Pinned diffusions and Markov bridges

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Abstract. In this article we consider a family of real-valued diffusion processes on the time interval $[0,1]$ indexed by their prescribed initial value $x \in \mathbb{R}$ and another point in space, $y \in \mathbb{R}$. We first present an *easy-to-check* condition on their drift and diffusion coefficients ensuring that the diffusion is pinned in y at time $t = 1$. Our main result then concerns the following question: can this family of pinned diffusions be obtained as the bridges either of a Gaussian Markov process or of an Itô diffusion? We eventually illustrate our precise answer with several examples.

Key words and phrases : pinned diffusion, α-Brownian bridge, α-Wiener bridge, Gaussian Markov process, reciprocal characteristics.

2010 Mathematics Subject Classification : 60 G 15, 60 H 10, 60 J 60.

1 Introduction

In this article we consider for any $x, y \in \mathbb{R}$ the stochastic process $(X_t^{xy})_{t\in[0,1)}$ defined as the solution to the stochastic differential equation (SDE)

$$\begin{cases} dX_t = h(t)(y - X_t)\,dt + \sigma(t)\,dB_t, & t \in [0,1) \\ X_0 = x \end{cases} \tag{1}$$

where h and σ are time functions and B is a standard Brownian motion.
We are concerned in particular with the situation where the diffusion is degenerated at time 1 in the sense that $\lim_{t\to 1} X_t^{xy} = y \quad \mathbf{P}$-a.s. for any $y \in \mathbb{R}$. In such a case, we say that the diffusion is *pinned*.
A celebrated example of such a family of diffusions is given by the Brownian bridges, obtained by setting $h(t) = \frac{1}{1-t}$, $t \in [0,1)$ and $\sigma \equiv 1$. Another slightly more sophisticated example studied during the last decade concerns the function $h(t) = \frac{\alpha}{1-t}$, $t \in [0,1)$, for

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some $\alpha > 0$. The resulting process is called *α-Wiener bridge* or *α-Brownian bridge* in the literature, see e.g. [1] or [2]. For reasons we will explain in Section 4 we prefer to introduce the new terminology *α-pinned Brownian diffusion* for these processes.

In Section 2 of this paper we present a set of conditions (A1)-(A3) on the functions h and σ that ensure that the solution of (1) is pinned in y at time $t = 1$. This result complements that of [3]. Moreover, our criteria seem to be easier to check than theirs.

Brownian bridges were originally obtained by *pinning* a Brownian motion at initial and terminal time. This means, using its Markovian characterization, the (well known) transition density of a Brownian bridge between x and y is given for any $0 \leq s < t < 1$ and $u, v \in \mathbb{R}$ by $p^y(s, u; t, v) = \dfrac{p(s, u; t, v)p(t, v; 1, y)}{p(s, u; 1, y)}$ where p is the transition density of Brownian motion. Using this property as definition for the bridges of a Markov process, it is natural to address the following question in our much more general context:

> Consider the family of pinned diffusions $\{X^{xy},\, x, y \in \mathbb{R}\}$ solving (1) with two given functions h and σ. Is it possible to find a continuous Markov process Z whose family of bridges coincides with $\{X^{xy},\, x, y \in \mathbb{R}\}$?

In Section 3 we present a complete answer while imposing on Z to belong to specific classes of processes. First we suppose in Theorem 3.1 that Z is a Gaussian process and use as tool the fact that each centered continuous Gaussian Markov process can be represented as a space-time scaled Brownian motion. Then, we treat in Theorem 3.5 the complementary diffusion setting supposing that Z is an Itô diffusion satisfying

$$dZ_t = b(t, Z_t)\, dt + \rho(t, Z_t)\, dB_t$$

where b and ρ are smooth functions. In this framework, our method relies on the computation of the so-called *reciprocal characteristics* $(F_{b,\rho}, \rho^2)$ associated to Z. These two space-time functions are a well adapted tool since they are invariant inside (and characterize in some sense) the whole family of bridges of Z (see Proposition 3.6 for more details).

Some related problems have been partially studied in the literature, but only in a Gaussian framework and only for particular pinned Brownian diffusions: Mansuy proved in [1] that the α-pinned Brownian diffusion from $x = 0$ to $y = 0$ is not the 0-0-bridge of a time-homogeneous Gaussian Markov process. Barczy and Kern studied a similar question for a general α-pinned Brownian diffusion from 0 to 0, see [4].

In Section 4 we eventually treat a number of examples of pinned diffusions with the methods developed in this article. These examples include α-pinned Brownian diffusions and F-Wiener bridges.

2 The framework: Families of pinned diffusions

Throughout, we work on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,1]}, \mathbf{P})$ satisfying the usual conditions. B denotes a standard Brownian motion with respect to $(\mathscr{F}_t)$.

In the whole article we suppose that h and σ are two functions satisfying the following assumption:

(A0) $\quad h : [0,1) \to \mathbb{R}$ is continuous and $\sigma : [0,1) \to (0,+\infty)$ is continuous.

We consider the family of processes $\{X^{xy},\, x,y \in \mathbb{R}\}$ where $(X_t^{xy})_{t\in[0,1)}$ is the unique strong solution to the SDE (1). It is straightforward to verify that

$$X_t^{xy} = \phi(t)x + (1-\phi(t))y + \phi(t)\int_0^t \frac{\sigma(r)}{\phi(r)}\,dB_r\,, \quad t \in [0,1), \tag{2}$$

where

$$\phi(t) := \exp\Big(-\int_0^t h(r)\,dr\Big). \tag{3}$$

Since the integrand in the stochastic integral in (2) is deterministic and locally bounded, X^{xy} is a Gaussian process with first and second moments

$$\mathbf{E}\,(\,X_t^{xy}\,) = \phi(t)\,x + (1-\phi(t))\,y, \quad t \in [0,1), \tag{4}$$

$$\mathbf{Cov}\,(X_s^{xy}, X_t^{xy}) = \phi(s)\,\phi(t)\int_0^s \frac{\sigma^2(r)}{\phi^2(r)}\,dr\,, \quad 0 \le s \le t < 1. \tag{5}$$

We are interested in the following property.

Definition 2.1. The family of processes $\{X^{xy},\, x,y \in \mathbb{R}\}$ given by (2) is called a *family of pinned diffusions* if for all $x,y \in \mathbb{R}$,

$$\mathbf{P}\Big(\lim_{t\to 1} X_t^{xy} = y\Big) = 1. \tag{6}$$

2.1 A simple condition ensuring pinning

We now show that the following additional assumptions on h and σ ensure the pinning property (6).

(A1) $\quad \displaystyle\lim_{t\to 1}\int_0^t h(r)\,dr = +\infty.$

(A2) $\quad \exists\, 0 < t_0 < 1, \quad h(t) > 0$ on $[t_0,1)$ and $\displaystyle\sup_{t\in[t_0,1)} \frac{\int_0^t h(r)\,dr}{h(t)} < +\infty.$

(A3) $\quad \sup_{t\in[0,1)} \sigma(t) < +\infty.$

Remark 2.2. In particular, (A2) is satisfied if h is non-decreasing and strictly positive on some interval $[t_0,1)$ where $t_0 \in (0,1)$.

Proposition 2.3. *If Assumptions* (A1)-(A3) *are satisfied then* (2) *defines a family of pinned diffusions.*

Proof. Due to Assumption (A1) we have $\lim_{t\to1}\phi(t)=0$. It remains to show that

$$\lim_{t\to1}\phi(t)\,M_t=0\quad\text{a.s.}\tag{7}$$

where $M_t=\displaystyle\int_0^t\frac{\sigma(r)}{\phi(r)}\,dB_r$, $t\in[0,1)$. The quadratic variation process of the local martingale M is given by

$$\langle M\rangle_t=\int_0^t\frac{\sigma^2(r)}{\phi^2(r)}\,dr=\int_0^t\sigma^2(r)\exp\Big(2\int_0^r h(u)\,du\Big)\,dr,\quad t\in[0,1).$$

Since $\langle M\rangle$ is deterministic, either $\lim_{t\to1}\langle M\rangle_t<+\infty$ or $\lim_{t\to1}\langle M\rangle_t=+\infty$ holds.
In the first case $\lim_{t\to1}M_t$ exists in $\mathbb{R}$ a.s. (see e.g. [5, problem 5.24]) and hence (7) holds.
Otherwise, if $\lim_{t\to1}\langle M\rangle_t=+\infty$, one can represent M as a time-changed Brownian motion: $M_t=\hat{B}_{\langle M\rangle_t}$, $t\in[0,1)$, where $\hat{B}$ is a Brownian motion, see e.g. [5, p. 174]. Hence

$$\limsup_{t\to1}\frac{|M_t|}{\sqrt{2\langle M\rangle_t\log\log\langle M\rangle_t}}=1\quad\text{a.s.}$$

and therefore, for any $\delta>0$,

$$\limsup_{t\to1}\phi(t)|M_t|\le\limsup_{t\to1}\phi(t)\,\sqrt{2\langle M\rangle_t\log\log\langle M\rangle_t}\le\limsup_{t\to1}\phi(t)\,\sqrt{2\langle M\rangle_t}\log^\delta\langle M\rangle_t.$$

The proof is complete as soon as we show that the right hand side above vanishes for some $\delta>0$. To that aim, we bound $\langle M\rangle$ from above: Define $H(t)=\int_0^t h(r)\,dr$, $t\in[0,1)$. Using (A1) and (A2) there exists $t_0\in(0,1)$ and $K>0$ such that

$$\frac{H'(t)}{H(t)}=\frac{h(t)}{\int_0^t h(r)\,dr}\ge K,\quad t\in[t_0,1).$$

Further, let $\|\sigma\|:=\sup_{t\in[0,1)}\sigma(t)$. Then, for any $t>t_1>t_0$,

$$\begin{aligned}\langle M\rangle_t&=\langle M\rangle_{t_1}+\int_{t_1}^t\sigma^2(r)\exp\Big(2\int_0^r h(u)\,du\Big)\,dr\\&\le\langle M\rangle_{t_1}+\frac{\|\sigma\|^2}{K}\int_{t_1}^t\exp\left(2H(r)\right)\frac{H'(r)}{H(r)}\,dr\\&=\langle M\rangle_{t_1}+\frac{\|\sigma\|^2}{K}\int_{H(t_1)}^{H(t)}\mathrm{e}^{2z}\,\frac{dz}{z}\\&=\langle M\rangle_{t_1}+\frac{\|\sigma\|^2}{2K}\Big(\frac{\mathrm{e}^{2z}}{z}\Big|_{H(t_1)}^{H(t)}+\int_{H(t_1)}^{H(t)}\frac{\mathrm{e}^{2z}}{z^2}\,dz\Big).\end{aligned}$$

Take t_1 sufficiently large in such a way that $z\mapsto\dfrac{\mathrm{e}^{2z}}{z^2}$ is increasing on $[H(t_1),\infty)$. Then,

$$\begin{aligned}\langle M\rangle_t&\le\langle M\rangle_{t_1}+\frac{\|\sigma\|^2}{2K}\Big(\frac{\mathrm{e}^{2H(t)}}{H(t)}-\frac{\mathrm{e}^{2H(t_1)}}{H(t_1)}+\big(H(t)-H(t_1)\big)\frac{\mathrm{e}^{2H(t)}}{H(t)^2}\Big)\\&\le\langle M\rangle_{t_1}+\frac{\|\sigma\|^2}{K}\frac{\mathrm{e}^{2H(t)}}{H(t)}.\end{aligned}$$

Therefore, there exists a positive constant C such that

$$\limsup_{t\to 1} \phi(t)\,\sqrt{2\langle M\rangle_t}\,\log^{\delta}\langle M\rangle_t \le C\ \limsup_{t\to 1} \mathrm{e}^{-H(t)}\sqrt{\frac{\mathrm{e}^{2H(t)}}{H(t)}}\,(2H(t))^{\delta} = 2^{\delta}C\ \lim_{t\to 1} H(t)^{\delta-1/2}$$

which vanishes for any $\delta < \frac{1}{2}$. □

Remark 2.4. Assumption (A1) is indeed a necessary condition to ensure that the diffusion is pinned: Recall that convergence in probability for Gaussian random variables implies L^1-convergence. Hence, due to (4), to ensure (6) it is necessary that $\lim_{t\to 1}\phi(t) = 0$ holds. The latter is equivalent to (A1).

Remark 2.5. Let us compare our result with the one from Barczy and Kern [3, Proposition 2.4]. They show that the diffusion X^{xy} is pinned if one replaces our Assumptions (A2) and (A3) by the following condition on h and σ:

(A2') There exists $\delta \in (0,1)$ such that

$$\sup_{t\in[0,1)} \mathrm{e}^{-2\delta\int_0^t h(s)\,ds}\int_0^t \mathrm{e}^{2\int_0^r h(s)\,ds}\sigma^2(r)\,dr < +\infty.$$

Clearly, for bounded σ, condition (A2) will be easier to check than condition (A2'). Furthermore, there are situations in which Assumption (A2) is satisfied while (A2') does not hold: Take $\sigma \equiv 1$ and $h(t) := \dfrac{2-t}{(1-t)^2},\ t\in[0,1)$.

(A1) is satisfied since $\int_0^t h(s)\,ds = \frac{1}{1-t} + \log\frac{1}{1-t} - 1 \overset{t\to 1}{\longrightarrow} +\infty$.
(A2) is satisfied since h is non-decreasing.
Therefore, we are dealing with a family of pinned diffusions. On the other hand

$$\int_0^t \mathrm{e}^{2\int_0^r h(s)\,ds}\,dr = \int_0^t \mathrm{e}^{\frac{2}{1-r}-2}\frac{1}{(1-r)^2}\,dr$$
$$= \int_1^{\frac{1}{1-t}} \mathrm{e}^{2r-2}\,dr = \frac{1}{2}\left(\mathrm{e}^{\frac{2t}{1-t}} - 1\right), \quad t\in[0,1).$$

For any $\delta\in(0,1)$, it follows that

$$\mathrm{e}^{-2\delta\int_0^t h(s)\,ds}\int_0^t \mathrm{e}^{2\int_0^r h(s)\,ds}\,dr = \frac{1}{2}(1-t)^{2\delta}\mathrm{e}^{-\frac{2\delta t}{1-t}}\left(\mathrm{e}^{\frac{2t}{1-t}} - 1\right) \overset{t\to 1}{\longrightarrow} +\infty.$$

Consequently (A2') is not satisfied.

3 Identification of bridge processes

In this section we examine for which choices of h and σ a family of pinned diffusions $\{X^{xy},\, x,y\in\mathbb{R}\}$ can be obtained as the family of bridges of a single Markov process Z. The following theorem answers this question if Z is assumed to be Gaussian.

Theorem 3.1. *Assume that the processes $\{X^{xy},\, x, y \in \mathbb{R}\}$ defined by (2) constitute a family of pinned diffusions (e.g. if (A1)-(A3) are satisfied). They correspond to the bridges of a non-degenerate Gaussian Markov process Z if and only if*

$$\Sigma := \lim_{t\to 1} \int_0^t \sigma^2(r)\, dr < +\infty \tag{8}$$

and the function h is related to σ as follows:

$$h(t) = \frac{\sigma^2(t)}{\Sigma - \int_0^t \sigma^2(r)\, dr}, \quad t \in [0, 1). \tag{9}$$

In this case the original process Z follows the dynamic $dZ_t = \sigma(t)\, dB_t$.

For the proof we need the following representation of continuous Gaussian Markov processes.

Lemma 3.2. *Let $Z = (Z_t)_{t\in[a,b]}$ be a non-degenerate continuous Gaussian Markov process. Then there exist three continuous functions $m, u, v : [a, b] \to \mathbb{R}$ such that*

$$Z \overset{(d)}{=} m(\cdot) + u(\cdot)\hat{B}_{v(\cdot)},$$

where u and v are strictly positive, v is non-decreasing and $\hat{B}$ is a Brownian motion.

Proof. Since a continuous Gaussian process is continuous in L^p for any $p \geq 1$ the functions $t \mapsto \mathbf{E}\,(Z_t)$ and $(s, t) \mapsto \mathbf{Cov}\,(Z_s, Z_t)$ are continuous. Define m as the first moment: $m(t) := \mathbf{E}\,(Z_t)\,,\ t \in [a, b]$. Then $Z - m$ is a centered and non-degenerate Gaussian Markov process having a continuous covariance function. The claimed representation now follows from [6, §3.2]. □

Proof of Theorem 3.1. Suppose first there exists a non-degenerate continuous Gaussian Markov process Z whose bridges are given by $\{X^{xy},\, x, y \in \mathbb{R}\}$. Let Z^x be the process Z started in x. Since we assume Z to be non-degenerate, $\mathbf{Var}\,(Z^x_t) > 0$ holds for all $t \in (0, 1]$. By Lemma 3.2, for any $\varepsilon \in (0, 1)$ the following representation holds:

$$Z^x|_{[\varepsilon,1]} \overset{(d)}{=} m(\cdot) + u(\cdot)\hat{B}_{v(\cdot)}$$

where $\hat{B}$ is a Brownian motion, $m, u, v : [\varepsilon, 1] \to \mathbb{R}$ are continuous, u and v are strictly positive and v is non-decreasing. Since $u(\cdot)\hat{B}_{cv(\cdot)}$ and $\sqrt{c}u(\cdot)\hat{B}_{v(\cdot)}$ have the same distribution for any $c > 0$ we can suppose $v(1) = 1$. Accordingly, the covariance of Z^x has the representation

$$c(s, t) := \mathbf{Cov}\,(Z^x_s, Z^x_t) = u(s)u(t)v(s), \quad s \leq t.$$

It follows that expectation and covariance of the bridge process Z^{xy} (corresponding to the conditional moments given $Z_1 = y$) are given by (see e.g. [7, p. 12])

$$\begin{aligned}\mathbf{E}\,(\,X_t^{xy}\,) &= m(t) + \frac{c(t,1)}{c(1,1)}(y - m(1)) \\ &= m(t) + \frac{u(t)v(t)}{u(1)}(y - m(1)), \quad t \in (0,1], \qquad (10)\\ \mathbf{Cov}\,(Z_s^{xy}, Z_t^{xy}) &= c(s,t) - \frac{c(s,1)c(t,1)}{c(1,1)} \\ &= u(s)v(s)u(t)\,(1 - v(t))\,, \quad s \le t. \qquad (11)\end{aligned}$$

Comparison with (4) and (5) yields

$$u(s)v(s) = \phi(s)\int_0^s \frac{\sigma^2(r)}{\phi^2(r)}\,dr, \quad \frac{u(s)v(s)}{u(1)} = 1 - \phi(s), \quad s \in (0,1).$$

This implies

$$\int_0^s \frac{\sigma^2(r)}{\phi^2(r)}\,dr = u(1)\,\frac{1 - \phi(s)}{\phi(s)}, \quad s \in (0,1).$$

Differentiating both sides of this equality yields $\phi' = -\frac{\sigma^2}{u(1)}$ and then

$$\exists C \in \mathbb{R}, \quad \phi(t) = C - \frac{1}{u(1)}\int_0^t \sigma^2(r)\,dr.$$

Since $\phi(0) = 1$ and $\lim_{t\to 1}\phi(t) = 0$ (see Remark 2.4), this implies $C = 1$ and $\Sigma = u(1) < +\infty$. Using (3) allows to deduce the desired expression for h.
Conversely, let Z^x be defined by $Z_t^x = x + \int_0^t \sigma(s)\,dB_s$, $t \in [0,1]$, and let h be given by (9). In this case we obtain as an alternative representation $Z_t^x \overset{(d)}{=} x + u(t)\hat{B}_{v(t)}$, $t \in [0,1]$, where $u(t) \equiv \sqrt{\Sigma}$ and $v(t) := \frac{1}{\Sigma}\int_0^t \sigma^2(r)\,dr$. A direct computation yields $\phi = 1 - v$ and

$$\int_0^t \frac{\sigma^2(r)}{\phi^2(r)}\,dr = \Sigma\,\frac{v(t)}{1 - v(t)}, \quad t \in [0,1).$$

Therefore, the moments defined by (4) and (5), respectively (10) and (11) agree. It remains to prove that the diffusion is pinned, i.e. to verify that (6) holds. This can be done e.g. by checking conditions (A1) and (A2'). (A1) holds since $\lim_{t\to 1}\phi(t) = 0$ and (A2') holds since

$$\mathrm{e}^{-2\delta\int_0^t h(s)\,ds}\int_0^t \mathrm{e}^{2\int_0^r h(s)\,ds}\sigma^2(r)\,dr = \Sigma\,(1 - v(t))^{2\delta - 1}v(t), \quad t \in [0,1),$$

is bounded for any $\delta \ge \frac{1}{2}$. □

Remark 3.3. In the particular case where σ extends to a continuous function on $[0,1]$, the second part of the above proof is a consequence of Theorem 3.1 and Theorem 3.2 in [8].

We continue looking for a process Z whose bridges match the family $\{X^{xy},\, x, y \in \mathbb{R}\}$ but we definitely complement the class of allowed Z considering now Itô diffusions which are a priori not Gaussian.

Definition 3.4. A weak solution $Z = (Z_t)_{t\in[0,1]}$ of a SDE of the form

$$dZ_t = b(t, Z_t)\, dt + \rho(t, Z_t)\, dB_t,$$

is called a *regular Itô diffusion* if the coefficients b and ρ are in $C^{1,2}([0,1] \times \mathbb{R})$, if ρ is strictly positive and if Z admits a strictly positive transition density p such that

(i) p is differentiable in all variables;

(ii) For any $(t_1, z_1) \in [0, 1) \times \mathbb{R}$, the partial derivatives of $p(t_1, z_1; \cdot, \cdot)$ exist up to order two and are continuous.

We are now able to state our main result.

Theorem 3.5. *Suppose that the functions h and σ satisfy, additionally to Assumption* (A0)*, the regularity condition $h, \sigma \in C^1([0, 1))$; assume that the associated processes $\{X^{xy},\, x, y \in \mathbb{R}\}$ given by* (2) *are pinned diffusions. Then $\{X^{xy},\, x, y \in \mathbb{R}\}$ correspond to the bridges of a regular Itô diffusion Z if and only if* (8) *and* (9) *are satisfied. In this case Z is indeed Gaussian and one can choose its drift coefficient $b \equiv 0$ and its diffusion coefficient $\rho^2(t, z) \equiv \sigma^2(t)$ as in Theorem 3.1.*

The proof is based on the notion of *reciprocal characteristics*, which was first introduced by Clark [9]: For any regular Itô diffusion Z with coefficients b and ρ define the space-time function

$$F_{b,\rho}(t, z) := \partial_t(b/\rho^2)(t, z) + \frac{1}{2}\partial_z\Big((b/\rho)^2(t, z) + \rho^2(t, z)\partial_z(b/\rho^2)(t, z)\Big).$$

The function $F_{b,\rho}$ together with ρ^2 are called the reciprocal characteristics associated with Z. Clark asserts that they are invariant inside the class of Itô diffusions which share the same bridges. For convenience of the reader we recall his precise result in the following proposition.

Proposition 3.6. *Let $X = (X_t)_{t\in[0,1]}$ and $\tilde{X} = (\tilde{X}_t)_{t\in[0,1]}$ be two regular Itô diffusions solving the SDEs*

$$\begin{aligned} dX_t &= b(t, X_t)\, dt + \rho(t, X_t)\, dB_t, \\ d\tilde{X}_t &= \tilde{b}(t, \tilde{X}_t)\, dt + \tilde{\rho}(t, \tilde{X}_t)\, d\tilde{B}_t. \end{aligned}$$

Then X and $\tilde{X}$ share the same bridges if and only if their reciprocal characteristics coincide, that is

$$\rho^2 \equiv \tilde{\rho}^2 \quad \textit{and} \quad F_{b,\rho} \equiv F_{\tilde{b},\tilde{\rho}}. \tag{12}$$

Proof. For a detailed proof under the above assumptions we refer to [10]. To briefly summarize, the main argument leads to the fact that X and $\tilde{X}$ share the same bridges if and only if they are $\mathfrak{h}$-tranforms in the sense of Doob. Equivalently, there exists a function $\mathfrak{h} \in C^{1,2}([0,1) \times \mathbb{R})$ such that

$$\tilde{\rho} \equiv \rho \quad \text{and} \quad \tilde{b} \equiv b + \rho^2 \, \partial_z \log \mathfrak{h}$$

where $\mathfrak{h}$ is space-time harmonic, i.e. satisfies

$$\partial_t \mathfrak{h} + \frac{1}{2} \rho^2 \, \partial_{zz} \mathfrak{h} + b \, \partial_z \mathfrak{h} = 0.$$

One of the calculations in this proof requires the validity of $\partial_t \partial_y p(s,x;t,y) = \partial_y \partial_t p(s,x;t,y)$ which justifies the presence of condition (ii) in Definition 3.4. The differentiability condition (i) on the first two variables is necessary to ensure regularity for the bridge transition density in the second two variables. □

Proof of Theorem 3.5. Assume that $\{X^{xy},\, x, y \in \mathbb{R}\}$ defined by (2) are the bridges of a regular Itô diffusion $Z = (Z_t)_{t\in[0,1]}$ which solves

$$dZ_t = b(t, Z_t)\, dt + \rho(t, Z_t)\, dB_t.$$

Define $b_y(t,z) := h(t)(y - z)$, $t \in [0,1), z \in \mathbb{R}$. Proposition 3.6 implies that for each $y \in \mathbb{R}$, the following system of equations holds on $[0,1) \times \mathbb{R}$:

$$\begin{cases} \rho \equiv \sigma \\ F_{b,\rho} \equiv F_{b_y,\sigma}. \end{cases}$$

In particular $F_{b_y,\sigma}(t,z) = \dfrac{h'(t)\sigma^2(t) - h(t)\partial_t \sigma^2(t) - h^2(t)\sigma^2(t)}{\sigma^4(t)} \big(y - z\big)$ should not depend on y. Hence,

$$h' = h^2 + hg \quad \text{where} \quad g = \partial_t \log \sigma^2.$$

Setting $G(t) := \int_0^t g(s)\, ds = \log \frac{\sigma^2(t)}{\sigma^2(0)}$, $t \in [0,1)$, we obtain as the unique solution of this ODE

$$h(t) = \frac{e^{G(t)}}{C - \int_0^t e^{G(s)}\, ds} = \frac{\sigma^2(t)}{\tilde{C} - \int_0^t \sigma^2(s)\, ds}$$

on its maximal interval of existence, where $C > 0$ and $\tilde{C} = \sigma^2(0)C$. Now,

$$\int_0^t h(s)\, ds = \log \frac{\tilde{C}}{\tilde{C} - \int_0^t \sigma^2(s)\, ds}$$

and by Remark 2.4, $\tilde{C} = \Sigma < +\infty$ is uniquely determined.
To prove the converse implication one follows the same argumentation as in the proof of Theorem 3.1. □

4 Examples

4.1 α-pinned Brownian diffusions

We first consider the family $\{X^{xy},\, x,y \in \mathbb{R}\}$ of pinned diffusions associated to

$$\sigma \equiv 1 \qquad \text{and} \qquad h(t) = \frac{\alpha(t)}{1-t}, \quad t \in [0,1),$$

where $\alpha : [0,1] \to \mathbb{R}$ is continuous and satisfies $\alpha(1) > 0$. They solve the SDE

$$\begin{cases} dX_t = \alpha(t)\, \dfrac{y - X_t}{1-t}\, dt + dB_t, \quad t \in [0,1), \\ X_0 = x. \end{cases} \tag{13}$$

In the particular case $x = y = 0$, they were introduced by Barczy and Kern under the name of *general α-Wiener bridges*. They generalize the so-called α-Wiener bridges ($\alpha(t) \equiv \alpha(0)$) which were first introduced by Brennan and Schwartz [11] to model the arbitrage profit associated to a given stock index future in absence of transaction costs. For a mathematical treatment of α-Wiener bridges see also Barczy and Pap [2] and Mansuy [1]. Barczy and Kern proved in [4] that the process X^{00} given by (13) is pinned in 0 at time $t = 1$. They then studied whether X^{00} can be obtained as the 0-0-bridge of an Ornstein-Uhlenbeck type process, i.e. a process Z satisfying $dZ_t = q(t)\, dt + \sigma(t)\, dB_t$ for continuous functions $q, \sigma : [0,1] \to \mathbb{R}$ where $\sigma(t) \neq 0$ for all $t \in [0,1]$.

Let us show that Assumptions (A1)-(A3) are satisfied. Since α is continuous on $[0,1]$ and $\alpha(1)$ is positive, there exists $t_0 \in (0,1)$ such that α is bounded und positive on $[t_0, 1]$:

$$\exists 0 < a < b,\ \forall t \in [t_0, 1], \quad a \le \alpha(t) \le b.$$

Then, for any $t \in [t_0, 1)$,

$$\int_0^{t_0} h(u)\, du + a \int_{t_0}^t \frac{du}{1-u} \le \int_0^t h(u)\, du \le \int_0^{t_0} h(u)\, du + b \int_{t_0}^t \frac{du}{1-u}.$$

The first inequality above implies that Assumption (A1) is satisfied and the second one yields

$$\limsup_{t\to 1} \frac{\int_0^t h(u)\, du}{h(t)} \le \limsup_{t \to 1} \frac{1-t}{a}\, b \int_{t_0}^t \frac{du}{1-u} \le \frac{b}{a} < +\infty,$$

which proves that Assumption (A2) is satisfied. Thus, due to Proposition 2.3, the pinning property of the solution X^{xy} of (13) is assured for any $x, y \in \mathbb{R}$.
Further, as a corollary of Theorems 3.1 and 3.5, one deduces that, if the function α differs from the constant 1, the family $\{X^{xy},\, x, y \in \mathbb{R}\}$ of such pinned diffusions does not coincide with the bridges of any Gaussian Markov process or of any regular Itô diffusion.
For this reason we propose to call the solution of (13) an *α-pinned Brownian diffusion* without using the word *bridges* which is not well adapted for this situation.

4.2 (α, γ)-pinned Brownian diffusions

As a modification of the previous example we consider for $\alpha > 0$ and $\gamma \geq 0$ the following SDE

$$\begin{cases} dX_t = \alpha \dfrac{y - X_t}{(1-t)^{1+\gamma}} \, dt + dB_t, \quad t \in [0, 1), \\ X_0 = x, \end{cases} \tag{14}$$

and its solution X^{xy}. Let us first verify that Assumptions (A1)-(A3) are satisfied: the function $h(t) = \frac{\alpha}{(1-t)^{1+\gamma}}$ clearly satisfies (A1); Assumption (A2) holds since h is increasing. It thus follows from Proposition 2.3 that $\{X^{xy},\, x, y \in \mathbb{R}\}$ are pinned diffusions and it makes sense to call them (α, γ)-*pinned Brownian diffusions*.
Theorem 3.1 applies and implies that for $(\alpha, \gamma) \neq (1, 0)$ the (α, γ)-pinned Brownian diffusions can not be obtained as the bridges of any Gaussian Markov process. Theorem 3.5 applies too: $\{X^{xy},\, x, y \in \mathbb{R}\}$ can not be obtained as the bridges of any regular Itô diffusion as soon as we are not in the very particular case $(\alpha, \gamma) = (1, 0)$.

4.3 F-Wiener bridges

Let $f : [0, 1) \to (0, +\infty)$ be a continuous probability density function and let F denote the corresponding cumulative distribution function. We now consider the SDE

$$\begin{cases} dX_t = \dfrac{f(t)}{1 - F(t)} (y - X_t) \, dt + \sqrt{f(t)} \, dB_t, \quad t \in [0, 1) \\ X_0 = x. \end{cases} \tag{15}$$

and its solution X^{xy}.
In the particular case $x = y = 0$ these processes are called F-Wiener bridges and play a role in statistics as weak limits of empirical processes, see e.g. [12]. It was shown in [3] that X^{00} is pinned in 0 at time $t = 1$.
We are indeed in the framework of Theorem 3.1 with $\sigma = \sqrt{f}$ and $\Sigma = 1$: The family $\{X^{xy},\, x, y \in \mathbb{R}\}$ coincides with the bridges of the Gaussian Markov process

$$Z_t = \int_0^t \sqrt{f(s)} \, dB_s \overset{(d)}{=} B_{F(t)}, \quad t \in [0, 1].$$

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