j+1} \int_{\mathbb{R}} \Big(\sum_{m=m_n+1}^{m_{n+1}} \sum_{m'=1}^{m_{n+1}} (\Delta_j a_m)^2(x)(\Delta_{k_1}^\varphi a_{m'})(x)(\Delta_{k_2}^\varphi a_{m'})(x)\Big)^{\frac{p}{4}} \,\mathrm{d}x\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= b_n^{-p/2} \sum_j \Big(2^{(2\beta+\gamma-2)jp/2} \\
&\qquad \sum_{j-1\leqslant k_1,k_2\leqslant j+1} \int_{\mathbb{R}} \Big(\sum_{m=m_n+1}^{m_{n+1}} (\Delta_j a_m)^2(x) \Big)^{\frac{p}{4}} \Big(\sum_{m'=1}^{m_{n+1}} (\Delta^{\varphi}_{k_1} a_{m'})(x)(\Delta^{\varphi}_{k_2} a_{m'})(x) \Big)^{\frac{p}{4}} \,\mathrm{d}x \Big).
\end{aligned}
$$

Hölder's inequality yields for the ℓ^1-sequence $c_k := \|a_k\|^2_{\beta-1,p,1}$, $k \geqslant 1$,

$$
\begin{aligned}
S_n &\leqslant \frac{1}{b_n^{p/2}} \sum_j 2^{(2\beta+\gamma-2)jp/2} \sum_{j-1\leqslant k_1,k_2\leqslant j+1} \int_{\mathbb{R}} \Big(\sum_{m=m_n+1}^{m_{n+1}} c_m \Big)^{\frac{p}{4}-1} \Big(\sum_{m=m_n+1}^{m_{n+1}} c_m^{-(\frac{p}{4}-1)} (\Delta_j a_m)^{\frac{p}{2}}(x) \Big) \\
&\quad \times \Big(\sum_{m'=1}^{m_{n+1}} c_{m'} \Big)^{\frac{p}{4}-1} \Big(\sum_{m'=1}^{m_{n+1}} c_{m'}^{-(\frac{p}{4}-1)} (\Delta^{\varphi}_{k_1} a_{m'})^{\frac{p}{4}}(x)(\Delta^{\varphi}_{k_2} a_{m'})^{\frac{p}{4}}(x) \Big) \,\mathrm{d}x \\
&\leqslant \|c_m\|_{\ell^1}^{\frac{p}{4}-1} b_n^{-p/2} \Big(\sum_{m=2^n+1}^{2^{n+1}} c_m \Big)^{\frac{p}{4}-1} \sum_j 2^{(2\beta+\gamma-2)jp/2} \\
&\qquad \sum_{j-1\leqslant k_1,k_2\leqslant j+1} \sum_{m=m_n+1}^{m_{n+1}} c_m^{-\frac{p}{4}+1} \sum_{m'=1}^{m_{n+1}} c_{m'}^{-\frac{p}{4}+1} \int_{\mathbb{R}} (\Delta_j a_m)^{\frac{p}{2}}(x) (\Delta^{\varphi}_{k_1} a_{m'})^{\frac{p}{4}}(x)(\Delta^{\varphi}_{k_2} a_{m'})^{\frac{p}{4}}(x) \,\mathrm{d}x.
\end{aligned}
$$

Writing $d_n := (\sum_{k=m_n+1}^{m_{n+1}} c_k)^{1/2} \in \ell^1$ and applying once again Hölder's inequality, we obtain

$$
\begin{aligned}
S_n &\lesssim \|c_m\|_{\ell^1}^{\frac{p}{4}-1} b_n^{-p/2} d_n^{p/2-2} \sum_j \sum_{j-1\leqslant k_1,k_2\leqslant j+1} \sum_{m=m_n+1}^{m_{n+1}} c_m^{-(p/4-1)} 2^{(\beta-1)jp/2} \|\Delta_j a_m\|_{L^p}^{p/2} \\
&\qquad \sum_{m'=1}^{2^{n+1}} c_{m'}^{-(p/4-1)} 2^{(\beta-1+\gamma)k_1 p/4} \|\Delta^{\varphi}_{k_1} a_{m'}\|_{L^p}^{p/4} 2^{(\beta-1+\gamma)k_2 p/4} \|\Delta^{\varphi}_{k_2} a_{m'}\|_{L^p}^{p/4} \\
&\lesssim \|c_m\|_{\ell^1}^{\frac{p}{4}-1} b_n^{-p/2} d_n^{p/2-2} \sum_{m=2^n+1}^{2^{n+1}} c_m^{-(p/4-1)} \|a_m\|^{p/2}_{\beta-1,p,p/2} \sum_{m'=1}^{2^{n+1}} c_{m'}^{-(\frac{p}{4}-1)} \|\varphi * a_{m'}\|^{p/2}_{\beta-1+\gamma,p,p/4} \\
&\leqslant \|c_m\|_{\ell^1}^{\frac{p}{4}} (d_n/b_n)^{p/2} \Big(\sup_{m'} c_{m'}^{-1/2} \|\varphi * a_{m'}\|_{\beta-1+\gamma,p,p/4} \Big)^{p/2}.
\end{aligned}
$$

With $\|\varphi * a_{m'}\|_{\beta-1+\gamma,p,p/4} \lesssim \|\varphi\|_{\gamma,1,1} \|a_{m'}\|_{\beta-1,p,p/4}$ by Young's inequality, we conclude $S_n \lesssim (d_n/b_n)^{p/2} \|\varphi\|^{p/2}_{\gamma,1,1}$. Since $d_n \lesssim n^{-3}$, we deduce $\sum_{n\geqslant 1} S_n < \infty$ for $b_n = n^{-3/2}$. □

Remark 4.7. For the special case where $\varphi * \xi$ is replaced by the antiderivative of ξ, alternative constructions of rough path and iterated integrals above stochastic processes defined by random Fourier or Schauder expansions were considered in [20, 25, 37].

5 Application to rough and stochastic differential equations

The general existence and uniqueness results for solutions to Volterra equations of the form (1.2) provided in Section 3 allow to recover well-known results in the paracontrolled distribution setting but additionally contain many novel results concerning differential equations driven by stochastic processes or convolutional rough paths. In the following we discuss some exemplary stochastic equations and explicitly state the particular existence and uniqueness results.

5.1 Stochastic and rough differential equations with possible delay

Ordinary stochastic differential equations and their pathwise counterparts given by rough differential equations constitute fundamental and well studied objects in stochastic analysis. These differential equations can typically written in their integral form

$$u(t) = u_0 + \int_0^t \sigma_1(u(s-r_1))\,\mathrm{d}\vartheta(s) + \int_0^t \sigma_2(u(s-r_2))\,\mathrm{d}s, \quad t \in [0,T], \tag{5.1}$$

where ϑ is a suitable driving signal, e.g., a (fractional) Brownian motion or a rough path, and $r_1, r_2 \geq 0$ are constant delay parameters. Thanks to the general regularity assumptions required on the kernel functions in Section 3, the differential equation (5.1) can be viewed as a special case of the Volterra equation (1.2), and we can recover for instance the following results. For this purpose, we denote by $\dot{\vartheta}$ the distributional derivative of $\vartheta \in \mathcal{B}^{\beta}_{p,\infty}$ and introduce a kernel function φ_T which is assumed to be compactly supported on $[0,2T]$, smooth on $\mathbb{R}\setminus\{0\}$ and satisfying $\varphi_T(t) = \mathbb{1}_{[0,\infty)}(t)$ for all $t \in [-T,T]$.

Corollary 5.1. *Let $u_0 \in \mathbb{R}$, $\sigma_1 \in C^3$, $\sigma_2 \in C^2$ with $\sigma_1(0) = \sigma_2(0) = 0$ and $r_1, r_2 \geq 0$.*

(i) If ϑ is an n-dimensional fractional Brownian motion with Hurst index $H > 1/2$ and $T > 0$ is sufficiently small, then there exists a unique solution to the stochastic differential equation (5.1).

(ii) If $(\dot{\vartheta}, \pi(\varphi_T * (\dot{\vartheta}\mathbb{1}_{[0,T]}), \dot{\vartheta}(\cdot + r_1)\mathbb{1}_{[0,T]}(\cdot + r_1))) \in \mathcal{B}^{\beta,1}_{p}(\varphi_T)$ for $\beta > 1/3$, $p \in [3,\infty)$ and $T > 0$ is sufficiently small, then there exists a unique solution to the rough differential equation (5.1).

Proof. Let χ be a smooth and compactly support function with $\chi(t) = 1$ for $t \in [0,T]$. Equation (5.1) coincides on the interval $[0,T]$ with

$$u(t) = u_0\chi(t) + \int_{\mathbb{R}} \varphi_T(t-s-r_1)\sigma_1(u(s))\dot{\Xi}(s)\,\mathrm{d}s + \int_{\mathbb{R}} \varphi_T(t-s-r_2)\sigma_2(u(s))\mathbb{1}_{[0,T]}(s+r_2)\,\mathrm{d}s$$

for $t \in \mathbb{R}$ and driving signal $\dot{\Xi}(\cdot) := \dot{\vartheta}(\cdot + r_1)\mathbb{1}_{[0,T]}(\cdot + r_1)$. Note that $\varphi_T(\cdot - r_1) \in \mathcal{B}^1_{1,\infty}$ and $(\cdot - r_1)\varphi_T(\cdot - r_1) \in \mathcal{B}^2_{1,\infty}$. Hence, (i) and (ii) follow by applying Theorem 3.13 and recalling that a fractional Brownian motion with Hurst index $H > 1/2$ has almost surely $(H-\varepsilon)$-Hölder continuous sample paths for every $\varepsilon > 0$. $\square$

Existence and uniqueness results for stochastic delay equations like (5.1) driven by a fractional Brownian motion with Hurst index $H > 1/2$ were first obtained by Ferrante and Rovira [18]. Differential equations driven by α-Hölder continuous rough paths with $\alpha \in (1/3, 1/2)$ and constant delay were first treated by Neuenkirch et al. [35]. Rough differential equations without delay but in the paracontrolled distribution setting were considered in [24] and [37]. Furthermore, we would like to point out that Corollary 5.1 (ii) can be applied to a fractional Brownian motion with Hurst index $H \in (1/3, 1/2)$ due to Theorem 4.6.

Remark 5.2. For stochastic and rough differential equations like (5.1), it is straightforward to obtain a solution on any arbitrary large interval $[0,T]$ applying iteratively Corollary 5.1 on small intervals and glueing the so obtained local solutions together.

5.2 Stochastic and rough Volterra equations

Stochastic integral equations of Volterra type appear in various areas of mathematical modelling such as in physics or mathematical finance and the treatment of such Volterra equations involving stochastic integration goes back to the pioneering works of Berger and Mizel [7, 8]. The pathwise counterparts of stochastic Volterra equations, namely, Volterra equations driven by rough paths were first considered by Deya and Tindel [14, 15]. More precisely, we consider Volterra equations of convolution type

$$u(t) = u_0(t) + \int_0^t \psi_1(t-s)\sigma_1\big(u(s)\big)\dot{\vartheta}(s)\,\mathrm{d}s + \int_0^t \psi_2(t-s)\sigma_2\big(u(s)\big)\,\mathrm{d}s, \tag{5.2}$$

for $t \in [0,T]$ and $\dot{\vartheta}$ denotes again the distributional derivative of the path $\vartheta \in \mathcal{B}^{\beta}_{p,\infty}$.

Corollary 5.3. *Let $p \in [3,\infty]$ and $\beta \in (1/3,1/2)$. Suppose that $\psi_1,\psi_2 \in \mathcal{B}^1_{\infty,\infty}$, $u_0 \in \mathcal{B}^{2\beta}_{p,\infty}$ and $\sigma_1 \in C^3$, $\sigma_2 \in C^2$ with $\sigma_1(0)=\sigma_2(0)=0$. If $(\dot{\vartheta},\pi((\varphi_T\psi_1)*(\dot{\vartheta}),\dot{\vartheta}) \in \mathcal{B}^{\beta,1}_p(\varphi_T\psi_1)$ and $T>0$ is sufficiently small, then there exists $u \in \mathcal{B}^{\beta}_{p,\infty}$ which is the unique solution to the rough Volterra equation (5.2) on $[0,T]$.*

Proof. We first observe that $\psi_i\varphi_T \in \mathcal{B}^1_{1,\infty}$ because $\psi_i \in \mathcal{B}^1_{\infty,\infty}$ and $\varphi_T \in \mathcal{B}^{1/p}_{p,\infty}$, for $i=1,2$. Moreover, the rough Volterra equation (5.2) coincides on the interval $[0,T]$ with

$$\begin{aligned} u(t) = u_0(t) &+ \int_{\mathbb{R}} \varphi_T(t-s)\psi_1(t-s)\sigma_1(u(s))\dot{\vartheta}(s)\,\mathrm{d}s \\ &+ \int_{\mathbb{R}} \varphi_T(t-s)\psi_1(t-s)\sigma_2(u(s))\,\mathrm{d}s, \quad t\in\mathbb{R}. \end{aligned}$$

Therefore, Theorem 3.13 and Remark 3.14 imply the assertion. □

Remark 5.4. Assuming $\psi_1(0)\neq 0$, it is not necessary to include the kernel function ψ_1 in the definition of the driving rough path. Indeed, one can take a generic rough path, i.e., independent of ψ_1, thanks to Lemma 4.1. Furthermore, notice that the kernel ψ_1 has only to be Lipschitz continuous. This Lipschitz assumption is a significant relaxation compared to the C^3-regularity of the kernel functions so far required for Volterra equations of convolutional type driven by rough paths, see Deya and Tindel [14, 15].

The previous pathwise existence and uniqueness result for Volterra equations can immediately be applied to a wide class of stochastic processes thanks to Theorem 4.6.

Corollary 5.5. *Let ϑ be a stochastic process such that $\dot{\vartheta}$ is of the form (4.1) satisfying Assumption (A) and (B) for $\beta\in(1/3,1)$ and $p\geq 3$. Suppose that $\psi_1,\psi_2\in\mathcal{B}^1_{\infty,\infty}$, $u_0\in\mathcal{B}^{2\beta}_{p,\infty}$ and $\sigma_1\in C^3$, $\sigma_2\in C^2$ with $\sigma_1(0)=\sigma_2(0)=0$. If $T>0$ is sufficiently small, then there exists $u\in\mathcal{B}^{\beta}_{p,\infty}$ which is the unique solution of the stochastic Volterra equation (5.2) on $[0,T]$.*

5.3 SDEs with fractional derivatives

Stochastic Volterra equations with singular kernels are of particular interest because of their applications to stochastic partial differential equations (e.g. [44]) and stochastic differential equations with fractional derivatives (e.g. [43]), but also because of recent developments in mathematical finance showing that Volterra equations with singular kernels serve as very suitable models for the probabilistic and irregular behaviour of volatility in financial markets, see e.g. [17].

In order to consider SDEs allowing for fractional derivatives, let us recall the definition of the Riemann-Liouville fractional integral operator (with base point 0), which is given by

$$I^r(f)(t) := \frac{1}{\Gamma(r)}\big((s^{r-1}\mathbb{1}_{(0,\infty)}(s))*f\big)(t) = \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}f(s)\,\mathrm{d}s$$

for $r\in(0,1)$, f a suitable function and the Gamma function

$$\Gamma(r) := \int_0^\infty t^{r-1}e^{-t}\,\mathrm{d}t, \qquad r>0.$$

The corresponding fractional derivative operator is defined by $D^r f := \frac{\mathrm{d}}{\mathrm{d}t}I^{1-r}(f)$. While there are many different fractional derivative operators, the Riemann-Liouville derivative can be considered as a natural extension of the classical derivative to fractional order. A (fairly simple) stochastic differential equation of fractional order $r\in(0,1)$ driven by a Brownian motion is

$$D^r u(t) = \sigma(u(t))\,\mathrm{d}W(t), \quad u(0)=u_0,$$

or equivalently expressed as a Volterra integral equation with singular kernel

$$u(t) = u_0 + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}\sigma(u(s))\dot{W}(s)\,\mathrm{d}s, \quad t\in[0,T], \tag{5.3}$$

where $\dot{W}$ is the distributional derivative of a Brownian motion W. For a more general treatment of fractional stochastic differential equations driven by Brownian motion, we refer for instance to Lototsky and Rozovsky [30]. Based on the results provided in Section 3, we obtain the following existence and uniqueness statement.

Corollary 5.6. *Let W be an n-dimensional Brownian motion and $r > 5/6$. Suppose that $u_0 \in \mathbb{R}$ and $\sigma \in C^3$ with $\sigma(0) = 0$. If $T > 0$ is sufficiently small, then there exists $u \in \mathcal{B}^{\alpha}_{3,\infty}$ for any $\alpha < r - 1/2$ which is the unique solution to the stochastic Volterra equation (5.3) on $[0, T]$.*

Proof. The proof works as the proof of Corollary 5.3 combined with the observations that the localised kernel function $\varphi(x) := x^{r-1}\varphi_T$ satisfies $\varphi \in \mathcal{B}^{\gamma}_{1,\infty}$ for every $\gamma < r$ and that the sample paths of a Brownian motion can be considered as convolutional rough paths with regularity $\beta < 1/2$ due to Theorem 4.6. □

5.4 SDEs with additive Lévy noise

Stochastic differential equations with an additive Lévy noise constitute appropriate models for dynamical systems which are subject to external shocks. Examples of such systems naturally appear in insurance mathematics, where for instance SDEs with long term memory and additive Lévy noise are used to model the general reserve process of an insurance company, cf. Rolski et al. [39]. More precisely, we consider the stochastic differential equation

$$u(t) = u_0 + \int_0^t \sigma_1(u(s))\,\mathrm{d}s + \int_0^t \sigma(u(s))\,\mathrm{d}\vartheta(s) + L(t), \quad t \in [0, T], \tag{5.4}$$

where ϑ is a fractional Brownian motion and L is a Lévy process. This type of stochastic differential equations were recently investigated, e.g., in Bai and Ma [3].

Corollary 5.7. *Let L be an n-dimensional Lévy process and ϑ be a fractional Brownian motion with Hurst index $H > 1/2$. Let $p \in [2, \infty]$ and $\beta \in (1/2, 1)$. Suppose that $p > 2$, $u_0 \in \mathbb{R}$ and $\sigma_1, \sigma_2 \in C^2$ with $\sigma_1(0) = \sigma_2(0) = 0$. If $T > 0$ is sufficiently small, then there exists $u \in \mathcal{B}^{1/p}_{p,\infty} \cap L^\infty$ which is the unique solution of the stochastic differential equation (5.4) on $[0, T]$.*

Proof. As in the proof of Corollary 5.1 one can reformulate the SDE (5.4) as a Volterra equation which coincides with (5.4) on the interval $[0, T]$. Furthermore, let us recall that the sample paths of a fractional Brownian motion and of a Lévy process are almost surely in $\mathcal{B}^{\beta}_{p,\infty}$ and $\mathcal{B}^{1/p}_{p,\infty}$ for every $\beta < H$ and $p > 2$, respectively, see for instance [40, Proposition 2] and [10, Proposition 5.31]. Hence, we deduce the assertion from Proposition 3.2 in combination with a scaling argument analogously the proof of Theorem 3.13 and Remark 3.14. □

5.5 Stochastic moving average processes driven by Lévy processes

Moving average processes driven by Lévy processes and in particular shot noise processes provide a modern toolbox for mathematical modelling of, e.g., turbulence, signal processing or shot prices on energy markets, see [5, 6] and the references therein. Allowing these types of models to possess a state dependent volatility, we consider the stochastic convolution equation

$$u(t) = u_0 + \int_{\mathbb{R}} \psi(t - s)\sigma(u(s))\,\mathrm{d}L(s), \quad t \in [0, T], \tag{5.5}$$

where L is a general Lévy process. Because of the desired averaging property generated by the kernel function, it is naturally to postulate the assumption of $\psi \in \mathcal{B}^{\gamma}_{1,\infty}$ for $\gamma > 1$. In this case we arrive at the following existence and uniqueness result.

Corollary 5.8. *Let L be an n-dimensional Lévy process, $p \in (2, \infty]$, $\gamma > 1$ and $\alpha = 1/p + \gamma - 1$. Suppose that $\psi \in \mathcal{B}^{\gamma}_{1,\infty}$ has compact support, $u_0 \in \mathbb{R}$ and $\sigma \in C^2$ with $\sigma(0) = 0$. If $T > 0$ is sufficiently small, then there exists $u \in \mathcal{B}^{\alpha}_{p,\infty}$ which is the unique solution of the stochastic convolution equations (5.5) on $[0, T]$.*

Proof. Due to the compact support assumption of ψ, we can localise the equation (5.5) such that we obtain a (localised) Volterra equation which coincides with (5.5) on the interval $[0,T]$. Since the sample paths of a Lévy process are almost surely in $\mathcal{B}^{1/p}_{p,\infty}$ for every $p > 2$, see again [10, Proposition 5.31], we conclude the assertion from Proposition 3.1 in combination with a scaling argument analogously the proof of Theorem 3.13 and Remark 3.14. $\square$

5.6 Relation to stochastic PDEs

In general, stochastic Volterra equations are known to have many links to stochastic partial differential equations. Here we would like to discuss this link in the case of (a slightly modified version of) a stochastic evolution equation studied by Mytnik and Salisbury [34]. We consider the differential operator $\Delta_\vartheta := \partial_x x^\vartheta \partial_x$ (in one space dimension) for a parameter $\vartheta < 2$ and the associated evolution equation

$$\begin{aligned} \partial_t u(t,x) &= \Delta_\vartheta u(t,x) + \sigma\big(u(t,x)\big)\,\xi(\mathrm{d}t,\mathrm{d}x), \\ u(0,x) &= g(x), \end{aligned} \tag{5.6}$$

with multiplicative noise, where ξ is the space-time derivative of $\vartheta(t,x) = W_t \mathbf{1}_{[\eta,\infty)}(x)$ for some $\eta \in \mathbb{R}$, that is

$$\xi(\mathrm{d}t,\mathrm{d}x) = \dot{W}(\mathrm{d}t)\,\delta_\eta(\mathrm{d}x).$$

with Dirac measure δ_η in $\eta \in \mathbb{R}$. Note that we recover the stochastic heat equation with multiplicative noise in the case $\vartheta = 0$ and the fundamental solution of (5.6) with $\xi = 0$ is

$$p_t(x) = \frac{c_\vartheta}{t^{1/(2-\vartheta)}} \exp\Big(- \frac{x^{2-\vartheta}}{(2-\vartheta)^2 t}\Big)$$

with normalising constant c_ϑ such that a mild solution of (5.6) is given by the formula

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}} p(t,x-y)g(y)\,\mathrm{d}y + \int_0^t \int_{\mathbb{R}} p(t-s,x-y)\sigma\big(u(t,y)\big)\,\xi(\mathrm{d}s,\mathrm{d}y) \\ &= \int_{\mathbb{R}} p(t,x-y)g(y)\,\mathrm{d}y + \int_0^t p(t-s,x-\eta)\sigma\big(u(t,\eta)\big)\,\dot{W}(\mathrm{d}s). \end{aligned}$$

In particular, the solution process $v(t) := u(t,\eta)$ along the edge $\{(t,\eta) : t \in \mathbb{R}_+\}$ solves the singular stochastic Volterra equation

$$\begin{aligned} v(t) &= \int_{\mathbb{R}} p(t,\eta-y)g(y)\,\mathrm{d}y + \int_0^t p(t-s,0)\sigma\big(v(t)\big)\,\dot{W}(\mathrm{d}s) \\ &= \int_{\mathbb{R}} p(t,\eta-y)g(y)\,\mathrm{d}y + \int_0^t \frac{c_\vartheta}{(t-s)^{1/(2-\vartheta)}}\sigma\big(v(t)\big)\,\dot{W}(\mathrm{d}s). \end{aligned}$$

For $\vartheta < -4$ Theorem 3.10 provides the existence of the pathwise solution process $v(t)$. In the case of the Laplace operator, i.e. $\vartheta = 0$, the singularity in the kernel is too severe to directly apply Theorem 3.10 and would require a further extension of the above theory.

A Auxiliary Besov estimates

The appendix provides (in the previous sections) frequently used, but fairly elementary lemmas concerning Besov spaces. The first one states the invariance of Besov norms under linear shifts.

Lemma A.1. *Let $\alpha \in \mathbb{R}$, $p \in [1,\infty]$ and $y \in \mathbb{R}^d$. If $f \in \mathcal{B}^\alpha_{p,\infty}$, then $f(\cdot + y) \in \mathcal{B}^\alpha_{p,\infty}$ with*

$$\|f\|_{\alpha,p,\infty} = \|f(\cdot + y)\|_{\alpha,p,\infty}.$$

Proof. For $y \in \mathbb{R}^d$ and $f \in \mathcal{B}^\alpha_{p,\infty}$, note that

$$\mathcal{F}f(\cdot + y)(z) = \int_{\mathbb{R}^d} e^{i\langle z, x-y\rangle} f(x)\,\mathrm{d}x = \mathcal{F}f(z)e^{i\langle z,y\rangle}, \quad z \in \mathbb{R}^d,$$

from which we deduce that

$$\Delta_j f(\cdot + y)(z) = \mathcal{F}^{-1}(\rho_j e^{-i\langle \cdot, y\rangle}\mathcal{F}f)(z) = \mathcal{F}^{-1}(\rho_j \mathcal{F}f)(z+y).$$

Therefore, $\|\Delta_j f(\cdot + y)\|_{L^p} = \|\Delta_j f\|_{L^p}$ for each $j \geq -1$ and thus $\|f\|_{\alpha,p,\infty} = \|f(\cdot + y)\|_{\alpha,p,\infty}$. ☐

For sufficiently regular distributions/functions the Besov norm of a product can be directly estimated and in particular the product is then a well-defined operation.

Lemma A.2.

(i) Let $p \in [2,\infty]$, $\alpha \in (1/p, 1)$ and $\beta \in (1-\alpha, 1)$. If $f \in \mathcal{B}^\alpha_{p,\infty}$ and $g \in \mathcal{B}^{\beta-1}_{p,\infty}$, then

$$\|fg\|_{\beta-1,p,\infty} \lesssim \|f\|_{\alpha,p,\infty}\|g\|_{\beta-1,p,\infty}.$$

(ii) Let $p \in [2,\infty]$ and $\beta \in [0,1)$ be such that $\frac{1}{p} + \beta > 1$. If $f \in \mathcal{B}^{\frac{1}{p}}_{p,\infty} \cap L^\infty$ and $g \in \mathcal{B}^{\beta-1}_{p,\infty}$, then

$$\|fg\|_{\beta-1,p,\infty} \lesssim \big(\|f\|_{\frac{1}{p},p,\infty} + \|f\|_\infty\big)\|g\|_{\beta-1,p,\infty}.$$

(iii) Let $p \in [3,\infty]$, $\alpha \in (1/p,1)$ and $\beta > 0$ such that $\alpha+\beta < 1$ and $2\alpha+\beta > 1$. If $f \in L^\infty \cup \mathcal{B}^\alpha_{p,\infty}$ and $g \in \mathcal{B}^{\alpha+\beta-1}_{p/2,\infty}$, then

$$\|fg\|_{\alpha+\beta-1,p/2,\infty} \lesssim \big(\|f\|_\infty\|g\|_{2\alpha+\beta-1,p/3,\infty}\big) \wedge \big(\|f\|_{\alpha,p,\infty}\|g\|_{\alpha+\beta-1,p/2,\infty}\big).$$

(iv) Let $p \in [2,\infty]$ and $\alpha \in (1/p,1)$. If $f \in \mathcal{B}^\alpha_{p,\infty}$ and $g \in \mathcal{B}^\alpha_{p,\infty}$, then

$$\|fg\|_{\alpha,p,\infty} \lesssim \|f\|_{\alpha,p,\infty}\|g\|_{\alpha,p,\infty}.$$

(v) If $f \in \mathcal{B}^{\frac{1}{p}}_{p,\infty} \cap L^\infty$ and $g \in \mathcal{B}^{\frac{1}{p}}_{p,\infty} \cap L^\infty$ with $p \in [2,\infty]$, then

$$\|fg\|_{\frac{1}{p},p,\infty} \lesssim \big(\|f\|_{\frac{1}{p},p,\infty} + \|f\|_\infty\big)\big(\|g\|_{\frac{1}{p},p,\infty} + \|g\|_\infty\big).$$

Proof. Applying Besov embedding ($\alpha > 1/p$) and Bony's estimates (Lemma 2.1) lead to:

$$
\begin{aligned}
&(i) & \|fg\|_{\beta-1,p,\infty} &\lesssim \|T_f g\|_{\beta-1,p,\infty} + \|\pi(f,g)\|_{\alpha+\beta-1,p/2,\infty} + \|T_g f\|_{\alpha+\beta-1,p/2,\infty}\\
&&&\lesssim \|f\|_{\alpha,p,\infty}\|g\|_{\beta-1,p,\infty},\\
&(ii) & \|fg\|_{\beta-1,p,\infty} &\lesssim \|T_f g\|_{\beta-1,p,\infty} + \|\pi(f,g)\|_{\frac{1}{p}+\beta-1,p/2,\infty} + \|T_g f\|_{\frac{1}{p}+\beta-1,p/2,\infty}\\
&&&\lesssim \big(\|f\|_{\frac{1}{p},p,\infty} + \|f\|_\infty\big)\|g\|_{\beta-1,p,\infty},\\
&(iii) & \|fg\|_{\alpha+\beta-1,p/2,\infty} &\lesssim \|T_f g\|_{\alpha+\beta-1,p/2,\infty} + \|\pi(f,g)\|_{2\alpha+\beta-1,p/3,\infty} + \|T_g f\|_{\alpha+\beta-1,p/2,\infty}\\
&&&\lesssim \|f\|_\infty\|g\|_{\alpha+\beta-1,p/2,\infty} + \|g\|_{\alpha+\beta-1,p/2,\infty}\|f\|_{0,\infty,\infty}\\
&&&\quad + \big(\|f\|_{0,\infty,\infty}\|g\|_{2\alpha+\beta-1,p/3,\infty} \wedge \|f\|_{\alpha,p,\infty}\|g\|_{\alpha+\beta-1,p/2,\infty}\big)\\
&&&\lesssim \big(\|f\|_\infty\|g\|_{2\alpha+\beta-1,p/3,\infty}\big) \wedge \big(\|f\|_{\alpha,p,\infty}\|g\|_{\alpha+\beta-1,p/2,\infty}\big),\\
&(vi) & \|fg\|_{\alpha,p,\infty} &\lesssim \|T_f g\|_{\alpha,p,\infty} + \|\pi(f,g)\|_{2\alpha,p/2,\infty} + \|T_g f\|_{a,p,\infty} \lesssim \|f\|_{\alpha,p,\infty}\|g\|_{a,p,\infty},\\
&(v) & \|fg\|_{\frac{1}{p},p,\infty} &\lesssim \|T_f g\|_{\frac{1}{p},p,\infty} + \|\pi(f,g)\|_{\frac{2}{p},p/2,\infty} + \|T_g f\|_{\frac{1}{p},p,\infty}\\
&&&\lesssim \big(\|f\|_{\frac{1}{p},p,\infty} + \|f\|_\infty\big)\big(\|g\|_{\frac{1}{p},p,\infty} + \|g\|_\infty\big).
\end{aligned}
$$

☐

The following estimates are crucial to obtain the existence of a solution to the Volterra equation (1.2) and the local Lipschitz continuity of the corresponding Itô-Lyons map (3.17).

Lemma A.3.

(i) Let $\alpha > 0$ and $p \in [1, \infty]$. If $f \in \mathcal{B}^{\alpha}_{p,\infty} \cap L^{\infty}$ and $F \in C^{\lceil \alpha \rceil}$ with $F(0) = 0$, then

$$\|F(f)\|_{\alpha,p,\infty} \lesssim \|F\|_{C^{\lceil \alpha \rceil}} \|f\|_{\alpha,p,\infty}.$$

(ii) Let $\alpha \in (1/p, 1]$ and $p \in [2, \infty]$. If $f, g \in \mathcal{B}^{\alpha}_{p,\infty}$ and $F \in C^2$, then

$$\|F(f) - F(g)\|_{\alpha,p,\infty} \lesssim \|F\|_{C^2}\big(1 + \|f\|_{\alpha,p,\infty} + \|g\|_{\alpha,p,\infty}\big)\|f - g\|_{\alpha,p,\infty}.$$

(iii) If $f, g \in \mathcal{B}^{\frac{1}{p}}_{p,\infty} \cap L^{\infty}$ and $F \in C^2$ with $p \in [2, \infty]$, then

$$\|F(f)-F(g)\|_{\frac{1}{p},p,\infty} \lesssim \|F\|_{C^2}\big(1+\|f\|_{\frac{1}{p},p,\infty}+\|f\|_{\infty}+\|g\|_{\frac{1}{p},p,\infty}+\|g\|_{\infty}\big)\big(\|f-g\|_{\frac{1}{p},p,\infty}+\|f-g\|_{\infty}\big).$$

Proof. (i) can be deduced from [2, Theorem 2.87].

For (ii) we apply Lemma A.2 (iv) and the first part of this lemma to obtain

$$\begin{aligned}
\|F(f) - F(g)\|_{\alpha,p,\infty} &\leq \int_0^1 \|F'(f + s(g - f))(f - g)\|_{\alpha,p,\infty}\, \mathrm{d}s \\
&\lesssim \|f - g\|_{\alpha,p,\infty} \int_0^1 \|F'(f + s(g - f))\|_{\alpha,p,\infty}\, \mathrm{d}s \\
&\lesssim \|F\|_{C^2}\big(1 + \|f\|_{\alpha,p,\infty} + \|g\|_{\alpha,p,\infty}\big)\|f - g\|_{\alpha,p,\infty}.
\end{aligned}$$

For (iii) we apply an analogous estimate, but use Lemma A.2 (v) instead of (iv). □

We also need this linearization lemma:

Lemma A.4. *Let $\sigma \in C^2$, $p \geqslant 1$ and $\alpha > 1/p$. Supposing $u = T_{u^{(1)}} w_1 + T_{u^{(2)}} w_2 + u^{\#} \in \mathcal{B}^{\alpha}_{p,\infty}$ with $u^{(1)}, u^{(2)}, w_1, w_2 \in \mathcal{B}^{\alpha}_{p,\infty}$ and $u^{\#} \in \mathcal{B}^{2\alpha}_{p/2,\infty}$, we have*

$$\sigma(u) = \sigma(0) + T_{\sigma'(u)} u + S_{\sigma}(u)$$

for a function $S_{\sigma}(u) \in \mathcal{B}^{2\alpha}_{p/2,\infty}$ satisfying

$$\|S_{\sigma}(u)\|_{2\alpha,p/2,\infty} \lesssim \|\sigma\|_{C^2}\Big(1 + \sum_{j=1,2} \|u^{(j)}\|_{\infty}\|w_j\|_{\alpha,p,\infty}\Big)\big(\|u\|_{\alpha,p,\infty} + \|u^{\#}\|_{2\alpha,p/2,\infty}\big).$$

Proof. The proof follows from Step 1 in the proof of [37, Proposition 5.6] with $\widetilde{u} = u$ and $v_u = T_{u^{(1)}} w_1 + T_{u^{(2)}} w_2$. □

A refinement of [37, Lemma 2.3] is given by the following result:

Lemma A.5. *Let $\lambda, \gamma > 0$, $p \geqslant 1$ and $f \in \mathcal{B}^{\gamma}_{p,\infty}$. We have for any $\gamma' \in [0, \gamma) \cap [0, 1/p]$:*

(i) If $\chi \in \mathcal{B}^{\gamma}_{1/\gamma',\infty}$, then

$$\|\chi \Lambda_{\lambda} f\|_{\gamma,p,\infty} \lesssim \lambda^{\gamma'-1/p} |\log \lambda| \|f\|_{\gamma,p,\infty} \|\chi\|_{\gamma,1/\gamma',\infty}.$$

(ii) If additionally $xf(x) \in \mathcal{B}^{\gamma+1+\varepsilon}_{p,\infty}$ for some $\varepsilon > 0$, then we have for any functions χ_1, χ_2 such that $C_{\chi} := \|\chi_1\|_{\gamma+1,\infty,\infty}(\|\chi_2\|_{\gamma+1,p,\infty} + \|x\chi_2(x)\|_{L^{1/\gamma'}})$ is finite and for any $\lambda \in (0, 1)$

$$\big\|\chi_1(x)\chi_2(x)\Lambda_{\lambda}\big(xf(x)\big)\big\|_{\gamma+1,p,\infty} \lesssim \lambda^{1+\gamma'-1/p} |\log \lambda| C_{\chi}\big(\|xf(x)\|_{\gamma+1+\varepsilon,p,\infty} + \|f\|_{\gamma,p,\infty}\big).$$

Proof. We decompose $\chi\Lambda_\lambda f$ into small and larger Littlewood-Paley blocks. Arguing as in [37, Lemma 2.3] for the Δ_{-1} block, we have for the small blocks

$$\begin{aligned}\Big\|\sum_{j\lesssim 1}\Delta_j(\chi\Lambda_\lambda f)\Big\|_{\gamma,p,\infty} &= \Big\|\sum_{j\lesssim 1}\Delta_j\Lambda_\lambda(\chi(\lambda^{-1}\cdot)f)\Big\|_{\gamma,p,\infty}\\ &\lesssim \sum_{j:2^j\lesssim\lambda^{-1}\vee 1}\lambda^{-1/p}\big\|\Delta_j\big(\chi(\lambda^{-1}\cdot)f\big)\big\|_{L^p}\\ &\lesssim \lambda^{-1/p}|\log\lambda|\|\chi(\lambda^{-1}\cdot)f\|_{0,p,\infty}\lesssim\lambda^{-1/p}|\log\lambda|\|\chi(\lambda^{-1}\cdot)f\|_{L^p}.\end{aligned}\tag{A.1}$$

For any $\gamma'\in[0,\gamma)\cap[0,1/p]$ and $q\geqslant p$ satisfying $\frac{1}{p}=\gamma'+\frac{1}{q}$ Hölder's inequality yields (with convention $1/0=:\infty$)

$$\|\chi(\lambda^{-1}\cdot)f\|_{L^p}\leqslant\|\chi(\lambda^{-1}\cdot)\|_{L^{1/\gamma'}}\|f\|_{L^q}\lesssim\lambda^{\gamma'}\|\chi\|_{L^{1/\gamma'}}\|f\|_{\gamma,p,\infty},$$

which gives the asserted bound for blocks Δ_j with j smaller than a fixed constant.

Hence, we are left to bound the higher Littlewood-Paley blocks. Using Bony's decomposition, we get

$$\begin{aligned}\Big\|\sum_{j\gtrsim 1}\Delta_j(\chi\Lambda_\lambda f)\Big\|_{\gamma,p,\infty} &\leqslant \Big\|\sum_{j\gtrsim 1}\Delta_j T_\chi(\Lambda_\lambda f)\Big\|_{\gamma,p,\infty}\\ &\qquad+\Big\|\sum_{j\gtrsim 1}\Delta_j T_{\Lambda_\lambda f}\chi\Big\|_{\gamma,p,\infty}+\Big\|\sum_{j\gtrsim 1}\Delta_j\pi(\chi,\Lambda_\lambda f)\Big\|_{\gamma,p,\infty}.\end{aligned}\tag{A.2}$$

We will estimate these three terms separately. By the support properties of the Littlewood-Paley blocks in the Fourier domain we have $\Delta_j T_\chi(\Lambda_\lambda f)=\Delta_j\sum_{k\sim j}S_{k-1}\chi\Delta_k(\Lambda_\lambda f)$. Therefore,

$$2^{j\gamma}\|\Delta_j T_\chi(\Lambda_\lambda f)\|_{L^p}\lesssim 2^{j\gamma}\sum_{k\sim j}\|S_{k-1}\chi\|_{L^\infty}\|\Delta_k(\Lambda_\lambda f)\|_{L^p}\lesssim\|\chi\|_\infty\big\|\big(2^{k\gamma}\|\Delta_k(\Lambda_\lambda f)\|_{L^p}\big)_{k\geqslant 0}\big\|_{\ell^\infty}.$$

The last norm in the previous display can be estimated as in [37, Lem. 2.3], which yields

$$\sup_{j\gtrsim 1}2^{j\gamma}\|\Delta_j T_\chi(\Lambda_\lambda f)\|_{L^p}\lesssim\lambda^{\gamma-1/p}|\log\lambda|\|\chi\|_\infty\|f\|_{\gamma,p,\infty}.$$

For the second term in (A.2) we note with γ' and q as above that

$$2^{j\gamma}\|\Delta_j T_{\Lambda_\lambda f}\chi\|_{L^p}\lesssim 2^{j\gamma}\sum_{k\sim j}\|S_{k-1}\Lambda_\lambda f\|_{L^q}\|\Delta_k\chi\|_{L^{1/\gamma'}}\lesssim\|f(\lambda\cdot)\|_{L^q}\|\chi\|_{\gamma,1/\gamma',\infty},$$

where $\|f(\lambda\cdot)\|_{L^q}=\lambda^{-1/q}\|f\|_{L^q}\lesssim\lambda^{\gamma'-1/p}\|f\|_{\gamma,p,\infty}$. Finally, the third term in (A.2) is bounded by

$$\begin{aligned}2^{j\gamma}\|\Delta_j\pi(\chi,\Lambda_\lambda f)\|_{L^p}&\lesssim 2^{j\gamma}\sum_{k\gtrsim j}\Big\|\sum_{|l|\leqslant 1}\Delta_{k-l}\chi\Delta_k\Lambda_\lambda f\Big\|_{L^p}\\ &\lesssim\sum_{k\gtrsim j}2^{-(k-j)\gamma}\sum_{|l|\leqslant 1}\|\Delta_{k-l}\chi\|_\infty 2^{k\gamma}\|\Delta_k\Lambda_\lambda f\|_{L^p}\\ &\lesssim\|\chi\|_\infty\big\|\big(2^{k\gamma}\|\Delta_k(\Lambda_\lambda f)\|_{L^p}\big)_{k\geqslant 0}\big\|_{\ell^\infty}\lesssim\lambda^{\gamma-1/p}|\log\lambda|\|\chi\|_\infty\|f\|_{\gamma,p,\infty}.\end{aligned}$$

For part (i) it remains to note that $\|\chi\|_{L^{1/\gamma'}}\leqslant\|\chi\|_{\gamma,1/\gamma',\infty}$ and $\|\chi\|_\infty\lesssim\|\chi\|_{\gamma-\gamma',\infty,\infty}\lesssim\|\chi\|_{\gamma,1/\gamma',\infty}$ due to Besov embeddings.

For (ii) we first note for the small blocks as in (A.1)

$$\Big\|\sum_{j\lesssim 1}\Delta_j\big(\chi_1(x)\chi_2(x)\Lambda_\lambda(xf(x))\big)\Big\|_{\gamma+1,p,\infty}\lesssim\sum_{j:\lambda 2^j\lesssim 1}\lambda^{-1/p}\big\|\Delta_j\big(\chi_1(x/\lambda)\chi_2(x/\lambda)xf(x)\big)\big\|_{L^p}$$

$$
\begin{aligned}
&\lesssim \lambda^{-1/p}|\log\lambda|\|\chi_1(x/\lambda)\chi_2(x/\lambda)xf(x)\|_{L^p}\\
&\lesssim \lambda^{-1/p}|\log\lambda|\|x\chi_1(x/\lambda)\chi_2(x/\lambda)\|_{L^{1/\gamma'}}\|f\|_{L^q}\\
&\lesssim \lambda^{\gamma'+1-1/p}|\log\lambda|\|x\chi_1(x)\|_{L^{1/\gamma'}}\|\chi_2\|_\infty\|f\|_{\gamma,p,\infty}.
\end{aligned}
$$

For the large blocks we obtain as in (i)

$$
\begin{aligned}
&\Big\|\sum_{j\gtrsim 1}\Delta_j\big(\chi_1(x)\chi_2(x)\Lambda_\lambda(xf(x))\big)\Big\|_{\gamma+1,p,\infty}\\
&\quad\lesssim \lambda^{\gamma+1-1/p}|\log\lambda|\|\chi_1\|_\infty\|\chi_2(\lambda^{-1}x)xf(x)\|_{\gamma+1,p,\infty}+\|\chi_2(x)\Lambda_\lambda(xf(x))\|_{L^p}\|\chi_1\|_{\gamma+1,\infty,\infty}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\|\chi_2(x)\Lambda_\lambda(xf(x))\|_{L^p}&=\lambda\|\chi_2(x)xf(\lambda x)\|_{L^p}\lesssim\lambda\|x\chi_2(x)\|_{L^{1/\gamma'}}\|f(\lambda x)\|_{L^q}\\
&\lesssim\lambda^{\gamma'+1-1/p}\|x\chi_2(x)\|_{L^{1/\gamma'}}\|f\|_{\gamma,p,\infty},
\end{aligned}
$$

we only need a uniform bound for $\|\chi_2(\lambda^{-1}x)xf(x)\|_{\gamma+1,p,\infty}$ for which we apply (i) with $\gamma'=p^{-1}-\varepsilon<1$:

$$
\begin{aligned}
\|\chi_2(\lambda^{-1}x)xf(x)\|_{\gamma+1,p,\infty}&=\|xf(x)\Lambda_{\lambda^{-1}}\chi_2\|_{\gamma+1,p,\infty}\\
&\lesssim\lambda^{\varepsilon}|\log\lambda|\|\chi_2\|_{\gamma+1,p,\infty}\|xf(x)\|_{\gamma+1,1/\gamma',\infty}\\
&\lesssim\|\chi_2\|_{\gamma+1,p,\infty}\|xf(x)\|_{\gamma+1+\varepsilon,p,\infty},
\end{aligned}
$$

where the last estimate follows from the embedding $\mathcal{B}^{1+\gamma+\varepsilon}_{p,\infty}\subseteq\mathcal{B}^{1+\gamma}_{\gamma',\infty}$. □

Finally, we estimate the Besov norms of the scaled resonant term.

Lemma A.6. *For $\alpha,\beta\in\mathbb{R}$, $p\geqslant 2$, $f,g\in\mathcal{S}$ we have uniformly in $\lambda\in(0,1]$ that*

$$
\big\|\Lambda_\lambda\pi(f,g)-\pi(\Lambda_\lambda f,\Lambda_\lambda g)\big\|_{\alpha+\beta,p/2,\infty}\lesssim\lambda^{-|\alpha+\beta|-p/2}\|f\|_{\alpha,p,\infty}\|g\|_{\beta,p,\infty}+\|\Lambda_\lambda f\|_{\alpha,p,\infty}\|\Lambda_\lambda g\|_{\beta,p,\infty}.
$$

Proof. We proceed by generalising the proofs of [24, Lem. B.1] and of [9, Theorem 2.1]. Let us choose $K=K(\lambda)\in\mathbb{N}$ such that $\lambda':=\lambda 2^K\in(1/2,1]$ and decompose

$$
\begin{aligned}
\Lambda_\lambda\pi(f,g)=&\sum_{j,k<K:|k-j|\leqslant 1}\Lambda_\lambda\Delta_i f\Delta_k g\\
&+\sum_{j,k\geqslant K:|k-j|\leqslant 1}\big(\mathcal{F}^{-1}[\rho(2^{-j+K}\lambda'^{-1}\cdot)]*\Lambda_\lambda f\big)\big(\mathcal{F}^{-1}[\rho(2^{-k+K}\lambda'^{-1}\cdot)]*\Lambda_\lambda g\big).
\end{aligned}\tag{A.3}
$$

The Fourier transform of the first term is spectrally supported in a ball with radius of order $2^K\sim\lambda^{-1}$ such that

$$
\begin{aligned}
\Big\|\sum_{j,k\leqslant K:|k-j|\leqslant 1}\Lambda_\lambda\Delta_i f\Delta_k g\Big\|_{\alpha+\beta,p/2,\infty}&\lesssim(2^{K(\alpha+\beta)}\vee 1)\sum_{j,k\leqslant K:|k-j|\leqslant 1}\big\|\Lambda_\lambda\Delta_j f\Delta_k g\big\|_{L^{p/2}}\\
&\lesssim(2^{K(\alpha+\beta)}\vee 1)\lambda^{-2/p}\sum_{j,k\leqslant K:|k-j|\leqslant 1}\|\Delta_j f\|_{L^p}\|\Delta_k g\|_{L^p}\\
&\lesssim(2^{K(\alpha+\beta)}\vee 1)\lambda^{-2/p}\sum_{j,k\leqslant K:|k-j|\leqslant 1}2^{-j\alpha-k\beta}\|f\|_{\alpha,p,\infty}\|g\|_{\beta,p,\infty}\\
&\lesssim(\lambda^{-(\alpha+\beta)}\vee 1)(\lambda^{(\alpha+\beta)}\vee 1)\lambda^{-2/p}\|f\|_{\alpha,p,\infty}\|g\|_{\beta,p,\infty}.
\end{aligned}
$$

The second term in (A.3) equals $\pi'(\Lambda_\lambda f,\Lambda_\lambda g)$ where π' is the resonant term corresponding to the modified partition of unity $(\chi(\cdot/\lambda'),\rho(\cdot/\lambda'))$. Note that the scaling parameter $\lambda'\in(1/2,1]$ is uniformly bounded from above and below. It remains to show

$$
\big\|\pi'(f,g)-\pi(f,g)\big\|_{\alpha+\beta,p/2,\infty}\lesssim\|f\|_{\alpha,p,\infty}\|g\|_{\beta,p,\infty}.
$$

Owing to $fg = T'_g f + T'_f g + \pi'(f,g)$ for the paraproduct operators $T'_g f$ associated to $(\chi(\cdot/\lambda'), \rho(\cdot/\lambda'))$, we have

$$\|\pi'(f,g) - \pi(f,g)\|_{\alpha+\beta,p/2,\infty} \leqslant \|T'_f g - T_f g\|_{\alpha+\beta,p/2,\infty} + \|T'_g f - T_g f\|_{\alpha+\beta,p/2,\infty}.$$

Generalising [9, Thm. 2.1], we will now prove

$$\|T_g f - T^*_g f\|_{\alpha+\beta,p/2,\infty} \lesssim \|f\|_{\alpha,p,\infty}\|g\|_{\beta,p,\infty} \tag{A.4}$$

for the operator

$$T^*_g f := \mathcal{F}^{-1}\Big[\int_{\mathbb{R}} \chi(u-v,v)\mathcal{F}g(u-v)\mathcal{F}f(v)\,\mathrm{d}v\Big]$$

where $\chi\colon \mathbb{R}^2 \setminus \{0\} \to [0,1]$ is a C^∞-function such that for sufficiently small constants $0 < \varepsilon_1 < \varepsilon_2$:

$$\chi(u,v) = \begin{cases} 1, & |u| \leqslant \varepsilon_1 |v| \\ 0, & |u| \geqslant \varepsilon_2 |v| \end{cases}.$$

The estimate (A.4) especially implies that $T_g f$ and thus $\pi(f,g)$ does not depend on the choice of the partition of unity up to a regular remainder, which concludes the proof.

To verify (A.4), we decompose

$$\mathcal{F}\big[T^*_g f\big] = \sum_{j,k} \int_{\mathbb{R}} \chi(u-v,v)\mathcal{F}[\Delta_k g](u-v)\mathcal{F}[\Delta_j f](v)\,\mathrm{d}v.$$

Due to the support assumption on χ, the terms with $2^k \gtrsim 2^{j-1}$ are zero and the integrands with $2^k \lesssim 2^{j-1}$ coincide with $\mathcal{F}[\Delta_k g] * \mathcal{F}[\Delta_j f](u)$. Therefore, for integers $N_1 < N_2$ depending only on ε_1, ε_1 and (χ,ρ), respectively, we have

$$T^*_g f = \sum_j \sum_{k<j-N_1} \Delta_k g \Delta_j f + R(g,f)$$

with

$$R(g,f) = \sum_j R_j(g,f), \quad R_j(g,f) := \sum_{k=j-N_1}^{j-N_2} \mathcal{F}^{-1}\Big[\int_{\mathbb{R}} \chi(u-v,v)\mathcal{F}[\Delta_k g](u-v)\mathcal{F}[\Delta_j f](v)\,\mathrm{d}v\Big].$$

Fubini's theorem yields

$$\begin{aligned} R_j(g,f)(x) &= \sum_{k=j-N_1}^{j-N_2} \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{ix(u+v)}\chi(u,v)\mathcal{F}[\Delta_k g](u)\mathcal{F}[\Delta_j f](v)\,\mathrm{d}v\,\mathrm{d}u \\ &= \sum_{k=j-N_1}^{j-N_2} \int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{F}^{-1}[\chi](s,t)\Delta_k g(x-s)\Delta_j f(x-t)\,\mathrm{d}s\,\mathrm{d}t. \end{aligned}$$

Since $\mathcal{F}^{-1}\chi \in L^1(\mathbb{R}^2)$ due to the regularity of χ, Young's inequality implies

$$\|R_j(g,f)\|_{L^{p/2}} \lesssim \sum_{k=j-N_1}^{j-N_2} \|\mathcal{F}^{-1}\chi\|_{L^1}\|\Delta_k g\|_{L^p}\|\Delta_j f\|_{L^p}.$$

Noting that $R_j(g,f)$ is spectrally supported in an annulus with radius of order $2^j C$ for some $C > 0$, we obtain

$$\|R(g,f)\|_{\alpha+\beta,p/2,\infty} \lesssim \sup_m 2^{m(\alpha+\beta)} \sum_{2^m \sim 2^j C} \sum_{k\sim j} \|\Delta_k g\|_{L^p}\|\Delta_j f\|_{L^p} \lesssim \|f\|_{\alpha,p,\infty}\|g\|_{\beta,p,\infty}$$

and thus (A.4). □

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