Topological defects in lattice models and affine Temperley–Lieb algebra

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Abstract

This paper is the first in a series where we attempt to define defects in critical lattice models that give rise to conformal field theory topological defects in the continuum limit. We focus mostly on models based on the Temperley-Lieb algebra, with future applications to restricted solid-on-solid (also called anyonic chains) models, as well as non-unitary models like percolation or self-avoiding walks. Our approach is essentially algebraic and focusses on the defects from two points of view: the "crossed channel" where the defect is seen as an operator acting on the Hilbert space of the models, and the "direct channel" where it corresponds to a modification of the basic Hamiltonian with some sort of impurity. Algebraic characterizations and constructions are proposed in both points of view. In the crossed channel, this leads us to new results about the center of the affine Temperley-Lieb algebra; in particular we find there a special subalgebra with non-negative integer structure constants that are interpreted as fusion rules of defects. In the direct channel, meanwhile, this leads to the introduction of fusion products and fusion quotients, with interesting mathematical properties that allow to describe representations content of the lattice model with a defect, and to describe its spectrum.

Contents

1 Introduction

A defect—or interface—in conformal field theory is generally defined as a non-contractible line separating two a priori different conformal field theories (CFTs), with matching conditions between the two sides of the line. Various situations can be encountered in this general context. We will restrict here to the case of so-called topological defects, where the two CFTs are identical, and the stress-energy tensor is continuous across the defect line. In this case, correlation functions for fields inserted away from the defect line are unchanged when the line is continuously deformed, as long as the line is not taken across the field insertions: hence the name "topological".

Defects in CFT appear in a variety of physical problems, both in two-dimensional statistical mechanics, e.g. in the context of Kramers–Wannier duality [1, 2], and in imaginary-time one-dimensional quantum mechanics, e.g. in the context of quantum impurity problems such as the Kondo problem [3, 4]. The problem of classifying topological defects has received considerable attention, in particular in the case of rational CFTs [5,6]. For such theories with diagonal modular invariants, for instance, it has been shown that the set of defects is isomorphic with the set of representations of the chiral algebra. Many results for non-diagonal invariants, or for non-rational unitary theories such as Liouville [7] are also known.

Meanwhile, the general question of relating structures within the CFTs with properties of underlying lattice models has also attracted much attention. Work in this direction has included attempts to define lattice versions of the Virasoro algebra [8–10], to define fusion of primary fields in terms or representation theory of lattice algebras [11–16], to calculate modular transformations from lattice partition functions [17], and to build topological defect lines directly on the lattice [18]. Many of these attempts drew from the pioneering work of Kadanoff and Ceva [19].

The present work is motivated by our interest in non-unitary (in particular, logarithmic) conformal field theory (LCFT) [20]. Decisive progress has been realized in this difficult subject by turning to lattice models—in particular, to understand better the indecomposable properties of the Virasoro-algebra representations involved. In view of the close relationship between defects, primary fields and fusion in the unitary case, it is natural to continue the program set out in [21] by trying to define topological defects on the lattice using an algebraic approach. While such endeavor has been partially completed in the case of restricted solid-on-solid models—whose associated CFTs are rational, and which are closely related to "anyonic" spin chains [18]—we will be interested here in the profoundly different case of loop models, which provide regularizations of the simplest known LCFTs. This paper will discuss the first part of our study, where we will focus on the definition and mathematical properties of a certain kind of lattice topological defects.

The correspondence between CFTs and lattice models is often best handled by thinking of the CFT in radial quantization, where, after the usual logarithmic mapping, (imaginary) time propagation occurs along the axis of a cylinder, and space is periodic. In this point of view, the non-contractible line for the topological defect can either run along the infinite cylinder, or be a non-contractible loop winding around it. We will refer to these two situations as a defect in the "direct" or in the "crossed" channel, respectively, see Fig. 1.

In the crossed channel, the defect can be associated with an operator X acting on the Hilbert space of the bulk CFT. The defect is topological if X commutes with the chiral Vir and the anti-chiral $\overline{Vir}$ Virasoro generators [6]:

$$[L_n, X] = 0 = [\bar{L}_n, X] \ . \tag{1}$$

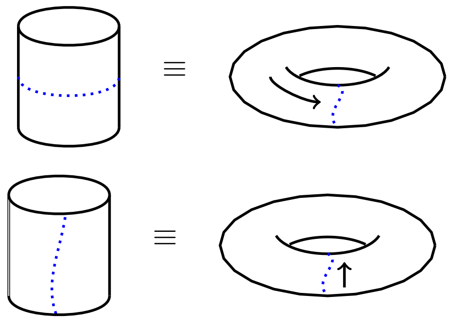


Figure 1: The two possible geometries for a defect line after mapping the plane to the cylinder.

Our strategy to identify the possible choices of operators X is based on the identification of (representations of) the Virasoro algebra via the continuum limit of the Temperley-Lieb (TL) algebra—an idea that has been used in several works on related topics [8, 22, 23]. This "identification" must be qualified. First, since we are dealing with bulk CFTs, we must think of the product of the chiral and anti-chiral Virasoro algebras, $Vir \otimes \overline{Vir}$. Similarly, since the lattice models are defined on a cylinder, the proper lattice algebra is a "periodicized" version—the affine Temperley-Lieb algebra aTL: strictly speaking, the continuum limit of this algebra is known to be larger than $Vir \otimes \overline{Vir}$, and has been identified as the "interchiral algebra" in [22].

In the typical physical interpretation of the (affine) Temperley-Lieb algebras on n sites, the nodes on the top and bottom of the TL diagrams should be interpreted as a chain of n subsystems whose interactions are determined by the TL generators, but whose internal sub-structure is not—it is determined by the specific model chosen, which also fixes the aTL representation corresponding to the chain. The simplest examples of these are the various kinds of spin-chains, like the twisted XXZ model. We will therefore start our search for lattice analogues of topological defects by demanding the closest lattice equivalent of (1), that is by looking for operators X on the lattice that commute with the interactions in the chain, or, in a model-independent setting, that are central in aTL. We will follow this model-independent point of view on lattice defects as central elements satisfying certain nice properties, e.g. having a well-defined fusion. This is discussed in Section 3 after the algebraic preliminaries of Section 2 where we recall the usual definition of aTL together with a less standard formulation using a blobbed set of generators. In this last formulation, the lattice meaning of X turns out to be very simple: it just consists in passing a line "above" or "below" the non-contractible loops by using solutions of the spectral-parameter independent Yang-Baxter equation exchanging spin-1/2 (the value relevant for bulk loops) and spin-j (the value relevant for the defect lines) representations. The topological nature of this defect is obvious, as the Yang-Baxter equation allows one to move and deform the defect line at will without changing neither the partition function, nor the correlation functions if operators are inserted. The simplest example of such a defect operator X is given by a diagram corresponding to a single non-contractible loop passing *over* the bulk, see Fig. 2 where we denote this operator by Y. This operator and its powers are manifestly in the center of the affine TL algebra. We define similarly operators $\bar{Y}$ where the non-contractible loop is passing *under* the bulk. The two operators generate an interesting algebra of defects.

Let us describe this type of defect operators in more precise mathematical terms. First of all, the aTL algebras depend on n (the number of sites) and a loop parameter $\mathfrak{q}+\mathfrak{q}^{-1}$. In this paper, we

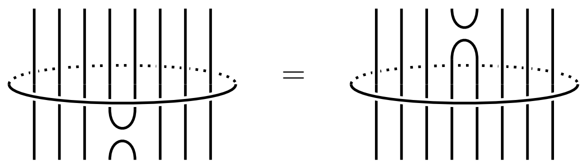


Figure 2: Commutativity of defect operator Y with e_j generators of aTL.

consider only the case of $\mathfrak{q}$ a generic complex number (not a root of unity, we leave the root of unity case discussion for a forthcoming paper). The aTL algebra can be obtained as a quotient of the so-called affine Hecke algebra of type $\hat{A}_{n-1}$ where all central elements are known—they form the algebra of symmetric Laurent polynomials in Jucys-Murphy elements J_i, for $1 \leq i \leq n$. One of our main mathematical results in this paper is that the image of this affine Hecke center inside aTL is generated by the two elements Y and $\bar{Y}$, i.e. their powers can be written as symmetric polynomials in the J_i, and vice versa. We shall call this natural subalgebra in the center of aTL the *symmetric center* Z_{sym}. Moreover, we show that products of Chebyshev polynomials in Y and $\bar{Y}$ provide a "canoncical" basis in Z_{sym} with *non-negative integer* structure constants, i.e. a product of two defect operators is decomposed onto defect operators again, and with non-negative multiplicities. The multiplicities are interpreted as fusion rules of the defects.

Of course, the line passing above or below the loops can as well be taken to run along the axis of the cylinder, i.e. along the time direction. This corresponds to having the defect in the direct channel. In this setting, the presence of the defect line leads to a modified Hilbert space where an extra representation of spin j is introduced, together with a Hamiltonian suitably modified by corresponding "defect" terms. This is discussed in Section 4, where we relate spectral properties of such a modified Hamiltonian (which is hard to study directly) to a clear and precise algebraic construction within the representation theory of aTL algebras—namely the fusion product and fusion quotient. The first is based on a certain induction, while the second is dual to it and practically very convenient for actual calculations. In simple terms, the spectrum of the spin-j defect Hamiltonian is given by the spectrum of the standard affine TL Hamiltonian with no defects however acting on the fusion quotient of an aTL representation by the spin-j standard TL representation. The advantage of this construction is that it allows us to perform precise calculations, as we demonstrate in several examples, including the case of the twisted XXZ model.

In the last section 5, we provide conclusions and discuss a CFT interpretation together with further steps that will be discussed in the next papers, like the analysis of modular S-transformation in infinite lattices and the continuum limit from a more physical point of view. In Section 5, we also make an attempt to give a precise mathematical definition of lattice defects studied in this work. Finally, several appendices contain proofs of our mathematical results and auxiliary calculations, such as examples of fusion products and fusion quotients.

2 Algebraic preliminaries: the affine TL algebra

In this section, we fix our notations and conventions. We first give a definition of the affine Temperley-Lieb algebra in terms of generators, and in terms of diagrams. We give the definition both in terms of the translation generator, which is very standard, and a new one in terms of the so-called blob and hoop generators; the blob formulation is significantly more convenient when discussing topological defects. We then discuss the standard modules and give the eigenvalues of the common central elements on them. The orbit classification shows that representations can be identified by the eigenvalues of the topological defect operators and the full translation operator u^n.

2.1 Two definitions

The affine Temperley-Lieb algebras $\{\mathsf{aTL}_n(\mathfrak{q})\}$ form a family of infinite dimensional associative $\mathbb{C}$-algebras, indexed by a positive integer n – number of sites – and a non-zero complex number $\mathfrak{q}$. They can be defined in many ways but we chose three particular presentations for their relevance in physics. Each of these are described in terms of generators with relations and were chosen because they lighten the notation in particular sub-sections of this work.

The first set of generators, which we shall refer to as the *periodic* set of generators, is the one appearing in the original literature on these algebras: two *shift* generators u, u^{-1}, and n *arc* generators $e_1, \dots, e_n$, with the defining relations ($n > 2$)

$$\begin{aligned} e_i e_i &= (\mathfrak{q} + \mathfrak{q}^{-1}) e_i, \\ e_i e_{i\pm1} e_i &= e_i, \\ e_i e_j &= e_j e_i \ \text{ if } |i-j| \geq 2, \\ u e_i &= e_{i+1} u, \\ u^2 e_{n-1} &= e_1 \dots e_{n-1}, \end{aligned} \tag{2}$$

which stands for all i, and we defined $e_0 \equiv e_n$, $e_{n+1} \equiv e_1$. If $n = 2$, one must remove the relations $e_i e_{i\pm1} e_i = e_i$, but the other relations are unchanged. If $n = 1$, one must remove all the arc generators, keeping only the shift generators with the defining relations $uu^{-1} = u^{-1}u = 1$. One notices immediately that this set of generators is not minimal, since for instance $e_i = u^{i-1} e_1 u^{1-i}$ for all $i \geq 1$. Furthermore, the elements $u^{\pm n}$ are both obviously central. The sub-algebra generated by $\{e_1, \dots, e_n\}$ is often called the *periodic* Temperley-Lieb algebra, while the one generated by $\{e_1, \dots, e_{n-1}\}$ is called the *regular* Temperley-Lieb algebra.

The second set of generators, which we shall refer to as the *blobbed* set of generators, is significantly less known: there are two *blob* generators b, b^{-1}, and $n-1$ *arc* generators e_i, $1 \leq i \leq n-1$ (so if $n = 1$ there are no arc generators), with defining relations

$$\begin{aligned} e_i e_i &= (\mathfrak{q} + \mathfrak{q}^{-1}) e_i, \\ e_i e_{i\pm1} e_i &= e_i, \\ e_i e_j &= e_j e_i \qquad \text{if } |i-j| \geq 2, \\ e_i b &= b e_i \qquad \text{if } i \geq 2, \\ e_1 b e_1 &= \underbrace{(\mathfrak{q} b + \mathfrak{q}^{-1} b^{-1})}_{\equiv -Y} e_1 = e_1 (\mathfrak{q} b + \mathfrak{q}^{-1} b^{-1}), \end{aligned} \tag{3}$$

which stands for all i such that these expressions make sense; note that in this case we have no generator e_n. We also note that the element $Y \equiv -\mathfrak{q} b - \mathfrak{q}^{-1} b^{-1}$ introduced in the above relations is central, it will be called the *hoop* operator[1]. We want to stress that in our formulation the

[1]The name will be justified via its diagrammatical presentation that we discuss below.

blob generator b is invertible. The epithet *blob* here denotes the relation with the so-called *blob algebra* [33], which is a finite-dimensional algebra where the blob element is not invertible but an idempotent. This latter algebra is obtained by taking the quotient of $\mathsf{aTL}_n(\mathfrak{q})$ by the two-sided ideal $\mathsf{aTL}_n(\mathfrak{q})\cdot(Y-y1)$ for some $y\in\mathbb{C}$, or in simple words the blob algebra is obtained via fixing the eigenvalue of Y. See for instance [29].

We note that the connection with the first description, i.e. in terms of "periodic" type generators is (here, we place periodic type generators in RHS)

$$\begin{aligned} e_i &= e_i, \qquad 1\le i\le n-1, \\ b &= (-\mathfrak{q})^{-3/2} g_1^{-1}\dots g_{n-1}^{-1}u^{-1}, \qquad (4)\\ b^{-1} &= (-\mathfrak{q})^{3/2} u\, g_{n-1}\dots g_1, \qquad (5)\end{aligned}$$

where we introduced the *braid* generators

$$g_i^{\pm 1} = (-\mathfrak{q})^{\pm 1/2}1 + (-\mathfrak{q})^{\mp 1/2} e_i. \qquad (6)$$

It is straightforward to check the braid relations

$$g_i g_{i\pm 1} g_i = g_{i\pm 1} g_i g_{i\pm 1}. \qquad (7)$$

The normalization[2] in (6) was chosen such that

$$\begin{aligned} g_i^{\pm 1} g_{i+1}^{\pm 1} e_i &= e_{i+1} g_i^{\pm 1} g_{i+1}^{\pm 1} = e_{i+1}e_i\,, \\ g_{i+1}^{\pm 1} g_i^{\pm 1} e_{i+1} &= e_i g_{i+1}^{\pm 1} g_i^{\pm 1} = e_i e_{i+1}\,. \qquad (8)\end{aligned}$$

These relations are used to prove the equivalence of the relations in equations (3) and (2), specifically when verifying those involving b, u, or e_n.

We note that an expression of periodic generators in terms of the blobbed ones is obtained as follows: the shift generators $u^{\pm 1}$ are obtained multiplying both sides of (4)-(5) with appropriate $g_i^{\pm 1}$'s, then the generator e_n is formally defined as $u^{-1}e_1u$. It is then rather straightforward, however tedious, to show that the defining relations (2) are equivalent to those in (3). We give one example of such computations as the others are all quite similar; recall the proposed form for b in (4), we then verify that

$$\begin{aligned} (\mathfrak{q}b)^2 e_1 &= (-\mathfrak{q})^{-1} u g_{n-1}^{-1}\dots g_1^{-1} u g_{n-1}^{-1}\dots \underbrace{g_1^{-1}e_1}_{=-(-\mathfrak{q})^{3/2}e_1}\,, \\ &= -u g_{n-1}^{-1}\dots g_2^{-1}\,(1-\mathfrak{q}e_1)\, u g_{n-1}^{-1}\dots g_2^{-1} e_1, \\ &= -u^2\,\underbrace{g_{n-2}^{-1}\dots g_1^{-1} g_{n-2}^{-1}\dots g_2^{-1} e_1}_{=e_{n-1}e_{n-2}\dots e_1} + \mathfrak{q} b e_1 b e_1, \\ &= -u^2 e_{n-1}e_{n-2}\dots e_1 + \mathfrak{q} b e_1 b e_1, \\ &= -e_1 + \mathfrak{q} b e_1 b e_1, \end{aligned}$$

and then multiplying both sides by b^{-1} from the left yields the identity $e_1 b e_1 = (\mathfrak{q}b + \mathfrak{q}^{-1}b^{-1})e_1$ from the list in (3). We also note that in the context of blob algebras (recalled above as the

[2] The normalization used here for g_i will also become useful when doing graphical calculations.

quotients), the relation between periodic and blobbed generators reflects what was called "braid translation" in [27, 33].

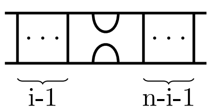
We will also use another relation between the periodic and blobbed set of generators: Because the algebra is invariant under the substitution $b \to b^{-1}$, $\mathfrak{q} \to \mathfrak{q}^{-1}$, i.e. it provides an algebra automorphism, there is a second way to write the blob generators in terms of the generators of "periodic" type:

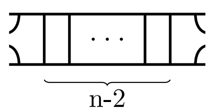
$$\begin{aligned} \bar{b} &= (-\mathfrak{q})^{-3/2} u\, g_{n-1}^{-1} \dots g_1^{-1}, \\ \bar{b}^{-1} &= (-\mathfrak{q})^{3/2} g_1 \dots g_{n-1} u^{-1}. \end{aligned} \tag{9}$$

We turn now to introduction of diagrammatical presentations for both types of generators, and it is much easier to check such an equivalence (or isomorphism of the two algebras) by doing standard diagram calculations.

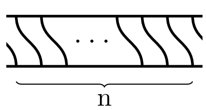
2.2 Diagrammatic presentation

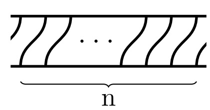
We now introduce the graphical presentation of the algebra, which can be used to write words in the algebra in a very compact and intuitive form. Each of the classical generators gets associated to a diagram with $2n$ nodes connected by n *strands*, or *lines*:

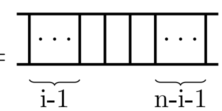
$e_i =$ 

, $e_n =$ 

, $i = 1, \dots, n-1,$ (10)

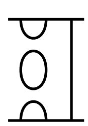
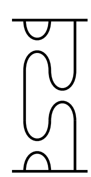
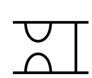
$u =$ 

, $u^{-1} =$ 

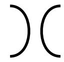
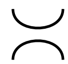
, $1_{\mathsf{aTL}_n} =$ 

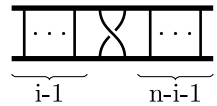
, (11)

where the opposing vertical sides are identified, so these drawings should be imagined as being drawn on a cylinder, with the top and bottom black lines resting on its top and bottom edge, respectively. Strands that connects both edges of the cylinder are called *through lines*. One can show that every diagram which can be drawn on this cylinder with n non-intersecting strings represents a non-zero element of the algebra, and every such element is represented by a unique diagram, up to isotopy of the strands which is ambient on the boundary. Sums of elements of the algebra can be understood as formal sums of diagrams, and products in the algebra are computed using *diagram composition*[3]: the diagrams ab is defined by putting the diagram for b on top of the diagram for a and joining the strands that meet. A closed arc that is homotopic to a point is simply removed and replaced by a factor $\mathfrak{q} + \mathfrak{q}^{-1}$. For instance, here are some of the defining relations of the algebra in the diagrammatic presentation (for $n = 3$):

$e_1 e_1 =$  $= (\mathfrak{q} + \mathfrak{q}^{-1}) e_1,$ $e_1 e_2 e_1 =$  $=$  $= e_1.$ (12)

[3]In this work, product of operators are read from left to right, and diagrams are read from bottom to top. In some the authors previous work, for instance in [13], the opposite convention is used so operators were multiplied right to left and diagrams read from top to bottom.

 $\equiv (-\mathfrak{q})^{\frac{1}{2}}$  $+ (-\mathfrak{q})^{-\frac{1}{2}}$ 

$g_i = (-\mathfrak{q})^{\frac{1}{2}}\mathbf{1} + (-\mathfrak{q})^{-\frac{1}{2}} e_i =$ 

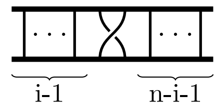
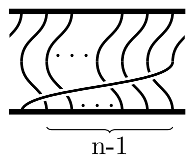
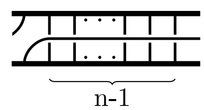
$g_i^{-1} = (-\mathfrak{q})^{-\frac{1}{2}}\mathbf{1} + (-\mathfrak{q})^{\frac{1}{2}} e_i =$ 

Figure 3: Braid notations

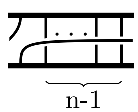
For graphical presentation of the blobbed generators, we introduce first the *braid notation* for the overlapping strands in Fig. 3, as well as the diagram presentation of $g_i^{\pm 1}$ introduced in (6). Then using (11) we get by stacking the diagrams:

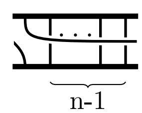
$g_1^{-1} \dots g_{n-1}^{-1} u^{-1} \quad =$ 

$=$ 

(13)

and similar calculation for $ug_{n-1} \dots g_1$. Therefore, the blob generators b and b^{-1} from (4)-(5) can be represented as

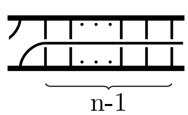
$b = (-\mathfrak{q})^{-3/2}$ 

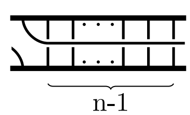
$, \qquad b^{-1} = (-\mathfrak{q})^{3/2}$ 

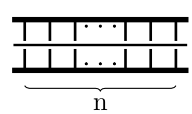
. (14)

It is then straightforward to check the relations (3) using the standard graphical manipulations together with the relations (8).

We recall the central element $Y = -(\mathfrak{q}b + \mathfrak{q}^{-1}b^{-1})$. In the diagram basis, it can be written as

$Y = (-\mathfrak{q})^{-\frac{1}{2}}$ 

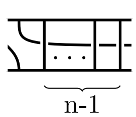
$+ (-\mathfrak{q})^{\frac{1}{2}}$ 

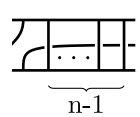
$=$ 

,

where for the last equality we also used the braid conventions in Fig. 3. That Y is central is easy to check using the diagrammatic calculation as in Fig. 2: generators e_j obviously commute with the insertion of a line going "above" or "under" the system, the same applies for the commutativity with the shift operators where one just uses the braid relations.

Recall now the algebra automorphism $b \to b^{-1}$, $\mathfrak{q} \to \mathfrak{q}^{-1}$ discussed above (9). The diagram presentation for the second set of blobbed generators is

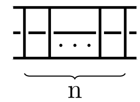
$\bar{b} = (-\mathfrak{q})^{-3/2}$ 

$, \qquad \bar{b}^{-1} = (-\mathfrak{q})^{3/2}$ 

. (15)

The second representative of the blob generators $\bar{b}$ and $\bar{b}^{-1}$ allows us to identify the second distinct central element $\bar{Y}$:

$$\bar{Y} \equiv -(\mathfrak{q}\bar{b} + \mathfrak{q}^{-1}\bar{b}^{-1}) =$$



. (16)

We will show below that for generic values of $\mathfrak{q}$ the two central elements Y and $\bar{Y}$ generate a natural subalgebra Z_{sym} in the center of $\mathsf{aTL}_n(\mathfrak{q})$. We call this subalgebra *the symmetric centre* of $\mathsf{aTL}_n(\mathfrak{q})$ and it has two interesting properties (that will be proven in the next section):

1. Z_{sym} is an image of the whole center of the affine Hecke algebra under the standard covering map: $\widehat{H}_n(q) \to \mathsf{aTL}_n(\mathfrak{q})$, $T_i \to g_i$, $J_i \to J_i$. Recall that the center of $\widehat{H}_n(q)$ is spanned by symmetric polynomials in the Jucy-Murphy elements J_i, for $1 \le i \le n$.

2. There is a special "canonical" basis (made of Chebyshev polynomials) such that the structure constants are non-negative integers, i.e. Z_{sym} endowed with this basis is a Verlinde algebra.

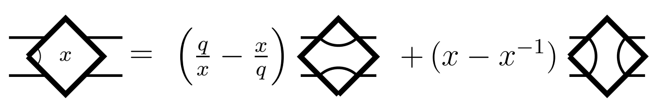
The second point is very important for our defect construction, and we will see below in Section 3 that the central elements in the canonical basis (on certain representations) provide operators that represent topological defects in the crossed channel.

It is however not clear to us whether the symmetric centre Z_{sym} generates the centre of $\mathsf{aTL}_n(\mathfrak{q})$ or not. We plan to come back to this important question in the next publication.

2.3 Tile formalism and the transfer matrix

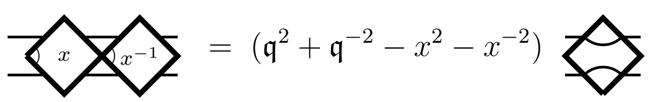
While we formulate most of our results in terms of diagrams with strings and arcs on a cylinder, a very significant body of work on this subject is written in terms of *planar tiles* (see for instance [34, 35]); we present here a brief translation between the two formalisms and use it to introduce the usual transfer matrix.

The planar tile with spectral parameter x is defined by[4]

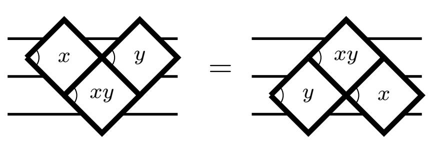


. (17)

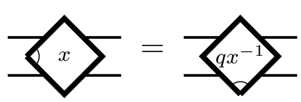
These satisfy three particular relations:



, (18)



, (19)

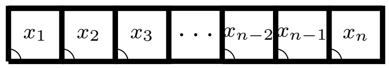


. (20)

[4] This tile is often divided by $(\mathfrak{q}-\mathfrak{q}^{-1})$ to normalize it, but then the natural defect operator would be $(\mathfrak{q}-\mathfrak{q}^{-1})^{-n}Y$ instead of Y.

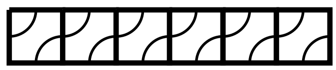
These are respectively called the inversion, Yang-Baxter, and crossing symmetry.

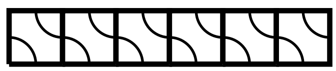
The transfer matrix $T_n(\vec{x})$ can then be defined as

$T_n(\vec{x}) =$  $, \qquad \vec{x} = \{x_1, x_2, \ldots, x_n\} , \qquad (21)$

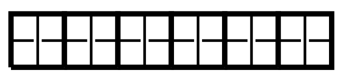
where there are n tiles and the opposing vertical sides are identified so that this defines an element of $\mathsf{aTL}_n(\mathfrak{q})$ for each n-dimensional vector $\vec{x}$. If $x_1 = x_2 = \ldots = x_n$ the transfer matrix is said to be *homogeneous* and is simply written $T_n(x_1)$. Using the three previous identities, one readily shows that homogeneous transfer matrices commute with each others, i.e. $[T_n(x), T_n(y)] = 0$, and thus define families of integrable lattice models.

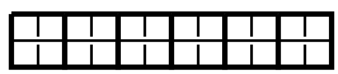
We note four specific cases of the homogeneous transfer matrix that are of importance in this work. Setting the spectral parameter x to 1 or $\mathfrak{q}$ gives the translation operators $u^{\mp 1}$:

$T_n(1) = (\mathfrak{q} - \mathfrak{q}^{-1})^n$  $= (\mathfrak{q} - \mathfrak{q}^{-1})^n u^{-1}, \qquad (22)$

$T_n(\mathfrak{q}) = (\mathfrak{q} - \mathfrak{q}^{-1})^n$  $= (\mathfrak{q} - \mathfrak{q}^{-1})^n u, \qquad (23)$

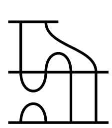
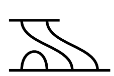
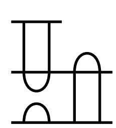
while taking the limits in the spectral parameter to zero or infinity produces the two hoop operators Y and $\bar{Y}$:

$\lim_{x\to 0}((-(-\mathfrak{q})^{-\frac{1}{2}}x)^n T_n(x)) =$  $= \bar{Y}, \qquad (24)$

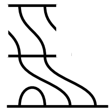
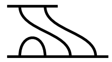
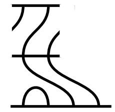
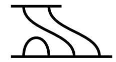
$\lim_{x\to\infty}(((-\mathfrak{q})^{-\frac{1}{2}}x)^{-n} T_n(x)) =$  $= Y. \qquad (25)$

2.4 Standard modules

We present a brief overview of the most common class of $\mathsf{aTL}_n \equiv \mathsf{aTL}_n(\mathfrak{q})$ modules: the *standard* modules $\mathsf{W}_{k,z}(n)$; these are indexed by a non-negative integer $2k \leq n$ (so k is a half-integer), of the same parity as n, and a non-zero complex number z. The simplest way of describing their basis is in terms of diagrams having n (k) nodes on their bottom (top) side, and having exactly k through lines. One simply takes the formal sums of every such diagrams, and use diagram composition to describe the action of the algebra (by stacking an algebra diagram on the bottom), understanding that if composition produces a diagram with less than k through lines, it is identified with the zero element. For instance,

 $=$  $, \qquad$  $= 0. \qquad (26)$

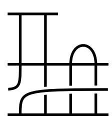
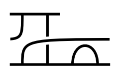
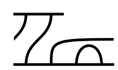
This is the way standard modules $\mathsf{S}_k(n)$ are defined for the regular Temperley-Lieb algebra $\mathsf{TL}_n(\mathfrak{q})$, by simply excluding the diagrams with strings crossing the imaginary boundary on each side of the diagrams; while for $\mathsf{TL}_n(\mathfrak{q})$ such diagrams form a finite dimensional module, it is not true for the affine version $\mathsf{aTL}_n(\mathfrak{q})$, as e.g. the translation generators $u^{\pm 1}$ produce states with arbitrary winding of through lines. To get a finite dimensional module for $\mathsf{aTL}_n(\mathfrak{q})$, one must also fix the eigenvalues of the two central elements identified in the previous section: $-Y = \mathfrak{q}b + \mathfrak{q}^{-1}b^{-1}$ and $-\bar{Y} = \mathfrak{q}\bar{b} + \mathfrak{q}^{-1}\bar{b}^{-1}$. The simplest way to do this is to define the right action of u (the action on through lines) as multiplication by z, i.e.

 $= z$  $,$  $= z^{-1}$  $,$ (27)

where LHS of the first equality is the right action of u while LHS of the second equality is the right action of u^{-1}. The eigenvalue of the central element u^n is thus z^n. It was shown in [32] that the endomorphism ring of standard modules is one dimensional, so any central element must act like a multiple of the identity on a standard module; finding the eigenvalue is then simply a matter of choosing a convenient element x such that computing Yx is easy. For example, using x which is filled by non-nested arcs from the right and the rest are the $2k$ through lines, we calculate that the choice (27) for the action of u also fixes the eigenvalues of the central elements Y and $\bar{Y}$, as follows:

$$\begin{aligned} Y &= -(\mathfrak{q}b + \mathfrak{q}^{-1}b^{-1}) = z(-\mathfrak{q})^{k} + z^{-1}(-\mathfrak{q})^{-k}, \\ \bar{Y} &= -(\mathfrak{q}\bar{b} + \mathfrak{q}^{-1}\bar{b}^{-1}) = z(-\mathfrak{q})^{-k} + z^{-1}(-\mathfrak{q})^{k}. \end{aligned} \tag{28}$$

To see this, we first recall the diagram presentation for b in (14). Applying then $-\mathfrak{q}b$ to the chosen x and expanding the braid-crossings according to the rules in Figure 3, only one configuration has a non-zero contribution that corresponds to the factor $z^{-1}(-\mathfrak{q})^{-k}$. As an example of such a calculation for $k = 1$, $n = 4$, we have

$(-\mathfrak{q})b$  $= (-\mathfrak{q})^{-\frac{1}{2}}$  $= (-\mathfrak{q})^{-\frac{1}{2}}$  $= (-\mathfrak{q})^{-1}$  . (29)

A similar calculation can be done for $\bar{b}^{\pm 1}$ confirming the result in (28).

It shall be convenient in what follows to use the notation

$$\mathsf{W}^{o}_{\pm|k|,\delta}(n) \equiv \mathsf{W}_{|k|,\delta^{\pm 1}(-\mathfrak{q})^{-k}}(n), \qquad \mathsf{W}^{u}_{\pm|k|,\mu}(n) \equiv \mathsf{W}_{|k|,\mu^{\pm 1}(-\mathfrak{q})^{k}}(n), \tag{30}$$

which fixes the eigenvalue of $Y = \delta + \delta^{-1}$, or $\bar{Y} = \mu + \mu^{-1}$, respectively, and the superscript o/u refers here to the central element being fixed: the one with a horizontal line going *over* (Y) or *under* ($\bar{Y}$) all others.

We conclude this section with a description of the structure of these modules at generic $\mathfrak{q}$. Based on the results [32], we observe that there exists a non-zero morphism[5] $f\colon \mathsf{W}_{s,w}(n) \to \mathsf{W}_{r,z}(n)$ if and only if $s \geq r$ and $Y, \bar{Y}$ have the same eigenvalues on both modules; furthermore any such morphism is proportional to the identity (for $s = r$) or to a unique injective map. The conditions on equality of the eigenvalues of Y and $\bar{Y}$ is equivalent to the Graham-Lehrer conditions [32]:

$$z = \begin{cases} w(-\mathfrak{q})^{r-s} & \text{if } (-\mathfrak{q})^{2(r-s)} = 1 \text{ or } w^2 = (-\mathfrak{q})^{2r}, \\ w^{-1}(-\mathfrak{q})^{r+s} & \text{if } (-\mathfrak{q})^{2(r+s)} = 1 \text{ or } w^2 = (-\mathfrak{q})^{-2r}. \end{cases} \tag{31}$$

Furthermore, each standard module has a unique simple quotient denoted by $\overline{\mathsf{W}}_{r,z}(n)$, and these form a complete set of irreducible modules.

[5]For brevity, we will use the term "morphism" instead of the more standard "homomorphism".

2.5 Tower structure

The family of affine Temperley-Lieb algebras admits inclusions of the form (we will often abbreviate $\mathsf{aTL}_n \equiv \mathsf{aTL}_n(\mathfrak{q})$)

$$\mathsf{aTL}_n \subset \mathsf{aTL}_{n+1} \subset \mathsf{aTL}_{n+2} \subset \dots \ , \qquad n \geq 1 \ ,$$

giving the structure of a tower of algebras [13, Sec. 3.3]. Some of these inclusions will play a role in our construction of topological defects so we describe them here. We now assume that k is a positive integer, and define a morphism of algebras

$$\phi^u_{n,k}\colon \ \mathsf{aTL}_n \to \mathsf{aTL}_{n+k}, \tag{32}$$

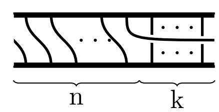
by its action on the various sets of generators of the algebra. For clarity, we add a superscript to the generators to indicate which algebra they belong to; for instance $u^{(n)}$ is the shift generator in aTL_n, while $u^{(n+2)}$ is the shift generator in aTL_{n+2}, etc. With this notation, the map $\phi^u_{n,k}$ on the blobbed set of generators is

$$\phi^u_{n,k}\colon \quad \big(b^{(n)}\big)^{\pm 1} \mapsto \big(b^{(n+k)}\big)^{\pm 1}, \tag{33}$$

$$e_i^{(n)} \mapsto e_i^{(n+k)}. \tag{34}$$

It is straightforward to verify that $\phi^u_{n,k}$ defines an inclusion of algebras. We note that this definition is parallel to what was done in affine Hecke algebra terms in [13, Sec. 4.4.2]. While the map is very simple with the blobbed generators, it is more complicated when expressed on the periodic set of generators, for instance

$$\phi^u_{n,k}\colon \quad u^{(n)} \mapsto u^{(n+k)} g^{(n+k)}_{n+k-1} g^{(n+k)}_{n+k-2} \dots g^{(n+k)}_{n} = \tag{35}$$



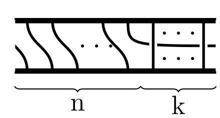
which agrees with [13, Eq. (3.9)], after one takes into account the difference in conventions. One therefore sees that, in terms of diagrams, the morphism $\phi^u_{n,k}$ consists in adding k through lines on the right side of each diagrams, going *under* every lines that wraps around the cylinder (hence the superscript u on the morphism).

Similarly, one defines another morphism of algebras

$$\phi^o_{n,k}\colon \ \mathsf{aTL}_n \to \mathsf{aTL}_{n+k} \tag{36}$$

by adding the k lines *over* the lines that wrap around the cylinder, i.e. on the periodic set of generators the map is

$$\phi^o_{n,k}\colon \quad u^{(n)} \mapsto u^{(n+k)} \big(g^{(n+k)}_{n+k-1}\big)^{-1} \big(g^{(n+k)}_{n+k-2}\big)^{-1} \dots \big(g^{(n+k)}_{n}\big)^{-1} = \tag{37}$$



,

which also agrees with [13, Eq. (3.10)]. On the blobbed set of generators, the map is simply

$$\phi^o_{n,k}\colon \quad \big(\bar{b}^{(n)}\big)^{\pm 1} \mapsto \big(\bar{b}^{(n+k)}\big)^{\pm 1}, \tag{38}$$

$$e_i^{(n)} \mapsto e_i^{(n+k)}. \tag{39}$$

Furthermore, while we placed the extra lines on the right side of the diagram, we could have put them on the left side instead; we name the resulting morphisms

$$\psi^{u/o}_{n,k}\colon\ \mathsf{aTL}_n \to \mathsf{aTL}_{n+k}, \tag{40}$$

for the corresponding *under* and *over* versions. We then notice that the two subalgebras $\phi^u_{n,k}(\mathsf{aTL}_n)$ and $\psi^o_{k,n}(\mathsf{aTL}_k)$ commute with each others; this can be seen by a direct calculation as in [13] or showing that

$$\phi^u_{n,k}\big(b^{(n)}\big) \propto J_1^{(n+k)}, \qquad \psi^o_{k,n}\big(b^{(k)}\big) \propto J_{n+1}^{(n+k)},$$

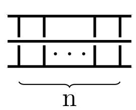
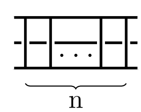
where J_i is the Jucys-Murphy element[6] of the affine Temperley-Lieb algebra (see Section 3.3). This fact can be exploited to define a monoidal structure on the affine Temperley-Lieb category [13], see also [14] for the corresponding fusion calculation. We note that $\phi^o_{n,k}(\mathsf{aTL}_n)$ and $\psi^u_{k,n}(\mathsf{aTL}_k)$ also commute.

3 Lattice topological defects: crossed channel

In this section, we formulate our lattice topological defects in terms of the affine TL algebra using the hoop operators – the central elements Y and $\bar{Y}$ introduced in the previous section – and describe their fusion rules. In more mathematical terms, we show that the two elements Y and $\bar{Y}$ generate an interesting subalgebra Z_{sym} in the centre of $\mathsf{aTL}_n(\mathfrak{q})$ – the so-called symmetric centre – we will show that Z_{sym} agrees with the algebra of symmetric Laurent polynomials in the famous Jucys-Murphy elements. We also show that it admits a certain basis with *non-negative integer* structure constants. Interestingly, at least for generic values of $\mathfrak{q}$, the structure constants do not depend on n or $\mathfrak{q}$.

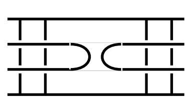
3.1 The algebra of defects Y and $\bar{Y}$

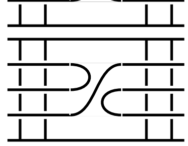
Recall that the hoop operators defined in Section 2.2 can be represented by diagrams with a single closed string wrapping over or under all the other strings:

$$Y = -(\mathfrak{q}b + \mathfrak{q}^{-1}b^{-1}) =$$  , $$\bar{Y} = -(\mathfrak{q}\bar{b} + \mathfrak{q}^{-1}\bar{b}^{-1}) =$$  ,   (41)

and these are central elements in $\mathsf{aTL}_n(\mathfrak{q})$. This wrapping string can be isotopically deformed at will without changing the spectrum of the transfer matrix from Section 2.3, and it thus can be thought of as a defect line (in the crossed channel). We are interested in the algebra generated by these hoop operators, and first study their powers.

Taking powers of the hoop operators will increase the width of the defects by increasing the number of lines going across the system; one can then imagine Temperley-Lieb operators acting horizontally on the defect. For instance

$$Y^2(e_1) =$$  $$= (\mathfrak{q} + \mathfrak{q}^{-1})1_{\mathsf{aTL}_n}, \tag{42}$$

$$Y^3(e_1e_2) =$$  $$= Y. \tag{43}$$

[6] The Jucys-Murphy elements form a commutative subalgebra.

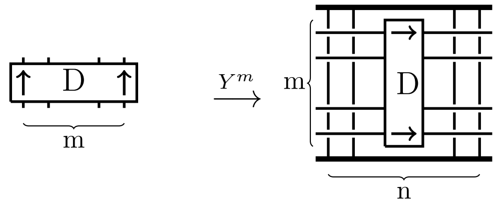


Figure 4: An illustration of the action of the map Y^m; the D box represent some diagram in TL_m and the arrows illustrate its orientation. The map then rotates the diagram 90 degrees clockwise, and insert it on the defect. The result is a central element of aTL_n.

One recognize that this corresponds to taking a Markov trace in the horizontal direction; in particular, the operator Y^m can be seen as a map from TL_m to the ring of endomorphisms of aTL_n:

$$Y^m\colon\ \mathsf{TL}_m \to \mathrm{End}_{\mathsf{aTL}_n}\big(\mathsf{aTL}_n\big)\ , \tag{44}$$

where a given element in TL_m considered as a diagram is just placed on the m horizontal strands, as in Fig. 4. It is easy to see that the image of this map lives in the center of aTL_n, and the central elements provide an endomorphism via the multiplication. We similarly define the mapping

$$\bar{Y}^m\colon \mathsf{TL}_m \to \mathrm{End}_{\mathsf{aTL}_n}\big(\mathsf{aTL}_n\big) \tag{45}$$

whose image is also in the center of aTL_n, and that can be represented graphically similarly to Fig. 4, however with horizontal lines going under the vertical ones. We show below that the images of the two maps Y^m and $\bar{Y}^m$ generate an algebra that we call $\mathsf{Z}_{\mathrm{sym}}$.

3.2 Higher-spin operators Y_j and $\bar{Y}_j$

Instead of applying the defect operators Y^m on individual elements of the Temperley-Lieb algebra, we can have them act on an entire ideal, sending each to a sub-ring of the ring of endomorphisms of aTL_n. If $\mathfrak{q}$ is generic, every indecomposable left-ideal of TL_m is isomorphic to one of the form $\mathsf{S}_j(m) = \mathsf{TL}_m P_j$, where P_j is an idempotent of spin j; when $j = m/2$ one can use the Jones-Wenzl projectors

$$P_{m/2} = W_1^{m+1}\ ,$$

defined recursively through the following formula:

$$\begin{aligned} &W_i^1(n) \equiv W_i^2(n) \equiv 1_{\mathsf{TL}_n},\\ &W_i^m(n) \equiv W_{i+1}^{m-1}(n)\left(1_{\mathsf{TL}_n} - \frac{\mathfrak{q}^{m-2}-\mathfrak{q}^{2-m}}{\mathfrak{q}^{m-1}-\mathfrak{q}^{1-m}} e_i\right) W_{i+1}^{m-1}(n), \end{aligned} \tag{46}$$

where the index m is related to the spin as above, and i is just the lattice position.

Recall that P_j is an idempotent, i.e. $P_jP_j = P_j$, and the map Y^m has the property of a trace, we then have

$$Y^m(xP_j) = Y^m(P_jxP_j)$$

for all $x \in \mathsf{TL}_m$. By construction, P_jxP_j is an endomorphism of the ideal $\mathsf{S}_j(m)$ (by multiplication on the right), which is simple whenever $\mathfrak{q}$ is generic; it follows that $P_jxP_j = \lambda_x P_j$ for some $\lambda_x \in \mathbb{C}$, and thus that

$$Y^m(\mathsf{S}_j(m)) = \mathbb{C}Y_j, \tag{47}$$

where we introduced a special central element

$$Y_j := Y^{2j}(W_1^{2j+1}). \tag{48}$$

Here, we used the fact that the trace of P_j is independent both of m, and of the particular choice of P_j we made (see Appendix A.1 for details of the proof). In particular, the identity (47) makes sense and is true for any valid value of m when the ideal $\mathsf{S}_j(m)$ is non-zero.

Using the recurrence relation for the Jones-Wenzl projectors, we find

$$Y_j = \mathsf{U}_{2j}\big(\frac{1}{2}Y\big), \tag{49}$$

where $\mathsf{U}_k(x)$ is the Chebyshev polynomial of the second kind, of order k. For instance, we have

$$\begin{aligned} Y_{1/2} &= Y\,, \\ Y_1 &= (Y_{1/2})^2 - 1\,, \\ Y_{3/2} &= (Y_{1/2})^3 - 2Y_{1/2}\,, \\ Y_2 &= (Y_{1/2})^4 - 3(Y_{1/2})^2 + 1\,. \end{aligned}$$

Recall that Y acts on $\mathsf{W}^o_{k,\delta}$ as $(\delta + \delta^{-1})$; writing $\delta = e^{i\theta}$, the higher-spin operator eigevalues are thus

$$Y_j = \frac{\sin((2j+1)\theta)}{\sin\theta}.$$

The important observation is that the properties of the Chebyshev polynomials allow us to decompose products of Y_js:

$$Y_j \cdot Y_k = \sum_{r=|j-k|}^{j+k} Y_r. \tag{50}$$

We finally note that the whole construction of this section would work equally well if the defect had been going under the strings instead of over them, by simply replacing Y with $\bar{Y}$ everywhere it appears. We begin with the map $\bar{Y}^m$ defined in (45). Its properties are identical to those of the map Y^m in every way; applying it to the ideals $\mathsf{S}_j(m)$ yields higher-spin defect operators $\bar{Y}_j$ whose eigenvalues on $\mathsf{W}^u_{k,\delta}$ are

$$\bar{Y}_j = \frac{\sin((2j+1)(\phi))}{\sin\phi},$$

where $\delta \equiv e^{i\phi}$. And they have similarly the fusion

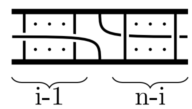
$$\bar{Y}_j \cdot \bar{Y}_k = \sum_{r=|j-k|}^{j+k} \bar{Y}_r. \tag{51}$$

The algebra generated by Y_j and $\bar{Y}_k$ will be called *the symmetric center* Z_{sym}, this name will be justified in the next subsection. In other words, the images of the two maps Y^m and $\bar{Y}^m$ generate Z_{sym} as claimed above.

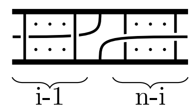
We finally note that for the "mixed" fusion $Y_j \cdot \bar{Y}_k$ there is no interesting decomposition, or rather a trivial one, and the element $Y_j \cdot \bar{Y}_k$ has to be thought of as one of the basis elements in Z_{sym}. Of course all the other products in the algebra Z_{sym} can be now decomposed over Y_j, $\bar{Y}_k$, and $Y_j \cdot \bar{Y}_k$ using (50) and (51).

3.3 Relation to symmetric polynomials.

While identifying the topological defect operators with the hoop operator is an intuitive choice, there are many other known central elements, which could also lead to topological defects. These are built from the so-called Jucys-Murphy elements; let

$$J_1 \equiv \bar{b}, \qquad J_i \equiv g_{i-1} J_{i-1} g_{i-1} = (-\mathfrak{q})^{-3/2}$$ 

$$, \qquad i = 2, \ldots n \tag{52}$$

$$M_1 \equiv b, \qquad M_i \equiv g_{i-1} M_{i-1} g_{i-1} = (-\mathfrak{q})^{-3/2}$$ 

$$, \qquad i = 2, \ldots n \tag{53}$$

It is straightforward, though tedious, to prove that the Js commute with each others and so do the Ms; furthermore if $P(x_1, \ldots, x_n)$ is a symmetric Laurent polynomial, then $P((-\mathfrak{q})J_1, \ldots, (-\mathfrak{q})^n J_n)$ and $P((-\mathfrak{q})^i M_1, \ldots, (-\mathfrak{q})^n M_n)$ are central in aTL_n. All of these can be generated from the power-sum symmetric polynomials

$$C_k(n) = \sum_{i=1}^{n} ((-\mathfrak{q})^{i+1} M_i)^k, \qquad \bar{C}_k(n) = \sum_{i=1}^{n} ((-\mathfrak{q})^{i+1} J_i)^k. \tag{54}$$

However, it turns out that these are related to the hoop operators through the following relations:

$$C_k(n) + C_{-k}(n) = (-\mathfrak{q})^{-nk} \bar{C}_k(n) + (-\mathfrak{q})^{nk} \bar{C}_{-k}(n) = 2[n]_k \mathsf{T}_k(\bar{Y}/2), \tag{55}$$

$$\bar{C}_k(n) + \bar{C}_{-k}(n) = (-\mathfrak{q})^{-nk} C_k(n) + (-\mathfrak{q})^{nk} C_{-k}(n) = 2[n]_k \mathsf{T}_k(Y/2), \tag{56}$$

where we defined

$$[n]_k \equiv \frac{(-\mathfrak{q})^{kn} - (-\mathfrak{q})^{-kn}}{(-\mathfrak{q})^k - (-\mathfrak{q})^{-k}},$$

where it is understood that $[n]_0 \equiv n$, and $\mathsf{T}_k(x)$ is the kth Chebyshev polynomial of the first kind. The proof of these identities can be found in Appendix A.2. If $(-\mathfrak{q})^{nk} \neq 1$, these relations can be combined to find

$$((-\mathfrak{q})^k - (-\mathfrak{q})^{-k}) C_k(n) = 2\left((-\mathfrak{q})^{kn} \mathsf{T}_{|k|}(Y/2) - \mathsf{T}_{|k|}(\bar{Y}/2)\right), \tag{57}$$

$$((-\mathfrak{q})^k - (-\mathfrak{q})^{-k}) \bar{C}_k(n) = 2\left((-\mathfrak{q})^{kn} \mathsf{T}_{|k|}(\bar{Y}/2) - \mathsf{T}_{|k|}(Y/2)\right). \tag{58}$$

Finally, using the properties of the Chebyshev polynomial it follows that

$$Y_{k/2} = \sum_{\substack{j=1-k \\ \text{step } =2}}^{k-1} \frac{1}{[n]_j} \bar{C}_j(n). \tag{59}$$

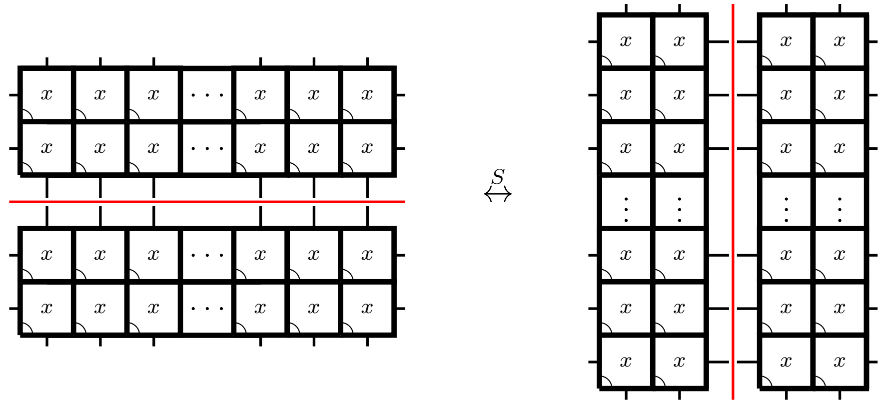


Figure 5: The modular S-transformation, which is the lattice rotation by 90^o, sends a defect Y (in red) in the crossed channel to a defect in the direct channel, and vice versa.

4 Lattice topological defects: direct channel

In this section, we are interested in interpretation of previously introduced defects Y_j and $\bar{Y}_j$ in the direct channel, or in their Hamiltonian realization. The action of the defect $Y_{1/2}$ in the direct channel can be inferred by a simple modular transformation - that is, a rotation by 90^o as in Fig. 5. What this means microscopically is that we should have a system where, on top of the usual TL interaction terms, we have an extra line that simply goes over/under the others, and this contributes to defect terms in the Hamiltonian.

The Hamiltonian with defects can be obtained as a logarithmic derivative evaluated at $x = 1$ of the transfer matrix $T_n(x; m)$ in Fig. 6 in the case of rotation of the defect $\bar{Y}_{m/2}$. In this case, we obtain the Hamiltonian on $n + m$ sites

$$H^u = \sum_{j=1}^{n-1} e_j^{(n+m)} + \mu_{n,m}^{-1} e_n^{(n+m)} \mu_{n,m} \rho,$$

where $\mu_{n,m} = g_n g_{n+1} \dots g_{n+m}$ and the idempotent ρ is the JW idempotent W_1^{m+1}. We are interested in the spectral problem of H^u. It is important to note that this Hamiltonian can be written as

$$H^u = \phi_{n,m}^u \Big(\sum_{j=1}^{n} e_j^{(n)} \Big),$$

i.e. as the image of the standard periodic TL Hamiltonian

$$H_n = \sum_{j=1}^{n} e_j$$

on n sites under the embedding map $\phi_{n,m}^u$. To solve the spectral problem, we present an algebraic construction linking the spectrum of H^u with the spectrum of the standard Hamiltonian H_n acting on a certain fusion quotient module of $\mathsf{aTL}_n(\mathfrak{q})$. This requires certain preparation and an algebraic discussion below. We then come back to the spectral problem in Section 4.4 with the final result

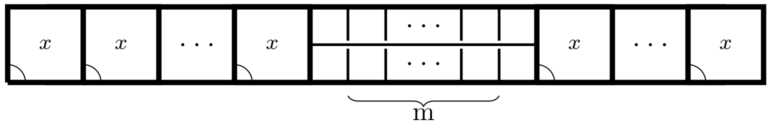
$T_n(x;k) =$ 

Figure 6: $T_n(x;m)$ is a transfer matrix carrying a defect of width m going under the other lines; taking its logarithmic derivative evaluated at $x = 1$ yields the Hamiltonian H_n^u (up to a normalization factor) with $\rho = 1$.

formulated in Theorem 4.1, and then provide an explicit example based on the twisted XXZ chains in Section 4.5.

Let us begin with the idea that stays behind the two algebraic constructions formulated below. Adding the extra lines/defects in the direct channel can be realised as a functor that combine a module of aTL (the bulk model) with a module of TL (the defect) into a new module of aTL (the bulk model with a defect); it turns out that there are (at least) two natural ways of doing this: one can *add* new strands carrying the defect to the module, a process we call the *fusion product*, or one can *impose* the defect on an existing part of the module, a process we call the *fusion quotient*.

4.1 The fusion product

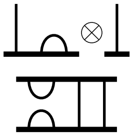
This section uses the notation introduced in section 2.5

Let m, k both be positive integers, we give aTL_{m+k} the structure of a $(\mathsf{aTL}_{m+k}, \mathsf{aTL}_m \otimes_{\mathbb{C}} \mathsf{TL}_k)$ bimodule by letting aTL_{m+k} act on the left through the natural representation, and $\mathsf{aTL}_m \otimes_{\mathbb{C}} \mathsf{TL}_k$ acts on the right by the morphism $\phi_{m,k}^{u/o} \otimes_{\mathbb{C}} \psi_{k,m}^{o/u}$, where we identified TL_k with its image in aTL_k. For M an aTL_m module, and V a TL_k-module, our definition of the fusion product can then be written

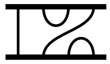
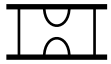
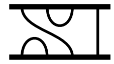
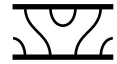
$$M \times_f^{u/o} V \equiv \mathsf{aTL}_{m+k} \otimes_{\mathsf{aTL}_m \otimes_{\mathbb{C}} \mathsf{TL}_k} (M \otimes_{\mathbb{C}} V)\,, \tag{60}$$

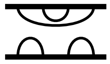
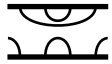
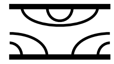
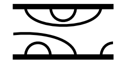
where the superscript u/o denotes which one of $\phi_{m,k}^{u/o}$ we used to define the bimodule structure of aTL_{m+k}. From a more physical point of view, this corresponds to having a bulk model described by M which contains an isolated sub-system V, such that they are both entirely blind to each others so that the Hilbert space of the system is simply the tensor product of the Hilbert spaces of $M \otimes_{\mathbb{C}} V$; at some point one then remove the barrier between the two sub-system and thus letting V *propagate* freely inside M. Note also that this fusion is related, though different, to ones introduced previously (see Appendix B for more details).

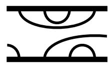
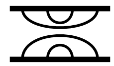
Before giving the general result we give a small example and compute the fusion product of two standard modules $\mathsf{W}_{1/2,z}(3) \times_f^o \mathsf{S}_{1/2}(1)$. Since the standard modules are cyclic, their fusion is also, and thus $\mathsf{W}_{1/2,z}(3) \times_f^o \mathsf{S}_{1/2}(1) = \mathsf{aTL}_4 x$, with

$x =$  , (61)

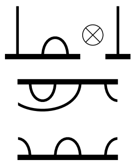
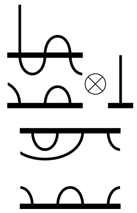
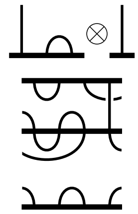
where we also introduced our diagram notation for the fusion product: the diagram at the bottom is the element of aTL_4, the one on the top left corner is the element of $\mathsf{W}_{1,z}(3)$, and the one on the top right corner is the element of $\mathsf{S}_{1/2}(1)$. Since this module is cyclic, we can choose a basis of the form $\{a_i x | i = 1, ...\}$ for some subset $\{a_i\} \subset \mathsf{aTL}_4$; in the case at hand the simplest choice is

$a_1 =$  $a_2 =$  $a_3 =$  $a_4 =$ 

$a_5 =$  $a_6 =$  $a_7 =$  $a_8 =$ 

$a_9 =$  $a_{10} =$  .

It is not trivial at all to show that this set is sufficient, for instance, can $e_4 a_2 x$ really be expressed as a linear combination of $a_i x$? Indeed it can:

$e_4 a_2 x =$  $= z^{-1}$  $= z^{-1}$  $= -(-\mathfrak{q})^{3/2} z^{-1} a_6 x,$ (62)

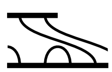
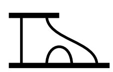
where the last equality was obtained by using the closed braid identity

 $= -(-\mathfrak{q})^{3/2}$  . (63)

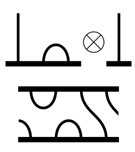
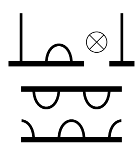
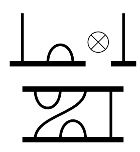
Using similar tricks, every elements can be brought to a linear combination of the $a_i x$.

Note that it is clear that any element $a \in \mathsf{aTL}_4$ acting on a linear combinations of $a_5 x, a_6 x, \dots, a_{10} x$ will result in another linear combination of those same basis elements; in other words, $\{a_i x | i = 5, 6, \dots, 10\}$ generate a submodule of $\mathsf{W}_{1/2,z}(3) \times_f^o \mathsf{S}_{1/2}(1)$, which we immediately recognize has $\mathsf{W}_{0,z_-}(4)$ for some $z_- \in \mathbb{C}^*$. By definition, $z_- + z_-^{-1}$ is the weight of the non-contractible loops in the standard module, which must be equal to the eigenvalue of $\bar{Y}$; however $\phi^o_{3,1}(\bar{Y}^{(3)}) = \bar{Y}^{(4)}$, so this eigenvalue must be $z(-\mathfrak{q})^{-1/2} + z^{-1}(-\mathfrak{q})^{1/2}$ (the eigenvalue of $\bar{Y}$ on $\mathsf{W}_{1/2,z}(3)$). We thus conclude[7] that $z_- = z(-\mathfrak{q})^{-1/2}$.

Similarly, the quotient of $\mathsf{W}_{1/2,z}(3) \times_f^o \mathsf{S}_{1/2}(1)$ by this submodule yields the standard module $\mathsf{W}_{1,z_+}(4)$. By definition, in $\mathsf{W}_{1,z_+}(4)$

 $\equiv z_+$  . (64)

By contrast, in $\mathsf{W}_{1/2,z}(3) \times_f^o \mathsf{S}_{1/2}(1)$

 $= \mathfrak{q}$  $+ z(-\mathfrak{q})^{1/2}$  , (65)

where we used the same trick as in equation (62). We thus conclude that $z_+ = (-\mathfrak{q})^{1/2} z$.

Finally, the eigenvalue of $Y^{(4)}$ on $\mathsf{W}_{0,z_-}(4)$ is $z_- + z_-^{-1}$, while on W_{1,z_+} it is $(-\mathfrak{q})z_+ + (-\mathfrak{q})^{-1} z_+^{-1}$, so this fusion product cannot be indecomposable unless

$$(-\mathfrak{q})z_+ + (-\mathfrak{q})^{-1} z_+^{-1} = z_- + z_-^{-1} \iff z^2 = (-\mathfrak{q})^2 \text{ or } (-\mathfrak{q})^2 = 1. \tag{66}$$

[7]By definition $\mathsf{W}_{0,z} = \mathsf{W}_{0,z^{-1}}$ so there is no ambiguity here.

It follows that, if $\mathfrak{q}$ and z are *generic*

$$\mathsf{W}_{1/2,z}(3) \times_f^o \mathsf{S}_{1/2} \simeq \mathsf{W}_{0,z(-\mathfrak{q})^{-1/2}}(4) \oplus \mathsf{W}_{1,z(-\mathfrak{q})^{1/2}}(4). \tag{67}$$

What if the parameters are not generic? If $z^2 = (-\mathfrak{q})^2$, a direct calculation shows that the defect operator Y has a Jordan block linking the two standard modules, and this fusion product is indecomposable. However, these new indecomposable modules are, for the moment, largely unclassified, and we plan to come back to this question in the close future.

More generally, we find that (for generic values of the parameters)

$$\mathsf{W}^u_{k,\delta} \times_f^o \mathsf{S}_T \simeq \bigoplus_{i=k-t}^{k+t} \mathsf{W}^u_{i,\delta} \simeq \bigoplus_{i=k-T}^{k+T} \mathsf{W}_{i,(-q)^{(i-k)}\delta}, \tag{68}$$

$$\mathsf{W}^o_{k,\delta} \times_f^u \mathsf{S}_T \simeq \bigoplus_{i=k-t}^{k+t} \mathsf{W}^o_{i,\delta} \simeq \bigoplus_{i=k-T}^{k+T} \mathsf{W}_{i,(-q)^{(k-i)}\delta}, \tag{69}$$

As shown in Appendix C, these results can also be derived by following the approach in [14], that is, by first establishing the branching rules from $\mathsf{aTL}_{n_1+n_2}$ to $\mathsf{aTL}_{n_1} \otimes \mathsf{TL}_{n_2}$ and then inferring the corresponding fusion product from Frobenius reciprocity.

4.2 The fusion quotient

The fusion product defined in the previous section implicitly assumed that the affine module M was only a left aTL_m-module. If M is an aTL_m-bimodule, then the fusion product $M \times_f^{u/o} V$ will also be a $(\mathsf{aTL}_{m+k}, \mathsf{aTL}_k)$ bimodule; given W a left aTL_{m+k} module, and V a left TL_k module, we define the fusion quotient by

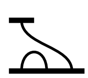
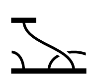
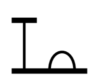
$$W \div_f^{u/o} V \equiv \mathsf{Hom}_{\mathsf{aTL}_{m+k}}\left(\mathsf{aTL}_m \times_f^{o/u} V, W\right). \tag{70}$$

Because $\mathsf{aTL}_m \times_f^{o/u} V$ is a *right* TL_m module, this Hom group is naturally a *left* aTL_m-module. Here's an example to show that the construction is actually quite natural despite its abstract definition. Let $V = \mathsf{TL}_k$ seen as a left module; one finds

$$\mathsf{aTL}_m \times_f V = \{a x_0 | a \in \mathsf{aTL}_{k+m}\}, \qquad x_0 = 1_{\mathsf{aTL}_{k+m}} \otimes_{\mathsf{aTL}_m \otimes_{\mathbb{C}} \mathsf{TL}_k} (1_{\mathsf{aTL}_m} \otimes_{\mathbb{C}} 1_{\mathsf{TL}_k}),$$
$$W \div_f \mathsf{TL}_k \simeq \{f_y : a x_0 \to a y | y \in U\}, \qquad a f_y \equiv f_{ay}. \tag{71}$$

One can now recognize that $W \div_f \mathsf{TL}_k$ is simply the restriction of W to aTL_m. More generally, if there exists an idempotent $a_0 \in \mathsf{TL}_k$ such that $V = \mathsf{TL}_k a_0$ then the fusion is then the subset $a_0 W$ of the restriction of W.

As a more concrete example we compute $\mathsf{W}_{1/2,z}(3) \div_f^o \mathsf{S}_{1/2}(1)$; since $\mathsf{S}_{1/2}(1) \simeq \mathsf{TL}_1$, this is simply the restriction of $\mathsf{W}_{1/2,z}(3)$ to aTL_2. We start by choosing a basis of the standard module:

$x_1 =$  $x_2 =$  $x_3 =$  . (72)

The element x_2 was chosen so that

$$\phi^o_{2,1}(e_1^{(2)}) x_2 = ((-\mathfrak{q})^{-1/2} z + (-\mathfrak{q})^{1/2} z^{-1}) x_1, \qquad \phi^o_{2,1}(e_2^{(2)}) x_2 = (\mathfrak{q} + \mathfrak{q}^{-1}) x_2, \qquad \phi^o_{2,1}(u^{(2)}) x_2 = x_1. \tag{73}$$

We thus recognize that $\{x_1, x_2\}$ span a submodule isomorphic to $\mathsf{W}_{0,z(-\mathfrak{q})^{-1/2}}(2)$. Furthermore,

$$\phi^o_{2,1}(e^{(2)}_1)x_3 = x_1, \qquad \phi^o_{2,1}(e^{(2)}_2)x_3 = -(-\mathfrak{q})^{-3/2}z^{-1}x_2, \phi^o_{2,1}(u^{(2)})x_3 = (-\mathfrak{q})^{1/2}zx_3 + \mathfrak{q}x_2, \tag{74}$$

so the quotient $\left(\mathsf{W}_{1/2,z}(3) \div^o_f \mathsf{S}_{1/2}(1)\right)/(\mathsf{W}_{0,z(-\mathfrak{q})^{-1/2}}(2))$ is isomorphic to $\mathsf{W}_{1,(-\mathfrak{q})^{1/2}z}(2)$. Finally, comparing the eigenvalues of Y on the two standard modules yields the conclusion: if $\mathfrak{q}$ and z are *generic*

$$\mathsf{W}_{1/2,z}(3) \div^o_f \mathsf{S}_{1/2}(1) \simeq \mathsf{W}_{0,z(-\mathfrak{q})^{-1/2}}(2) \oplus \mathsf{W}_{1,z(-\mathfrak{q})^{1/2}}(2). \tag{75}$$

More generally, we find that for generic values of the parameters

$$\mathsf{W}^u_{k,\delta}(n+m) \div^o_f \mathsf{S}_T(m) \simeq \bigoplus_{i=k-t}^{k+t} \mathsf{W}^u_{i,\delta}(n) \simeq \bigoplus_{i=k-T}^{k+T} \mathsf{W}_{i,(-q)^{(i-k)}\delta}(n) \tag{76}$$

$$\mathsf{W}^o_{k,\delta}(n+m) \div^u_f \mathsf{S}_T(m) \simeq \bigoplus_{i=k-t}^{k+t} \mathsf{W}^o_{i,\delta}(n) \simeq \bigoplus_{i=k-T}^{k+T} \mathsf{W}_{i,(-q)^{(k-i)}z}(n). \tag{77}$$

In these expressions it should be understood that all modules of the form $\mathsf{W}_{k,z}(n)$ with $n < k$ should be identified with the zero module.

4.3 Dualities between the two fusions

Aside from their possible interpretation as algebraic realisations of topological defects, the two types of fusion are of independent interest for the representation theory of aTL_n. As such, we mention here certain properties which they have, and which can be used to compute them. The first such property is that the fusion product and the fusion quotients are duals as functors, i.e. for any aTL_{n+m} module W, aTL_n module V and TL_m module U, there is a natural isomorphism

$$\mathsf{Hom}_{\mathsf{aTL}_{n+m}}\left(V \times^{u/o}_f U, W\right) \simeq \mathsf{Hom}_{\mathsf{aTL}_n}\left(V, W \div^{u/o}_f U\right). \tag{78}$$

It follows in particular that if one knows every fusion product, one can get back all the fusion quotient by using this duality, and vice versa.

The second property we mention is the associativity: for all aTL_n module W, TL_k module V and TL_m module U,

$$(W \times^{u/o}_f V) \times^{u/o}_f U \simeq W \times^{u/o}_f (V \times^r_f U) \simeq (W \times^{u/o}_f U) \times^{u/o}_f V, \tag{79}$$

where $\times^r_f$ is the fusion product in the regular Temperley-Lieb algebra, which was studied in detail in [12, 16]. Similarly, for all aTL_{n+k+m} module W, TL_k module V and TL_m module U,

$$(W \div^{u/o}_f V) \div^{u/o}_f U \simeq W \div^{u/o}_f (V \times^r_f U) \simeq (W \div^{u/o}_f U) \div^{u/o}_f V. \tag{80}$$

As an example, if we assume that $\mathfrak{q}$ is generic then for all $k \geq 0$

$$\mathsf{S}_k(n) \times^r_f \mathsf{S}_0(2) \simeq \mathsf{S}_k(n+2), \qquad \mathsf{S}_k(n) \times^r_f \mathsf{S}_{1/2}(1) \simeq \mathsf{S}_{k-1/2}(n+1) \oplus \mathsf{S}_{k+1/2}(n+1). \tag{81}$$

It follows that for a given aTL_n module W, knowing its fusion product (or quotient) with $\mathsf{S}_0(2m)$ and $\mathsf{S}_{1/2}(1)$ is enough to compute the fusion with all other standard modules by recurrence. Equations (68)-(69) and (76)-(77) were obtained in this manner.

4.4 Fusion and the Hamiltonian

We now go back to the problem of studying the defects in the direct channel, and the Hamiltonian H^u introduced in the beginning of this section 4.

Let W be a aTL_{n+m}-module, ρ be a non-zero idempotent of TL_m, and consider the classical Hamiltonian living in aTL_n:

$$H_n = \sum_{j=1}^{n} e_j^{(n)}. \tag{82}$$

Because the fusion quotient $\mathsf{W} \div_f^{u/o} (\mathsf{TL}_m \rho)$ can be seen as a restriction of W, one can express this Hamiltonian as an operator acting directly on W:

$$H_n^u = \phi_{n,m}^u\big(H_n\big) = \sum_{j=1}^{n-1} e_j^{(n+m)} + \mu_{n,m}^{-1} e_n^{(n+m)} \mu_{n,m} \rho, \tag{83}$$

$$H_n^o = \phi_{n,m}^o\big(H_n\big) = \sum_{j=1}^{n-1} e_j^{(n+m)} + \nu_{n,m} e_n^{(n+m)} \nu_{n,m}^{-1} \rho, \tag{84}$$

where $\mu_{n,m} = g_n g_{n+1} \dots g_{n+m}$, $\nu_{n,m} = g_{n+m} \dots g_n$. Choosing the idempotent ρ to correspond to a representation of spin $m/2$, or the standard module on m strands with m through lines, these are precisely the expressions obtained from spin chains with impurities.

As a corollary of the preceding discussion we formulate our main result on the spectral problem of the defect Hamiltonians:

Theorem 4.1 *Let $\rho \in \mathsf{TL}_m$ be an idempotent such that $\mathsf{TL}_m \rho \simeq V$, then for any aTL_{n+m}-module M the Hamiltonian $H_n^{u/o}$ is similar (as a matrix) to the direct sum of the classical Hamiltonian H_n acting on $M \div_f^{u/d} V$ and a zero matrix of dimension $dim((1-\rho)M)$.*

Similarly, the transfer matrix acting on this fused module is precisely the one in Fig. 6 obtained by adding a cluster of lines going under (or over) the other lines in the lattice. This strongly suggests that the fusion quotient is indeed the right algebraic construction for these defects. However it should be mentioned that for generic values of the parameters the fusion product and quotients are equivalent in the limit; it follows that while the Hamiltonian acting on the fusion product does not have such a simple interpretation it will produce the same spectrum in the limit.

4.5 Example of quotient: the twisted XXZ spin chain

The twisted XXZ spin chain on n sites can be realized by the Hamiltonian $H_n(Q)$ expressed in terms of the usual Pauli matrices acting on $(\mathbb{C}_2)^n$:

$$H_n(Q) = \sum_{j=1}^{n} \left(\sigma_j^- \sigma_{j+1}^+ + \sigma_{j+1}^- \sigma_j^+ + \frac{\mathfrak{q} + \mathfrak{q}^{-1}}{4} \left(\sigma_j^z \sigma_{j+1}^z - 1 \right) \right) = -\sum_{j=1}^{n} e_j, \tag{85}$$

where $\sigma^\pm = 1/2(\sigma_j^x \pm i\sigma_j^y)$ are the usual ladder operators, Q is a non-zero complex number, and the boundary conditions are

$$\sigma_{n+1}^z \equiv \sigma_1^z, \qquad \sigma_{n+1}^\pm \equiv Q^{\mp 2} \sigma_1^\pm. \tag{86}$$

The model is unitary if Q is on the unit circle in $\mathbb{C}$. The Temperley-Lieb generators are

$$-e_j \equiv \sigma_j^- \sigma_{j+1}^+ + \sigma_{j+1}^- \sigma_j^+ + \frac{\mathfrak{q}+\mathfrak{q}^{-1}}{4}\left(\sigma_j^z \sigma_{j+1}^z - 1\right) + \frac{\mathfrak{q}-\mathfrak{q}^{-1}}{4}(\sigma_j^z - \sigma_{j+1}^z), \tag{87}$$

with the twist

$$u = (-1)^{n/2} Q^{-\sigma_1^z} s_1 \dots s_{n-1}, \qquad s_j = \sigma_j^- \sigma_{j+1}^+ + \sigma_j^+ \sigma_{j+1}^- + \frac{1}{2}(\sigma_j^z \sigma_{j+1}^z + 1). \tag{88}$$

A quick calculation shows that the hoop operators are[8]

$$Y = (-1)^n \left(\mathfrak{q}^{S_z} Q^{-1} + \mathfrak{q}^{-S_z} Q\right), \qquad \bar{Y} = \mathfrak{q}^{S_z} Q + \mathfrak{q}^{-S_z} Q^{-1}, \tag{89}$$

with $S_z = \frac{1}{2}\sum_{j=1}^n \sigma_j^z$ the total spin. Our goal is now to impose a defect of spin 1/2 on this chain, which according to our formalism consist in computing the fusion quotient of its Hilbert space by $\mathsf{S}_1(1) = \mathsf{TL}_1$. This specific defect corresponds to a simple restriction from aTL_n to aTL_{n+1}, so the new Hamiltonian with a defect is either

$$H_{n-1}^u(Q) = -\sum_{j=1}^{n-1} \phi_{n-1,1}^u(e_j^{(n-1)}) = -\sum_{j=1}^{n-2} e_j^{(n)} - g_n^{(n)} e_{n-1}^{(n)} (g_n^{(n)})^{-1}, \tag{90}$$

or

$$H_{n-1}^o(Q) = -\sum_{j=1}^{n-1} \phi_{n-1,1}^o(e_j^{(n-1)}) = -\sum_{j=1}^{n-2} e_j^{(n)} - (g_n^{(n)})^{-1} e_{n-1}^{(n)} g_n^{(n)}, \tag{91}$$

for a defect that goes under or over the other lines, respectively. Using the explicit construction of e_{n-1}, g_n one finds

$$H_{n-1}^u(Q) = \sum_j^{n-1} (a_j^- a_{j+1}^+ + a_{j+1}^- a_j^+ + \frac{\mathfrak{q}+\mathfrak{q}^{-1}}{4}\left(a_j^z a_{j+1}^z - 1\right)) + \left((1-\mathfrak{q}^{2a_1^z})a_{n-1}^- + Q^2(1-\mathfrak{q}^{-2a_{n-1}^z})a_1^-\right)\sigma_n^+,$$

where we defined new operators $a_j^k = \sigma_j^k$, $k = z, \pm$, $j = 1, 2, \dots n-1$, with boundary conditions

$$a_n^z \equiv a_1^z, \qquad a_n^\pm \equiv (Q^2 \mathfrak{q}^{-\sigma_n^z})^{\mp 1} a_1^\pm. \tag{92}$$

It follows that

$$H_{n-1}^u(Q) \sim \begin{pmatrix} & (\dots)\otimes|\uparrow\rangle & (\dots)\otimes|\downarrow\rangle \\ & H_{n-1}(-Q\mathfrak{q}^{-1/2}) & \Delta \\ & 0 & H_{n-1}(-Q\mathfrak{q}^{1/2}) \end{pmatrix}, \qquad \Delta = (1-\mathfrak{q}^{2a_1^z})a_{n-1}^- + Q^2(1-\mathfrak{q}^{-2a_{n-1}^z})a_1^-. \tag{93}$$

A straightforward calculation then shows the defect operators:

$$Y = (-1)^n(\mathfrak{q}^{S_z} Q^{-1} + \mathfrak{q}^{-S_z} Q) = (-1)^{n-1}\left(\mathfrak{q}^{S_z - \frac{1}{2}\sigma_n^z}\left(-Q\mathfrak{q}^{-\frac{1}{2}\sigma_n^z}\right)^{-1} + \mathfrak{q}^{-S_z + \frac{1}{2}\sigma_n^z}\left(-Q\mathfrak{q}^{-\frac{1}{2}\sigma_n^z}\right)\right), \tag{94}$$

[8] In everything that follows, one should understand that for all matrix A, $\mathfrak{q}^A \equiv (-\mathfrak{q})^A(-1)^{-A}$. We simplify these expressions to lighten the notation but one should be careful when verifying these results numerically.

$$\bar{Y} \sim \begin{array}{c} \quad (\ldots)\otimes|\uparrow\rangle \qquad\qquad (\ldots)\otimes|\downarrow\rangle \\ \begin{pmatrix} Q_-\mathfrak{q}^{\tilde{S}_z}+Q_-^{-1}\mathfrak{q}^{-\tilde{S}_z} & Q(\mathfrak{q}-\mathfrak{q}^{-1})^2\tilde{S}_- \\ 0 & Q_+\mathfrak{q}^{\tilde{S}_z}+Q_+^{-1}\mathfrak{q}^{-\tilde{S}_z} \end{pmatrix} \end{array}, \tag{95}$$

where $Q_\pm \equiv -Q\mathfrak{q}^{\pm 1/2}$, and $\tilde{S}_-$, $\mathfrak{q}^{\pm\tilde{S}_z}$ are the standard $U_\mathfrak{q}(\mathfrak{sl}_2)$ generators on $n-1$ spins

$$\tilde{S}_- = \sum_{i=1}^{n-1} (\mathfrak{q})^{\sum_{j=1}^{i-1}\sigma_j^z}\sigma_i^-(\mathfrak{q})^{-\sum_{j=i+1}^{n-1}\sigma_j^z}, \qquad \mathfrak{q}^{\pm\tilde{S}_z} = \mathfrak{q}^{\sum_{j=1}^{n-1}\sigma_j^z/2}. \tag{96}$$

Note that $\bar{Y}$ can be diagonalized if and only if $(Q-\mathfrak{q}^{-S_z})(Q+\mathfrak{q}^{-S_z})$ is an invertible matrix, which can be verified by comparing its eigenvalues in the $\sigma_n^z = \pm 1$ sectors. It follows in particular that the Hamiltonian (93) cannot have a Jordan block linking the $\sigma_n^z = \pm 1$ sectors if Q is generic.

Similarly, one finds

$$H_{n-1}^o(Q) \sim \begin{array}{c} (\ldots)\otimes|\uparrow\rangle \qquad\qquad (\ldots)\otimes|\downarrow\rangle \\ \begin{pmatrix} H_{n-1}(Q\mathfrak{q}^{1/2}) & 0 \\ \Delta & H_{n-1}(Q\mathfrak{q}^{-1/2}) \end{pmatrix} \end{array}, \qquad \Delta = (1-\mathfrak{q}^{2a_1^z})a_{n-1}^+ + Q^{-2}(1-\mathfrak{q}^{-2a_{n-1}^z})a_1^+. \tag{97}$$

Note that in each of these expressions, the off-diagonal term Δ can only link sectors of $H_{n-1}(Q\mathfrak{q}^{\pm 1/2})$ corresponding to different total spin ($\sum_{j=1}^{n-1} a_j$) because of the ladder operators appearing in it.

5 Conclusion: connection to CFT

In order to provide a lattice analogue of CFT topological defects X satisfying (1), we have defined and studied in a model-independent way operators on the lattice that commute with the local interactions given by the TL elements—the central elements Y and $\bar{Y}$ in aTL_n—and have demonstrated their interesting properties. From the crossed-channel point of view, these defect operators generate an algebra spanned by Y_j, $\bar{Y}_j$, and their products, that has structure constants or fusion rules (50) and (51) resembling the chiral and anti-chiral fusion rules of Virasoro Kac modules of type $(1, s)$ where $s = 2j+1$. We recall that the Kac modules are obtained as quotients of Verma modules of the conformal weight $h_{1,s}$ by the submodule generated by the singular vector at the level $h_{1,s} + s$.

The analogy with CFT goes further: Recall that at least in rational CFT a topological defect can be seen as a map from the set of chiral primary fields to the ring of endomorphisms of the Hilbert space of the full non-chiral CFT. In much the same way our maps Y^m and $\bar{Y}^m$ from Fig. 4 defining the defect operators send ideals in the open or regular TL algebra (which are known to correspond to chiral primary fields of conformal weight $h_{1,s}$) to central elements in affine TL algebra which are realized as endomorphisms of the bulk lattice model, e.g. of periodic spin-chains.

We saw that the higher-spin defects Y_j and $\bar{Y}_j$ (48)-(49) carry some sort of internal structure “living” on the horizontal non-contractible loops. From the direct-channel point of view, or after a modular transformation, this internal structure was realized in Section 4 as some sort of impurities in the spatial direction. Therefore, we have just rewritten the defects Y_j and $\bar{Y}_j$ in the Hamiltonian formulation. Interestingly, the problem of spectrum with impurities was reformulated in algebraic terms as a rather simple fusion product of affine and regular TL representations which is a sort of combination of the constructions in [13, 14] and [15] that we review in Appendix B.

So far we have defined and studied lattice defects that do not depend on a spectral parameter. Let us call these defects of *first type*. However, there is some evidence that there should be a *second type* of (lattice) defects that do depend on a spectral parameter. Though they are not central in aTL_n, but possibly become topological defects X, i.e. they satisfy (1), in the continuum limit only. We will address studying these defects of the second type in our next paper where an identification with Virasoro Kac modules of the type $(r, 1)$ is expected.

It is important for several reasons to try to define what we call lattice defects in a precise mathematical way and in higher generality, for a possible application to more general lattice models not necessarily based on TL interactions. For the first kind of defects, from the results obtained in this work, we are approaching a mathematical definition of (an algebra of) defects for general lattice algebras (e.g. $\mathsf{aTL}_n(\mathfrak{q})$, Birman-Wenzl-Murakami, Brauer algebras, etc):

Definition: *In a lattice algebra A, a space of defects D of the first type is a subspace in the center of A such that it forms a Verlinde algebra.*

Note that not any central element in a lattice algebra corresponds to a defect operator; it should also have nice properties that reflect know properties from the CFT side. That is why we demand that the space of defects forms a Verlinde algebra. First of all this implies the presence of a special basis in this algebra with structure constants being non-negative integers. Secondly, the idea is that these integer numbers should correspond to fusion rules of corresponding representations of an (anti-)chiral algebra, e.g. Virasoro.

We have indeed recovered these two aspects in our case of $A = \mathsf{aTL}_n(\mathfrak{q})$, where we identified D as the symmetric center Z_{sym} of $\mathsf{aTL}_n(\mathfrak{q})$, and the latter as a Verlinde algebra generated by Y_j and $\bar{Y}_j$ where the structure constants do not depend on n and correspond to fusion rules of chiral and anti-chiral Virasoro representations of type $(1, s)$.[9] This is here shown to be true for the generic $\mathfrak{q}$ case where the fusion rules might look rather trivial, since they are $sl(2)$ type fusion after all. The situation is not so trivial in degenerate cases (where $\mathfrak{q}$ is a root of unity) that we will describe in one of our forthcoming papers on the subject, with applications to minimal models as well as LCFTs. There, a connection to Virasoro fusion rules also holds, although it is much less evident due to more involved representation theory.

However, this is not the end of the story. Any Verlinde algebra has the third aspect: it admits a modular S-transformation that "diagonalizes" the fusion rules. For the moment we have concentrated on the first two aspects only. It is, of course, an important problem to properly define and analyze such S-transformations in a precise algebraic way, and hopefully it will reflect the modular transformation on the lattice. We hope to come back to this problem soon.

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[9] Strictly speaking our algebra of defects is the product of two Verlinde algebras, for chiral and anti-chiral Virasoro representations, modulo non-linear algebraic relations between Y and $\bar{Y}$.

A Proofs and rigors

We collect in this appendix the proofs of certain technical results used in this work.

A.1 Topological defects with a higher spin.

We show here how to obtain the expressions for topological defects with higher-spins given in section 3.2, i.e.

$$Y_{j/2} = \mathsf{U}_{2j}(Y/2). \tag{98}$$

First we show that the result is independent of the choice of idempotent we make.

Let A be some finite dimensional $\mathbb{C}$-algebra, ρ_1, ρ_2 be two idempotents such that $A\rho_1 \simeq A\rho_2$ as left A modules, and let F be any function defined on A such that for all $a, b \in A$, $F(ab) = F(ba)$ (in other words F is a trace). We know that

$$\mathsf{Hom}_A(A\rho_1, A\rho_2) \simeq \rho_1 A \rho_2, \qquad \mathsf{Hom}_A(A\rho_2, A\rho_1) \simeq \rho_2 A \rho_1, \tag{99}$$

where the isomorphism is obtained by right-multiplication. For instance,

$$(f : A\rho_1 \to A\rho_2) \to \rho_1 f(\rho_1)\rho_2, \qquad \rho_1 a \rho_2 \to (\underbrace{b\rho_1 \to b\rho_1 a \rho_2}_{\in \mathsf{Hom}_A(A\rho_1, A\rho_2)}). \tag{100}$$

Because $A\rho_1 \simeq A\rho_2$, it follows that there exists $a, b \in A$ such $\rho_1 a \rho_2$ is the function that sends $A\rho_1 \to A\rho_2$ and $\rho_2 b \rho_1$ is the function that sends $A\rho_2 \to A\rho_1$. In particular, if $A\rho_1$ is irreducible this means that $\rho_1 a \rho_2 b \rho_1 = \alpha \rho_1$ and $\rho_2 b \rho_1 a \rho_2 = \gamma \rho_2$, for some non-zero complex numbers α, γ. However one quickly verifies that

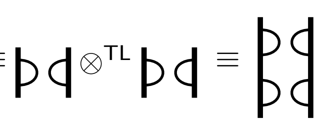
$$\gamma^2 \rho_2 = (\rho_2 b \rho_1 a \rho_2)^2 = \rho_2 b(\rho_1 a \rho_2 b \rho_1) a \rho_2 = \alpha\gamma\rho_2,$$

so $\alpha = \gamma$ and we can thus choose a, b such that $\alpha = \gamma = 1$. Now by hypothesis the function F is cyclic and ρ_1 and ρ_2 are idempotents so

$$F(\rho_1) = F(\rho_1 a \rho_2 b \rho_1) = F(\rho_2 b \rho_1 a \rho_2) = F(\rho_2). \tag{101}$$

Next, we remark that for any elements $a \in \mathsf{TL}_n$, $b \in \mathsf{TL}_m$, $Y^{n+m}(a \otimes^{\mathsf{TL}} b) = Y^n(a) Y^m(b)$ where $\otimes^{\mathsf{TL}}$ is the tensor product in the Temperley-Lieb category, obtained by joining diagrams side by side. For instance,

$$e_1 \otimes^{\mathsf{TL}} e_1 \equiv$$



(102)

It follows in particular that for any idempotent $a \in \mathsf{TL}_n$,

$$Y^{n+1}(a \otimes^{\mathsf{TL}} 1_{\mathsf{TL}_1}) = Y_{1/2} Y^n(a). \tag{103}$$

Now if the idempotent a is such that $\mathsf{TL}_n a \simeq \mathsf{S}_k(n)$, one can show that there exists a decomposition of the idempotent $(a \otimes^{\mathsf{TL}} 1_{\mathsf{TL}_1}) = a_- + a_+$, where $a_\pm$ are orthogonal idempotents such that $\mathsf{TL}_{n+1} a_\pm \simeq \mathsf{S}_{k\pm 1/2}(n+1)$. This is done by using the fusion rules in the regular Temperley-Lieb family of algebras (see [12, 16]). It thus follows that

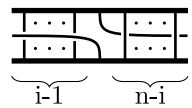
$$Y_{(k+1)/2} = Y_{1/2} Y_{k/2} - Y_{(k-1)/2}, \qquad Y_0 \equiv 1_{\mathsf{aTL}_n}, \tag{104}$$

which is the Chebyshev recurrence relation, giving (98).

A.2 The Jucys-Murphy elements

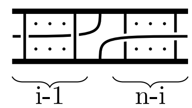
Recall that aTL_n admits two non-equivalent sets of Jucys-Murphy elements defined by (up to a normalization)

$$J_1 \equiv \bar{b}, \qquad J_i \equiv g_{i-1} J_{i-1} g_{i-1} = (-\mathfrak{q})^{-3/2}$$



$$, \qquad i = 2, \ldots n \tag{105}$$

$$M_1 \equiv b, \qquad M_i \equiv g_{i-1} M_{i-1} g_{i-1} = (-\mathfrak{q})^{-3/2}$$



$$, \qquad i = 2, \ldots n. \tag{106}$$

The normalization chosen is such that for any symmetric Laurent polynomials $p(x_1, x_2, \ldots, x_n)$, the element $p((-q)^1 J_1, (-q)^2 J_2, \ldots, (-q)^n J_n)$ is central in aTL_n. We wish to prove the identities (55) and (56):

$$C_k(n) + C_{-k}(n) = (-\mathfrak{q})^{-nk} \bar{C}_k(n) + (-\mathfrak{q})^{nk} \bar{C}_{-k}(n) = 2[n]_k \mathsf{T}_k(\bar{Y}/2), \tag{107}$$

$$\bar{C}_k(n) + \bar{C}_{-k}(n) = (-\mathfrak{q})^{-nk} C_k(n) + (-\mathfrak{q})^{nk} C_{-k}(n) = 2[n]_k \mathsf{T}_k(Y/2), \tag{108}$$

where

$$C_k(n) = \sum_{i=1}^{n} ((-\mathfrak{q})^{i+1} M_i)^k, \qquad \bar{C}_k(n) = \sum_{i=1}^{n} ((-\mathfrak{q})^{i+1} J_i)^k. \tag{109}$$

However, the proofs for the identities involving M are identical to those involving the Js, so we shall only prove the two identities involving $C_k(n)$. The proof of this result relies on two key observations; the first is the identity

$$[i]_k (-\mathfrak{q})^{\pm k} - [i-1]_k = (-\mathfrak{q})^{\pm ki}, \qquad i = 0, 1, \ldots. \tag{110}$$

The second observation is that for all $i = 1, \ldots, n-1$ we have the relations in $\mathsf{aTL}_n(\mathfrak{q})$:

$$\bar{X}_{i+1} \equiv (-\mathfrak{q})^2 J_i + (-\mathfrak{q})^{-2} J_i^{-1} = (-\mathfrak{q}) J_{i+1} + (-\mathfrak{q})^{-1} J_{i+1}^{-1}, \tag{111}$$

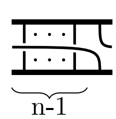
$$X_{i+1} \equiv (-\mathfrak{q})^2 M_i + (-\mathfrak{q})^{-2} M_i^{-1} = (-\mathfrak{q}) M_{i+1} + (-\mathfrak{q})^{-1} M_{i+1}^{-1}, \tag{112}$$

$$Y = (-\mathfrak{q})^2 J_n + (-\mathfrak{q})^{-2} J_n^{-1}, \tag{113}$$

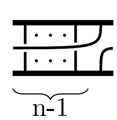
$$\bar{Y} = (-\mathfrak{q})^2 M_n + (-\mathfrak{q})^{-2} M_n^{-1}. \tag{114}$$

These can all be proven in the same way, by showing that both sides of these equality correspond to the same diagram. For instance

$$(-\mathfrak{q})^2 J_n + (-\mathfrak{q})^{-2} J_n^{-1} = (-\mathfrak{q})^{1/2}$$



$$+ (-\mathfrak{q})^{-1/2}$$



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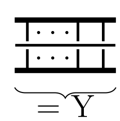


$$, \tag{115}$$

where we used the definition of the braids. Putting the two observations together gives the following relations, for all $i = 1, \dots, n-1$,

$$\begin{aligned}
((-\mathsf{q})^{i+1}J_i)^k + ((-\mathsf{q})^{i+1}J_i)^{-k} &= [i]_k \left(((-\mathsf{q})^2 J_i)^k + ((-\mathsf{q})^2 J_i)^{-k}\right) - [i-1]_k \left(((-\mathsf{q})J_i)^k + ((-\mathsf{q})J_i)^{-k}\right), \\
&= 2[i]_k \mathsf{T}_k \left(\frac{(-\mathsf{q})^2 J_i + (-\mathsf{q})^{-2} J_i^{-1}}{2}\right) - 2[i-1]_k \mathsf{T}_k \left(\frac{(-\mathsf{q}) J_i + (-\mathsf{q})^{-1} J_i^{-1}}{2}\right), \\
&= 2[i]_k \mathsf{T}_k \left(\frac{\bar{X}_{i+1}}{2}\right) - 2[i-1]_k \mathsf{T}_k \left(\frac{\bar{X}_i}{2}\right), \\
((-\mathsf{q})^{n+1}J_n)^k + ((-\mathsf{q})^{n+1}J_n)^{-k} &= 2[n]_k \mathsf{T}_k \left(\frac{Y}{2}\right) - 2[n-1]_k \mathsf{T}_k \left(\frac{\bar{X}_n}{2}\right),
\end{aligned}$$

where we used the fact that for all non-zero x

$$2\mathsf{T}_k((x + x^{-1})/2) = x^k + x^{-k}.$$

Then, it follows that

$$\begin{aligned}
C_k(n) + C_{-k}(n) &= \sum_{i=1}^{n} \left(((-\mathsf{q})^{i+1}J_i)^k + ((-\mathsf{q})^{i+1}J_i)^{-k}\right) \\
&= 2[1]_k \mathsf{T}_k(\bar{X}_2/2) + \sum_{i=2}^{n} \left(((-\mathsf{q})^{i+1}J_i)^k + ((-\mathsf{q})^{i+1}J_i)^{-k}\right) \\
&= 2[2]_k \mathsf{T}_k(\bar{X}_3/2) + \sum_{i=3}^{n} \left(((-\mathsf{q})^{i+1}J_i)^k + ((-\mathsf{q})^{i+1}J_i)^{-k}\right) \\
&= \dots \\
&= 2[n]_k \mathsf{T}_k(Y/2).
\end{aligned}$$

Using very similar arguments, we have

$$\begin{aligned}
((-\mathsf{q})^{i+1-n}J_i)^k + ((-\mathsf{q})^{i+1-n}J_i)^{-k} &= -[n-i]_k \left(((-\mathsf{q})^2 J_i)^k + ((-\mathsf{q})^2 J_i)^{-k}\right) + [n+1-i]_k \left(((-\mathsf{q})J_i)^k + ((-\mathsf{q})J_i\right. \\
&= 2[n+1-i]_k \mathsf{T}_k(\bar{X}_i/2) - 2[n-i]_k \mathsf{T}_k(\bar{X}_{i+1}/2) \\
((-\mathsf{q})^1 J_n)^k + ((-\mathsf{q})^1 J_n)^{-k} &= 2\mathsf{T}_k(\bar{X}_n/2),
\end{aligned}$$

which give

$$\begin{aligned}
(-\mathsf{q})^{-nk} C_k[n] + (-\mathsf{q})^{nk} C_{-k}[n] &= \sum_{i=1}^{n} \left(((-\mathsf{q})^{i+1-n}J_i)^k + ((-\mathsf{q})^{i+1-n}J_i)^{-k}\right) \\
&= 2[n+1-n]_k \mathsf{T}_k(\bar{X}_n/2) + \sum_{i=1}^{n-1} \left(((-\mathsf{q})^{i+1-n}J_i)^k + ((-\mathsf{q})^{i+1-n}J_i)^{-k}\right) \\
&= 2[n+1-(n-1)]_k \mathsf{T}_k(\bar{X}_{n-1}/2) + \sum_{i=1}^{n-2} \left(((-\mathsf{q})^{i+1-n}J_i)^k + ((-\mathsf{q})^{i+1-n}J_i)^{-k}\right) \\
&= \dots \\
&= 2[n+1-1]_k \mathsf{T}_k(\bar{X}_1/2) \equiv 2[n]_k \mathsf{T}_k(\bar{Y}/2),
\end{aligned}$$

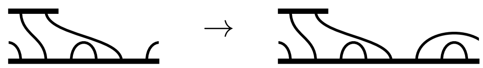
where we used the fact that $X_1 \equiv \bar{Y}$ by definition.

A.3 Fusion with standard modules

We explain here how to compute the fusion product/quotient of standard modules.

A.3.1 $\mathsf{W}_{k,z}(n+2) \div^{u/o} \mathsf{S}_0(2)$

Assuming that $\mathfrak{q}^2 \neq -1$, the primitive idempotent corresponding to the projective module $\mathsf{S}_0(2)$ is $\rho_0 \equiv (\mathfrak{q}+\mathfrak{q}^{-1})^{-1} e_1$. According to our definition of the fusion quotient, we must consider the subspace $W_0 \equiv \rho_0 \mathsf{W}_{k,z}(n+2)$ with the action of aTL_n obtained from the morphism of algebras $\phi^{u/o}_{n,2}$; however, in this case there is a map $\psi : \mathsf{W}_{k,z}(n) \to W_0$ which consists in adding two positions linked with an arc on the right of every diagram in $\mathsf{W}_{k,z}(n)$, for instance

 (116)

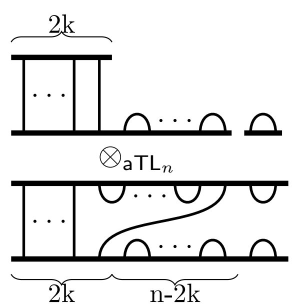
One sees directly that this map defines a morphism of modules, and that the resulting sub-module of W_0 is the same for both types of fusion. Furthermore, any diagram in $\mathsf{W}_{k,z}(n+2)$ is sent to one of the form $\psi(x)$ by the action of the idempotent ρ_0, so this morphism is surjective. Because the map ψ is obviously injective as well, it must be an isomorphism and we thus get

$$\mathsf{W}_{k,z}(n+2) \div^{u/o}_{f} \mathsf{S}_0(2) \simeq \mathsf{W}_{k,z}(n), \tag{117}$$

where it is understood that $k \leq n/2$ because otherwise the e_{n+1} would acts as zero on $\mathsf{W}_{k,z}(n+2)$.

A.3.2 $\mathsf{W}_{k,z}(n) \times^{u/o}_{f} \mathsf{S}_0(2)$

Because both $\mathsf{W}_{k,z}(n)$ and $\mathsf{S}_0(2)$ are cyclic, so is their fusion product; in particular one can write $\mathsf{W}_{k,z}(n) \times^{u/o}_{f} \mathsf{S}_0(2) = \mathsf{aTL}_{n+2} x$ with

$x \equiv$  . (118)

We also recall our diagram notation for the fusion product: the diagram at the bottom is the element of aTL_{n+2}, the one on the top left corner is the element of $\mathsf{W}_{k,z}(n)$, and the one on the top right corner is the element of $\mathsf{S}_0(2)$. Note that closing together any of the through lines in the bottom diagram of x will yield the zero element, because it will be able to pass through the tensor product. It follows that this fusion product is isomorphic to a standard module of the form $\mathsf{W}_{k,z'}(n+2)$ for some z'. Note also that because of the duality between the two types of fusion

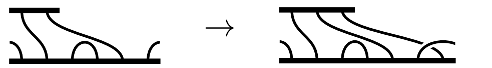
$$\mathrm{Hom}_{\mathsf{aTL}_{n+2}}\Big(\mathsf{W}_{k,z}(n) \times^{u/o}_{f} \mathsf{S}_0(2), \mathsf{W}_{k,z}(n+2)\Big) \simeq \mathrm{Hom}_{\mathsf{aTL}_n}\big(\mathsf{W}_{k,z}(n), \underbrace{\mathsf{W}_{k,z}(n+2) \div^{u/o}_{f} \mathsf{S}_0(2)}_{\simeq \mathsf{W}_{k,z}(n)}\big) \simeq \mathbb{C}, \tag{119}$$

so we conclude that $z' = z$, and thus

$$\mathsf{W}_{k,z}(n) \times^{u/o}_{f} \mathsf{S}_0(2) \simeq \mathsf{W}_{k,z}(n+2). \tag{120}$$

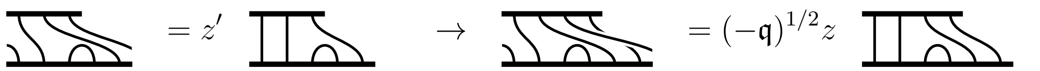
A.3.3 $\mathsf{W}_{k,z}(n+1) \div_f^{u/o} \mathsf{S}_{1/2}(1)$, ($k \neq 0$)

Both types of fusion are very similar so we focus on the u-type. The primitive idempotent corresponding to $\mathsf{S}_{1/2}(1)$ is simply the identity so the fusion quotient is the full restriction of $\mathsf{W}_{k,z}(n+1)$. There is then a map $\psi : \mathsf{W}_{k-1/2,z'}(n) \to \mathsf{W}_{k,z}(n+1)$ which consists in adding a single position on the right of every diagram in $\mathsf{W}_{k-1/2,z'}(n)$ and adding a through line to it which passes under every other line. For instance,



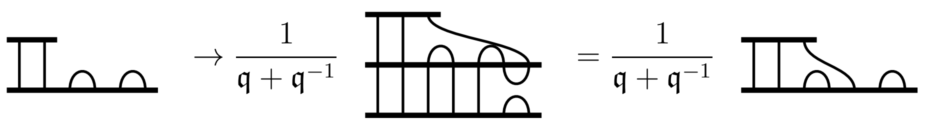
(121)

One can see that this map (extended linearly) indeed defines an injective morphism of aTL_n module if $z' = (-\mathfrak{q})^{1/2} z$; this condition on z can be seen by observing that



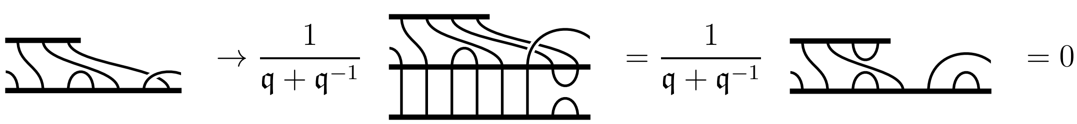
(122)

Next, consider the map $\phi : \mathsf{W}_{k,z}(n+1) \to \mathsf{W}_{k+1/2,z(-\mathfrak{q})^{-1/2}}(n+2)$, defined by $\phi = (\mathfrak{q}+\mathfrak{q}^{-1})^{-1} e_{n+1}\psi$. In other words we add an extra through line, going under all others, at the right of each diagram in $\mathsf{W}_{k,z}(n+1)$ then multiply the result by the idempotent $(\mathfrak{q}+\mathfrak{q}^{-1})^{-1} e_{n+1}$. For instance, we get



(123)

Based on the previous results of this section, we recognize the the image of this map is $\mathsf{W}_{k+1/2,z(-\mathfrak{q})^{-1/2}}(n+2) \div_f^u \mathsf{S}_0(2) \simeq \mathsf{W}_{k+1/2,z(-\mathfrak{q})^{-1/2}}(n)$. It can then be shown that this map (extended linearly) defines a surjective morphism of aTL_n modules. Furthermore, one can see that the image of the first map ψ is contained in the kernel of the second map ϕ:



(124)

where we used the fact that diagrams with less than $k/2$ through lines are equivalent to the zero element in $\mathsf{W}_{k,z}(n)$.

Finally, note that

$$\mathsf{Dim}(\mathsf{W}_{k,z}(n+1) \div_f^u \mathsf{S}_1(1)) = \underbrace{\mathsf{Dim}(\mathsf{W}_{k,z}(n+1))}_{\binom{n+1}{(n+1)/2-k}} = \underbrace{\mathsf{Dim}(\mathsf{W}_{k-1/2,z'}(n))}_{\binom{n}{(n+1)/2-k}} + \underbrace{\mathsf{Dim}(\mathsf{W}_{k+1/2,z''}(n))}_{\binom{n}{(n-1)/2-k}}, \tag{125}$$

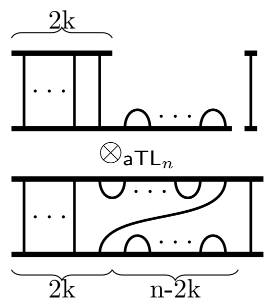
so the image of ψ is exactly the kernel of ϕ. Since the eigenvalues of $\bar{Y}$ are different on $\mathsf{W}_{k\pm1/2,z(-\mathfrak{q})^{\mp1/2}}(n)$, if z is generic, it follows that

$$\mathsf{W}_{k,z}(n+1) \div_f^u \mathsf{S}_{1/2}(1) \simeq \mathsf{W}_{k-1/2,z(-\mathfrak{q})^{1/2}}(n) \oplus \mathsf{W}_{k+1/2,z(-\mathfrak{q})^{-1/2}}(n). \tag{126}$$

What about when z is not generic? The eigenvalues of $\bar{Y}$ will be the same on the modules appearing in the previous direct sum if and only if $(-\mathfrak{q})^2 = 1$ or $z^2 = (-\mathfrak{q})^{k/2}$; one can verify directly that $\bar{Y}$ has a Jordan block when acting on the fusion quotient only in the later case. The modules appearing in the fusion quotient is then the indecomposable extension of two standard modules; while these are not yet classified, we shall show that they are indeed unique in our forthcoming work.

A.3.4 $\mathsf{W}_{k,z}(n+1) \times_f^{u/o} \mathsf{S}_{1/2}(1)$, $(k \neq 0)$

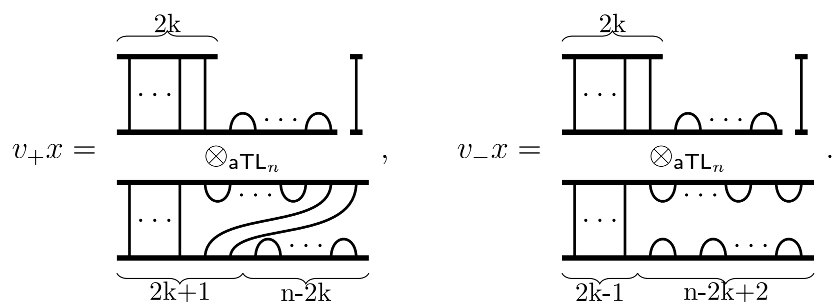
Both types of fusion are very similar so we again focus on the u-type. Since standard modules are always cyclic, so is their fusion and $\mathsf{W}_{k,z}(n) \times_f^u \mathsf{S}_{1/2}(1) = \mathsf{aTL}_{n+1} x$ with

$x \equiv$  . (127)

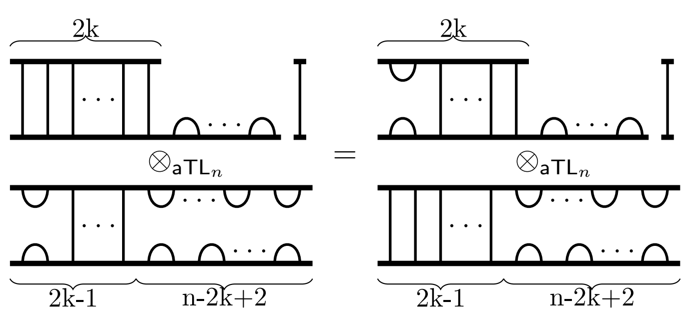
The module can be decomposed by defining diagrams

$$v_+ = e_{2k+2} e_{2k+4} \dots e_n, \qquad v_- = e_{2k} e_{2k+2} \dots e_n. \tag{128}$$

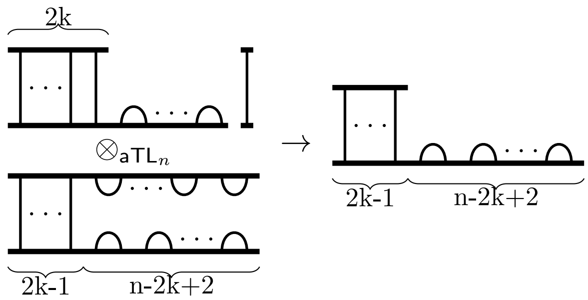
These were chosen such that



One verifies easily that $\mathsf{aTL}_{n+1} v_- x$ is a sub-module: any diagram acting on $v_- x$ will either contract two (or more) through lines together or shuffle around the arcs on the bottom boundary. However, the former will be trivial because, for instance,

$e_1 v_- x =$  $= 0,$ (129)

where we used the definition of the tensor product, and the definition of the standard modules. We thus conclude that $\mathsf{aTL}_{n+1} v_- x$ is a sub module isomorphic to $\mathsf{W}_{k-1/2,z'}(n+1)$ for some z', where the isomorphism is obtained by simply *cutting off* the top of the diagrams, i.e.



. (130)

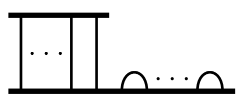
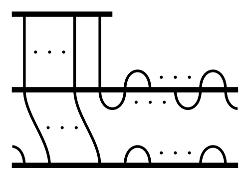
To find z' we use the duality between the two fusion:

$$\mathrm{Hom}_{\mathsf{aTL}_{n+1}}\Big(\mathsf{W}_{k,z}(n)\times_f^u \mathsf{S}_{\frac{1}{2}}(1),\mathsf{W}_{k-\frac{1}{2},z'}(n+1)\Big)\simeq \mathrm{Hom}_{\mathsf{aTL}_n}\big(\mathsf{W}_{k,z}(n),\ \underbrace{\mathsf{W}_{k-\frac{1}{2},z'}(n+1)\div_f^u \mathsf{S}_{\frac{1}{2}}(1)}_{\simeq \mathsf{W}_{k,z'(-\mathfrak{q})^{-1/2}}(n)\oplus \mathsf{W}_{k-1,z'(-\mathfrak{q})^{1/2}}(n)}\ \big)$$

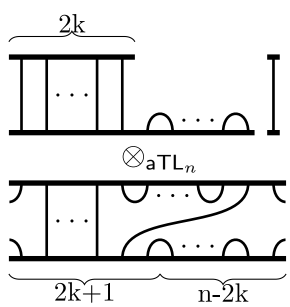
$$\simeq \delta_{z,z'(-\mathfrak{q})^{-\frac{1}{2}}}\mathbb{C},$$

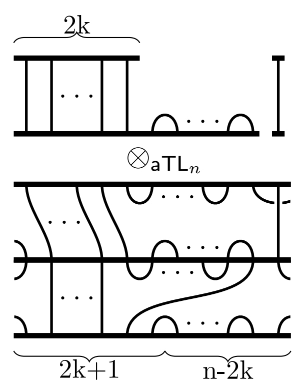
so it follows that $z'=z(-\mathfrak{q})^{1/2}$.

Note now that acting on v_+x with some diagram a can do two things: it can shuffle the arcs on the bottom boundary, and it can close pairs of through lines. One can verify that closing the two rightmost through lines together will produce an element of the form av_-x, and the same thing happens when closing the leftmost line with the rightmost one. To see this last assertion, observe that in $\mathsf{W}_{k,z}$

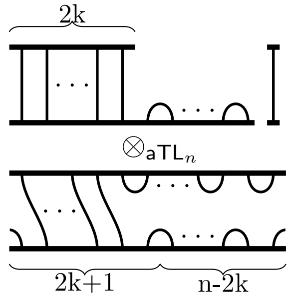
 $= z^{-1}$  , (131)

it thus follows that

$e_{2k+1}\dots e_{n-1}e_{n+1}v_+x =$ 

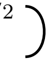
$= z^{-1}$ 

(132)

$= -(-\mathfrak{q})^{3/2}z^{-1}$ 

, (133)

where we used the closed braid identity

 $= -(-\mathfrak{q})^{3/2}$  . (134)

Repeating the arguments leading to the identification of $\mathsf{aTL}_{n+1}v_-x$, we finally obtain

$$(\mathsf{W}_{k,z}(n) \times_f^u \mathsf{S}_{1/2}(1))/(\mathsf{aTL}_{n+1}v_-x) = \mathsf{aTL}_{n+1}(v_+x + \mathsf{aTL}_{n+1}v_-x) \simeq \mathsf{W}_{k+1/2,z(-\mathfrak{q})^{-1/2}}(n+1). \quad (135)$$

To complete the decomposition, we need to figure out if this quotient splits, i.e. if the fusion product is the direct sum of two standard modules, or if it's their indecomposable extension. However, if z is generic then $\bar{Y}$ has distinct eigenvalues on $\mathsf{W}_{k\pm1/2,z(-\mathfrak{q})^{\mp1/2}}(n+1)$ so the quotient must split, and thus

$$\mathsf{W}_{k,z}(n) \times_f^u \mathsf{S}_{1/2}(1) \simeq \mathsf{W}_{k-1/2,z(-\mathfrak{q})^{1/2}}(n+1) \oplus \mathsf{W}_{k+1/2,z(-\mathfrak{q})^{-1/2}}(n+1). \quad (136)$$

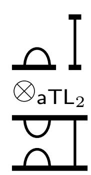
If $z^2 = (-\mathfrak{q})^k$ then the quotient does not split and the fusion product is then the indecomposable extension of the two standard modules appearing in the previous direct sum.

A.3.5 $\mathsf{W}_{0,z}(n) \times_f^u \mathsf{S}_1(1)$

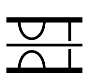
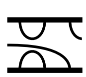
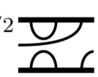
By the results in the previous sections, we know that

$$\begin{aligned}\mathsf{W}_{0,z}(2m) \times_f^u \mathsf{S}_{1/2}(1) &\simeq (\mathsf{W}_{0,z}(2) \times_f^u \mathsf{S}_0(2(m-1))) \times_f^u \mathsf{S}_{1/2}(1)\\ &\simeq (\mathsf{W}_{0,z}(2) \times_f^u \mathsf{S}_{1/2}(1)) \times_f^u \mathsf{S}_0(2(m-1)),\end{aligned}$$

so we focus on the case $n = 2$ and the other cases will follow directly from it. Using the same reasoning as in the $k \neq 0$ cases, we find that $\mathsf{W}_{0,z}(2) \times_f^u \mathsf{S}_{1/2}(1) = \mathsf{aTL}_3 x$, with

$x \equiv$  . (137)

Now acting on x with any diagram can only move around the arc at the bottom of x, so the fusion product is generated by elements of the form $u^i x$ for $i \in \mathbb{Z}$. However,

$Ye_1 =$  $= (-\mathfrak{q})^{1/2}$  $+ (-\mathfrak{q})^{-1/2}$  $= ((-\mathfrak{q})^{-1/2}u^3 + (-\mathfrak{q})^{1/2}u^{-3})e_1.$ (138)

Since by construction $Yx = (z + z^{-1})x$, it follows that $(u^3 - (-\mathfrak{q})^{-1/2}z)(u^3 - (-\mathfrak{q})^{-1/2}z^{-1})x = 0$, and thus that the fusion product has dimension six. Furthermore, if $z^2 \neq 1$ then u^3 must have the two eigenvalues $(-\mathfrak{q})^{-1/2}z^{\pm1}$ so

$$\mathsf{W}_{0,z}(2) \times_f^u \mathsf{S}_{1/2}(1) \simeq \mathsf{W}_{1/2,(-\mathfrak{q})^{-1/2}z}(3) \oplus \mathsf{W}_{1/2,(-\mathfrak{q})^{-1/2}z^{-1}}(3). \quad (139)$$

Note that if $z^2 = 1$ then the two standard modules $\mathsf{W}_{1/2,(-\mathfrak{q})^{-1/2}z^{\pm1}}(3)$ are isomorphic, and one can show that the fusion product is then the self-extension of $\mathsf{W}_{1/2,(-\mathfrak{q})^{-1/2}z}(3)$.

A.3.6 $\mathsf{W}_{0,z}(n+1) \div_f^{u/o} \mathsf{S}_{1/2}(1)$

Here again the two fusion types are similar so we focus only on the u-type. Here we must restrict to z generic right from the start; it is possible to present a unified proof for the z generic or not cases, but doing so requires more sophisticated tools which we haven't introduced here. We thus start by noticing that

$$\begin{aligned}\mathrm{Hom}_{\mathsf{aTL}_n}\big(\mathsf{W}_{k,z'}(n), \mathsf{W}_{0,z}(n+1) \div_f^u \mathsf{S}_{1/2}(1)\big) &\simeq \mathrm{Hom}_{\mathsf{aTL}_{n+1}}\big(\mathsf{W}_{k,z'}(n) \times_f^u \mathsf{S}_{1/2}(1), \mathsf{W}_{0,z}(n+1)\big)\\ &\simeq \delta_{k,1/2}(\delta_{z',(-\mathfrak{q})^{1/2}z^1} + \delta_{z',(-\mathfrak{q})^{1/2}z^{-1}})\mathbb{C},\end{aligned}$$

where we used the duality between the fusion product and the fusion quotient together with the formulas for the fusion product of standard modules obtained in the previous section. Furthermore, for generic values of z the standard modules $\mathsf{W}_{1/2,(-\mathfrak{q})^{1/2}z^{\pm 1}}(n)$ are simple and non-isomorphic, so these morphisms must be injective. Finally, we have

$$\underbrace{\mathrm{Dim}(\mathsf{W}_{0,z}(n+1))}_{\binom{n+1}{(n+1)/2}} = 2\,\underbrace{\mathrm{Dim}(\mathsf{W}_{1/2,z'}(n))}_{\binom{n}{(n-1)/2}}, \tag{140}$$

and we thus conclude that

$$\mathsf{W}_{0,z}(n+1) \div_f^u \mathsf{S}_{1/2}(1) \simeq \mathsf{W}_{1/2,z(-\mathfrak{q})^{1/2}}(n) \oplus \mathsf{W}_{1/2,z^{-1}(-\mathfrak{q})^{1/2}}(n). \tag{141}$$

B Comparison with the other fusion types

We discuss briefly how the defect operators – the hoop elements Y and $\bar{Y}$ – introduced in this work behaves with respect to the various previously defined fusion products, in particular the one introduced by Gainutdinov and Saleur in [13] and the one introduced by Belletête and Saint-Aubin [15].

B.1 The GS fusion

For this fusion, we endow aTL_{n+m} (n, m positive integers) with the structure of a left $(\mathsf{aTL}_n, \mathsf{aTL}_m)$ module through the injection $\Phi \equiv \phi^u_{n,m} \otimes \psi^o_{m,n} : \mathsf{aTL}_n \otimes \mathsf{aTL}_m \to \mathsf{aTL}_{n+m}$. In terms of diagrams, this corresponds to gluing the two cylinders on which aTL_n and aTL_m lives into a pair of *pants*; Fig. 7 illustrates how the hoop operators behave under this gluing.

Given an aTL_n-module U and an aTL_m-module V, their fusion is defined as

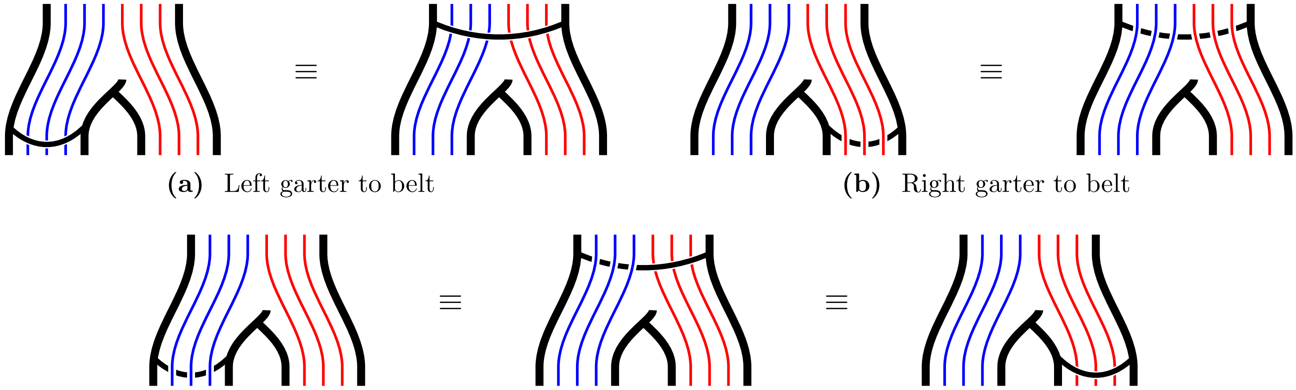
$$U \times_{GS} V \equiv \mathsf{aTL}_{n+m} \otimes_{\mathsf{aTL}_n \otimes \mathsf{aTL}_m} U \otimes_{\mathbb{C}} V. \tag{142}$$

Note that by construction (see Figs. 7a and 7b)

$$Y^{(n+m)} = \phi^u_{n,m}(Y^{(n)}), \qquad \bar{Y}^{(n+m)} = \psi^o_{m,n}(\bar{Y}^{(m)}), \tag{143}$$

so that the central elements $Y, \bar{Y}$ are fully determined from their value on U and V, respectively. It follows in particular that this fusion product is not commutative in general. Furthermore, the construction also gives (see Fig. 7c)

$$\phi^u_{n,m}(\bar{Y}^{(n)}) = \psi^o_{m,n}(Y^{(m)}), \tag{144}$$



(a) Left garter to belt **(b)** Right garter to belt

(c) Garters to uncomfortable belt

Figure 7: The behaviour of the hoop operators under the fusion product.

which imposes severe constraints on the combinations of modules leading to non-trivial fusion products.

To illustrate just how restrictive equations (143) and (144) are, we compute the fusion of two standard modules $U \simeq \mathsf{W}^u_{r,\delta}$, and $V \simeq \mathsf{W}^o_{s,\mu}$. First, equation (144) imposes $\mu = \delta^{\pm 1}$. Second, if the fusion product is non-zero then it must have at least one non-trivial simple quotient[10] and all of the simple modules of aTL_n are isomorphic to a standard modules $\mathsf{W}^o_{k,\nu}$ for some integer k and $\nu \in \mathbb{C}^*$. Combining this observation with equation (143) gives the conditions

$$\nu = (\delta(-\mathfrak{q})^{2r})^{\pm 1}, \qquad \text{and} \qquad \nu = (\mu(-\mathfrak{q})^{-2s})^{\pm 1}(-\mathfrak{q})^{2k}, \tag{145}$$

where all the $\pm$ are independent. In the generic cases where δ^2 is not an integer power of $(-\mathfrak{q})$, and $\mathfrak{q}$ is not a root of unity, this leaves the possibilities:

$$k = (r + \epsilon s), \qquad \delta = \mu^{\epsilon}, \tag{146}$$

or, using a more usual notation

$$\mathsf{W}_{r,z}(n) \times_{GS} \mathsf{W}_{s,w}(m) \simeq \begin{cases} a_+ \mathsf{W}_{r+s,z(-\mathfrak{q})^{-s}}(n) & \text{if } z = w(-\mathfrak{q})^{r+s} \\ a_- \mathsf{W}_{r-s,z(-\mathfrak{q})^{s}}(n) & \text{if } z = w^{-1}(-\mathfrak{q})^{r-s} \\ 0 & \text{otherwise} \end{cases}, \tag{147}$$

where $a_\pm$ are (unknown) non-negative integers (because each module could appear multiple times) and it is understood that if $s > r$, $\mathsf{W}_{r-s,z(-\mathfrak{q})^s}(n) \equiv \mathsf{W}_{s-r,z^{-1}(-\mathfrak{q})^{-s}}(n)$. It therefore simply remains to find the value of $a_\pm$; more extensive calculations [13] yields $a_\pm = 1$.

B.2 The BSA fusion

For this fusion, we endow aTL_{n+m} (n, m positive integers) with the structure of a left $(\mathsf{TL}_n, \mathsf{TL}_m)$ module through the injection $\Phi \equiv \phi^u_{n,m} \otimes \psi^o_{m,n} : \mathsf{TL}_n \otimes \mathsf{TL}_m \to \mathsf{aTL}_{n+m}$, where we identified the regular algebras with their images inside the affine algebra. In terms of diagrams, this corresponds

[10]This follows because the fusion of cyclic modules is also cyclic and the Blob algebra is Noetherian.

to taking the two strips on which the regular algebras live and stitching them into a cylinder, introducing no particular relations for the hoop operators. Given U, V a TL_n and a TL_m module, respectively, their fusion is defined as

$$U \times_{BSA} V \equiv \mathsf{aTL}_{n+m} \otimes_{\mathsf{TL}_n \otimes \mathsf{TL}_m} U \otimes_{\mathbb{C}} V. \tag{148}$$

Note that because no relations were introduced for the hoop operators, the fusion of any non-zero module is always infinite dimensional. For the particular case of standard modules (and $\mathfrak{q}$ generic), one finds [15]

$$\mathsf{S}_r \times_{BSA} \mathsf{S}_s \simeq \bigoplus_{k=|r-s|}^{r+s} \mathsf{P}_k, \tag{149}$$

where the P_k are the projective indecomposable modules of the affine Temperley-Lieb algebra. The algebraic structure of these modules is quite complicated and described in some details in [15]. We simply mention that there is a family of inclusions $\mathsf{P}_k \subset \mathsf{P}_{k+2} \subset \mathsf{P}_{k+4} \subset \ldots$ that is such that $\mathsf{P}_k/\mathsf{P}_{k-2}$ is an indecomposable submodule of the direct product of the standard modules $\mathsf{W}_{k,z}$ for all non-zero z. In other (more heuristic) words, P_k is an indecomposable *collage* of all $\mathsf{W}_{r,z}$ with $r \leq k$, $z \in \mathbb{C}^*$.

C Alternative proof of the relations (68)-(69)

We first establish the branching rules from $\mathsf{aTL}_{n_1+n_2}$ to $\mathsf{aTL}_{n_1} \otimes \mathsf{TL}_{n_2}$, by an adaptation of the working of [14] that dealt with the case of $\mathsf{aTL}_{n_1} \otimes \mathsf{aTL}_{n_2}$. The algebras aTL_{n_1} and TL_{n_2} can be embedded into aTL_n on $n = n_1 + n_2$ sites, by defining the periodic generator $e_0^{(1)}$ and the shift operator $u^{(1)}$ for the first subalgebra by braid translation in aTL_n (see [14]), while all other generators simply carry over.

Using the techniques of [14], and in particular the operator $\tau_j^{(1)}$ defined there, we then find the branching rules (actually, one has just to restrict the second tensor factor in branching rules of [14] to the subalgebra $\mathsf{TL}_{n_2} \subset \mathsf{aTL}_{n_2}$)

$$\mathsf{W}_{j,z}(n) = \bigoplus_{j_1,j_2} \mathsf{W}_{j_1,z_1}(n_1) \otimes \left(\bigoplus_{k \geq j_2} \mathsf{S}_k(n_2) \right) , \tag{150}$$

with the following values of the momenta:

- For $j = j_1 + j_2$ and any values of j_1, j_2: $z_1 = (i\sqrt{\mathfrak{q}})^{-2j_2} z^{+1}$.
- For $j = j_1 - j_2$ and either $j = 0$ or $j_2 > 0$: $z_1 = (i\sqrt{\mathfrak{q}})^{+2j_2} z^{+1}$.
- For $j = j_2 - j_1$ and either $j = 0$ or $j_1 > 0$: $z_1 = (i\sqrt{\mathfrak{q}})^{+2j_2} z^{-1}$.

Notice that this closely parallels the main result of [14], the only difference being that the right tensorands W_{j_2,z_2} there have been repaced by $\bigoplus_{k \geq j_2} \mathsf{S}_k$, which is exactly the restriction of W_{j_2,z_2} to the subalgebra $\mathsf{TL}_{n_2} \subset \mathsf{aTL}_{n_2}$. In particular, these modules have the same dimensions.

We can now read off the corresponding fusion rules by Frobenius reciprocity:

$$\mathrm{Hom}_{\mathsf{aTL}_{n+m}}\big(\mathsf{W}_{k,w}(n) \times_f \mathsf{S}_j(m), \mathsf{W}_{j,z}(n+m)\big) \simeq \mathrm{Hom}_{\mathsf{aTL}_n \otimes \mathsf{TL}_m}\big(\mathsf{W}_{k,w}(n) \otimes \mathsf{S}_j(m), \mathsf{W}_{j,z}(n+m)\big),$$

where on the right side of the equation $\mathsf{W}_{j,z}(n+m)$ is seen as a $\mathsf{aTL}_n \otimes \mathsf{TL}_m$-module, i.e. is identified with the branching rules (150). If q and z are generic, the standard modules $\mathsf{W}_{j,z}(n+m)$ are simple, so the dimension of this homomorphism group is the number of copies of the standard module appearing as direct summands of the fusion product.[11] There is however a slight subtlety in computing these dimensions, which is conveniently illustrated in the example $(n_1, n_2, n) = (2, 4, 6)$. The branching rules read in this case

$$\begin{aligned}
\mathsf{W}_{0,z}(6) &= \mathsf{W}_{0,z}(2) \otimes \big(\mathsf{S}_2(4) \oplus \mathsf{S}_1(4) \oplus \mathsf{S}_0(4)\big) \oplus \\
&\quad \mathsf{W}_{1,(i\sqrt{\mathfrak{q}})^2 z}(2) \otimes \big(\mathsf{S}_2(4) \oplus \mathsf{S}_1(4)\big) \oplus \mathsf{W}_{1,\frac{(i\sqrt{\mathfrak{q}})^2}{z}}(2) \otimes \big(\mathsf{S}_2(4) \oplus \mathsf{S}_1(4)\big)\,, \\
\mathsf{W}_{1,z}(6) &= \mathsf{W}_{0,\frac{z}{(i\sqrt{\mathfrak{q}})^2}}(2) \otimes \big(\mathsf{S}_2(4) \oplus \mathsf{S}_1(4)\big) \oplus \\
&\quad \mathsf{W}_{1,z}(2) \otimes \big(\mathsf{S}_2(4) \oplus \mathsf{S}_1(4) \oplus \mathsf{S}_0(4)\big) \oplus \mathsf{W}_{1,\frac{(i\sqrt{\mathfrak{q}})^4}{z}}(2) \otimes \mathsf{S}_2(4)\,, \\
\mathsf{W}_{2,z}(6) &= \mathsf{W}_{0,\frac{z}{(i\sqrt{\mathfrak{q}})^4}}(2) \otimes \mathsf{S}_2(4) \oplus \mathsf{W}_{1,\frac{z}{(i\sqrt{\mathfrak{q}})^2}}(2) \otimes \big(\mathsf{S}_2(4) \oplus \mathsf{S}_1(4)\big) \\
\mathsf{W}_{3,z}(6) &= \mathsf{W}_{1,\frac{z}{(i\sqrt{\mathfrak{q}})^4}}(2) \otimes \mathsf{S}_2(4)\,.
\end{aligned} \tag{151}$$

At first sight it appears that one would have fusion rules like

$$\begin{aligned}
\mathsf{W}_{0,z} \times_f \mathsf{S}_1 &= \mathsf{W}_{0,z} \oplus \mathsf{W}_{1,(-\mathfrak{q})z}\,, \\
\mathsf{W}_{0,z} \times_f \mathsf{S}_2 &= \mathsf{W}_{0,z} \oplus \mathsf{W}_{1,(-\mathfrak{q})z} \oplus \mathsf{W}_{2,(-\mathfrak{q})^2 z}\,.
\end{aligned} \tag{152}$$

This is however not quite correct. Indeed, we should be careful when the left tensorand in the fusion product is $\mathsf{W}_{0,z}$, since we have to take into account the isomorphism $\mathsf{W}_{0,z} \simeq \mathsf{W}_{0,z^{-1}}$. Therefore the corresponding terms in the branching rules (151) can also be written

$$\begin{aligned}
\mathsf{W}_{1,z}(6) &= \mathsf{W}_{0,\frac{(i\sqrt{\mathfrak{q}})^2}{z}}(2) \otimes \big(\mathsf{S}_2(4) \oplus \mathsf{S}_1(4)\big) \oplus \dots\,, \\
\mathsf{W}_{2,z}(6) &= \mathsf{W}_{0,\frac{(i\sqrt{\mathfrak{q}})^4}{z}}(2) \otimes \mathsf{S}_2(4) \oplus \dots\,.
\end{aligned} \tag{153}$$

This implies that we have a few extra terms, and (152) should be corrected into

$$\begin{aligned}
\mathsf{W}_{0,z} \times_f \mathsf{S}_1 &= \mathsf{W}_{0,z} \oplus \mathsf{W}_{1,(-\mathfrak{q})z} \oplus \mathsf{W}_{1,(-\mathfrak{q})z^{-1}}\,, \\
\mathsf{W}_{0,z} \times_f \mathsf{S}_2 &= \mathsf{W}_{0,z} \oplus \mathsf{W}_{1,(-\mathfrak{q})z} \oplus \mathsf{W}_{2,(-\mathfrak{q})^2 z} \oplus \mathsf{W}_{1,(-\mathfrak{q})z^{-1}} \oplus \mathsf{W}_{2,(-\mathfrak{q})^2 z^{-1}}\,.
\end{aligned}$$

Taking into account this subtlety, the general result comes out as

$$\mathsf{W}_{j_1,z} \times_f \mathsf{S}_{j_2} = \bigoplus_{j=\max(j_1-j_2,j_{12}^\star)}^{j_1+j_2} \mathsf{W}_{j,(-\mathfrak{q})^{j-j_1} z} \oplus \bigoplus_{j=j_{21}^\star}^{j_2-j_1} \mathsf{W}_{j,(-\mathfrak{q})^{j+j_1} z^{-1}}\,, \tag{154}$$

where we have defined $j_{12}^\star = (j_1 - j_2) \bmod 1$, and $j_{21}^\star = (j_2 - j_1) \bmod 1$. After some amount of rewriting, this can be shown to lead to (68)-(69) in the main text, as claimed.

[11] Note that this implicitly assumes that the fusion product of standard modules is semisimple, this can be shown by using the fact that the fusion product preserves the spectrum of at least one of the hoop operators. It follows that if a module factors through the blob algebra, then so will its fusion with any other module; for generic values of $\mathfrak{q}, w$, the blob algebra through which the standard module $\mathsf{W}_{k,w}(n)$ factors through is semisimple, therefore so is its fusion.

References

[1] J. Fröhlich, J. Fuchs, I. Runkel and C. Schweigert, *Kramers-Wannier duality from conformal defects*, Phys. Rev. Lett. **93**, 070601 (2004).

[2] J. Fröhlich, J. Fuchs, I. Runkel and C. Schweigert, *Duality and defects in rational conformal field theory*, arXiv:hep-th/0607247.

[3] C. Bachas and M. Gaberdiel, *Loop operators and the Kondo problem*, JHEP 0411 (2004) 065.

[4] M. Oshikawa and I. Affleck, *Boundary conformal field theory approach to the two-dimensional critical Ising model with a defect line*, Nucl. Phys. B **495**, 533–582 (1997).

[5] V.B. Petkova and J.-B. Zuber, *Generalized twisted partition functions*, Phys. Lett. B **504**, 157 (2001).

[6] V. Petkova, *Topological defects in CFT*, invited talk at SYMPHYS XV, Yerevan, 25–29 July 2011.

[7] G. Sarkissian, *Defects and permutation branes in the Liouville field theory*, Nucl. Phys. B **821**, 607 (2009).

[8] W.M. Koo and H. Saleur, *Representations of the Virasoro algebra from lattice models*, Nucl. Phys. B **426**, 459 (1994).

[9] A. Milsted and G. Vidal, *Extraction of conformal data in critical quantum spin chains using the Koo-Saleur formula*, Phys. Rev. B **96**, 245105 (2017).

[10] M. Zini and Z. Wang, *Conformal field theories as scaling limit of anyonic chains*, arXiv:1706.08497.

[11] N. Read and H. Saleur, *Associative-algebraic approach to logarithmic conformal field theories*, Nucl. Phys. B **777**, 316 (2007).

[12] A.M. Gainutdinov and R. Vasseur, *Lattice fusion rules and logarithmic operator product expansions*, Nucl. Phys. B **868**, 223–270 (2013).

[13] A.M. Gainutdinov and H. Saleur, *Fusion and braiding in finite and affine Temperley-Lieb categories*, arXiv:1606.04530, preprint Hamburger Beiträge zur Mathematik **596**.

[14] A.M. Gainutdinov, J.L. Jacobsen and H. Saleur, *A fusion for the periodic Temperley-Lieb algebra and its continuum limit*, arXiv:1712.07076, to appear in JHEP.

[15] J. Belletête and Y. Saint-Aubain, *On the computation of fusion over the affine Temperley-Lieb algebra*, Nucl. Phys. B **937**, 333–370 (2018).

[16] J. Belletête, *The fusion rules for the Temperley-Lieb algebra and its dilute generalization*, J. Phys. A: Math. Theor. **48**, 395205 (2015).

[17] V. Pasquier and H. Saleur, *Common structures between finite systems and conformal field theories through quantum groups*, Nucl. Phys. B **330**, 52 (1990).

[18] D. Aasen, R.S.K. Mong and P. Fendley, *Topological defects on the lattice. I: The Ising model*, J. Phys. A: Math. Theor. **49**, 354001 (2016).

[19] L.P. Kadanoff and H. Ceva, *Determination of an operator algebra for the two-dimensional Ising model*, Phys. Rev. B **3**, 3918 (1971).

[20] A. Gainutdinov, D. Ridout and I. Runkel (guest editors), *Special issue on logarithmic conformal field theory*, J. Phys. A **46** (2013).

[21] A.M. Gainutdinov, J.L. Jacobsen, N. Read, H. Saleur and R. Vasseur, *Logarithmic conformal field theory: a lattice approach*, J. Phys. A: Math. Theor. **46**, 494012 (2013).

[22] A.M. Gainutdinov, N. Read and H. Saleur, *Associative algebraic approach to logarithmic CFT in the bulk: the continuum limit of the $gl(1|1)$ periodic spin chain, Howe duality and the interchiral algebra*, Comm. Math. Phys. **341**, 35–103 (2016).

[23] A.M. Gainutdinov, N. Read, H. Saleur and R. Vasseur, *The periodic $sl(2|1)$ alternating spin chain and its continuum limit as a bulk LCFT at $c = 0$*, JHEP (2015) 114.

[24] A.N. Kirillov and N.Yu. Reshetikhin, *Representations of the algebra $U_q sl2$, q-orthogonal polynomials and invariants of links*, in T. Kohno (ed.), *New developments in the theory of knots* (World Scientific, Singapore, 1990).

[25] L. Alvarez-Gaumé, C. Gomez and G. Sierra, *Quantum group interpretation of some conformal field theories*, Phys. Lett. B **220**, 142 (1989).

[26] D. Bernard and A. Leclair, *Residual quantum symmetries of the restricted sine-Gordon theories*, Nucl. Phys. B **340**, 721 (1990).

[27] J. Belletête, A.M. Gainutdinov, J.L. Jacobsen, H. Saleur and R. Vasseur, *On the correspondence between boundary and bulk lattice models and (logarithmic) conformal field theories*, J. Phys. A: Math. Theor. **50**, 484002 (2017).

[28] A. Feiguin, S. Trebst, A.W.W. Ludwig, M. Troyer, A. Kitaev, Z. Wang and M.H. Freedman, Phys. Rev. Lett. **98**, 160409 (2007); C. Gils, E. Ardonne, S. Trebst, A.W.W. Ludwig, M. Troyer and Z. Wang, Phys. Rev. Lett. **101**, 050401 (2008).

[29] T. Halverson, M. Mazzocco and A. Ram, *Commuting families in Hecke and Temperley-Lieb algebras*, Nagoya Math. J. **195**, 125–152 (2009).

[30] C. Chui, C. Merkat, P. Orrick and P.A. Pearce, *Integrable lattice realizations of conformal twisted boundary conditions*, Phys. Lett. B **517**, 429–435 (2001).

[31] F.C. Alcaraz, U. Grimm and V. Rittenberg, *The XXZ Heisenberg chain, conformal invariance and the operator content of $c < 1$ systems*, Nucl. Phys. B **316**, 735 (1989).

[32] J.J. Graham and G.I. Lehrer, *The representation theory of affine Temperley-Lieb algebras*, L'Ens. Math. **44**, 173 (1998).

[33] P. Martin and H. Saleur, *The blob algebra and the periodic Temperley-Lieb algebra*, Lett. Math. Phys. **30**, 189 (1994).

[34] P.A. Pearce, J. Rasmussen and S.P. Villani, *Solvable critical dense polymers on the cylinder*, J. Stat. Mech. **1002**:P02010,(2010).

[35] A. Morin-Duchesne, P.A. Pearce and J. Rasmussen, *Fusion hierarchies, T-systems and Y-systems of logarithmic minimal models*, J. Stat. Mech. P05012 (2014).