PARTITIONING EDGE-COLOURED INFINITE COMPLETE BIPARTITE GRAPHS INTO MONOCHROMATIC PATHS

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ABSTRACT. In 1978, Richard Rado showed that every edge-coloured complete graph of countably infinite order can be partitioned into monochromatic paths of different colours. He asked whether this remains true for uncountable complete graphs and a notion of *generalised paths*. In 2016, Daniel Soukup answered this in the affirmative and conjectured that a similar result should hold for complete bipartite graphs with bipartition classes of the same infinite cardinality, namely that every such graph edge-coloured with r colours can be partitioned into $2r - 1$ monochromatic generalised paths with each colour being used at most twice.

In the present paper, we give an affirmative answer to Soukup's conjecture.

1. INTRODUCTION

Throughout this paper, the term *colouring* always refers to edge colourings of graphs with finitely many colours.

In the 1970s, Erdős proved (unpublished) that every 2-coloured complete graph of countably infinite order, i.e. every 2-coloured $K_{\aleph_0}$, can be partitioned into monochromatic paths of different colours, where 'path' means either a finite path or a one-way infinite ray. Rado subsequently extended Erdős result to any finite number of colours [7, Theorem 2].

In the same paper, Rado then asked whether a similar result holds for all infinite complete graphs, even the uncountable ones. Clearly, it is not possible to partition such a graph into finitely many 'usual' paths, as graph-theoretic paths and rays are inherently countable. Hence, Rado introduced the following notion of *generalised path*: A generalised path is a graph P together with a well-order $\prec$ on $V(P)$ (called the *path order* on P) satisfying that the set $\{w \in N(v) \colon w \prec v\}$ of down-neighbours of v is cofinal below v for every vertex $v \in V(P)$, i.e. for every $v' \prec v$ there is a neighbour w of v with $v' \preceq w \prec v$ (cf. Figure 1).
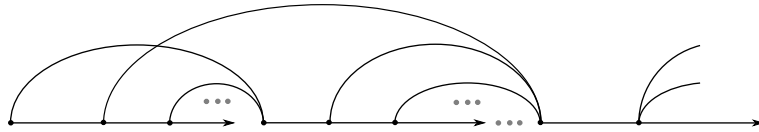

FIGURE 1. A generalised path.

In particular, every successor element is adjacent to its predecessor in the well-order. Calling such a graph P a 'generalised path' is justified by the fact that between any two vertices $v \prec w$ of P there exists a *finite* path from v to w strictly increasing with respect to $\prec$, see e.g. [3, Observation 5.2]. If the situation is clear, we write P instead of $(P, \prec)$ and treat P as a graph. By $\Lambda(P, \prec) = \Lambda(P)$ we denote the limit elements of the well-order $(P, \prec)$. If necessary, the path-order $\prec$ on $V(P)$ will be referred to as $\prec_P$. If $v, v' \in P$, then we denote by (v, v') and $[v, v']$ the open and closed intervals with respect to $\prec$, and by $[v, v + \omega)$ the ray of P starting at v compatible with the path order. Note that a one-way infinite ray can be viewed quite naturally as a generalised path of order type ω, and conversely, every generalised path of order type ω contains a spanning one-way infinite ray. Thus, partitioning a graph into monochromatic generalised path of order type ω is equivalent to partitioning it into monochromatic rays.

From now on, the term *path* is used in the extended sense of a generalised path.

Elekes, Soukup, Soukup and Szentmiklòssy [3] have recently answered a special case of Rado's question for $\aleph_1$-sized complete graphs and two colours in the affirmative. Shortly after, Soukup [9] gave a complete answer to Rado's question for any finite number of colours and complete graphs of arbitrary infinite cardinality.

Theorem 1 (Soukup, [9, Theorem 7.1])**.** *Let r be a positive integer. Every r-edge-coloured complete graph of infinite order can be partitioned into monochromatic generalised paths of different colours.*

In [9, Conjecture 8.1], Soukup conjectures that a similar result holds for complete bipartite graphs, namely that every r-coloured complete bipartite graph with bipartition classes of cardinality $\kappa \geq \aleph_0$ can be partitioned into $2r - 1$ monochromatic generalised paths, and has proven his conjecture in the countable case $\kappa = \aleph_0$ [8, Theorem 2.4.1]. If true, this bound would be best possible in the sense that there are r-colourings of $K_{\kappa, \kappa}$ for which the graph cannot be partitioned into $2r - 2$ monochromatic paths, see [8, Theorem 2.4.1].

We remark that Soukup's conjecture is inspired by the corresponding conjecture in the finite case, due to Prokovskiy [6, Conjecture 4.5]. In contrast to the infinite case, the finite conjecture is only known for two colours [5, p. 169 (footnote)].

The main result of this paper is to prove Soukup's conjecture for all uncountably cardinalities and any (finite) number of colours.

Theorem 2. *Let r be a positive integer. Every r-edge-coloured complete bipartite graph with bipartition classes of the same infinite cardinality can be partitioned into $2r - 1$ monochromatic generalised paths with each colour being used at most twice.*

The first uncountable case of Theorem 2, where the bipartition classes have size $\aleph_1$, was proved by the first author in his Master's thesis [1]. In this paper, we extend these ideas to give a proof for all uncountable cardinalities.

Our proof relies on the methods developed by Soukup in his original paper [9]. However, we re-introduce in this paper the new, helpful notion of X-*robust* paths from [1]—generalised paths which are resistant against the deletion of vertices from

X. After introducing such paths, we will state in Section 2 three high-level results relying on this new notion, and then provide a proof of Theorem 2 from these auxiliary results. In fact, our discussion will also lead to a new, conceptually simpler closing argument for a proof of Soukup's Theorem 1.

In Sections 3 and 4, we then provide proofs of the auxiliary results. For the second of these auxiliary results, to be proved in Section 3, we need to strengthen a result by Soukup [9, §5] to give the statement that any edge-coloured complete bipartite graph with bipartition classes (A, B) of cardinality $\kappa > \aleph_0$ contains a monochromatic path P of order type κ in colour k (say) covering a large subsets $X \subseteq A$ which itself is κ-star-linked in colour k, where it is precisely the κ-*star-linked-property* (to be defined below) which is new. We remark that while our statement is slightly stronger, our proof very much relies on Soukup's proof [9, §5] and does not give an independent proof of Soukup's result. A discussion how one obtains the strengthened version of Soukup's result is provided in Section 5.

Finally, in Section 4 we prove our third auxiliary result. This part contains a crucial new idea how to directly construct an X-robust path Q of order type $\kappa > \aleph_0$ with $X \in [V(Q)]^\kappa$ from a given generalised path P with the star-linked property as above, using nothing but countable combinatorics and avoiding intricate set theoretical arguments using elementary submodels as employed in [9] and [1].

Notation. For graph theoretic notation we follow the text book *Graph Theory* [2] by Diestel. For a natural number $n \in \mathbb{N}$ we write $[n] = \{1, 2, \ldots, n\}$ and if $m \leq n$, we write $[m, n] = \{m, m+1, \ldots, n\}$.

Let $G = (V, E)$ be a graph, $r \geq 1$ and $k \in [r]$. An *r-edge-colouring* (or simply *r-colouring*) of G is a map $c \colon E \to [r]$. A path $P \subseteq G$ is *monochromatic* (in colour k with regard to the colouring c) if P is also a path in the graph induced by the edges of colour k, i.e. if P is a path in $G[\bigcup c^{-1}(k)]$. More generally, suppose that $\mathcal{P}$ is a graph property. We say that G *has property $\mathcal{P}$ in colour k* if $G[\bigcup c^{-1}(k)]$ has property $\mathcal{P}$. For a vertex v of G we write $N(v, k)$ for the neighbourhood of v in $G[\bigcup c^{-1}(k)]$. As a shorthand, we also write $N(v, \neq k) := N(v) \setminus N(v, k)$ for the neighbourhood of v in all colours but k. Let $A \subseteq V$. The common neighbourhood $\bigcap \{N(v) \colon v \in A\}$ of vertices in A is written as $N[A]$. The common neighbourhood of A in colour k is written as $N[A, k]$. For a cardinal κ, we say that A is κ-*star-linked in B*, if $N[F] \cap B$ has cardinality κ for every finite $F \subseteq A$. In the case where $B = V(G)$ we simply say that A is κ-star-linked.

When talking about *partitions of G* we always mean vertex partitions and we allow empty partition classes. If $A, B \subseteq V(G)$ are disjoint sets of vertices, then $G[A, B]$ denotes the bipartite graph on $A \cup B$ given by all the edges between A and B.

For a set X, we write $[X]^\kappa = \{Y \subseteq X \colon |Y| = \kappa\}$ and $[X]^{<\kappa} = \{Y \subseteq X \colon |Y| < \kappa\}$.

2. A HIGH-LEVEL PROOF OF THE MAIN RESULT

The aim of this section is to give an overview of the proof of Theorem 2. We shall start with a rough idea, inspired by Soukup's work in [9, Theorem 7.1]. After

that, we present three main ingredients for our proof of Theorem 2: Lemma 4, Lemma 5 and Lemma 7. For the moment, we will skip the latter two and discuss them below in Section 3 and Section 4. We conclude this section with a proof of Theorem 2 and a proof of Theorem 1—also based on the three lemmas.

2.1. **A rough outline.** First, let us have a look at an important idea in Soukup's proof of Theorem 1. In [9, Lemma 4.6], Soukup provides some *conditions* which guarantee the existence of a spanning generalised path in a graph. Let us refer to these conditions by (†). Let κ be an infinite cardinal and $G = (V, E)$ the complete graph of order κ. Suppose that the edges of G are coloured with $r \geq 1$ many colours. In [9, Claim 7.1.2], Soukup shows that one can find sets $X \subseteq W \subseteq V$ and a colour $k \in [r]$, such that

(1) $G[W \setminus X']$ satisfies (†) in colour k for every $X' \subseteq X$, and
(2) $V \setminus W$ is covered by disjoint monochromatic paths of different colours not equal to k in the graph $G[V \setminus W, X]$.

Once such W, X and k are found, we just have to find $r - 1$ disjoint monochromatic paths of different colours $\neq k$ covering $V \setminus W$ in $G[V \setminus W, X]$ as in (2), let $X' \subset X$ be the vertices of X covered by these $r - 1$ paths, and apply (1) to guarantee the existence of a monochromatic path in colour k disjoint from all previous ones and covering the remaining vertices.

Whilst it is difficult to work with the conditions from (†) in the bipartite setting directly, the use of (†) in (1) motivates the following definition:

Definition 3. Let P be a path and $X \subseteq V(P)$. We say that P is X-*robust* iff for every $X' \subseteq X$ the graph $P - X'$ admits a well-order for which $P - X'$ is a path of the same order type as P.

Our strategy for the bipartite case can now by summarised as follows. Let κ be an infinite cardinal and $G = (V, E)$ the complete bipartite graph with bipartition classes of cardinality κ, where the edges of G are coloured with $r \geq 1$ many colours. Assume that we find $X \subseteq W \subseteq V$ and a colour $k \in [r]$, such that

(1′) $G[W]$ has a spanning X-robust path in colour k, and
(2′) $V \setminus W$ is covered by $2r - 2$ disjoint monochromatic paths in the graph $G[V \setminus W, X]$ in colours not equal to k with every colour appearing at most twice.

Then it is clear that we can complete a proof of Theorem 2 in a similar way as above.

2.2. **The three ingredients.** To prove our main theorem, we shall need the following three ingredients.

Theorem 4 (Soukup, [9, Thm 6.2])**.** *Let G be an infinite bipartite graph with bipartition classes A and B. Suppose that $|A| \leq |B|$ and that $|B \setminus N(a)| < |B|$ for every vertex $a \in A$. Then for every finite edge colouring of G, there are disjoint monochromatic paths of different colours in G covering A.*
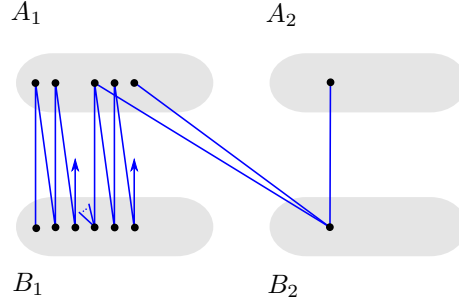
A_1 A_2

B_1 B_2

FIGURE 2. The colour 1 is indicated blue in the figure.

Proof. By the argument in [9, p. 271, l. 17-20], the theorem follows from [9, Theorem 6.2].   □

The next main lemma, which is a strengthening of a similar result by Soukup [9, §5], helps to find a monochromatic path P which has some desirable additional properties.

Lemma 5. *Let κ be an infinite cardinal and G the complete bipartite graph with bipartition classes A, B both of cardinality κ. Suppose that $c \colon E(G) \to [r]$ is a colouring of G with $r \geq 1$ many colours. Then there are disjoint sets $A_1, A_2 \in [A]^\kappa$, $B_1, B_2 \in [B]^\kappa$ such that (up to renaming the colours):*

- *$G[A_1, B_1]$ has a spanning path P of order type κ in colour 1 all whose limits are contained in B_1, and*
- *$A_1 \sqcup A_2$ is κ-star-linked in B_2 in colour 1. (cf. Figure 2)*

Our final ingredient converts the path P from above into a new path Q that has two additional properties: first, Q will be X-robust for some large X, and secondly, Q will be able to additionally cover certain highly inseparable sets of vertices.

Definition 6 (cf. Diestel, [2, p. 354])**.** *Let G be a graph and κ a cardinal. A set $U \subseteq V(G)$ of vertices is $<\kappa$-inseparable if distinct vertices $v, w \in U$ cannot be separated by less than κ many vertices, i.e. v and w are contained in the same component of $G - W$ for every $W \in [V(G) \setminus \{v, w\}]^{<\kappa}$.*

Lemma 7. *Let κ be an uncountable cardinal and G a bipartite graph with bipartition classes A, B both of size κ. Suppose there are disjoint sets $A_1, A_2 \in [A]^\kappa$, $B_1, B_2 \in [B]^\kappa$ such that*

- *$G[A_1, B_1]$ has a spanning path P of order type κ with $\Lambda(P) \subset B_1$, and*
- *$A_1 \sqcup A_2$ is κ-star-linked in B_2.*

Then there is a set $X \in [A_2]^\kappa$ and an X-robust path Q covering $A_1 \sqcup A_2$ with $\Lambda(P) = \Lambda(Q)$. Moreover, if $C \subseteq (A \setminus A_1) \sqcup (B \setminus \Lambda)$ covers A_2 and is $<\kappa$-inseparable in $G[A \setminus A_1, B \setminus \Lambda]$, then Q can be chosen to cover C.

2.3. **Combining the ingredients.** Our three main ingredients can be applied to yield a proof of Theorem 2 as follows:

Theorem 2. *Let r be a positive integer. Every r-edge-coloured complete bipartite graph with bipartition classes of the same infinite cardinality can be partitioned into $2r - 1$ monochromatic generalised paths with each colour being used at most twice.*

Proof of Theorem 2. Let κ be an infinite cardinal and G the complete bipartite graph with bipartition classes A, B both of cardinality κ. Suppose that $c\colon E(G) \to [r]$ is a colouring of G. Since the countable case has been solved in [8, Theorem 2.4.1] already, we may assume that κ is uncountable.

We will construct a partition $\mathcal{A} = \{A_1, \ldots, A_4\}$ of A and a partition $\mathcal{B} = \{B_1, \ldots, B_4\}$ of B such that, up to renaming the colours,

(i) $G[A_1, B_1]$ has a spanning path P of order type κ in colour 1 with $\Lambda(P) \subset B_1$, and $|A_2| = \kappa$,

(ii) $A_1 \sqcup A_2$ is κ-star-linked in B_2 in colour 1,

(iii) $A_2 \sqcup A_3$ is $<\kappa$-inseparable in $G[A_2 \sqcup A_3, B_2 \sqcup B_3]$ in colour 1, and

(iv) $A_4 \sqcup B_4$ can be partitioned into $r - 1$ monochromatic paths $P_2, \ldots, P_r$ in $G[A_4, B_4]$ with distinct colours in $[2, r]$.
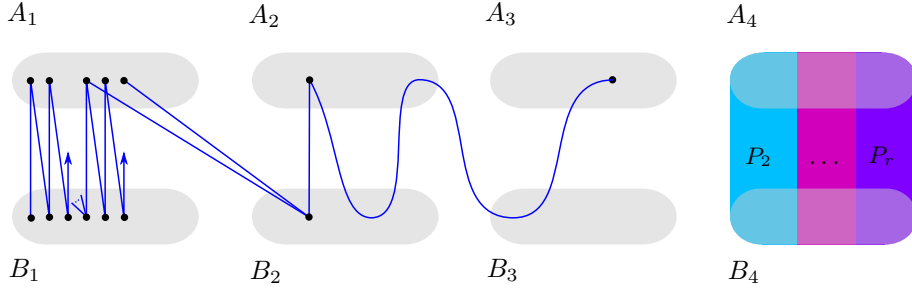

FIGURE 3. The colour 1 is indicated blue in the figure.

Let us first see how to complete the proof with these partitions established:

Let C be the set of vertices with $A \cap C = A_2 \cup A_3$ and where $B \cap C$ consist of those vertices in $B \setminus (\Lambda \cup B_4)$ that send κ many edges in colour 1 to A_2, and observe that (iii) implies that C is $<\kappa$-inseparable in $G[A \setminus (A_1 \cup A_4), B \setminus (\Lambda \cup B_4)]$ in colour 1. Hence, by (i) and (ii), we may apply Lemma 7 in $G[A \setminus A_4, B \setminus B_4]$ induced by the edge of colour 1 in order to obtain a set $X \in [A_2]^\kappa$ and an X-robust, monochromatic path Q in colour 1 with limits $\Lambda = \Lambda(Q) = \Lambda(P)$, covering $A_1 \cup A_2 \cup A_3 \cup C \cup \Lambda$.

Next, note that since $X \subseteq A_2$, it follows by choice of C that $|X \setminus N(b, \neq 1)| = |X \cap N(b, 1)| < \kappa = |X|$ for every vertex b in $B \setminus (Q \cup B_4)$. Therefore, we may apply Lemma 4 to the bipartite graph $G[B \setminus (Q \sqcup B_4), X]$ with the edges in colour

$\neq 1$ to obtain disjoint monochromatic paths $P_{r+1}, \ldots, P_{2r-1}$ with different colours in $[2, r]$ covering $B \setminus (Q \sqcup B_4)$.

Let P_1 be the path that results by deleting $X' = X \cap \left(\bigcup_{i=r+1}^{2r-1} P_i \right)$ from Q, using that Q is X-robust. Together with the paths $P_2, \ldots, P_r$ provided by (iv), we have found a partition of G into $2r-1$ disjoint monochromatic paths $P_1, \ldots, P_{2r-1}$ using every colour at most twice.

To complete the proof, it remains to construct the partitions $\mathcal{A}$ and $\mathcal{B}$.

Claim. There are disjoint sets $A_1, A_2 \in [A]^\kappa$ and $B_1, \tilde{B}_2 \in [B]^\kappa$ such that (up to renaming the colours)

- $G[A_1, B_1]$ has a spanning path P of order type κ in colour 1 all whose limits are contained in B_1 and
- $A_1 \sqcup A_2$ is κ-star-linked in $\tilde{B}_2$ in colour 1.

Proof. Apply Lemma 5 to the graph G and the colouring c. $\Diamond$

Claim. There is a partition $\{B_2, \tilde{B}_3\}$ of $\tilde{B}_2$ such that

- $A_1 \sqcup A_2$ is κ-star-linked in B_2 in colour 1 and
- $G[A_1 \sqcup A_2, \tilde{B}_3]$ has a perfect matching in colour 1.

Proof. Take an increasing cofinal sequence of regular cardinals $(\kappa_\alpha)_{\alpha < \mathrm{cf}(\kappa)}$ in κ and write $A_1 \sqcup A_2$ as an increasing union of sets $\{A_\alpha : \alpha < \kappa\}$ with $|A_\alpha| = \kappa_\alpha$. Simultaneously define in $\mathrm{cf}(\kappa)$ many steps increasing sets $\{B'_\alpha : \alpha < \mathrm{cf}(\kappa)\}$ and $\{B''_\alpha : \alpha < \mathrm{cf}(\kappa)\}$ such that

- B'_α and B''_α are disjoint subsets of $\tilde{B}_2$ with $|B'_\alpha| = |B''_\alpha| = \kappa_\alpha$ for $\alpha < \mathrm{cf}(\kappa)$,
- $G[A_\alpha, B'_\alpha]$ has a perfect matching in colour 1 and
- $|N_G[F, 1] \cap B''_\alpha| = \kappa_\alpha$ for every finite $F \subseteq A_1 \sqcup A_2$.

Letting $\tilde{B}_3 := \bigcup\{B'_\alpha : \alpha < \mathrm{cf}(\kappa)\}$ and $B_2 := \tilde{B}_2 \setminus \tilde{B}_3$ completes the construction. $\Diamond$

Let A_3 consist of those vertices in $A \setminus (A_1 \sqcup A_2)$ that send κ many edges in colour 1 to $\tilde{B}_3$. Note that since A_2 is κ-star linked in B_2 in colour 1, it follows that $A_2 \sqcup A_3$ is $<\kappa$-inseparable in $G[A_2 \sqcup A_3, B_2 \sqcup \tilde{B}_3]$ in colour 1.

Claim. There is a partition $\{\hat{B}_3, \tilde{B}_4\}$ of $\tilde{B}_3$ such that

- $A_2 \sqcup A_3$ is $<\kappa$-inseparable in $G[A_2 \sqcup A_3, B_2 \sqcup \hat{B}_3]$ in colour 1, and
- $\tilde{B}_4$ has cardinality κ.

Proof. If A_3 is empty, then just take a balanced partition $\{\hat{B}_3, \tilde{B}_4\}$ of $\tilde{B}_3$. Otherwise, fix a sequence $(a_\alpha)_{\alpha < \kappa}$ of vertices in A_3 such that every vertex in A_3 appears κ many times. Then fix a vertex in $N_G(a_\alpha, 1) \cap \tilde{B}_3$ for $\hat{B}_3$ and another in $N_G(a_\alpha) \cap \tilde{B}_3$ for $\tilde{B}_4$ for every $\alpha < \kappa$ (all distinct). This can be done recursively in κ many steps using that every vertex in A_3 sends κ many edges in colour 1 to $\tilde{B}_3$. It is easy to check that sets $\hat{B}_3$ and $\tilde{B}_4$ that arise in this manner fulfil our requirements. $\Diamond$

The last partition class of $\mathcal{A}$ is just $A_4 := A \setminus (A_1 \sqcup A_2 \sqcup A_3)$. Applying Lemma 4 to the spanning subgraph of $G[A_4, \tilde{B}_4]$ induced by the colours $2, \ldots, r$ (and the induced

colouring) gives rise to disjoint monochromatic paths $P_2, \ldots, P_r$ of different colours in $[2, r]$. Let $B_4 := \bigcup \{B \cap P_i : i \in [2, r]\}$ and $B_3 := B \setminus (B_1 \sqcup B_2 \sqcup B_4)$.

We claim that $\mathcal{A} = \{A_1, \ldots, A_4\}$ and $\mathcal{B} = \{B_1, \ldots, B_4\}$ are as desired. Indeed, it is clear by construction that they are partitions of A and B respectively. From the first and second claim, it follows that (i) and (ii) is satisfied respectively. By the definition of A_4 and B_4 in the last paragraph, we have (iv). And by the third claim, since $B_3 \supseteq \hat{B}_3$, it follows that (iii) holds. $\square$

Finally, we demonstrate that our approach for the proof of Theorem 2, which itself of course relies in many ways on ideas and results from Soukup's [9], can be used to give a conceptually simple closing argument for a proof of Theorem 1:

Theorem 1 (Soukup, [9, Theorem 7.1])**.** *Let r be a positive integer. Every r-edge-coloured complete graph of infinite order can be partitioned into monochromatic generalised paths of different colours.*

Proof. Let κ be an infinite cardinal, G the complete graph on κ and $c\colon E(G) \to [r]$ a colouring for some $r \geq 1$. Since the countable case has been solved in [7, Theorem 2], we may assume that κ is uncountable. Fix a partition $\{A, B\}$ of $V(G)$ such that both partition classes have cardinality κ. Apply Lemma 5—to the graph $G[A, B]$ and the colouring induced by c—in order to get disjoint sets $A_1, A_2 \in [A]^{\kappa}$, $B_1, B_2 \in [B]^{\kappa}$ and a path P as in the lemma. Let Λ be the set of limits of P and write $A' := A_1 \sqcup A_2$, $B' := V(G) \setminus A'$. Furthermore, let C consist of A' together with all those vertices in $V(G) \setminus (A' \cup \Lambda)$ that send κ many edges in colour 1 to A_2. Apply Lemma 7—to the graph induced by the edges of colour 1 in $G[A', B']$ and the set C—in order to find a set $X \in [A_2]^{\kappa}$ and an X-robust path Q as in the lemma. Next, apply Lemma 4—to the graph induced by the edges of colour $\neq 1$ in $G[X, B' \setminus Q]$ and the colouring induced by c—in order to find paths $P_2, \ldots, P_r$ of different colours in $[2, r]$. The last path in colour 1 is $Q \setminus \bigcup_i P_i$, which is a path due to the X-robustness of Q. $\square$

3. Monochromatic paths covering a κ-star-linked set

In this section, we prove Lemma 5. A partial result of Soukup's [9] will assist us: It implies that an edge-coloured complete bipartite graph with bipartition classes (A, B) both of cardinality $\kappa > \aleph_0$ contains a monochromatic path P of order type κ covering a large $<\kappa$-inseparable subset of A (cf. [9, Theorem 5.10]). By modifying the proof, we obtain a strengthened version where this $<\kappa$-inseparable subset is even κ-star-linked, Theorem 8 below. As the main result of this section, we explain how to establish Lemma 5 as a consequence of Theorem 8. The detailed proof of Theorem 8 we defer until the end of this paper.

3.1. **Finding a monochromatic path covering a κ-star-linked set.** First we remind the reader of a few concepts from Soukup's [9]: Let κ be a cardinal. Then $H_{\kappa, \kappa}$ denotes the graph $(\kappa \times \{0\} \cup \kappa \times \{1\}, E)$ where

$$\{(\alpha, i), (\beta, j)\} \in E \text{ iff } i = 1, j = 2 \text{ and } \alpha < \beta < \kappa$$

(cf. [9, p.250, l.10–13]). Furthermore, a graph $G = (V, E)$ is of *type* $H_{\kappa,\kappa}$ if there are (not necessarily disjoint) subsets $A, B \subset V$ with $V = A \cup B$, and enumerations $A = \{a_\xi \colon \xi < \kappa\}$ and $B = \{b_\xi \colon \xi < \kappa\}$ such that

$$\{a, b\} \in E(G) \text{ if } a = a_\xi, \ b = b_\zeta \text{ for some } \xi \leq \zeta < \kappa.$$

The vertex set A is called the *main class* of G and B is called the *second class* of G (cf. [9, Definition 5.3]). Informally speaking, a type $H_{\kappa,\kappa}$ graph is just a copy of $H_{\kappa,\kappa}$ where the bipartition classes are allowed to intersect.

Another concept that we need is those of a *concentrated path* [9, Definition 4.1]: Let G be a graph and $A \subseteq V(G)$. A path $P \subseteq G$ is concentrated on A if and only if

$$N(v) \cap A \cap V(P \restriction [x, v)) \neq \emptyset$$

for all $v \in \Lambda(P)$ and $x \prec_P v$.

Theorem 8. *If G is an r-edge coloured graph of type $H_{\kappa,\kappa}$ with main class A, then there is a colour $k \in [r]$ and $X \in [A]^\kappa$ which is κ-star-linked in colour k, such that X is covered by a monochromatic path of size κ in colour k concentrated on X.*

Proof. Theorem 8 follows from Theorem 15 which is a strengthening of [9, Theorem 5.10], to be proved in our last Section 5 below. □

3.2. Finding a monochromatic path covering an improved κ-star-linked set (proof of Lemma 5).

We need two more lemmas before we can prove Lemma 5.

Lemma 9. *Let G be a bipartite graph with bipartition classes A, B. A path $P \subseteq G$ is concentrated on A if and only if all limits of P are contained in B.* □

Lemma 10 (cf. [9], Lemma 3.4). *Let κ be an infinite cardinal, $G = (V, E)$ a graph and $A, B \subseteq V(G)$. Suppose A is κ-star-linked in B. Moreover, let $c \colon E(G) \to [r]$ be a colouring of G with $r \geq 1$ many colours. Then there is a partition $\{A_i \colon i \in [r]\}$ such that A_i is κ-star-linked in B in colour i for every $i \in [r]$.*

Proof. Take a uniform ultrafilter $\mathcal{U}$ on B with $B \cap N[F] \in \mathcal{U}$ for every finite $F \subseteq A$ and write $A_i = \{v \in A \colon N(v, i) \in \mathcal{U}\}$. Then for $i \in [r]$ and $F \subseteq A_i$, we have $N[F, i] \cap B \in \mathcal{U}$ and thus $N[F, i] \cap B$ has cardinality κ. □

We are now ready to provide the proof for Lemma 5 which we restate here for convenience of the reader.

Lemma 5. *Let κ be an infinite cardinal and G the complete bipartite graph with bipartition classes A, B both of cardinality κ. Suppose that $c \colon E(G) \to [r]$ is a colouring of G with $r \geq 1$ many colours. Then there are disjoint sets $A_1, A_2 \in [A]^\kappa$, $B_1, B_2 \in [B]^\kappa$ such that (up to renaming the colours):*

- *$G[A_1, B_1]$ has a spanning path P of order type κ in colour 1 all whose limits are contained in B_1, and*
- *$A_1 \sqcup A_2$ is κ-star-linked in B_2 in colour 1. (cf. Figure 2)*

Proof. Fix a set $A' \in [A]^\kappa$ that is κ-star-linked in as many colours as possible and let I be the set of those colours. By Lemma 10, the set I is non-empty and we may assume that colour 1 is contained in I.

Claim. There are disjoint sets $B_1', B_2' \subseteq B$ such that A' is κ-star-linked in B_1' and B_2', in all colours in I.

Proof. Let $(\kappa_\alpha)_{\alpha < \mathrm{cf}(\kappa)}$ be a strictly increasing cofinal sequence of regular cardinals in κ and write A' as an increasing union of sets $(A_\alpha)_{\alpha < \mathrm{cf}(\kappa)}$ such that A_α has cardinality κ_α for $\alpha < \mathrm{cf}(\kappa)$. The sets B_1' and B_2' can be constructed simultaneously in $\mathrm{cf}(\kappa)$ many steps. Indeed—using that A' is κ-star-linked in all colours in I and the fact that A_α has only κ_α many finite subsets—it is straightforward to find two increasing sequences $(B_\alpha^1)_{\alpha < \mathrm{cf}(\kappa)}$ and $(B_\alpha^2)_{\alpha < \mathrm{cf}(\kappa)}$ of subsets of B such that for all $\alpha < \mathrm{cf}(\kappa)$

- B_α^i has cardinality κ_α for both $i = 1, 2$,
- B_α^1 and B_α^2 are disjoint, and
- A_α is κ_α-star-linked in B_α^i for both $i = 1, 2$ in all colours in I.

Letting $B_1' = \bigcup \{B_\alpha^1 : \alpha < \mathrm{cf}(\kappa)\}$ and $B_2' = \bigcup \{B_\alpha^2 : \alpha < \mathrm{cf}(\kappa)\}$ completes our construction. $\diamond$

Fix B_1' and B_2' as in the above claim and let $\{A_1', A_2'\}$ be a partition of A' such that both partition classes have cardinality κ. Since $G[A_1', B_1']$ is complete bipartite, it is in particular of type $H_{\kappa,\kappa}$, and so we may apply Theorem 8 to $G[A_1', B_1']$ to find a colour $k \in [r]$ and $X \in [A_1']^\kappa$ which is κ-star-linked in colour k such that X is covered by a monochromatic path P (say) of size κ in colour k concentrated on X.

By the maximality of I we have $k \in I$ and we may assume $k = 1$. Furthermore, by Lemma 9 we know that all limits of P are contained in B_1'. Hence, letting $A_1 := A \cap P$, $A_2 := A_2'$, $B_1 := B \cap P$ and $B_2 := B_2'$ completes the proof. $\square$

4. Constructing robust paths

In this section we will prove Lemma 7. There are two major steps: First, we show how to find a ray R that is $\{x\}$-robust for a single vertex x. Second, we will construct the path Q as a concatenation of rays each including a copy of R. The set X for which Q is X-robust will be the set of vertices x in the various copies of R.

4.1. **Constructing countable robust paths.** Consider the one-way infinite ladder on the positive integers shown in Figure 4. The well-order $\leq$ on the positive integers together with this ladder then forms a (generalised) path R, and it is easy to see that R is $\{2\}$-robust. Indeed, the graph $R' = R - \{2\}$ has the one-way infinite path $R' = 1436587\ldots$ as a spanning subgraph. Note that additionally, the first vertices of R' and of R coincide.

As we work in the bipartite setting, it is of importance that generalised paths that we want to install are bipartite. Our ray R is bipartite as shown in Figure 5.
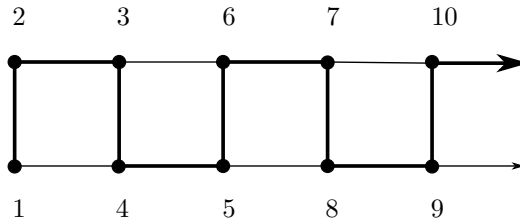
FIGURE 4. The fat edges indicate the path order of the one-way ladder on the positive integers.
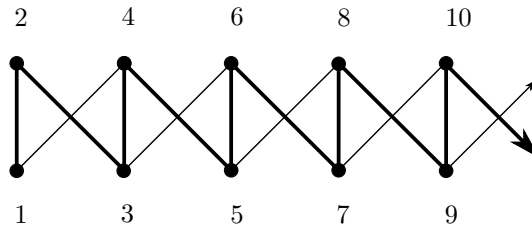
FIGURE 5. The top and bottom vertices in the ladder define the two bipartition classes of the one-way infinite ladder. The path order from Figure 4 is indicated fat again.

All countably infinite robust paths we construct will always consist of some finite path Q followed by a copy of R, where we denote this concatenation by $Q^\frown R$. It is easy to see that the path $Q^\frown R$ is then $\{x\}$-robust, where x is the vertex corresponding to the vertex $2 \in V(R)$. The following lemma is our key lemma for constructing paths of that kind:

Lemma 11. *Let G be a bipartite graph with bipartition classes A, B such that A is countably infinite. Suppose further that A is $\aleph_0$-star-linked and $a \in A$ is some fixed vertex. Then for any vertex $x \in A \setminus \{a\}$ there is an x-robust ray R in G starting in the vertex a and covering A. Moreover, there is a path order of $R - x$ with first vertex a.*

Proof. Fix an enumeration of $(a_n)_{n \geq 1}$ of A with $a_1 = a$ and $a_2 = x$. For $n \geq 1$ fix distinct vertices $(b_n)_{n \geq 1}$ such that b_n is contained in the common neighbourhood of $\{a_n, a_{n+1}, a_{n+2}\}$ for $n \geq 1$. This can be done since A is $\aleph_0$-star-linked.

Let us write $B' = \{b_n \colon n \geq 1\}$. Then $G[(A \setminus \{a_1\}) \cup B']$ has a copy[1] of the one-way infinite ladder L on ω as a spanning subgraph where b_1 corresponds to the vertex 1 and x corresponds to the vertex 2 of L. Let us write R' for this copy of L and endow R' with the path order induced by the path order $\leq$ on L. By our

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[1]The vertices in $A \setminus \{a_1\}$ correspond to the upper vertices in Figure 5 and the vertices in B' to the bottom vertices. The enumerations of $A \setminus \{a_1\}$ and respectively B' are the enumeration which 'go from left to the right'.

observations at the beginning of this subsection, the ray $R = a_1 {}^\frown R'$ is $\{x\}$-robust and starts with a. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.2. Constructing uncountable robust paths.

Lemma 7. *Let κ be an uncountable cardinal and G a bipartite graph with bipartition classes A, B both of size κ. Suppose there are disjoint sets $A_1, A_2 \in [A]^\kappa$, $B_1, B_2 \in [B]^\kappa$ such that*

- *$G[A_1, B_1]$ has a spanning path P of order type κ with $\Lambda(P) \subset B_1$, and*
- *$A_1 \sqcup A_2$ is κ-star-linked in B_2.*

Then there is a set $X \in [A_2]^\kappa$ and an X-robust path Q covering $A_1 \sqcup A_2$ with $\Lambda(P) = \Lambda(Q)$. Moreover, if $C \subseteq (A \setminus A_1) \sqcup (B \setminus \Lambda)$ covers A_2 and is $<\kappa$-inseparable in $G[A \setminus A_1, B \setminus \Lambda]$, then Q can be chosen to cover C.

Proof. Let us write λ_0 for the first vertex on P and let $\{\lambda_\alpha \colon 1 \le \alpha < \kappa\}$ be the enumeration of the limits of P along the path order of P, i.e. we have $\lambda_\alpha \prec_P \lambda_\beta$ whenever $1 \le \alpha < \beta$. Fix an enumeration $\{c_\alpha \colon \alpha < \kappa\}$ of C, (choose $C = A_2$ if C is not specified). Note that C has indeed cardinality κ as A_2 is included in C. We construct a sequence of pairwise disjoint paths $\mathcal{S} = (S_\alpha)_{\alpha < \kappa}$ and a sequence of distinct vertices $(x_\alpha)_{\alpha < \kappa}$ from A_2 satisfying the following:

(1) S_α has order type ω,
(2) S_α has first vertex λ_α and doesn't meet any other limits of P,
(3) S_α is x_α-robust and there is a path order $\prec_{S_\alpha - x_\alpha}$ of $S_\alpha - x_\alpha$ that has first vertex λ_α,
(4) $S_\alpha \cap A \cap P = A \cap P \restriction [\lambda_\alpha, \lambda_\alpha + \omega)$ and
(5) $\bigcup_{\beta \le \alpha} S_\alpha$ contains c_α.

Once $\mathcal{S}$ is defined we obtain Q as the concatenation $Q = S_0 {}^\frown S_1 {}^\frown S_2 {}^\frown \cdots$ (formally, the path order is given by the lexicographic order on $\bigcup_{\alpha < \kappa} \{\alpha\} \times S_\alpha$). Indeed, conditions (1) and (2) guarantee that the limits of Q and the limits of P coincide, and so it follows from (4) that Q is indeed a generalised path. By condition (5), the path Q covers C. Finally, put $X = \{x_\alpha \colon \alpha < \kappa\}$.

Claim. The path Q is X-robust.

Proof. In order to see that Q is X-robust, let $X' \subseteq X$ be arbitrary. Let S'_α be the path $(S_\alpha - x_\alpha, \prec_{S_\alpha - x_\alpha})$, if S_α meets X' and $S'_\alpha = S_\alpha$ otherwise. Then $Q' = S'_0 {}^\frown S'_1 {}^\frown S'_2 {}^\frown \cdots$ is a path of order type κ covering $Q - X'$. $\qquad\qquad\lozenge$

It remains to define $\mathcal{S} = (S_\alpha)_{\alpha < \kappa}$. Suppose that S_α has already been defined for $\alpha < \beta$. Write $\Sigma_\beta := \bigcup_{\alpha < \beta} S_\alpha$, a set of cardinality $<\kappa$. We first find a finite path T that

- starts in λ_β,
- ends in a vertex $a \in A_2$ (say),
- contains c_β (unless $c_\beta \in \Sigma_\beta$ already),
- avoids Σ_β and meets P only in $P \restriction [\lambda_\beta, \lambda_\beta + \omega)$.

Claim. A path T as above exists.

Proof. Let T_1 be the path of (edge-)length 1 or 0, that starts in λ_β and is followed by the successor of λ_β on P if λ_β is not already contained[2] in A_1.

Let w_1 denote the last vertex on T_1, and note that $w_1 \notin \Sigma_\beta$ by (4). Since $A_1 \sqcup A_2$ is κ-star-linked in B_2, we may chose any $w_2 \in A_2 \setminus \Sigma_\beta$ and find a vertex $w_3 \in (B_2 \cap N[\{w_1, w_2\}]) \setminus \Sigma_\beta$ so that $w_1 w_3 w_2$ forms a path of (edge-)length two.

If c_β is not yet covered by Σ_β, as C is $<\kappa$-inseparable, we find a finite path T_3 that contains c_β, starts in the vertex w_2 and ends in a vertex $a \in A_2$ and avoids

$$A_1 \cup \Lambda(P) \cup \Sigma_\beta \cup V(T_1) \cup V(T_2).$$

Otherwise, we put $T_3 = \emptyset$ and $a = w_2$. Then T can be chosen as $T_1{}^\frown T_2{}^\frown T_3$. $\Diamond$

To complete the proof, we now find a path R of order type ω such that it

- starts in the vertex a and avoids T everywhere else,
- is $\{x_\beta\}$-robust for a vertex $x_\beta \in A_2 \setminus \{a\}$ and there is a path order of $R - x_\beta$ that starts with a,
- avoids Σ_β and meets P only and in $P \upharpoonright [\lambda_\beta, \lambda_\beta + \omega)$.

Claim. A path R as above exists.

Proof. Choose $x_\beta \in A_2 \setminus (\Sigma_\beta \cup V(T))$ arbitrary. Apply Lemma 11 inside the bipartite graph $G[A', B']$ with the vertex a and the vertex $x = x_\beta$, where

$$A' = \{a, x_\beta\} \cup ((A \cap P \upharpoonright [\lambda_\beta, \lambda_\beta + \omega)) \setminus V(T))$$

is countable, and $B' = B_2 \setminus (\Sigma_\beta \cup V(T))$. $\Diamond$

Letting $S_\beta = T{}^\frown R$ completes the construction of S_β and thereby our proof is complete.   □

5. A result extracted from Soukup's [9]

The following lemma of Soukup is the main tool of constructing large generalised paths. To state the lemma, we need the following definition.

Definition 12 ([9, Definition 4.4]). Suppose that $G = (V, E)$ is a graph and $A \subseteq V$. We say that A satisfies $\spadesuit_\kappa$ if for each $\lambda < \kappa$ there are κ many disjoint paths concentrated on A each of order type λ.

Moreover, if we have a fixed edge-colouring $c \colon E \to [r]$ in mind, we write $\spadesuit_{\kappa,i}$ for "$\spadesuit_\kappa$ in colour i".

Lemma 13 ([9, Lemma 4.6]). *Suppose that $G = (V, E)$ is a graph, κ an infinite cardinal, and $A \in [V]^\kappa$. If*

(1) A is $<\kappa$-inseparable
and if κ is uncountable, then
(2) A satisfies $\spadesuit_\kappa$, and

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[2]Since all limits of P are contained in B_1, we have $\lambda_\beta \notin A_1$ as soon as $\beta \geq 1$. In the case where $\beta = 0$, we might have $\lambda_\beta \in A_1$.

(3) there is a nice sequence of elementary submodels $(M_\alpha)_{\alpha < cf(\kappa)}$ for $\{A, G\}$ covering A so that there is $x_\beta \in A \setminus M_\beta$, $y_\beta \in V \setminus M_\beta$ with $x_\beta y_\beta \in E$ and

$$|N_G(y_\beta) \cap A \cap M_\beta \setminus M_\alpha| \geq \omega$$

for all $\alpha < \beta < cf(\kappa)$,

then A is covered by a generalised path P concentrated on A.

Recall that Soukup considers for fixed κ and $r \geq 1$ the following statements:

$(\mathrm{IH})_{\kappa, r}$ Let H be a graph of type $H_{\kappa, \kappa}$ with main class A and second class B and $r \geq 1$. Then for every r-colouring of H, there is a colour k and an $X \in [A]^\kappa$ so that X satisfies all three conditions of Lemma 13 in colour k.

$(\mathrm{IH})_\kappa$ The statement $(\mathrm{IH})_{\kappa, r}$ holds for every $r \geq 1$.

Soukup's main result is then

Theorem 14 ([9, Theorem 5.10]). $(\mathrm{IH})_{\kappa, r}$ *holds for all* κ. *In particular, if G is a graph of type $H_{\kappa, \kappa}$ with a finite-edge colouring, then we can find a monochromatic path of size κ concentrated on the main class of G.*

We now strengthen Soukup's results as follows, and consider the statements:

$(\mathrm{IH})'_{\kappa, r}$ As the statement $(\mathrm{IH})_{\kappa, r}$ with the additional requirement that X is also κ-star-linked in colour k.

$(\mathrm{IH})'_\kappa$ The statement $(\mathrm{IH})'_{\kappa, r}$ holds for every $r \geq 1$.

The corresponding version of theorem [9, Theorem 5.10] then reads:

Theorem 15. $(\mathrm{IH})'_\kappa$ *holds for all* κ. *In particular, if G is a graph of type $H_{\kappa, \kappa}$ with main class A with an r-edge colouring, then there is a colour $k \in [r]$ and $X \in [A]^\kappa$ which is κ-star-linked in colour k, such that X is covered by a monochromatic path of size κ in colour k concentrated on X.*

The proof of Theorem 15 relies on the following lemma.

Lemma 16 (cf. [9, Lemma 5.9]). *Let κ be an infinite cardinal. Suppose that c is an r-edge colouring of a graph $G = (V, E)$ of type $H_{\kappa, \kappa}$ with main class A and second class B. Let $I \subsetneq [r]$, $X \in [A]^\kappa$ and suppose that X is κ-star linked in all colours $i \in I$. If $(\mathrm{IH})_\lambda$ holds for all $\lambda < \kappa$ then either*

(a) *there is an $i \in I$ such that X satisfies $\spadesuit_{\kappa, i}$, or*

(b) *there is $\tilde{X} \in [X]^{<\kappa}$ and a partition $\{X_j : j \in [r] \setminus I\}$ of $X \setminus \tilde{X}$ such that X_j is κ-star-linked in B in colour j for each $j \in [r] \setminus I$.*

Proof of Lemma 16. Follow the proof of [9, Lemma 5.9, p. 261] and in the last line apply the following Claim A instead of [9, Claim 5.9.3].

Claim A (cf. [9, Claim 5.9.3]). *Suppose that c is an r-edge colouring of a graph $G = (V, E)$ of type $H_{\kappa, \kappa}$ with main class A and second class B. Let $I \subset r$ and $X \subset A$. If for each finite subset $F \subset A$ we have*

$$\left| B \setminus \bigcup \{N(x, i) : x \in F, i \in I\} \right| = \kappa,$$

then there is a partition $\{X_j : j \in r \setminus I\}$ of X such that X_j is κ-star-linked in B in colour j for each $j \in [r] \setminus I$.

Proof. Take a uniform ultrafilter U on B so that $B \setminus \bigcup \{N(x,i) : x \in F, i \in I\} \in U$ for all finite subsets $F \subset A$. Define $X_j = \{x \in X \setminus \tilde{X} : N(x,j) \in U\}$ for each colour j and note that $\{X_j : j \in [r] \setminus I\}$ partitions X. Since ultrafilters are closed under finite intersections, it follows that

$$N[F,j] \in U$$

for all finite subsets $F \subset X_j$ and $j \in [r] \setminus I$, and since the filter U is uniform, we have $|N[F,j]| = \kappa$ and therefore that X_j is κ-star-linked in B for each such j. $\quad\diamond$

Indeed, by applying Claim A to the set $X \setminus \left(X^* \cup \tilde{A} \right)$ (defined in Soukup's proof), we readily obtain the stronger conclusion that the X_j are not only $<\kappa$-inseparable, but even κ-star-linked. $\hfill\square$

Proof of Theorem 15. We prove $(\text{IH})'_{\kappa,r}$ by induction on κ and r.

Note that $(\text{IH})'_\omega$ holds by [9, Lemma 3.4], so we may suppose that κ is uncountable. Also, $(\text{IH})'_{\kappa,1}$ holds: From [9, Observation 5.7], we know that for any graph G of type $H_{\kappa,\kappa}$, the main class of G satisfies all conditions of Lemma 13 (and so $(\text{IH})_{\kappa,1}$ holds). However, it is clear that the main class A is automatically κ-star-linked in G, and hence we have $(\text{IH})'_{\kappa,1}$.

Now fix an r-edge colouring of a graph G of type $H_{\kappa,\kappa}$ with main class A and second class B. As in Soukup's proof of [9, Theorem 5.10] (Theorem 14 above), we may assume by the induction assumption $(\text{IH})'_\lambda$ for $\lambda < \kappa$ that every $X \in [A]^\kappa$ satisfies condition (3) in Lemma 13.

Next, Soukup fixes a maximal $I \subseteq [r]$ with the property that there is a set $X \in [A]^\kappa$ such that X is $<\kappa$-inseparable in all colours $i \in I$ and he fixes such I and X. Instead, we now fix I maximal with the property that there is a set $X \in [A]^\kappa$ such that X is κ-star-linked in all colours $i \in I$. Then fix such I and X.

Claim B (cf. [9, Claim 5.10.1]). *There is $k \in I$ such that $\spadesuit_{\kappa,k}$ holds for X.*

Proof. Suppose that X fails $\spadesuit_{\kappa,i}$ for all $i \in I$. If $I \subsetneq [r]$, then apply Lemma 16 in G to the set X and the set of colours I. As X fails X fails $\spadesuit_{\kappa,i}$ for all $i \in I$, condition (b) of Lemma 16 must hold (note that by induction assumption, $(\text{IH})'_\lambda$ and hence $(\text{IH})_\lambda$ hold for all $\lambda < \kappa$, so we may apply Lemma 16): However, this means there is a colour $j \in [r] \setminus I$ and a set $X_j \in [X]^\kappa$ such that X_j is κ-star-linked in colour j as well. But the fact that X_j is then κ-star-linked in all colours $i \in I \cup \{j\}$ contradicts the maximality of I.

Therefore, $I = [r]$ must hold. From this, however, we may obtain a contradiction precisely as in the second half of the proof of [9, Claim 5.10.1]. $\quad\diamond$

Hence, the "in particular" part of the theorem, and hence Theorem 8 follows by applying Lemma 13 to the set X provided by $(\text{IH})'_\kappa$. The proof is complete. $\hfill\square$

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