

Conditional Extreme Value Models: Fallacies and Pitfalls

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the date of receipt and acceptance should be inserted later

Abstract Conditional extreme value models have been introduced by Heffernan and Resnick (2007) to describe the asymptotic behavior of a random vector as one specific component becomes extreme. Obviously, this class of models is related to classical multivariate extreme value theory which describes the behavior of a random vector as its norm (and therefore at least one of its components) becomes extreme. However, it turns out that this relationship is rather subtle and sometimes contrary to intuition. We clarify the differences between the two approaches with the help of several illuminative (counter)examples. Furthermore, we discuss marginal standardization, which is a useful tool in classical multivariate extreme value theory but, as we point out, much less straightforward and sometimes even obscuring in conditional extreme value models. Finally, we indicate how, in some situations, a more comprehensive characterization of the asymptotic behavior can be obtained if the conditions of conditional extreme value models are relaxed so that the limit is no longer unique.

Keywords Conditional extremes · Hidden regular variation · Multivariate extreme value models

Mathematics Subject Classification (2010) 60G70 · 60F05

1 Introduction

1.1 Motivation and Overview

The analysis of the extremal behavior of random vectors constitutes the central topic of multivariate extreme value theory (MEVT). Its historical development started with the extension of results from univariate extreme value theory to the multivariate setting by describing the limiting behavior of componentwise maxima of iid random vectors under suitable linear normalization. This behavior is determined by the distribution of (X_1, \dots, X_d) on the set of points with at least one large component. However, there are situations where this approach does not work, e.g., because the normalized maxima of some of the components do not converge.

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In such a case, if one is interested in the behavior of the other components as one component becomes extreme, a so-called conditional extreme value model (CEVM) introduced by Heffernan and Resnick (2007) may be appropriate. This model describes (certain aspects of) the behavior of (X_1, \dots, X_d) on the set of points where a *given* component is large. More concretely, it is assumed that one marginal distribution function, say the i th, belongs to the domain of attraction of some extreme value distribution, and that the conditional distribution of the other components, given that X_i exceeds a high threshold converges after a suitable linear normalization. Das and Resnick (2011a) and Das and Resnick (2011b) explored some connections between a CEVM and the aforementioned model from classical MEVT by comparing vague convergence of certain measures related to the distribution of (X_1, \dots, X_d) on different spaces. We use this framework of vague convergence in Section 2 below, where we recall the basics of both concepts.

A somewhat related model has been suggested by Heffernan and Tawn (2004). Here the basic assumption is that the conditional distribution of the other normalized components converge given that X_i is equal to a large value. Resnick and Zeber (2014) rephrased this assumption in terms of convergence of Markov kernels and examined its relation to a CEVM. They showed that, in the general case of non-standardized components, additional assumptions about the normalizing functions have to be satisfied in order to ensure that a CEVM follows from the convergence of Markov kernels, cf. Resnick and Zeber (2014), Proposition 4.1. Since our analysis will be based on vague convergence of measures, we will take the CEVM assumption as a starting point instead of the model from Heffernan and Tawn (2004).

In order to separate the marginal tail behavior from the dependence structure between extreme values of the components, it is common in classical MEVT to analyze the asymptotic behavior of the standardized vector $(X_1^*, \dots, X_d^*) = (1/(1 - F_1(X_1)), \dots, 1/(1 - F_d(X_d)))$, whose marginal distributions are standard Pareto if the marginal distributions of (X_1, \dots, X_d) are continuous. It is known that the componentwise maxima of the standardized vector converge if and only if $tP\{(X_1^*/t, \dots, X_d^*/t) \in \cdot\}$ converges vaguely to a non-degenerate limit measure, which means that (X_1^*, \dots, X_d^*) exhibits standard multivariate regular variation. Note that here the normalization is particularly simple and does not depend on the distribution of (X_1^*, \dots, X_d^*) anymore.

In the context of a CEVM, Heffernan and Resnick (2007) and Das and Resnick (2011a) also discuss the issue of standardizing the marginal distributions. However, their aim was merely to ensure that the same simple normalization can be used, whereas they do not try to transform the marginal distribution to a prescribed one. We will see in Section 3 that the meaning and the interpretation of standardization in a CEVM is so far not well understood. In particular it turns out that often a standardization in the sense of Heffernan and Resnick (2007) and Das and Resnick (2011a) completely changes the information about the extremal dependence structure. We thus think that there does not exist a “natural” way to standardize the margins in a general CEVM. In some cases, however, when the CEVM describes the asymptotic behavior of the not necessarily extreme components locally in a one-sided neighborhood of a fixed point, standardization is feasible.

In Section 4, we discuss the aforementioned connections between the CEVM and classical models from MEVT in detail and point out that the relations are much more intricate than realized so far. We show by (counter)examples that despite apparent similarities between the two models both concepts are too different to generally infer the convergence assumed in one model from the convergence in the other model - often contrary to intuition. For example, if the vector (X_1, X_2) is in the domain of attraction of a bivariate extreme value distribution and the two components are not asymptotically independent, then one may conjecture that this vector satisfies a CEVM as well. However, since the treatment of both the left tail and the central part of the distribution of X_1 differs in the two models, this needs not be the case. The counterexamples that we present show that, more fundamentally, both types of models may convey very different information about the behavior of X_1 for large values of X_2 , because in the classical extreme value models the normalizing functions for X_1 are determined by its tail behavior while in a CEVM they must match the overall

behavior of X_1 for large values of X_2 , which need not be related to large values of X_1 . One may try to overcome these differences by additional assumptions on the normalizing functions; cf. Das and Resnick (2011a). However, our counterexamples also show that some of the assumptions stated in the literature so far are in fact too weak in certain cases.

Finally, in Section 5 we put possible modifications of the CEVM forward which could help to overcome some of the drawbacks of the model identified in the preceding sections.

2 Extreme Value Models

2.1 Classical MEVT

Classical MEVT describes the limiting behavior of the joint distribution of componentwise maxima of iid random vectors after a suitable linear standardization of the margins. For notational simplicity, we focus on the bivariate case where for iid random vectors (X_i, Y_i) , $1 \leq i \leq n$, it is assumed that

$$P\left\{\left(\frac{\max_{1 \leq i \leq n} X_i - b_n}{a_n}, \frac{\max_{1 \leq i \leq n} Y_i - d_n}{c_n}\right) \in \cdot\right\} \rightarrow G \quad \text{weakly} \quad (2.1)$$

for suitable normalizing constants $a_n, c_n > 0$ and $b_n, d_n \in \mathbb{R}$ and a bivariate distribution function G with non-degenerate marginal distributions.

In particular, the marginal distribution functions F_X and F_Y of a random vector (X, Y) with the same distribution as (X_1, Y_1) belong to the domain of attraction of some extreme value distribution G_{γ_X} resp. G_{γ_Y} , where $G_\gamma(x) := \exp(-(1 + \gamma x)^{-1/\gamma})$ for all x such that $1 + \gamma x > 0$. This means that for suitable normalizing functions $a, c > 0$ and $b, d \in \mathbb{R}$

$$tP\left\{\frac{X - b(t)}{a(t)} > x\right\} \rightarrow (1 + \gamma_X x)^{-1/\gamma_X} \quad \forall x \in E^{(\gamma_X)} \quad (2.2)$$

$$tP\left\{\frac{Y - d(t)}{c(t)} > y\right\} \rightarrow (1 + \gamma_Y y)^{-1/\gamma_Y} \quad \forall y \in E^{(\gamma_Y)} \quad (2.3)$$

as $t \rightarrow \infty$. (Throughout the paper, $(1 + \gamma x)^{-1/\gamma}$ is interpreted as e^{-x} if $\gamma = 0$; likewise $(x^\gamma - 1)/\gamma := \log x$ for $\gamma = 0$.) Here

$$E^{(\gamma)} := \{x \in \mathbb{R} \mid 1 + \gamma x > 0\} = \begin{cases} (-\infty, -1/\gamma) & \gamma < 0, \\ (-\infty, \infty) & \text{if } \gamma = 0, \\ (-1/\gamma, \infty) & \gamma > 0 \end{cases}$$

denotes the interior of the support of G_γ . Moreover, let

$$q_\gamma := \inf E^{(\gamma)} = \begin{cases} -\infty & \gamma \leq 0, \\ -1/\gamma & \gamma > 0 \end{cases} \quad \text{and} \quad q^\gamma := \sup E^{(\gamma)} = \begin{cases} -1/\gamma & \gamma < 0, \\ \infty & \gamma \geq 0 \end{cases}$$

be the left and right endpoint of $E^{(\gamma)}$. Let $\bar{E}^{(\gamma)} = E^{(\gamma)} \cup \{q^\gamma\}$ denote the closure to the right of $E^{(\gamma)}$ (considered as a subset of the compactification $\bar{\mathbb{R}} = [-\infty, \infty]$ of \mathbb{R}), and let $\bar{\bar{E}}^{(\gamma)} = E^{(\gamma)} \cup \{q_\gamma, q^\gamma\}$ denote the topological closure of $E^{(\gamma)}$ in $\bar{\mathbb{R}}$. Define the one-point uncompactification of $\bar{E}^{(\gamma_X)} \times \bar{E}^{(\gamma_Y)}$ by $E^{(\gamma_X, \gamma_Y)} := (\bar{E}^{(\gamma_X)} \times \bar{E}^{(\gamma_Y)}) \setminus \{(q_{\gamma_X}, q_{\gamma_Y})\}$. See Resnick (2007), Section 3.3, and Das and Resnick (2011a) for details about the topology on these sets.

Convergence (2.1) is equivalent to the vague convergence

$$tP\left\{\left(\max\left(\frac{X - b(t)}{a(t)}, q_{\gamma_X}\right), \max\left(\frac{Y - d(t)}{c(t)}, q_{\gamma_Y}\right)\right) \in \cdot\right\} \xrightarrow{v} \mu \quad \text{as } t \rightarrow \infty \quad (2.4)$$

on $([q_{\gamma_X}, \infty] \times [q_{\gamma_Y}, \infty]) \setminus \{(q_{\gamma_X}, q_{\gamma_Y})\}$ for some normalizing functions $a, c > 0$ and $b, d \in \mathbb{R}$ to a Radon measure μ with non-degenerate margins satisfying $\mu(\{\infty\} \times [q_{\gamma_Y}, \infty]) = 0 = \mu([q_{\gamma_X}, \infty] \times \{\infty\})$ (see Beirlant et al. (2004), (8.71)). This is equivalent to

$$tP\left\{\left(\max\left(\frac{X-b(t)}{a(t)}, q_{\gamma_X}\right), \max\left(\frac{Y-d(t)}{c(t)}, q_{\gamma_Y}\right)\right) \in A\right\} \rightarrow \mu(A) < \infty \quad (2.5)$$

for all Borel sets $A \subset E^{(\gamma_X, \gamma_Y)}$ bounded away from $(q_{\gamma_X}, q_{\gamma_Y})$ such that $\mu(\partial A) = 0$. Note that for $A = (x, \infty) \times \bar{E}^{(\gamma_Y)}$ and $A = \bar{E}^{(\gamma_X)} \times (y, \infty)$ we recover the convergence of the left-hand sides of (2.2) and (2.3), respectively. Hence, for the above choice of the normalizing functions, $\mu((x, \infty) \times \bar{E}^{(\gamma_Y)}) = (1 + \gamma_X x)^{-1/\gamma_X}$ for all $x \in E^{(\gamma_X)}$, and $\mu(\bar{E}^{(\gamma_X)} \times (y, \infty)) = (1 + \gamma_Y y)^{-1/\gamma_Y}$ for all $y \in E^{(\gamma_Y)}$. Moreover, (2.4) is equivalent to

$$tP\left\{\frac{X-b(t)}{a(t)} > x \text{ or } \frac{Y-d(t)}{c(t)} > y\right\} \rightarrow \mu(E^{(\gamma_X, \gamma_Y)} \setminus ([q_{\gamma_X}, x] \times [q_{\gamma_Y}, y])) \quad (2.6)$$

for all $(x, y) \in E^{(\gamma_X)} \times E^{(\gamma_Y)}$. Therefore, in a slight abuse of notation, one may also write

$$tP\left\{\left(\frac{X-b(t)}{a(t)}, \frac{Y-d(t)}{c(t)}\right) \in \cdot\right\} \xrightarrow{v} \mu \quad \text{as } t \rightarrow \infty$$

on $E^{(\gamma_X, \gamma_Y)}$.

Sometimes it is useful to examine the marginal distributions and the dependence structure separately. To this end, define marginally normalized random variables

$$X^* := \frac{1}{1 - F_X(X)}, \quad Y^* := \frac{1}{1 - F_Y(Y)}.$$

If F_X is continuous, then X^* has a standard Pareto distribution; more generally, under (2.1), X^* is tail-equivalent to a standard Pareto random variable, i.e. $xP\{X^* > x\} \rightarrow 1$ as $x \rightarrow \infty$. Now convergence (2.1) is equivalent to the marginal convergences (2.2) and (2.3) together with the vague convergence

$$tP\left\{\left(\frac{X^*}{t}, \frac{Y^*}{t}\right) \in \cdot\right\} \xrightarrow{v} \mu^* \quad \text{as } t \rightarrow \infty \quad (2.7)$$

in $[0, \infty]^2 \setminus \{(0, 0)\}$ to the non-degenerate Radon measure μ^* given by

$$\mu^*([0, \infty]^2 \setminus ([0, x] \times [0, y])) := \mu\left(E^{(\gamma_X, \gamma_Y)} \setminus \left(\left[q_{\gamma_X}, \frac{x^{\gamma_X} - 1}{\gamma_X}\right] \times \left[q_{\gamma_Y}, \frac{y^{\gamma_Y} - 1}{\gamma_Y}\right]\right)\right)$$

for all $x, y > 0$. The latter convergence means that $tP\{(X^*, Y^*) \in tA\} \rightarrow \mu^*(A)$ for all Borel sets bounded away from the origin that are continuity sets of μ^* . Hence, the limit relations (2.4) and (2.7) describe the asymptotic behavior of (X, Y) and (X^*, Y^*) , respectively, when *at least one* component of the random vector is large. Note that the limit measure μ^* is homogeneous of order -1 , i.e. $\mu^*(tA) = t^{-1}\mu^*(A)$ for all $t > 0$ and $A \in \mathbb{B}([0, \infty)^2)$.

2.2 Conditional Extreme Value Models

In some applications, though, one is interested in the behavior of the random vector if a *given* component, say Y , is large. For example, the marginal expected shortfall of an asset which is defined as the expected loss from this investment given that some other quantity (e.g., the total loss of a portfolio) exceeds a high threshold is an important risk measure for a manager who must decide whether to add the asset to a given portfolio. Another example is given by internet traffic data, as analyzed in Das and Resnick (2011b), who looked at both file sizes and transfer rates of

transmissions in a network. They argued that the distribution of the transfer rates may not be in the domain of attraction of an extreme value distribution, in contrast to the file size distribution. Still, the asymptotic behavior of the transmission rate, given that the size of this transmitted file is large, is an important performance measure of the network. See also Heffernan and Tawn (2004) for an example concerning air pollutants.

In situations like these, the asymptotic behavior of the random vector may be described by a so-called conditional extreme value model (CEVM), as introduced in Heffernan and Resnick (2007) and further developed in Das and Resnick (2011a).

Here we start with a clarification of the model assumptions given by Resnick and Zeber (2014).

Definition 2.1 The vector (X, Y) follows a CEVM if there exist normalizing functions $\alpha, c > 0$ and $\beta, d \in \mathbb{R}$ such that (2.3) holds and, as $t \rightarrow \infty$,

$$tP\left\{\left(\frac{X - \beta(t)}{\alpha(t)}, \frac{Y - d(t)}{c(t)}\right) \in \cdot\right\} \xrightarrow{v} \mu_{X, Y>}(\cdot) \quad (2.8)$$

vaguely on $\bar{\mathbb{R}} \times \bar{E}^{(\gamma_Y)}$, and the limit measure $\mu_{X, Y>}$ satisfies the following non-degeneracy conditions:

- (i) $\mu_{X, Y>}(\cdot \times (y, \infty])$ is a non-degenerate measure on $\bar{\mathbb{R}}$ for all $y \in E^{(\gamma_Y)}$, i.e. $\mu_{X, Y>}((\bar{\mathbb{R}} \setminus \{x\}) \times (y, \infty]) > 0$ for all $x \in \bar{\mathbb{R}}$ and $y \in E^{(\gamma_Y)}$;
- (ii) $\mu_{X, Y>}(\{\infty\} \times \bar{E}^{(\gamma_Y)}) = 0$.

Then we write in short $(X, Y) \in CEV(\alpha, \beta, c, d, \gamma_Y, \mu_{X, Y>})$.

Since $[-\infty, \infty]$ is compact, the vague convergence (2.8) describes the behavior of the random vector (X, Y) when only Y needs to be extreme.

Remark 2.2 (i) Here and in the sequel, we extend a measure ν defined on a Borel subset M of \mathbb{R}^2 to a measure $\tilde{\nu}$ on $\bar{\mathbb{R}}^2$ by $\tilde{\nu}(A) := \nu(A \cap M)$ if necessary. For simplicity, in what follows we denote the extension again by ν .

- (ii) Similarly as above, (2.8) is equivalent to

$$tP\left\{\left(\frac{X - \beta(t)}{\alpha(t)}, \frac{Y - d(t)}{c(t)}\right) \in A\right\} \rightarrow \mu_{X, Y>}(A)$$

for all Borel sets $A \subset \bar{\mathbb{R}} \times (q_{\gamma_Y}, \infty]$ with $\inf\{y | (x, y) \in A\} > q_{\gamma_Y}$ such that $\mu_{X, Y>}(\partial A) = 0$. This is equivalent to

$$tP\left\{\frac{X - \beta(t)}{\alpha(t)} \leq x, \frac{Y - d(t)}{c(t)} > y\right\} \rightarrow \mu_{X, Y>}([-\infty, x] \times (y, \infty])$$

for all $x \in \bar{\mathbb{R}}$ and $y \in E^{(\gamma_Y)}$ with $\mu_{X, Y>}(\partial([-\infty, x] \times (y, \infty])) = 0$.

- (iii) Das and Resnick (2011a) and Resnick and Zeber (2014) do not assume (2.3), but they *conclude* this convergence for *some* extreme value index $\tilde{\gamma}_Y \in \mathbb{R}$ from the remaining assumptions. Note, however, that this index need not coincide with the value γ_Y used in the definition of the CEVM which only determines the support of $\mu_{X, Y>}$. In particular, if the assumptions of Definition 2.1 are fulfilled for some $\gamma_Y > 0$, then the convergence of measures (2.8) obviously holds true on $\bar{E}^{(\tilde{\gamma}_Y)}$ for all $\tilde{\gamma}_Y' > \gamma_Y$, too. Therefore, we have chosen the present formulation to avoid ambiguities concerning the meaning of the parameter γ_Y . Moreover, this way it seems more natural to consider vague convergence on the space $[-\infty, \infty] \times \bar{E}^{(\gamma_Y)}$. The additional normalizing condition $\mu_{X, Y>}(\bar{\mathbb{R}} \times (0, \infty]) = 1$ required by Das and Resnick (2011a) and Resnick and Zeber (2014) follows readily from convergence (2.3).

Condition (ii) has been added by Resnick and Zeber (2014) to the original set of conditions used by Heffernan and Resnick (2007) and Das and Resnick (2011a) to ensure uniqueness of the type of limit measure (i.e. uniqueness up to scale and shift transformations). Yet, as the following example shows, this condition has to be strengthened to rule out that different normalizations lead to completely different limit measures.

Example 2.3 Let Y be a standard Pareto random variable (i.e. $P\{Y > x\} = 1/x$ for all $x > 1$) so that (2.3) holds with $\gamma_Y = 1$. Let B be an independent discrete random variable that is uniformly distributed on $\{0, 1\}$,

$$X := B + (1 - B)(2 - 1/Y),$$

and $c(t) := d(t) := t$.

– For $\beta(t) = 0$ and $\alpha(t) = 1$ one has, for all $y \in E^{(1)} = (-1, \infty)$, $x \in \mathbb{R}$ and sufficiently large t ,

$$\begin{aligned} & tP\left\{\frac{X - \beta(t)}{\alpha(t)} \leq x, \frac{Y - c(t)}{d(t)} > y\right\} \\ &= \frac{t}{2}(P\{1 \leq x, Y > t(1 + y)\} + P\{2 - 1/Y \leq x, Y > t(1 + y)\}) \\ &= \frac{t}{2}\left(\frac{1}{t(1 + y)}1_{[1, \infty)}(x) + \frac{1}{t(1 + y)}1_{[2, \infty)}(x)\right) \\ &= \frac{1}{2}(1_{[1, \infty)}(x) + 1_{[2, \infty)}(x))(1 + y)^{-1} \\ &= \mu_{X, Y>}([-\infty, x] \times (y, \infty]). \end{aligned}$$

Here $\mu_{X, Y>} = (\frac{1}{2}\delta_1 + \frac{1}{2}\delta_2) \otimes \nu_1$, where δ_x denotes the Dirac-measure at x and ν_1 is given by $\nu_1(y, \infty] = 1/(1 + y)$ for all $y > -1$. This verifies (2.8), and the limit measure satisfies the conditions (i) and (ii) of Definition 2.1.

– Choosing $\beta(t) = 2$ and $\alpha(t) = 1/t$ instead yields for all $x \in \mathbb{R}$ and $y \in E^{(1)} = (-1, \infty)$

$$\begin{aligned} & tP\left\{\frac{X - \beta(t)}{\alpha(t)} \leq x, \frac{Y - d(t)}{c(t)} > y\right\} \\ &= \frac{t}{2}(P\{-1 \leq x/t, Y > t(1 + y)\} + P\{1/Y \geq -x/t, Y > t(1 + y)\}) \\ &\rightarrow \frac{1}{2}(1 + y)^{-1} + \frac{1}{2}((1 + y)^{-1} - x^-)^+ \\ &= \tilde{\mu}_{X, Y>}([-\infty, x] \times (y, \infty]) \end{aligned}$$

as $t \rightarrow \infty$, with $x^- := \max(-x, 0)$. Here, $\tilde{\mu}_{X, Y>}$ is the sum of $\frac{1}{2}\delta_{-\infty} \otimes \nu_1$ and the measure $\bar{\mu}$ which is concentrated on $\{(-(1 + y)^{-1}, y) | y > -1\}$ with $\bar{\mu}(\{(-(1 + y)^{-1}, y) | y > r\}) = 1/(2(1 + r))$ for all $r > -1$. Again all conditions of Definition 2.1 are met, but $\tilde{\mu}_{X, Y>}(\{-\infty\} \times (y, \infty]) = 1/(2(1 + y)) > 0$ for all $y \in E^{(1)}$.

So while the first limit measure describes the coarse behavior of X for large values of Y , namely that X can attain only values near the points 1 and 2, the second limit specifies the behavior of X in a neighborhood of 2 in greater detail, but it loses almost all information on the behavior of X on $(-\infty, 2 - \varepsilon]$ for any $\varepsilon > 0$. \square

To ensure that the convergence to types theorem can be applied which implies the uniqueness of the limit measure (up to scaling and shifts), we replace condition (ii) of Definition 2.1 with

$$(ii^*) \quad \mu_{X, Y>}(\{-\infty, \infty\} \times E^{(\gamma_Y)}) = 0.$$

If (X, Y) satisfies this modified set of conditions, we write $(X, Y) \in CEV^*(\alpha, \beta, c, d, \gamma_Y, \mu_{X, Y>})$. In Section 4 we discuss why in some situations it might be advantageous to drop condition (ii) (and (ii*)) although then the uniqueness of the limit measure is lost.

Remark 2.4 One can avoid any assumption ruling out mass in lines through infinity by using the framework of M -convergence instead of vague convergence, cf. Hult and Lindskog (2006) and Lindskog et al. (2014). Since assumption (ii*) is sufficient for our subsequent analysis, here we stick to the more conventional approach in order to facilitate the comparison with the previous literature on CEVM.

3 Standardization of CEVM

In classical MEVT one often considers the marginally standardized random vector (X^*, Y^*) , which has standard Pareto margins if the distribution functions of X and Y are continuous. This allows for a separation of the marginal tail behavior (investigated in univariate extreme value theory) and the extremal dependence structure which is inherent to MEVT. It seems reasonable to ask whether a similar separation can be achieved in CEVM, too.

It is easily seen that the component Y assumed to be large can be standardized in the same way as before, because its distribution function belongs to the domain of attraction of G_{γ_Y} . Hence, convergence (2.8) holds if and only if

$$tP\left\{\left(\frac{X - \beta(t)}{\alpha(t)}, \frac{Y^*}{t}\right) \in \cdot\right\} \xrightarrow{v} \mu_{X, Y>}^*(\cdot) \quad (3.1)$$

vaguely in $[-\infty, \infty] \times (0, \infty]$ with $\mu_{X, Y>}^*$ defined by

$$\mu_{X, Y>}^*([-\infty, x] \times (y, \infty]) := \mu_{X, Y>} \left([-\infty, x] \times \left(\frac{y^{\gamma_Y} - 1}{\gamma_Y}, \infty \right] \right)$$

for all $x \in \mathbb{R}$ and $y > 0$; see Das and Resnick (2011a), Section 3, for details. (Note that in this context the normalizing functions for Y^* are chosen slightly differently so that the limit in (2.3) equals y^{-1} for all $y > 0$; to obtain the usual limit, one has to consider $(Y^* - t)/t$ instead.)

In contrast, the distribution function of X does in general not belong to the domain of attraction of any extreme value distribution. Indeed, not the tail behavior of X is of interest, but the conditional behavior of X when Y attains large values. Heffernan and Resnick (2007) and Das and Resnick (2011a) thus aimed at the more modest goal of finding a function f such that $(f(X), Y^*)$ follows a CEVM with normalizing functions $\beta(t) = d(t) = 0$ and $\alpha(t) = c(t) = t$, i.e.

$$tP\left\{\left(\frac{f(X)}{t}, \frac{Y^*}{t}\right) \in \cdot\right\} \xrightarrow{v} \mu_{X, Y>}^{**}(\cdot) \quad (3.2)$$

vaguely in $[0, \infty] \times (0, \infty]$ with $\mu_{X, Y>}^{**}$ satisfying the non-degeneracy conditions

$$\mu_{X, Y>}^{**}(\cdot \times (y, \infty]) \text{ is a finite non-degenerate measure for all } y > 0, \text{ and} \quad (3.3)$$

$$\mu_{X, Y>}^{**}(\{\infty\} \times (0, \infty]) = 0 = \mu_{X, Y>}^{**}([0, \infty] \times \{\infty\}). \quad (3.4)$$

Note that, unlike X^* in classical MEVT, $f(X)$ does not have any pre-specified distribution. Hence, the notion “standardization” only refers to the resulting normalizing functions, but not to the marginal distributions.

Das and Resnick (2011a) required f to meet the following conditions:

(F1) $f : \text{range}(X) \rightarrow (0, \infty)$ (with $\text{range}(X)$ denoting the range of X),

- (F2) f is monotone,
(F3) f is unbounded.

The latter condition seems superfluous, because for bounded f the limit measure in (3.2) is necessarily concentrated on $\{0\} \times (0, \infty]$, violating (3.3).

The main claim in Section 3 of Das and Resnick (2011a) is that such a function f exists if and only if $\mu_{X, Y >}$, and thus $\mu_{X, Y >}^*$, is not a product measure. We show by counterexamples that in general neither of both implications hold. Moreover, even if both (2.8) and (3.2) hold, the respective limit measures may convey very different information about the conditional distribution of X given that Y is large.

Example 3.1 Let Y be a standard Pareto random variable (and thus $Y = Y^*$), and for some independent discrete random variable B that is uniformly distributed on $\{-1, 1\}$ define $X := 2 + B/Y$. Then, for all $x \in \mathbb{R}$ and $y > 0$

$$\begin{aligned} & tP\left\{\frac{X-2}{t-1} \leq x, \frac{Y^*}{t} > y\right\} \\ &= t\left(P\left\{B = 1, Y \geq \frac{t}{x}, Y > ty\right\}1_{(0, \infty)}(x)\right. \\ &\quad \left.+ P\left\{B = -1, Y \leq \frac{t}{|x|}, Y > ty\right\}1_{(-\infty, 0)}(x) + P\{B = -1, Y > ty\}1_{[0, \infty)}(x)\right) \\ &\rightarrow \frac{1}{2}\left(\min\left(x, \frac{1}{y}\right)1_{(0, \infty)}(x) + \left(\frac{1}{y} - |x|\right)^+ 1_{(-\infty, 0)}(x) + \frac{1}{y}1_{[0, \infty)}(x)\right) \\ &= \mu_{X, Y >}^*([-\infty, x] \times (y, \infty]), \end{aligned}$$

i.e., (X, Y^*) satisfies the conditions of Definition 2.1 (except for the normalized form of the limit in (2.3)) and (ii*) as well. Here $\mu_{X, Y >}^*$ denotes the measure that is concentrated on $\{(y^{-1}, y) | y > 0\} \cup \{(-y^{-1}, y) | y > 0\}$ with $\mu_{X, Y >}^*(\{(y^{-1}, y) | y > r\}) = 1/(2r) = \mu_{X, Y >}^*(\{(-y^{-1}, y) | y > r\})$, $r > 0$; hence it is not a product measure.

Now consider an arbitrary function $f : (1, 3) \rightarrow (0, \infty)$ satisfying (F1) and (F2). Since f is monotone, $u_0 := \sup_{3/2 \leq x \leq 5/2} f(x)$ is finite. Thus, for $x, y > 0$ and $t > \max(2/y, u_0/x)$,

$$tP\left\{\frac{f(X)}{t} > x, \frac{Y^*}{t} > y\right\} = tP\{f(2 + B/Y) > tx > u_0, Y > ty > 2\} = 0.$$

So, if (3.2) holds, then $\mu_{X, Y >}^{**}((0, \infty) \times (y, \infty]) = 0$ for all $y > 0$, that is, $\mu_{X, Y >}^{**}(\cdot \times (y, \infty])$ is degenerated, with all its mass concentrated at 0.

We have hence shown that X cannot be standardized in the sense used by Das and Resnick (2011a), despite the fact that $\mu_{X, Y >}^*$ is not a product measure. \square

The next example shows that in some cases X can even be standardized if $\mu_{X, Y >}^*$ is a product measure, and that the limit measures may describe completely different aspects of the stochastic behavior of X , given that Y is large.

Example 3.2 Let $X := B(1 - U/Y)$, for Y and B as in Example 3.1 and an independent random variable U that is uniformly distributed on $(0, 1)$. Then, for all $x \neq -1$, $y > 0$ and $t > 1/y$,

$$\begin{aligned} tP\left\{X \leq x, \frac{Y^*}{t} > y\right\} &= \frac{t}{2}\left(P\{1 - U/Y \leq x, Y > ty\} + P\{U/Y - 1 \leq x, Y > ty\}\right) \\ &= \frac{t}{2}E\left(P\left(ty < Y \leq \frac{U}{1-x} \mid U\right)1_{(-\infty, 1)}(x) + P\{Y > ty\}1_{[1, \infty)}(x)\right) \\ &\quad + P\left(Y > \max\left(\frac{U}{1+x}, ty\right) \mid U\right)1_{(-1, 0)}(x) + P\{Y > ty\}1_{[0, \infty)}(x) \end{aligned}$$

$$\begin{aligned}
&= \frac{t}{2} \left(E \left(\left(\frac{1}{ty} - \frac{1-x}{U} \right)^+ \right) 1_{(-\infty, 1)}(x) + \frac{1}{ty} 1_{[1, \infty)}(x) \right. \\
&\quad \left. + E \left(\min \left(\frac{1}{ty}, \frac{1+x}{U} \right) \right) 1_{(-1, 0)}(x) + \frac{1}{ty} 1_{[0, \infty)}(x) \right) \\
&= E \left(\left(\frac{1}{2y} - \frac{t(1-x)}{2U} \right)^+ \right) 1_{(-\infty, 1)}(x) + \frac{1}{2y} 1_{[1, \infty)}(x) \\
&\quad + E \left(\min \left(\frac{1}{2y}, \frac{t(1+x)}{2U} \right) \right) 1_{(-1, 0)}(x) + \frac{1}{2y} 1_{[0, \infty)}(x) \\
&\rightarrow \frac{1}{2y} 1_{[1, \infty)}(x) + \frac{1}{2y} 1_{(-1, 0)}(x) + \frac{1}{2y} 1_{[0, \infty)}(x) \\
&= \frac{1}{2y} (1_{[1, \infty)}(x) + 1_{[-1, \infty)}(x)) \\
&= \mu_{X, Y>}^*([-\infty, x] \times (y, \infty])
\end{aligned}$$

as $t \rightarrow \infty$. Thus, (3.1) holds with limit measure $\mu_{X, Y>}^* = (\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1) \otimes \tilde{\nu}_1$, where $\tilde{\nu}_1$ is given by $\tilde{\nu}_1(y, \infty] = 1/y$ for all $y > 0$.

Now define $f : (-1, 1) \rightarrow (0, \infty)$ by $f(x) = 1/(1-x) \in (1/2, \infty)$, which obviously satisfies (F1)–(F3). For all $x, y > 0$ and $t > \max(1/x, 1/y)$, one has

$$\begin{aligned}
tP \left\{ \frac{f(X)}{t} \leq x, \frac{Y^*}{t} > y \right\} &= tP \{ X \leq 1 - 1/(tx), Y > ty \} \\
&= \frac{t}{2} \left(P \left\{ 1 - \frac{U}{Y} \leq 1 - \frac{1}{tx}, Y > ty \right\} + P \{ Y > ty \} \right) \\
&= \frac{t}{2} E(P(ty < Y \leq tUx | U)) + \frac{1}{2y} \\
&= \frac{1}{2} \int_{(y/x, 1]} \frac{1}{y} - \frac{1}{ux} du + \frac{1}{2y} \\
&= \frac{1}{2} \left(\frac{1}{y} - \frac{1}{x} + \frac{\log(y/x)}{x} \right) 1_{\{x > y\}} + \frac{1}{2y}.
\end{aligned}$$

Direct calculations show that the last expression equals $\mu_{X, Y>}^{**}([0, x] \times (y, \infty])$ where the measure $\mu_{X, Y>}^{**}$ is the sum of $\frac{1}{2}\delta_0 \otimes \tilde{\nu}_1$ and the measure with Lebesgue density $g(x, y) = (2x^2y)^{-1} 1_{\{x > y\}}$. Hence, (3.2) holds with a non-degenerate limit measure $\mu_{X, Y>}^{**}$, although $\mu_{X, Y>}^*$ is a product measure.

Note that the limit measures $\mu_{X, Y>}^*$ and $\mu_{X, Y>}^{**}$ convey very different information on the behavior of X . While the former shows that, for large values of Y , the random variable X may only attain values close to ± 1 (with probability $1/2$ in each case), but does not reveal any more detailed information on its behavior in the vicinity of these points, $\mu_{X, Y>}^{**}$ describes the fine structure of the conditional distribution of X near 1, but does not give any information about its behavior on sets bounded away from 1, beyond the fact that half of its mass is concentrated on these sets. \square

Remark 3.3 In Example 3.2 the measure $\mu_{X, Y>}^{**}(\cdot \times (y, \infty])$ has half of its mass concentrated in 0. In a sense, this point corresponds to $-\infty$ in the original setting for (X, Y) . Hence, one might think of strengthening condition (3.4) to $\mu_{X, Y>}^{**}(\{0, \infty\} \times (0, \infty]) = 0 = \mu_{X, Y>}^{**}([0, \infty] \times \{\infty\})$, similarly as we have replaced condition (ii) in Definition 2.1 with (ii*). Then counterexamples of the type considered in Example 3.2 are ruled out, and the conclusion that a standardization of X is impossible if $\mu_{X, Y>}^*$ is a product measure may be correct. Note, however, that (ii*) has been introduced to ensure that the convergence to types theorem is applicable and the limit measure is unique (up to scale and shift transformations). In the situation of (3.2), where we consider convergence on a space with finite left endpoint of the x -component, uniqueness of the limit is already ensured under the weaker condition (3.4).

Conversely, one may construct random vectors (X, Y) which do not fulfill the conditions of a CEVM (incl. condition (ii*)), but the relations (3.2)–(3.4) hold for a suitable function f which satisfies (F1)–(F3), and the limit measure $\mu_{X, Y>}^{**}$ is not a product measure. The following example of this type is a modification of Example 4.1 of Resnick and Zeber (2014).

Example 3.4 Let Y be standard Pareto, $X := e^Y$ and $f := \log$. Then obviously (3.2) holds with a measure $\mu_{X, Y>}^{**}((x, \infty] \times (y, \infty]) = \min(x^{-1}, y^{-1})$ for all $x \geq 0, y > 0$, which is concentrated on the main diagonal and hence cannot be a product measure.

Suppose $(X, Y) = (X, Y^*)$ satisfies convergence (3.1) for some normalizing functions $\alpha > 0$ and $\beta \in \mathbb{R}$ and some measure $\mu_{X, Y>}^*$ such that $\mu_{X, Y>}^*(\{-\infty, \infty\} \times (0, \infty]) = 0$. Heffernan and Resnick (2007) have shown in their Proposition 1 that then there exist $C, \rho \in \mathbb{R}$ such that, as $t \rightarrow \infty$,

$$\frac{\alpha(\lambda t)}{\alpha(t)} \rightarrow \lambda^\rho, \quad \frac{\beta(\lambda t) - \beta(t)}{\alpha(t)} \rightarrow C \frac{\lambda^\rho - 1}{\rho}. \quad (3.5)$$

Here one may assume without loss of generality that $C \neq 0$; else replace $\beta(t)$ with $\beta(t) + \alpha(t)$ for which (3.1) also holds true (with a different limit measure). A combination of Theorem B.2.2 and Corollary B.2.13 of de Haan and Ferreira (2006) shows that $|\beta|$ is regularly varying with index ρ or 0. Thus $\alpha(t)x + \beta(t) = o(t^{\max(\rho, 0)+\varepsilon})$ for all $x \in \mathbb{R}$, by Proposition B.1.9 of de Haan and Ferreira (2006). Then, however,

$$\begin{aligned} tP\left\{\frac{X - \beta(t)}{\alpha(t)} \leq x, \frac{Y^*}{t} > y\right\} &= tP\{Y \in (ty, \log(\alpha(t)x + \beta(t)))\} 1_{[e, \infty)}(\alpha(t)x + \beta(t)) \\ &= \left(\frac{1}{y} - \frac{t}{\log(\alpha(t)x + \beta(t))}\right)^+ 1_{[e, \infty)}(\alpha(t)x + \beta(t)) \end{aligned}$$

converges to 0 for all $x \in \mathbb{R}$ and all $y > 0$, contradicting the assumptions of a CEVM. \square

The essential reason why standardization of X fails in Example 3.1 is that convergence (3.2) only describes the conditional behavior of $f(X)$, given Y is large, on a *left* neighborhood of the “point” ∞ , while convergence (3.1) specifies the conditional behavior of X on a *two-sided* neighborhood of ∞ , which obviously cannot be mapped onto a one-sided neighborhood (of ∞) by a monotone transformation. In Example 3.2 the problem arises, because convergence (3.1) for (X, Y^*) describes the conditional behavior of X at the two different points 1 and -1 (though only in a very crude way), while (3.2) can only convey information on the conditional behavior on a one-sided neighborhood of a single point (on a more detailed scale).

So the type of information given by the limit measure in a CEVM may be of qualitatively different nature from the one given by the limit measure (2.4) in classical multivariate extreme value theory (or in the type of models considered by Ledford and Tawn (1997)). The concept of standardization considered by Heffernan and Resnick (2007) and Das and Resnick (2011a) does not make allowance for this crucial difference. It is therefore not suitable for CEVM in their full generality.

However, when the normalizing constants in the definition of the CEVM are such that the limit model focusses on the behavior of X in a one-sided neighborhood of a single point (possibly $\pm\infty$), then a standardization is often possible. We do not aim at the most general results of that type, but merely discuss two important cases.

Proposition 3.5 *Suppose that $(X, Y) \in CEV^*(\alpha, \beta, c, d, \gamma_Y, \mu_{X, Y>}^*)$. If one of the following sets of conditions is fulfilled, then X can be standardized, i.e. there is a function f satisfying (F1) and (F2) such that (3.2) holds:*

- (i) $\alpha(t) \rightarrow \infty$ as $t \rightarrow \infty$, $\beta(t) = 0$ and $\mu_{X, Y>}^*((0, \infty) \times (y, \infty)) > 0$ for all $y > 0$, or
- (ii) $\alpha(t) \rightarrow 0$ as $t \rightarrow \infty$, $\beta(t) = \beta_0$ for some $\beta_0 \in \mathbb{R}$, and $\text{range}(X) \subset (-\infty, \beta_0)$ or $\text{range}(X) \subset (\beta_0, \infty)$.

Proof In the case (i), α is regularly varying with some index $\rho > 0$. (The case $\rho = 0$ is excluded, since β is not Π -varying; see Heffernan and Resnick (2007), Subsection 2.2.) W.l.o.g. we may assume that α is increasing and invertible with $\alpha(1) = 1$ and increasing inverse function α^\leftarrow , because there exists an asymptotically equivalent function $\tilde{\alpha}$ with these properties and $(X, Y) \in CEV^*(\tilde{\alpha}, 0, c, d, \gamma_Y, \mu_{X, Y>})$. Let $f(x) := \alpha^\leftarrow(x)$ for $x \geq 1$ and $f(x) = 1/(2-x)$ for $x < 1$ which is an increasing function. Then, for $x > 0$ and $t > 1/x$,

$$\begin{aligned} tP\left\{\frac{f(X)}{t} > x, \frac{Y^*}{t} > y\right\} &= tP\left\{X > \alpha(tx), \frac{Y^*}{t} > y\right\} \\ &= tP\left\{\frac{X}{\alpha(t)} > \frac{\alpha(tx)}{\alpha(t)}, \frac{Y^*}{t} > y\right\} \\ &\rightarrow \mu_{X, Y>}^*(x^\rho, \infty] \times (y, \infty], \end{aligned}$$

if $\mu_{X, Y>}^*(\{x^\rho\} \times (y, \infty]) = 0$. Hence, (3.2) holds with limit measure $\mu_{X, Y>}^{**} = (\mu_{X, Y>}^*)^T$ induced by the transformation $T: \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R} \times (0, \infty)$, $T(x, y) = ((\max(x, 0))^{1/\rho}, y)$ (i.e. $\mu_{X, Y>}^{**}(\cdot) = \mu_{X, Y>}^*(T^{-1}(\cdot))$). The assumption $\mu_{X, Y>}^*((0, \infty) \times (y, \infty)) > 0$ ensures that $\mu_{X, Y>}^{**}$ is not degenerate.

The arguments are similar in the case (ii). Here, α is regularly varying with some index $\rho < 0$ and we may assume w.l.o.g. that α is decreasing and invertible. If $\text{range}(X) \subset (-\infty, \beta_0)$, let $f(x) = \alpha^\leftarrow(\beta_0 - x)$ on a left neighborhood of β_0 , else $f(x) := \alpha^\leftarrow(x - \beta_0)$ for x in a right neighborhood of β_0 . In the former case, f is increasing, and, for $x \geq 0$ and sufficiently large t , we have

$$\begin{aligned} tP\left\{\frac{f(X)}{t} \leq x, \frac{Y^*}{t} > y\right\} &= tP\left\{X - \beta_0 \leq -\alpha(tx), \frac{Y^*}{t} > y\right\} \\ &= tP\left\{\frac{X - \beta_0}{\alpha(t)} \leq -\frac{\alpha(tx)}{\alpha(t)}, \frac{Y^*}{t} > y\right\} \\ &\rightarrow \mu_{X, Y>}^*([-\infty, -x^\rho] \times (y, \infty]) \end{aligned}$$

if $\mu_{X, Y>}^*(\{-x^\rho\} \times (y, \infty]) = 0$. Note that the assumptions imply that $\mu_{X, Y>}^*$ is concentrated on $(-\infty, 0] \times (0, \infty)$. Hence, the limit defines a non-degenerate measure $\mu_{X, Y>}^{**} = (\mu_{X, Y>}^*)^T$ with $T: [-\infty, 0] \times (0, \infty) \rightarrow \mathbb{R} \times (0, \infty)$, $T(x, y) = (|x|^{1/\rho}, y)$. The case $\text{range}(X) \subset (\beta_0, \infty)$ can be treated analogously. \square

While in case (ii) the standardized CEVM conveys the same information about the conditional behavior of X as the original CEVM, in case (i) the information about the behavior of negative values of X with large modulus is lost. (Of course, an analogous result holds if $\mu_{X, Y>}^*((-\infty, 0) \times (y, \infty)) > 0$ and one uses a suitable decreasing standardizing function f , so that the information about large values of X is lost.) If one relaxes the conditions on the function f in that one does not restrict its range, but allows f to be \mathbb{R} -valued, then one can standardize X in the case $\alpha(t) \rightarrow \infty$ and $\beta(t) = 0$ without additional assumptions and without any loss of information. (To this end, define $f(x) = \alpha^\leftarrow(|x|x/|x|)$ for $x \neq 0$ and $f(0) = 0$ for a version of the normalizing function α satisfying $\alpha(0, \infty) = (0, \infty)$.)

4 Relationship to other extreme value models

Recall that the classical multivariate extreme value model describes the asymptotic behavior of (X, Y) on sets where at least one component is large, while the CEVM for (X, Y) and for (Y, X) describes the behavior of (X, Y) for large values of Y and of X , respectively. Therefore, one might expect that the former implies the latter pair, and vice versa. However, it turns out that the relationship is much more intricate than this crude reasoning suggests. The following lemma gives a simple result in this spirit. Recall Remark 2.2(i) about the extension of measures.

Proposition 4.1 Define the map $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $S(x, y) := (y, x)$.

(i) Suppose (2.2)–(2.4) hold with $\gamma_X, \gamma_Y \leq 0$ and a non-degenerate limit measure μ satisfying $\mu(\bar{E}^{(\gamma_X)} \times \bar{E}^{(\gamma_Y)}) > 0$, i.e. X and Y are asymptotically dependent. Then $(X, Y) \in \text{CEV}(a, b, c, d, \gamma_Y, \mu_{X, Y>})$ and $(Y, X) \in \text{CEV}(c, d, a, b, \gamma_X, \mu_{Y, X>})$ with $\mu_{X, Y>}(\cdot) := \mu(\cdot \cap (\bar{\mathbb{R}} \times \bar{E}^{(\gamma_Y)}))$ and $\mu_{Y, X>}(\cdot) := \mu^S(\cdot \cap (\bar{\mathbb{R}} \times \bar{E}^{(\gamma_X)})) = \mu(S^{-1}(\cdot \cap (\bar{\mathbb{R}} \times \bar{E}^{(\gamma_X)})))$.

If $\mu(\{-\infty\} \times \mathbb{R}) = 0 = \mu(\mathbb{R} \times \{-\infty\})$, then also condition (ii*) is met for both CEVM.

(ii) Conversely, if $(X, Y) \in \text{CEV}(a, b, c, d, \gamma_Y, \mu_{X, Y>})$ and $(Y, X) \in \text{CEV}(c, d, a, b, \gamma_X, \mu_{Y, X>})$, then (X, Y) follows a classical extreme value model and (2.4) holds for the non-degenerate limit measure

$$\mu(\cdot) := \mu_{X, Y>}^{pr}(\cdot \cap (\bar{E}^{(\gamma_X)} \times \bar{E}^{(\gamma_Y)})) + \mu_{Y, X>}^{Sopr}(\cdot \cap (\bar{E}^{(\gamma_X)} \times \{q_{\gamma_Y}\})) \quad (4.1)$$

with $pr(x, y) := (x \vee q_{\gamma_X}, y \vee q_{\gamma_Y})$ denoting the projection of $\bar{\mathbb{R}}^2 \setminus ([-\infty, q_{\gamma_X}] \times [-\infty, q_{\gamma_Y}])$ onto $([q_{\gamma_X}, \infty] \times [q_{\gamma_Y}, \infty]) \setminus \{(q_{\gamma_X}, q_{\gamma_Y})\}$. Hence, μ is given by

$$\mu([q_{\gamma_X}, x] \times (y, \infty]) = \mu_{X, Y>}([-\infty, x] \times (y, \infty]) \quad \forall x \geq q_{\gamma_X}, y > q_{\gamma_Y}, \quad (4.2)$$

$$\mu((x, \infty] \times [q_{\gamma_Y}, y]) = \mu_{Y, X>}([-\infty, y] \times (x, \infty]) \quad \forall x > q_{\gamma_X}, y \geq q_{\gamma_Y}. \quad (4.3)$$

If condition (ii*) holds for both CEVM, then $\mu(\{q_{\gamma_X}\} \times [q_{\gamma_Y}, \infty]) = 0 = \mu([q_{\gamma_X}, \infty] \times \{q_{\gamma_Y}\})$.

Proof Recall that (2.4) is equivalent to (2.6), which is in turn equivalent to

$$tP\left\{\frac{X-b(t)}{a(t)} \leq x, \frac{Y-d(t)}{c(t)} > y\right\} \rightarrow \mu([q_{\gamma_X}, x] \times (y, \infty]) \quad \forall x \geq q_{\gamma_X}, y > q_{\gamma_Y}, \quad (4.4)$$

$$tP\left\{\frac{X-b(t)}{a(t)} > x, \frac{Y-d(t)}{c(t)} \leq y\right\} \rightarrow \mu((x, \infty] \times [q_{\gamma_Y}, y]) \quad \forall x > q_{\gamma_X}, y \geq q_{\gamma_Y}, \quad (4.5)$$

with suitable extensions of measures as in Remark 2.2(i).

Proof of (i): We only prove $(X, Y) \in \text{CEV}(a, b, c, d, \gamma_Y, \mu_{X, Y>})$, as the second assertion follows by completely analogous arguments and the assertion on condition (ii*) is obvious.

Note that $q_\gamma = -\infty$ for $\gamma \leq 0$ so that the maxima in (2.5) can be omitted. Moreover, by (2.3), $\mu(\bar{\mathbb{R}} \times [q^{\gamma_Y}, \infty]) = 0$, and so $\mu_{X, Y>}(\cdot) = \mu(\cdot \cap (\bar{\mathbb{R}} \times (-\infty, \infty]))$.

Now consider an arbitrary Borel set $A \subset \bar{\mathbb{R}} \times \bar{E}^{(\gamma_Y)}$ such that $\inf\{y \mid (x, y) \in A\} > q_{\gamma_Y} = -\infty$ and $\mu_{X, Y>}(\partial A) = 0$. Then $\mu(\partial A) = \mu_{X, Y>}(\partial A) = 0$ and by (2.5)

$$tP\left\{\left(\frac{X-b(t)}{a(t)}, \frac{Y-d(t)}{c(t)}\right) \in A\right\} \rightarrow \mu(A) = \mu_{X, Y>}(A),$$

which proves (2.8) and $\mu_{X, Y>}(\bar{\mathbb{R}} \times (y, \infty]) < \infty$ for all $y \in \bar{E}^{(\gamma_Y)}$. Moreover, (2.2) ensures condition (ii) of Definition 2.1.

Suppose $0 = \mu_{X, Y>}((\bar{\mathbb{R}} \setminus \{x\}) \times (y, \infty]) = \mu((\bar{\mathbb{R}} \setminus \{x\}) \times (y, \infty])$ for some $x \in \bar{\mathbb{R}}$ and $y \in \bar{E}^{(\gamma_Y)}$. If $\gamma_X < 0$ and $x \geq q^{\gamma_X}$, then $\mu((\bar{\mathbb{R}} \setminus \{x\}) \times (y, \infty]) = \mu(\bar{\mathbb{R}} \times (y, \infty]) = (1 + \gamma_Y y)^{-1/\gamma_Y} > 0$, contradicting the last assumption. Else, for μ^* as in (2.7),

$$0 = \mu((\bar{\mathbb{R}} \setminus \{x\}) \times (y, \infty]) = \mu^*\left(\left([0, \infty] \setminus \{(1 + \gamma_X x)^{1/\gamma_X}\}\right) \times \left((1 + \gamma_Y y)^{1/\gamma_Y}, \infty\right]\right) = 0,$$

and, by the homogeneity of μ^* , even $\mu(\bar{E}^{(\gamma_X)} \times \bar{E}^{(\gamma_Y)}) = \mu^*((0, \infty]^2) = 0$, contradicting the assumptions on μ . Hence, in any case one has $\mu_{X, Y>}((\bar{\mathbb{R}} \setminus \{x\}) \times (y, \infty]) > 0$, which shows that condition (i) of Definition 2.1 is fulfilled, too.

Proof of (ii): Standard arguments show that $D_X := \{x \in \bar{\mathbb{R}} \mid \mu_{X,Y>}(\{x\} \times \bar{E}^{(\gamma_Y)}) > 0\}$ and $D_Y := \{y \in \bar{\mathbb{R}} \mid \mu_{Y,X>}(\{y\} \times \bar{E}^{(\gamma_X)}) > 0\}$ are (at most) countable. For all $x \in E^{(\gamma_X)} \setminus D_X$ and $y \in E^{(\gamma_Y)} \setminus D_Y$ one has

$$\mu_{X,Y>}((x, \infty] \times (y, \infty]) = \lim_{t \rightarrow \infty} tP\left\{\frac{X - b(t)}{a(t)} > x, \frac{Y - d(t)}{c(t)} > y\right\} = \mu_{Y,X>}((y, \infty] \times (x, \infty]),$$

which shows that $\mu_{X,Y>}$ and $\mu_{Y,X>}^S$ coincide on $(q_{\gamma_X}, \infty] \times (q_{\gamma_Y}, \infty]$. It also implies that

$$\mu_{X,Y>}([q^{\gamma_X}, \infty] \times (q_{\gamma_Y}, \infty]) = 0 = \mu_{X,Y>}((q_{\gamma_X}, \infty] \times [q^{\gamma_Y}, \infty]),$$

and analogous equations for $\mu_{Y,X>}$. Let now first $x \geq q_{\gamma_X}$ and $y > q_{\gamma_Y}$. Then

$$\begin{aligned} tP\left\{\frac{X - b(t)}{a(t)} \leq x, \frac{Y - d(t)}{c(t)} > y\right\} &\rightarrow \mu_{X,Y>}([-\infty, x] \times (y, \infty]) \\ &= \mu_{X,Y>}([-\infty, \min(x, q^{\gamma_X})] \times (y, q^{\gamma_Y})) \\ &= \mu_{X,Y>}^{pr}([q_{\gamma_X}, \min(x, q^{\gamma_X})] \times (y, q^{\gamma_Y})) \\ &= \mu([q_{\gamma_X}, x] \times (y, \infty]), \end{aligned}$$

which shows (4.2) and (4.4). Furthermore, for $x > q_{\gamma_X}$ and $y \geq q_{\gamma_Y}$ we have

$$\begin{aligned} tP\left\{\frac{X - b(t)}{a(t)} > x, \frac{Y - d(t)}{c(t)} \leq y\right\} &\rightarrow \mu_{Y,X>}([-\infty, y] \times (x, \infty]) \\ &= \mu_{Y,X>}([-\infty, q_{\gamma_Y}] \times (x, q^{\gamma_X})) + \mu_{Y,X>}((q_{\gamma_Y}, \min(y, q^{\gamma_Y})) \times (x, q^{\gamma_X})) \\ &= \mu_{Y,X>}^{opr}((x, q^{\gamma_X}] \times \{q_{\gamma_Y}\}) + \mu_{Y,X>}^{pr}((x, q^{\gamma_X}] \times (q_{\gamma_Y}, \min(y, q^{\gamma_Y}))) \\ &= \mu((x, \infty] \times [q_{\gamma_Y}, y]), \end{aligned}$$

which shows (4.3) and (4.5). Moreover, our assumptions imply that (2.2) and (2.3) hold, and hence $\mu(\{\infty\} \times [q_{\gamma_Y}, \infty]) = 0 = \mu([q_{\gamma_Y}, \infty] \times \{\infty\})$. Therefore, (2.4) holds.

Again the assertion about condition (ii*) is trivial. \square

The conditions under which we have proved the relation between the classical multivariate extreme value model and the two CEVM are quite restrictive, in that we assume $\gamma_X, \gamma_Y \leq 0$ in Proposition 4.1 (i) and we require that the same normalizing functions are used in both CEVM in assertion (ii). We will show in two examples that in general one cannot dispense with these assumptions.

Example 4.2 Let $g(y) := y(2 + \sin \log y)$ for $y \geq 1$. Then g has a positive derivative and it is thus strictly increasing and invertible with a strictly increasing inverse $g^{\leftarrow} : [2, \infty) \rightarrow [1, \infty)$. Let B and Y be independent random variables such that B is uniformly distributed on $\{0, 1\}$ (i.e. it is Bernoulli(1/2)) and Y is standard Pareto distributed. Define $X := BY + (1 - B)(-g^{\leftarrow}(2Y))$. Then Y fulfills (2.3) with $c(t) = d(t) = t$ and $\gamma_Y = 1$, and X satisfies (2.2) for $a(t) = b(t) = t/2$ and $\gamma_X = 1$, because $tP\{(X - b(t))/a(t) > x\} = tP\{B = 1, Y > (t/2)(x + 1)\} = (1 + x)^{-1}$ for $x > -1$ and t sufficiently large.

Moreover, convergence (2.4) holds, because for all $x, y > -1$ and sufficiently large t

$$\begin{aligned} tP\left\{\max\left(\frac{X - b(t)}{a(t)}, -1\right) > x \text{ or } \max\left(\frac{Y - d(t)}{c(t)}, -1\right) > y\right\} \\ &= tP\left\{\frac{X - b(t)}{a(t)} > x\right\} + tP\left\{\frac{Y - d(t)}{c(t)} > y\right\} - tP\left\{B = 1, Y > \frac{t}{2}(1 + x), Y > t(1 + y)\right\} \\ &= (1 + x)^{-1} + (1 + y)^{-1} - \min((1 + x)^{-1}, (2(1 + y))^{-1}) \end{aligned}$$

$$\begin{aligned}
&= (2(1+y))^{-1} + \max\left((1+x)^{-1}, (2(1+y))^{-1}\right) \\
&= \mu([-1, \infty]^2 \setminus ([-1, x] \times [-1, y]))
\end{aligned}$$

where μ denotes the measure concentrated on $\{(t, (t-1)/2) | t > -1\} \cup \{(-1, t) | t > -1\}$ such that $\mu(\{(t, (t-1)/2) | t > r\}) = 1/(1+r)$ and $\mu(\{(-1, t) | t > r\}) = 1/(2(1+r))$ for all $r > -1$.

However, (X, Y) does not fulfill the conditions of any CEVM incl. (ii*). Suppose $(X, Y) \in CEV^*(\alpha, \beta, t, 1, \mu_{X, Y})$ for some normalizing functions $\alpha > 0$ and $\beta \in \mathbb{R}$. Recall that Heffernan and Resnick (2007) have shown that then the normalizing functions satisfy the relations (3.5) where without loss of generality one may assume $C \neq 0$. (Note that one needs condition (ii*) in order to apply the convergence to types theorem as it is done in that paper.) In particular, α is regularly varying with index ρ , $|\beta|$ is regularly varying with index ρ or 0, and, by Theorem B.2.2 of de Haan and Ferreira (2006), $\beta(t)/\alpha(t) \rightarrow C/\rho$ if $\rho > 0$.

Let $g(y) := 2$ for $y < 1$. For all $y > -1$ and sufficiently large t , we have

$$\begin{aligned}
& {}_tP\left\{\frac{X - \beta(t)}{\alpha(t)} \leq x, \frac{Y - t}{t} > y\right\} \\
&= \frac{t}{2}P\{Y \leq \beta(t) + \alpha(t)x, Y > t(1+y)\} + \frac{t}{2}P\{g^{\leftarrow}(2Y) \geq -(\beta(t) + \alpha(t)x), Y > t(1+y)\} \\
&= \frac{1}{2}\left((1+y)^{-1} - \frac{t}{\beta(t) + \alpha(t)x}\right)^+ 1_{(1, \infty)}(\beta(t) + \alpha(t)x) \\
&\quad + \min\left((2(1+y))^{-1}, \frac{t}{g(-(\beta(t) + \alpha(t)x))}\right). \tag{4.6}
\end{aligned}$$

First assume $\alpha(t_n) = o(t_n)$ for some sequence $t_n \rightarrow \infty$. Then by the above remarks also $\beta(t_n) = o(t_n)$, and the first term of (4.6) vanishes asymptotically for all $x \in \mathbb{R}$. Together with (2.3), this implies $\mu(\{\infty\} \times (y, \infty]) \geq (2(1+y))^{-1}$, in contradiction to condition (ii) of Definition 2.1.

Hence $t = O(\alpha(t))$, which implies $\rho \geq 1$ and thus $\beta(t)/\alpha(t) \rightarrow C/\rho$ and $\beta(t) + \alpha(t)x = \alpha(t)(x + C/\rho + o(1))$.

Now, for $x > -C/\rho$ and sufficiently large t , (4.6) equals $((1+y)^{-1} - t/(\alpha(t)(x + C/\rho + o(1))))^+ / 2 + (2(1+y))^{-1}$. For all $y > -1$, the first summand of this expression has to converge to a positive limit for some $x > -C/\rho$, since otherwise (2.8) or (ii) of Definition 2.1 does not hold. Therefore, $t/\alpha(t) \rightarrow \alpha_0 \in [0, \infty)$.

Consider first the case $\alpha_0 = 0$, i.e. $t = o(\alpha(t))$. Since $g(z) \in [z, 3z]$ for all $z \geq 1$, $t/g(-(\beta(t) + \alpha(t)x))$ tend to 0 for $x < -C/\rho$, and so does (4.6). On the other hand, if $x > -C/\rho$, then by similar arguments one sees that (4.6) tends to $(1+y)^{-1} = \mu_{X, Y>}(\mathbb{R} \times (y, \infty])$. We conclude that $\mu_{X, Y>}((\mathbb{R} \setminus \{-C/\rho\}) \times (y, \infty]) = 0$, contradicting condition (i) of Definition 2.1.

Therefore, $\alpha(t)/t$ tends to some $\alpha_0 \in (0, \infty)$. Then, however, for $x < -C/\rho$, (4.6) equals

$$\min\left((2(1+y))^{-1}, \frac{t}{g(-\alpha_0 t(x + C/\rho)(1 + o(1)))}\right).$$

Direct calculations show that along the sequence $t_n = -\exp(2\pi n)/(\alpha_0(x + C/\rho))$ this expression converges to $\min((2(1+y))^{-1}, (-2\alpha_0(x + C/\rho))^{-1})$, while along the sequence $t_n = -\exp(2\pi n + \pi/2)/(\alpha_0(x + C/\rho))$ it converges to $\min((2(1+y))^{-1}, (-3\alpha_0(x + C/\rho))^{-1})$.

We have thus proved that (4.6) cannot converge for any choice of $\alpha(t)$ and $\beta(t)$ on a dense set of x -values to a non-degenerate limit, that is, (X, Y) does not satisfy the conditions of a CEVM incl. (ii*). \square

A closer inspection of Example 4.2 reveals that the problem arises from an irregular behavior of $(X - b(t))/a(t)$ on $(-\infty, -1) = (-\infty, q_{\gamma_X})$ for large values of Y . Convergence (2.4) mainly describes

the behavior of $((X - b(t))/a(t), (Y - d(t))/c(t))$ on $([q_{\gamma_X}, \infty] \times [q_{\gamma_Y}, \infty]) \setminus \{(q_{\gamma_X}, q_{\gamma_Y})\}$, while on $(-\infty, q_{\gamma_X})$ essentially only the total mass of $(X - b(t))/a(t)$ is known (for large values of Y). Therefore, an irregular behavior as in Example 4.2 can only be ruled out in the case $q_{\gamma_X} = -\infty$.

Remark 4.3 (i) In Example 4.2, the assumptions of the CEVM fail only because of the behavior of $(X - b(t))/a(t)$ below q_{γ_X} . Hence one may establish a limit for

$$tP\left\{\max\left(\frac{X - b(t)}{a(t)}, q_{\gamma_X}\right) \leq x, \frac{Y - d(t)}{c(t)} > y\right\}.$$

However, this convergence cannot readily be interpreted in terms of a CEVM for a modified vector (\tilde{X}, Y) .

- (ii) If $\gamma_X, \gamma_Y > 0$ and $X, Y \geq 0$, the assumptions (2.2)–(2.4) imply that the vector $(X^{1/\gamma_X}, Y^{1/\gamma_Y})$ is standard regularly varying on $[0, \infty]^2 \setminus \{(0, 0)\}$ with some limit measure ν . If ν is not concentrated on the axes, i.e. if there is asymptotic dependence between the two components, then $(X^{1/\gamma_X}, Y^{1/\gamma_Y})$ also satisfies a CEVM and so does (X, Y) , cf. Heffernan and Resnick (2007), Section 5. Similarly, one can also handle the case when X and Y are not necessarily non-negative but they have a finite lower bound.

Concerning the converse implication, the following example shows that assertion (ii) of Proposition 4.1 does not hold if one does not require quite strong restrictions on the relation between the normalizing functions for the conditional extreme value models for (X, Y) and (Y, X) . In particular, it follows that Theorem 2.1 of Das and Resnick (2011a) is not correct in its present form.

Example 4.4 For $0 < u \leq 1$, let $g_c(u) = u(1 + c \sin \log u)$. Direct calculations show that g_c is strictly increasing for all $|c| < 1/\sqrt{2}$ with $g_c(1) = 1$ and $\lim_{u \downarrow 0} g_c(u) = 0$. Denote its inverse by $g_c^{\leftarrow} : (0, 1] \rightarrow (0, 1]$, and define a strictly decreasing function by $\psi_c(z) = g_c^{\leftarrow}(1/z)$, $z \geq 1$. Let Z be a standard Pareto random variable and B an independent discrete random variable that is uniformly distributed on $\{1, \dots, 4\}$. Define

$$(X, Y) := \begin{cases} (2 - Z^{-1}, 2 - \psi_{1/2}(Z)), & B = 1, \\ (2 - Z^{-1/2}, 2 - \psi_{-1/2}(Z)), & B = 2, \\ (2 - Z^{-1}, 1 - Z^{-1}), & B = 3, \\ (1 - Z^{-1}, 2 - Z^{-1}), & B = 4. \end{cases}$$

Obviously, $X, Y \in (0, 2)$ almost surely.

Assertion 1: $(X, Y) \in CEV^*(1, 0, c, d, -1, \mu_{X, Y})$ with $c(t) = 4/(3t)$, $d(t) = 2 - 4/(3t)$ and the measure $\mu_{X, Y} = (\frac{1}{3}\delta_1 + \frac{2}{3}\delta_2) \otimes \nu_{-1}$, where ν_{-1} is given by $\nu_{-1}((r, \infty]) = 1 - r$ for all $r \leq 1$ (i.e. ν_{-1} is the Lebesgue measure restricted to $(-\infty, 1]$).

For all $x \in \mathbb{R}$, $y \leq 1$ and sufficiently large t , one has

$$\begin{aligned} & tP\left\{X \leq x, \frac{Y - d(t)}{c(t)} > y\right\} \\ &= \frac{t}{4} \left(P\left\{2 - Z^{-1} \leq x, \psi_{1/2}(Z) < \frac{4}{3t}(1 - y)\right\} + P\left\{2 - Z^{-1/2} \leq x, \psi_{-1/2}(Z) < \frac{4}{3t}(1 - y)\right\} \right. \\ &\quad \left. + P\left\{1 - Z^{-1} \leq x, Z^{-1} < \frac{4}{3t}(1 - y)\right\} \right) \\ &= \frac{t}{4} \left(P\left\{Z > \frac{1}{g_{1/2}(4(1 - y)/(3t))}\right\} + P\left\{Z > \frac{1}{g_{-1/2}(4(1 - y)/(3t))}\right\} \right) 1_{[2, \infty)}(x) \\ &\quad + \frac{t}{4} P\left\{Z > \frac{3t}{4(1 - y)}\right\} 1_{[1, \infty)}(x) \end{aligned}$$

$$\begin{aligned}
&= \frac{t}{4} (g_{1/2}(4(1-y)/(3t)) + g_{-1/2}(4(1-y)/(3t))) 1_{[2,\infty)}(x) + \frac{1-y}{3} 1_{[1,\infty)}(x) \\
&= \frac{1-y}{3} 1_{[1,\infty)}(x) + \frac{2(1-y)}{3} 1_{[2,\infty)}(x) \\
&= \mu_{X,Y>}((-\infty, x] \times (y, \infty]),
\end{aligned}$$

which proves (2.8). In particular, (2.3) holds with $\gamma_Y = -1$. Note that the limit measure is a product measure and that the marginal measure corresponding to X has mass concentrated at the points 1 and 2; hence the non-degeneracy conditions are fulfilled.

Assertion 2: $(Y, X) \in CEV^*(1, 0, a, b, -1, \mu_{Y,X>})$ with $a(t) = 2/t, b(t) = 2 - 2/t$ and $\mu_{Y,X>} = \nu_{-1} \otimes (\frac{1}{2}\delta_1 + \frac{1}{2}\delta_2)$.

This assertion follows by similar arguments from

$$\begin{aligned}
&tP\left\{Y \leq y, \frac{X - b(t)}{a(t)} > x\right\} \\
&= \frac{t}{4} \left(P\left\{2 - \psi_{1/2}(Z) \leq y, Z^{-1} < \frac{2}{t}(1-x)\right\} + P\left\{2 - \psi_{-1/2}(Z) \leq y, Z^{-1/2} < \frac{2}{t}(1-x)\right\} \right. \\
&\quad \left. + P\left\{1 - Z^{-1} \leq y, Z^{-1} < \frac{2}{t}(1-x)\right\} \right) \\
&\rightarrow \frac{1-x}{2} 1_{[2,\infty)}(y) + \frac{1-x}{2} 1_{[1,\infty)}(y) \\
&= \mu_{Y,X>}((-\infty, y] \times (x, \infty]),
\end{aligned}$$

for all $x \leq 1$ and $y \in \mathbb{R}$. In particular, (2.2) holds with $\gamma_X = -1$.

Assertion 3: Convergence (2.4) does not hold true.

If (2.4) holds, then because of (2.2) and (2.3) also the following expression must converge for all $x, y < 1$:

$$\begin{aligned}
&tP\left\{\frac{X - b(t)}{a(t)} > x, \frac{Y - d(t)}{c(t)} > y\right\} \\
&= \frac{t}{4} \left(P\left\{Z > \frac{t}{2(1-x)}, \psi_{1/2}(Z) < \frac{4}{3t}(1-y)\right\} + P\left\{Z > \left(\frac{t}{2(1-x)}\right)^2, \psi_{-1/2}(Z) < \frac{4}{3t}(1-y)\right\} \right) \\
&= \frac{t}{4} P\left\{Z > \max\left(\frac{t}{2(1-x)}, \frac{1}{g_{1/2}(4(1-y)/(3t))}\right)\right\} + o(1) \\
&= \frac{t}{4} \min\left(\frac{2(1-x)}{t}, g_{1/2}(4(1-y)/(3t))\right).
\end{aligned}$$

However, it is easily seen that the limit inferior of the last expression equals $\min((1-x)/2, (1-y)/6)$, whereas its limit superior is $\min((1-x)/2, (1-y)/2)$. (Choose $t = 4(1-y) \exp((2n+1/2)\pi)/3$ and $t = 4(1-y) \exp((2n-1/2)\pi)/3$, respectively.) \square

Note that in this example the conditional extreme value models convey rather limited information on the non-extreme component, in that they only specify the two points in the neighborhood of which this component will lie for large values of the other component. This allows for a very irregular behavior of the non-extreme component, say Y , on a finer scale (see case $B = 1$). If this irregular behavior for large values of X is balanced by the behavior for smaller values of X (see case $B = 2$), then the marginal tail behavior of Y can be regular. In the classical multivariate extreme value model, the behavior of Y for large values of X is examined in much more detail in the vicinity of the higher point of mass, so that the irregular behavior rules out the convergence required by this model.

To avoid such effects, one may require that the normalizations in both CEVM are identical, or, formally weaker, that they are equivalent in the sense of the convergence to types theorem, i.e., $(X, Y) \in CEV^*(\alpha, \beta, c, d, \gamma_Y, \mu_{X,Y>})$ and $(Y, X) \in CEV^*(\chi, \delta, a, b, \gamma_X, \mu_{Y,X>})$ where

$$\begin{aligned} \frac{\alpha(t)}{a(t)} &\rightarrow A \in (0, \infty), & \frac{\beta(t) - b(t)}{a(t)} &\rightarrow B \in \mathbb{R} \\ \frac{\chi(t)}{c(t)} &\rightarrow C \in (0, \infty), & \frac{\delta(t) - d(t)}{c(t)} &\rightarrow D \in \mathbb{R} \end{aligned} \quad (4.7)$$

Das and Resnick (2011a) used in Proposition 4.1 somewhat weaker conditions on the normalizing functions when they concluded convergence (2.4) from (2.2) and $(X, Y) \in CEV(\alpha, \beta, c, d, \gamma_Y, \mu_{X,Y>})$, provided $\lim_{t \rightarrow \infty} \alpha(t)/a(t) = A < \infty$ and $\beta(t)$ and $b(t)$ converge to the same limit in $(-\infty, \infty]$. The following example shows that in some cases these conditions are too weak to ensure that the conclusion holds, because they do not rule out that the normalizing functions β and b behave quite differently in the Gumbel case $\gamma_X = 0$.

Example 4.5 For $x \geq e$, let $\psi(x) := \log x + \sin \log \log x$, which is a continuous and increasing function with increasing inverse $\psi^{\leftarrow} : [1, \infty) \rightarrow [e, \infty)$. Observe that, for a sufficiently large $x_0 > 1$, the function $g(x) := 4/(3\psi^{\leftarrow}(x)) - e^{-x}/3$ is decreasing on $[x_0, \infty)$ with values in $(0, 1]$, because

$$\frac{d}{dx}g(\psi(x)) = -\frac{4}{3x^2} + \frac{1}{3x^2} \exp(-\sin \log \log x) \left(1 + \frac{\cos \log \log x}{\log x}\right) \leq \frac{1}{3x^2} \left(-4 + e\frac{4}{3}\right) < 0$$

for $x \geq e^3$.

Let Z be a random variable with survival function $P\{Z > x\} = g(x)1_{[x_0, \infty)}(x) + 1_{(-\infty, x_0)}(x)$. Let Y be a standard Pareto random variable independent of Z and define

$$X := \begin{cases} \log(Y/4), & Y > 4, \\ Z, & Y \leq 4. \end{cases}$$

Obviously, (2.3) holds with $c(t) = d(t) = t$ and $\gamma_Y = 1$.

Moreover, (2.2) is satisfied with $b = \psi$ and $a(t) = 1$. To see this, check that for $x > x_0$

$$P\{X > x\} = P\{\log(Y/4) > x, Y > 4\} + P\{Z > x\} \cdot P\{Y \leq 4\} = \frac{1}{4}e^{-x} + \frac{3}{4}g(x) = \frac{1}{\psi^{\leftarrow}(x)},$$

which implies that the quantile function F_X^{\leftarrow} of X (i.e. the generalized inverse of the cdf F_X) is given by $U_X(t) := F_X^{\leftarrow}(1 - 1/t) = \psi(t)$ for sufficiently large t . Hence

$$\begin{aligned} U_X(tx) - U_X(t) &= \log x + \sin \log(\log t + \log x) - \sin \log \log t \\ &= \log x + \sin(\log \log t + o(1)) - \sin \log \log t \\ &\rightarrow \log x \end{aligned}$$

by the uniform continuity of the sine function. This convergence is known to imply (2.2) with $\gamma_X = 0$, $b = U_X = \psi$ and $a(t) = 1$ (see, e.g., de Haan and Ferreira (2006), Theorem 1.1.2).

For $\alpha(t) = 1$ and $\beta(t) = \log t$, $x \in \mathbb{R}$ and $y > -1$, we have eventually

$$\begin{aligned} tP\left\{\frac{X - \beta(t)}{\alpha(t)} > x, \frac{Y - d(t)}{c(t)} > y\right\} &= tP\{\log(Y/4) > x + \log t, Y > t(1 + y)\} \\ &= tP\{Y > \max(4e^x, 1 + y)t\} \\ &= (\max(4e^x, 1 + y))^{-1} \\ &= \mu_{X,Y>}((x, \infty] \times (y, \infty]), \end{aligned}$$

where the measure $\mu_{X,Y>}$ is concentrated on the set $\{(\log(t/4), t-1) | t > 0\}$ with $\mu_{X,Y>} \{(\log(t/4), t-1) | t > r\} = r^{-1}$. This shows $(X, Y) \in CEV^*(\alpha, \beta, c, d, 1, \mu_{X,Y>})$. Note that $\alpha = a$ and that both $\beta(t)$ and $b(t)$ tend to ∞ , but they are not equivalent in the sense of (4.7).

Finally, we show that, in contrast to what is claimed in Proposition 4.1 of Das and Resnick (2011a), convergence (2.4) does not hold for the above choices of the normalizing functions. In view of (2.2) and (2.3), it suffices to show that for some $x \in \mathbb{R}$ and $y > -1$

$$\begin{aligned} tP\left\{\frac{X-b(t)}{a(t)} > x, \frac{Y-d(t)}{c(t)} > y\right\} &= tP\{\log(Y/4) > x + \psi(t), Y > t(1+y)\} \\ &= (\max(4 \exp(x + \sin \log \log t), 1+y))^{-1} \end{aligned}$$

does not converge. This, however, is obvious if one considers the sequences $t_n = \exp(\exp(2\pi n))$ and $\tilde{t}_n = \exp(\exp(2\pi n + \pi/2))$. \square

We conclude this section with an example about the connection between a CEVM and so-called hidden regular variation. Here we consider a random vector $(X, Y) \in [0, \infty)^2$ that is standard regularly varying on $[0, \infty]^2 \setminus \{(0, 0)\}$, i.e.

$$tP\left\{\left(\frac{X}{t}, \frac{Y}{t}\right) \in \cdot\right\} \xrightarrow{v} \mu^*(\cdot) \quad (4.8)$$

vaguely in $[0, \infty]^2 \setminus \{(0, 0)\}$ with a non-degenerate limit measure μ^* satisfying $\mu^*(\{\infty\} \times [0, \infty]) = 0 = \mu^*([0, \infty] \times \{\infty\})$; cf. (2.7). The random vector is called hidden regularly varying if, in addition, for a normalizing function λ_0 with $t = o(\lambda_0(t))$ and a non-degenerate limit measure μ_0 on $(0, \infty]^2$ with $\mu_0(\{\infty\} \times (0, \infty]) = 0 = \mu_0((0, \infty] \times \{\infty\})$,

$$\lambda_0(t)P\left\{\left(\frac{X}{t}, \frac{Y}{t}\right) \in \cdot\right\} \xrightarrow{v} \mu_0(\cdot) \quad (4.9)$$

vaguely in $(0, \infty]^2$, i.e.

$$\lambda_0(t)P\left\{\left(\frac{X}{t}, \frac{Y}{t}\right) \in B\right\} \rightarrow \mu_0(B)$$

for all μ_0 -continuous Borel sets $B \subset (0, \infty)^2$ which are bounded away from both axes. Hidden regular variation is only possible for a vector with asymptotically independent components, i.e. in the case $\mu^*((0, \infty)^2) = 0$, since else the normalizing function $\lambda_0(t)$ is of the order t .

Das and Resnick (2011a) conjectured in Section 5 that (X, Y) is hidden regularly varying if it fulfills (4.8) with $\mu^*((0, \infty)^2) = 0$ and the conditions of a CEVM as well. The following counterexample shows that in general this implication does not hold.

Example 4.6 Let Z_1, Z_2 and B be independent random variables, Z_1 be standard Pareto and B uniformly distributed on $\{1, 2, 3\}$. Moreover, $P(Z_2 > x) = x^{-2}(2 + \sin \log(x))/2$ for all $x \geq 1$, which is easily seen to be a decreasing function. For some $\tau \in (0, 1/2)$, define

$$(X, Y) := \begin{cases} (Z_1^\tau, Z_1) & B = 1, \\ (Z_1, Z_1^\tau) & \text{if } B = 2, \\ (Z_2, Z_2) & B = 3. \end{cases}$$

Let $c(t) = d(t) = t/3$. Then, for all $y > -1$,

$$\begin{aligned} tP\left\{\frac{Y-d(t)}{c(t)} > y\right\} &= \frac{t}{3}P\{Z_1 > t(1+y)/3\} + \frac{t}{3}P\{Z_1^\tau > t(1+y)/3\} + \frac{t}{3}P\{Z_2 > t(1+y)/3\} \\ &\rightarrow (1+y)^{-1} \end{aligned} \quad (4.10)$$

as $t \rightarrow \infty$, since both $P\{Z_1^\tau > t(1+y)/3\} = o(t^{-1})$ and $P\{Z_2 > t(1+y)/3\} = o(t^{-1})$. Therefore, (2.3) holds with $\gamma_Y = 1$.

Furthermore, with $\alpha(t) = \beta(t) = (t/3)^\tau$, we have for all $x \in \mathbb{R}, y > -1$

$$\begin{aligned} & tP\left\{\frac{X - \beta(t)}{\alpha(t)} \leq x, \frac{Y - d(t)}{c(t)} > y\right\} \\ &= tP\left\{\frac{Y - d(t)}{c(t)} > y\right\} - \frac{t}{3}P\left\{Z_1^\tau > \left(\frac{t}{3}\right)^\tau(1+x), Z_1 > \frac{t}{3}(1+y)\right\} \\ &\quad - \frac{t}{3}P\left\{Z_1 > \left(\frac{t}{3}\right)^\tau(1+x), Z_1^\tau > \frac{t}{3}(1+y)\right\} - \frac{t}{3}P\left\{Z_2 > \left(\frac{t}{3}\right)^\tau(1+x), Z_2 > \frac{t}{3}(1+y)\right\} \\ &\rightarrow (1+y)^{-1} - \min\left(\left((1+x)^+\right)^{-1/\tau}, (1+y)^{-1}\right) \\ &= \mu_{X,Y>}((-\infty, x] \times (y, \infty)) \end{aligned}$$

as $t \rightarrow \infty$. Here $\mu_{X,Y>}$ denotes the measure that is concentrated on $\{(t^\tau - 1, t - 1) | t > 0\}$ such that $\mu_{X,Y>}(\{(t^\tau - 1, t - 1) | t > r\}) = r^{-1}$ for all $r > 0$. Hence $(X, Y) \in CEV^*(\alpha, \beta, c, d, 1, \mu_{X,Y>})$. By symmetry of the construction, it follows immediately that (2.2) holds and that (Y, X) follows a CEVM as well.

Next, note that for $x, y > 0$

$$\begin{aligned} tP\{X > tx \text{ or } Y > ty\} &= tP\{X > tx\} + tP\{Y > ty\} - \frac{t}{3}P\{Z_1^\tau > tx, Z_1 > ty\} \\ &\quad - \frac{t}{3}P\{Z_1 > tx, Z_1^\tau > ty\} - \frac{t}{3}P\{Z_2 > t \max(x, y)\} \\ &\rightarrow x^{-1}/3 + y^{-1}/3 \\ &=: \mu^*([0, \infty]^2 \setminus ([0, x] \times [0, y])), \end{aligned}$$

i.e. (4.8) holds. Moreover, according to (4.10), $\mu^*([0, \infty] \times (y, \infty]) = \lim_{t \rightarrow \infty} P\{Y > ty\} = y^{-1}/3$, and likewise $\mu^*((x, \infty] \times [0, \infty]) = \lim_{t \rightarrow \infty} tP\{X > tx\} = x^{-1}/3$. Hence,

$$\mu^*((x, \infty]^2) = \mu^*((x, \infty] \times [0, \infty]) + \mu^*([0, \infty] \times (x, \infty]) - \mu^*([0, \infty]^2 \setminus [0, x]^2) = 0$$

for all $x > 0$. Thus $\mu^*([0, \infty]^2) = 0$, too, i.e. μ^* is concentrated on the axes with $\mu^*((r, \infty) \times \{0\}) = \mu^*({0} \times (r, \infty)) = (3r)^{-1}$ for all $r > 0$.

However, the vector (X, Y) is not hidden regularly varying. To show this, suppose that there exists a function λ_0 such that (4.9) holds for a non-degenerate limit measure μ_0 . Then there exists $x > 0$ such that $\mu_0((x, \infty)^2) > 0$ and $(x, \infty)^2$ is a μ_0 -continuity set, because μ_0 is homogeneous (see Resnick (2007), Section 9.4.1). Therefore,

$$\begin{aligned} & \lambda_0(t)P\{X > tx, Y > tx\} \\ &= \frac{\lambda_0(t)}{3} [P\{Z_1^\tau > tx, Z_1 > tx\} + P\{Z_1 > tx, Z_1^\tau > tx\} + P\{Z_2 > tx\}] \\ &= \frac{\lambda_0(t)}{3} \left[2(tx)^{-1/\tau} + (tx)^{-2}(2 + \sin \log(tx))/2 \right] \\ &= \frac{\lambda_0(t)}{3} (tx)^{-2}(2 + \sin \log(tx))(1 + o(1))/2 \\ &\rightarrow \mu_0((x, \infty]^2), \end{aligned}$$

which implies that $\lambda_0(t)t^{-2}(2 + \sin \log(tx))$ converges to a finite positive constant (depending on x). However, convergence (4.9) implies that λ_0 is a regularly varying function; cf. Resnick (2007), Section 9.4.1. Because $t^{-2}(2 + \sin \log(tx))$ is not a regularly varying function of t , this leads to a contradiction. \square

In this example, the extremal behavior of the vector (X, Y) is dominated by the regularly varying random variable Z_1 in the following sense: if at least one component of (X, Y) is large, then this

is typically due to the fact that Z_1 is large. However, if we are interested in extremal events when both components of (X, Y) exceed the same threshold (or, more general, thresholds of the same order of magnitude), then this happens typically because the random variable Z_2 is large, which is not regularly varying. Therefore, the assumptions of hidden regular variation are not met.

5 Conclusion

The discussions of the preceding sections show that the CEVM exhibits specific features which are not shared by other extreme value models. These aspects have not been appropriately addressed in the previous literature. (For instance, Das and Resnick (2011b) state that “The CEV model primarily differs from the multivariate extreme value model in the domain of attraction condition.”) This may be due to the fact that in the standard case where (X, Y) attains values in $[0, \infty)^2$ and both components are divided by the same factor, CEVM, classical extreme value models and the additional convergence (4.9) considered in models with hidden regular variation can all be considered special cases of a general concept of regular variation on cones; see e.g. Das et al. (2013).

However, as the above examples demonstrate, even for vectors (X, Y) that satisfy both the assumptions of a CEVM and of a model from classical MEVT, in general these models focus on very different information about the behavior of X for large values of Y . This is partly due to the fact that, after a linear normalization, the classical model does not discriminate values of X below q_{γ_X} , whereas in the limit a CEVM may retain more information about the not necessarily extreme component in that region. A more fundamental difference, though, is that in classical bivariate extreme value models the normalizing functions for X are completely determined by the marginal tail behavior, whereas in a CEVM they must be adapted to the overall behavior of X for large values of Y , which need not correspond to large values of X (or $-X$). While this allows for a more flexible relationship between X and Y , in some situations all normalizing functions that meet the conditions of Definition 2.1 (including (ii*)) result in a very coarse description of the behavior of X .

This problem is most easily illustrated by the example of a mixture. Suppose two random vectors (X_1, Y) and (X_2, Y) with the same second component both fulfill a CEVM. If we first pick one of the vectors at random and consider the conditional extreme value behavior of the resulting mixture model $P\{(X, Y) \in \cdot\} = (P\{(X_1, Y) \in \cdot\} + P\{(X_2, Y) \in \cdot\})/2$, then in some cases the approximating model specified by the limit measure $\mu_{X, Y>}$ and the normalizing functions neither retain the information encoded in the CEVM for (X_1, Y) nor the information given by the CEVM for (X_2, Y) .

Example 5.1 Let Y be regularly varying with index $-1/\gamma_Y$, i.e. (2.3) holds with $\gamma_Y > 0$, and let $X_i = \omega_i - g_i(Y)$, $i \in \{1, 2\}$, for some $\omega_1 < \omega_2$ and some invertible functions $g_i > 0$ which are regularly varying with negative index $-\tau_i$. Then the inverse g_i^{\leftarrow} is regularly varying at 0 with index $-1/\tau_i$ and with $U_Y(t) := F_Y^{\leftarrow}(1 - 1/t)$ one has $g_i^{\leftarrow}(-g_i(U_Y(t))x) = |x|^{-1/\tau_i} U_Y(t)(1 + o(1))$ for $x \leq 0$. Thus, for $y > q_{\gamma_Y} = -1/\gamma_Y$ and $x \leq 0$,

$$\begin{aligned} & tP\left\{\frac{X_i - \omega_i}{g_i(U_Y(t))} > x, \frac{Y - U_Y(t)}{\gamma_Y U_Y(t)} > y\right\} \\ &= tP\{g_i(Y) < -g_i(U_Y(t))x, Y > U_Y(t)(1 + \gamma_Y y)\} \\ &= tP\{Y > \max(g_i^{\leftarrow}(-g_i(U_Y(t))x), U_Y(t)(1 + \gamma_Y y))\} \\ &= t \min(1 - F_Y(|x|^{-1/\tau_i} U_Y(t)(1 + o(1))), 1 - F_Y(U_Y(t)(1 + \gamma_Y y))) \\ &\rightarrow \min(|x|^{1/(\tau_i \gamma_Y)}, (1 + \gamma_Y y)^{-1/\gamma_Y}) \\ &= \mu_{X, Y>}((x, \infty] \times (y, \infty]), \end{aligned}$$

as $t \rightarrow \infty$. Here $\mu_{X,Y>}$ denotes the measure that is concentrated on $\{(-t^{-\tau_i\gamma_Y}, (t^{\gamma_Y} - 1)/\gamma_Y) | t > 0\}$ such that $\mu_{X,Y>}\{(-t^{-\tau_i\gamma_Y}, (t^{\gamma_Y} - 1)/\gamma_Y) | t > r\} = r^{-1}$ for all $r > 0$. Therefore, $(X_i, Y) \in \text{CEV}^*(g_i \circ U_Y, \omega_i, \gamma_Y U_Y, U_Y, \gamma_Y, \mu_{X,Y>})$.

The mixture also follows a CEVM, because

$$\begin{aligned} tP\left\{X \leq x, \frac{Y - U_Y(t)}{\gamma_Y U_Y(t)} > y\right\} &= \frac{t}{2} \sum_{i=1}^2 P\{g_i(Y) \geq \omega_i - x, Y > U_Y(t)(1 + \gamma_Y y)\} \\ &\rightarrow \frac{1}{2} \sum_{i=1}^2 1_{[\omega_i, \infty)}(x)(1 + \gamma_Y y)^{-1/\gamma_Y} \end{aligned}$$

for all $x \in \mathbb{R}$ and $y > q_{\gamma_Y}$. However, only ω_1 and ω_2 can be recovered from this CEVM, whereas the more detailed behavior of X near these points for large values of Y , which is specified by the CEVM for (X_1, Y) and (X_2, Y) , is lost.

In contrast, the classical bivariate extreme value model for the mixture $P\{(X, Y) \in \cdot\}$ keeps all the information from the model for the component $P\{(X_2, Y) \in \cdot\}$ with the larger point of accumulation $\omega_2 > \omega_1$, while the information about the other component is lost. To see this, note that for all $x < 1/(\tau_2\gamma_Y)$, $y > q_{\gamma_Y}$ and sufficiently large t

$$\begin{aligned} tP\left\{\frac{X - (\omega_2 - g_2(U_Y(t/2)))}{\gamma_Y \tau_2 g_2(U_Y(t/2))} > x \text{ or } \frac{Y - U_Y(t)}{\gamma_Y U_Y(t)} > y\right\} \\ &= \frac{t}{2} (P\{g_1(Y) < \omega_1 - \omega_2 + g_2(U_Y(t/2))(1 - \gamma_Y \tau_2 x) \text{ or } Y > U_Y(t)(1 + \gamma_Y y)\} \\ &\quad + P\{g_2(Y) < g_2(U_Y(t/2))(1 - \gamma_Y \tau_2 x) \text{ or } Y > U_Y(t)(1 + \gamma_Y y)\}) \\ &= \frac{t}{2} (P\{Y > U_Y(t)(1 + \gamma_Y y)\} \\ &\quad + P\{Y > \min(U_Y(t/2)(1 - \gamma_Y \tau_2 x)^{-1/\tau_2}(1 + o(1)), U_Y(t)(1 + \gamma_Y y)\}) \\ &\rightarrow \frac{1}{2}(1 + \gamma_Y y)^{-1/\gamma_Y} + \max\left(\frac{1}{2}(1 - \gamma_Y \tau_2 x)^{1/(\gamma_Y \tau_2)}, \frac{1}{2}(1 + \gamma_Y y)^{-1/\gamma_Y}\right) \\ &= \mu([\infty, \infty] \times [q_{\gamma_Y}, \infty]) \setminus ([\infty, x] \times [q_{\gamma_Y}, y]). \end{aligned}$$

where μ denotes the measure concentrated on $\{(\gamma_Y^{-1}\tau_2^{-1}(1 - t^{-\tau_2}), (t^{2-\gamma_Y} - 1)/\gamma_Y) | t > 0\} \cup \{(-\infty, (t-1)/\gamma_Y) | t > 0\}$ such that $2\mu(\{(-\infty, (t-1)/\gamma_Y) | t > r\}) = \mu(\{(\gamma_Y^{-1}\tau_2^{-1}(1 - t^{-\tau_2}), (t^{2-\gamma_Y} - 1)/\gamma_Y) | t > r\}) = r^{-1/\gamma_Y}$, $r > 0$. \square

Similar effects occur if, for large values of Y , the random variable X may attain values in the neighborhood of different points (possibly including ∞ and $-\infty$). The behavior of X in the neighborhood of each finite point can be captured by a CEVM only if the conditions (ii) and (ii*) of Definition 2.1, which rule out mass on $\{\pm\infty\} \times \bar{E}^{(\gamma_Y)}$, are dropped. Then in general the limit measure is not unique anymore (up to shift and scaling). In Example 5.1, one would get three fundamentally distinct limits: the one derived in the example, and two further limit measures which describe the behavior of X near ω_1 and ω_2 , respectively, on a more detailed scale. The two latter ones, which arise if one uses e.g. the normalizing functions $\alpha_i(t) = g_i(U_Y(t))$ and $\beta_i(t) = \omega_i$, have mass on $\{\infty\} \times \bar{E}^{(\gamma_Y)}$ and $\{-\infty\} \times \bar{E}^{(\gamma_Y)}$, respectively.

In general, of course, there may be arbitrarily many (even an infinite countable number of) accumulation points of X for large values of Y , and hence of possible limits. Moreover, even if there is only one point of accumulation, different multiplicative normalizing functions α may lead to different limits if mass on $\{\pm\infty\} \times \bar{E}^{(\gamma_Y)}$ is allowed; see e.g. Example 5.4 of Das et al. (2013). So while in some cases omitting the condition (ii) in Definition 2.1 allows for a more detailed analysis of X for large values of Y , one pays the price of an enormously increased complexity.

An alternative ad-hoc solution in the case when X is concentrated on several accumulation points ω_i may be to separately consider a CEVM (incl. condition (ii*)) for X restricted to a neighborhood N_i of each of these points, that is, to require vague convergence of $tP(\{((X - \beta_i(t))/\alpha_i(t), (Y - d(t))/c(t)) \in \cdot\} \cap \{X \in N_i\})$ for each i . One would then have two layers of CEVM. The first is obtained by the present definition. If the marginal measure corresponding to the first coordinate is discrete (or, more generally, has a discrete part), then in addition one may consider ‘localized’ CEVM models on suitable neighborhoods of each point of mass. Although such an approach would yield a much more complete picture, it seems practically feasible only in those cases when the number of accumulation points is small.

Acknowledgements This project was partly supported by the German Research Foundation DFG, Grant no JA 2160/1. We thank Jutta Vollstädt for fruitful discussions. The constructive and helpful remarks by a referee and an associate editor led to an improvement of the presentation.

References

- Beirlant, J., Goegebeur, Y., Segers, J., Teugels, J., de Waal, D. and Ferro, C. (2004). *Statistics of Extremes*. Wiley.
- Das, B., Mitra, A. and Resnick, S.I. (2013). Living on the multidimensional edge: seeking hidden risks using regular variation. *Adv. in Appl. Probab.* **45**, 139–163.
- Das, B. and Resnick, S.I. (2011a). Conditioning on an extreme component: Model consistency with regular variation on cones. *Bernoulli* **17**, 226–252.
- Das, B. and Resnick, S.I. (2011b). Detecting a conditional extreme value model. *Extremes* **14**, 29–61.
- de Haan, L. and Ferreira, A. (2006). *Extreme Value Theory*. Springer, New York.
- Heffernan, J.E. and Resnick, S.I. (2007). Limit laws for random vectors with an extreme component. *Ann. Appl. Probab.* **17**, 537–571.
- Heffernan, J.E. and Tawn, J.A. (2004). A conditional approach for multivariate extreme values. *J. R. Statist. Soc. B* **66**, 497–546.
- Hult, H. and Lindskog, F. (2006). Regular variation for measures on metric spaces. *Publ. Inst. Math. (Beograd) (N.S.)* **80**, 121–140.
- Ledford, A.W. and Tawn, J.A. (1997). Modelling dependence within joint tail regions. *J. R. Statist. Soc. B* **59**, 475–499.
- Lindskog, F., Resnick, S.I. and Roy, J. (2014). Regularly varying measures on metric spaces: hidden regular variation and hidden jumps. *Probab. Surveys* **11**, 270–314.
- Resnick, S.I. (2007). *Heavy-tail Phenomena*. Springer, New York.
- Resnick, S.I. and Zeber, D. (2014). Transition kernels and the conditional extreme value model. *Extremes* **17**, 263–287.