Tangles are decided by weighted vertex sets

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We show that, given a k-tangle τ in a graph G, there always exists a weight function $w\colon V(G) \to \mathbb{N}$ such that a separation (A, B) of G of order $< k$ lies in τ if and only if $w(A) < w(B)$, where $w(U) \coloneqq \sum_{u \in U} w(u)$ for $U \subseteq V(G)$.

Introduction

Tangles in graphs have played a central role in graph minor theory ever since their introduction by Robertson and Seymour in [4]. Formally, a tangle in a graph G is an orientation of all low-order separations of G satisfying certain consistency assumptions. Tangles can be used to locate, and thereby capture the essence of, highly connected substructures in G in that every such substructure defines a tangle in G by orienting each low-order separation of G towards the side containing most or all of that substructure. In view of this, if some tangle in G contains the separation (A, B), we think of A and B as the 'small' and the 'big' side of (A, B) in that tangle, respectively. Our main result will make this intuition concrete.

As a concrete example, if G contains an $n \times n$-grid for large n, then the vertex set of that grid defines a tangle τ in G as follows. Take note that no separation of low order can divide the grid into two parts of roughly equal size: If the grid is large enough then at least 90% of its vertices, say, will lie on the same side of such a separation. Orienting all separations of order $< k$ for some fixed k much smaller than n then gives a tangle τ. In this way, the vertex set of the $n \times n$-grid 'defines τ by majority vote'.

In [1] Diestel raised the question whether all tangles in graphs arise in the above fashion, that is, whether all graph tangles are decided by majority vote by some subset of the vertices:

Problem 1. *Given a k-tangle τ in a graph G, is there always a set X of vertices such that a separation (A, B) of order $< k$ lies in τ if and only if $|A \cap X| < |B \cap X|$?*

A partial answer to this was given in [2], where Elbracht showed that such a set X always exists if G is $(k-1)$-connected and has at least $4(k-1)$ vertices. The general problem appears to be hard.

If a tangle in G is decided by some vertex set X by majority vote, this set X can be used as an oracle for that tangle, allowing one to store complete information about the complex structure of a tangle using a set of size at most $|V|$. On the other hand, if there were tangles without such a decider set, this would mean that tangles are a fundamentally more general concept than concrete highly cohesive subsets, not just an indirect way of capturing them.

In this paper, we consider a fractional version of Diestel's question and answer it affirmatively, making precise the notion that B is the 'big' side of a separation $(A, B) \in \tau$: given a k-tangle τ in G, rather than finding a vertex set X which decides τ by majority vote, we find a weight function $w\colon V(G) \to \mathbb{N}$ on the vertices such that for all separations (A, B) of order $< k$ we have $(A, B) \in \tau$ if and only if the vertices in B have higher total weight than those in A.

Thus we show that every graph tangle is decided by some *weighted* set of vertices. This weight function, or weighted set of vertices, can then serve as an oracle for that tangle in the same way that a vertex set deciding the tangle by majority vote would. For any tangle, the existence of such a weight function with values in $\{0, 1\}$ is equivalent to the existence of a vertex set X deciding that tangle by majority vote.

Geelen [3] pointed out that the analogue of Diestel's question for tangles in matroids is false: there are matroid tangles which cannot be decided by majority vote, not even when considering a fractional version of the problem.

In the main section of this paper we will formally define separations and tangles, and formulate and prove our main theorem.

Weighted deciders

Formally, a *separation* of a graph $G = (V, E)$ is a pair (A, B) with $A \cup B = V$ such that G contains no edge between $A \setminus B$ and $B \setminus A$, and the *order* of a separation (A, B) is the size $|A \cap B|$ of its separator $A \cap B$. Furthermore, for an integer k, a *k-tangle* in G is a set consisting of exactly one of (A, B) and (B, A) for every separation (A, B) of G of order $< k$, with the additional property that no three 'small' sides of separations in τ cover G, that is, that there are no $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \tau$ for which $G = G[A_1] \cup G[A_2] \cup G[A_3]$.

Our main result is the following:

Theorem 2. *Let $G = (V, E)$ be a finite graph and τ a k-tangle in G. Then there exists a function $w\colon V \to \mathbb{N}$ such that a separation (A, B) of G of order $< k$ lies in τ if and only if $w(A) < w(B)$, where $w(U) \coloneqq \sum_{u \in U} w(u)$ for $U \subseteq V$.*

We shall prove Theorem 2 in the remainder of this section.

For a graph G there is a partial order on the separations of G given by letting $(A, B) \le (C, D)$ if and only if $A \subseteq C$ and $B \supseteq D$. One of the main ingredients for the proof of Theorem 2 is the following observation about those separations in a tangle τ that are maximal in τ with respect to this partial order. It says, roughly, that they divide each other's separators so that, on average, those separators lie more on the big side of the separation than on the small side, according to the tangle.

Lemma 3. *For every k-tangle τ in a graph G and distinct maximal elements $(A,B),(C,D)$ of τ we have $|B\cap(C\cap D)|+|D\cap(A\cap B)|>|A\cap(C\cap D)|+|C\cap(A\cap B)|$.*

Proof. The separation $(A\cup C,\, B\cap D)$ of G is strictly larger than the maximal elements (A,B) and (C,D) and hence cannot lie in τ. Since A, C and $B\cap D$ together cover G, by the tangle axioms τ cannot contain the inverse $(B\cup D,\, A\cap C)$ either. Thus we must have $|(A\cup C)\cap(B\cap D)|\geq k$, from which it follows that $|(A\cap C)\cap(B\cup D)|<k$. Combining these two inequalities and adding $|A\cap B\cap C\cap D|$ to both sides proves the claim. □

Additionally we shall use a result from linear programming. For a vector $x\in\mathbb{R}^n$ we use the usual shorthand notation $x\geq 0$ to indicate that all entries of x are non-negative, and similarly write $x>0$ if all entries of x are strictly greater than zero.

Lemma 4 ([5])**.** *Let $K\in\mathbb{R}^{n\times n}$ be a skew-symmetric matrix, i.e. $K^T=-K$. Then there exists a vector $x\in\mathbb{R}^n$ such that*

$$Kx\geq 0 \quad \text{and} \quad x\geq 0 \quad \text{and} \quad x+Kx>0.$$

We are now ready to prove Theorem 2.

Proof of Theorem 2. Let a finite graph $G=(V,E)$ and a k-tangle τ in G be given. Since G is finite it suffices to find a weight function $w\colon V\to\mathbb{R}_{\geq 0}$ such that a separation (A,B) of order $<k$ lies in τ precisely if $w(A)<w(B)$; this can then be turned into such a function with values in $\mathbb{N}$.

For this it is enough to find a function $w\colon V\to\mathbb{R}_{\geq 0}$ such that $w(A)<w(B)$ for all maximal elements (A,B) of τ: for if $w(A)<w(B)$ and $(C,D)\leq(A,B)$ then

$$w(C)\leq w(A)<w(B)\leq w(D).$$

So let us show that such a weight function w exists.

To this end let $(A_1,B_1),\dots,(A_n,B_n)$ be the maximal elements of τ and set

$$m_{ij}\coloneqq|B_i\cap(A_j\cap B_j)|-|A_i\cap(A_j\cap B_j)|$$

for $i,j\leq n$. Let M be the matrix $\{m_{ij}\}_{i,j\leq n}$. Observe that, by Lemma 3, we have $m_{ij}+m_{ji}>0$ for all $i\neq j$ and hence the matrix $M+M^T$ has positive entries everywhere but on its diagonal (where it has zeros). We further define

$$K'\coloneqq\frac{M+M^T}{2} \qquad \text{and} \qquad K\coloneqq M-K'.$$

Then K is skew-symmetric, that is, $K^T=-K$. Let $x=(x_1,\dots,x_n)^T$ be the vector obtained by applying Lemma 4 to K. We define a weight function $w\colon V\to\mathbb{R}$ by

$$w(v)\coloneqq\sum_{i\colon v\in A_i\cap B_i}x_i\,.$$

Note that w has its image in $\mathbb{R}_{\geq 0}$ and observe further that, for $Y \subseteq V$, we have

$$w(Y) = \sum_{y \in Y} w(y) = \sum_{i=1}^{n} x_i \cdot |Y \cap (A_i \cap B_i)| \, .$$

With this, for $i \leq n$, we have

$$\begin{aligned} w(B_i) - w(A_i) &= \sum_{j=1}^{n} x_j \cdot (|B_i \cap (A_j \cap B_j)| - |A_i \cap (A_j \cap B_j)|) \\ &= \sum_{j=1}^{n} x_j \cdot m_{ij} \\ &= (Mx)_i \, , \end{aligned}$$

where $(Mx)_i$ denotes the i-th coordinate of Mx. Thus w is the desired weight function if we can show that $Mx > 0$, that is, if all entries of Mx are positive.

From $x + Kx > 0$ we know that at least one entry of x is positive. Let us first consider the case that x has two or more positive entries. Then $K'x > 0$ since K' has positive values everywhere but on the diagonal, and hence

$$Mx = (K + K')\, x > 0$$

since $Kx \geq 0$. Therefore, in this case, w is the desired weight function.

Consider now the case that exactly one entry of x, say x_i, is positive, and that x is zero in all other coordinates. Then for $j \neq i$ we have $(Mx)_j \geq (K'x)_j > 0$ and thus $w(B_j) - w(A_j) = (Mx)_j > 0$. However $(Mx)_i = 0$ and thus $w(A_i) = w(B_i)$, so w is not yet as claimed. To finish the proof it remains to modify w such that $w(A_i) < w(B_i)$ while ensuring that we still have $w(A_j) < w(B_j)$ for $j \neq i$. This can be achieved by picking a sufficiently small $\varepsilon > 0$ such that $w(A_j) + \varepsilon < w(B_j)$ for all $j \neq i$, picking any $v \in B_i \smallsetminus A_i$, and increasing the value of $w(v)$ by ε. □

We conclude with the remark that Theorem 2 and its proof extend to tangles in hypergraphs without any changes. Even more generally, the following version of Theorem 2, which is formulated in the language of [1], can be established with exactly the same proof as well:

Theorem 5. *Let U be a universe of set separations of a finite ground-set V with order function $|(A,B)| := |A \cap B|$. Then for any regular k-profile P in U there exists a function $w\colon V \to \mathbb{N}$ such that a separation (A,B) of order $< k$ lies in P if and only if $w(A) < w(B)$.*

Theorem 5 then applies to tangles in graphs or hypergraphs by taking for U the universe of separations of a (hyper-)graph and for P the given k-tangle. (See [1] for more on the relation between graph tangles and profiles.) Theorem 5 holds with the same proof as Theorem 2, since Lemma 3 holds in this setting too, using the definition of profile rather than the tangle axioms.

References

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