The Little Bundles Operad

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Abstract

Hurwitz spaces are homotopy quotients of the braid group action on the moduli space of principal bundles over a punctured plane. By considering a certain model for this homotopy quotient we build an aspherical topological operad that we call the *little bundles operad*. As our main result, we describe this operad as a groupoid-valued operad in terms of generators and relations and prove that the categorical little bundles algebras are precisely Turaev's crossed categories. Moreover, we prove that the evaluation on the circle of a homotopical two-dimensional equivariant topolological field theory yields a little bundles algebra up to coherent homotopy.

Keywords: operad, topological field theory, braid group, monoidal category, crossed category

Contents

1 Introduction

Consider for $r \geq 0$ an r-ary operation $f \in E_2(r)$ of the little disks operad E_2 [BV68, BV73], i.e. an affine embedding of r disks into another disk, and the groupoid $\mathsf{PBun}_G(\mathsf{C}(f))$ of G-bundles over the closed complement $\mathsf{C}(f)$ of the image of the embedding f. Then the (pure) braid group on r strands acts on the space $\mathsf{PBun}_G(\mathsf{C}(f))$. The homotopy quotient is known as a Hurwitz space. We consider a model $W_2(r)$ for this homotopy quotient which, by restriction to the boundary circles, comes with a Serre fibration $W_2(r) \longrightarrow \mathrm{Map}(\mathbb{S}^1, BG)^{r+1}$ to the $r+1$-fold product of the free loop space of the classifying space of G. This allows us to prove that the fibers of this Serre fibration, considered for varying $r \geq 0$, combine into a topological $\mathrm{Map}(\mathbb{S}^1, BG)$-colored operad E_2^G, that we call the *operad of little G-bundles*.

The operad E_2^G of little bundles is aspherical and, as our main result, we exhibit a presentation as groupoid-valued operad in terms of generators and relations (Section 4) using so-called G-crossed parenthesized braids. This allows us to prove in Corollary 4.8 that the categorical little bundles algebras are precisely Turaev's G-crossed categories [Tur00] – a G-equivariant version of a braided category which, however, is not a braided category itself in the usual sense. These categories have been extensively studied e.g. in [Müg04, Kir04, GNN09, ENOM10, Tur10]. One ingredient of the proof is a recent coherence result for G-equivariant categories [Gal17]. In the existing literature the bookkeeping of the coherence data of a G-crossed category is done manually. Our operadic approach encodes this data in a compact way and allows to define G-crossed algebras up to coherent homotopy beyond the category-valued case. In particular, it naturally leads to a notion of a G-crossed differential graded algebra or ∞-category.

The little bundles operad is closely related to the study of G-equivariant topological field theory [Tur10, TuVi12, TuVi14], a flavor of topological field theory featuring bordisms equipped with principal G-bundles. The introduction of the bundle decoration leads to interesting phenomena such as a certain non-commutativity of the algebraic structures that can be extracted from the field theory.

Certain aspects of the little bundles operad introduced in the present paper are implicit in the literature on equivariant topological field theories because it captures the genus zero part of surfaces decorated with G-bundles. This becomes manifest in the following observations: The extraction of G-crossed algebras from two-dimensional G-equivariant topological field theories valued in vector spaces via evaluation on the circle [Kau02, Tur10] can be understood as a statement about the path connected components of the little bundles operad (Remark 4.6). Similarly, the interpretation of the coherence diagrams for G-crossed categories [Tur00] in terms of low-dimensional topology as carried out in [MNS12] for category-valued field theories constructed from Drinfeld doubles and in [SW18] in the general case can be phrased in a unified way via the little bundles operad.

The operadic approach is particularly well-suited for the study of $(\infty, 1)$ equivariant topological field theories [MW18-II]. As a main application, we prove that in dimension two the value of such a theory on the circle produces a homotopy little bundles algebra (Theorem 5.2), i.e. an algebra over the Boardman-Vogt resolution of the little bundles operad. This is a generalization of the well-known result that the value of a two-dimensional topological field theory on the circle is an E_2-algebra [Ge94, BZFN10].

Our definition of the little bundles operad is in fact obtained as special case of a more general construction allowing both for higher dimensional disks and non-aspherical target spaces. Although the present work mainly focuses on the little bundles operad, the applications of its generalization to the study of topological field theories with non-aspherical target are indicated in Proposition 5.3.

2 Maps on complements of little disks

For $n \geq 1$ let E_n be the little n-disks operad. Recall that the operations $E_n(r)$ in arity $r \geq 0$ for this topological operad are given as follows: Denote by $\mathbb{D}^n$ the closed n-dimensional disk $\mathbb{D}^n := \{x \in \mathbb{R}^n \,|\, |x|^2 \leq 1\}$. Then $E_n(r)$ is given by the space of all maps $f : \coprod_{k=1}^r \mathbb{D}^n \longrightarrow \mathbb{D}^n$ such that

- the restriction $f_k : \mathbb{D}^n \longrightarrow \mathbb{D}^n$ of f to the k-th disk is an affine embedding, i.e. given by a rescaling of the radius and a translation,

- for $1 \leq j < k \leq r$ the interiors of the images of f_j and f_k do not intersect, i.e. $\operatorname{im} \overset{\circ}{f}_j \cap \operatorname{im} \overset{\circ}{f}_k = \emptyset$.

For details we refer e.g. to [Fre17-I, Chapter 4].

We write $\mathsf{C}(f) := \mathbb{D}^n \setminus \operatorname{im} \overset{\circ}{f}$ for the complement of the interior of the image of f in the closed n-disk. This allows us to the define the subspace

$$W_n(r) := \{(f,x) \in E_n(r) \times \mathbb{D}^n \,|\, x \in \mathsf{C}(f)\} \subset E_n(r) \times \mathbb{D}^n \ .$$

The boundary $\partial \mathsf{C}(f)$ of $\mathsf{C}(f)$ consists of r spheres $\mathbb{S}^{n-1}$ (the *ingoing boundary* or *ingoing spheres*) possibly wedged together and sitting inside a bigger sphere $\mathbb{S}^{n-1}$ (the *outgoing boundary* or *outgoing sphere*). The goal of this section is to define an operad E_n^T whose colors are maps from $\mathbb{S}^{n-1}$ to some fixed topological space T and whose operations from an r-tuple $\underline{\varphi} = (\varphi_1, \ldots, \varphi_r)$ of maps $\mathbb{S}^{n-1} \longrightarrow T$ (generally, we will use an underline to indicate tuples) to a single map $\psi : \mathbb{S}^{n-1} \longrightarrow T$ will be given by all operations $f \in E_n(r)$ equipped with maps $\mathsf{C}(f) \longrightarrow T$ whose restriction to $\partial \mathsf{C}(f)$ is $(\underline{\varphi}, \psi)$.

We will introduce the operad E_n^T formally in Section 2.2. The definition will not be as a homotopy quotient of a braid group action (as mentioned in the introduction), but will make use of the auxiliary constructions in Section 2.1. The description as a homotopy quotient is then discussed in Section 3.1.

2.1 The auxiliary spaces $W_n^T(r)$

For a topological space T we define $W_n^T(r)$ as the set of pairs (f, φ), where $f \in E_n(r)$ and $\varphi : \mathsf{C}(f) \longrightarrow T$ is a continuous map. Projection onto the first factor yields a map

$$p : W_n^T(r) \longrightarrow E_n(r)$$

of sets. We equip $W_n^T(r)$ with the final topology with respect to all maps of sets $g : Y \longrightarrow W_n^T(r)$ from arbitrary topological spaces Y to the set W_n^T such that

(I) $p \circ g$ is a continuous map,

(II) and the natural map $W_n(r) \times_{E_n(r)} Y \longrightarrow T$ is continuous.

The further investigation of the spaces W_n^T involves some elementary, but tedious point-set topology and is carried out in Appendix A.1. We only mention the crucial properties necessary in the sequel:

Proposition 2.1 (Appendix, Proposition A.4). *The map $p : W_n^T(r) \longrightarrow E_n(r)$ is a Serre fibration.*

Next we define the subspace

$$\partial W_n(r) := \{(f,x) \in E_n(r) \times \mathbb{D}^n \,|\, x \in \partial \mathsf{C}(f)\} \subset W_n(r)$$

and note that there is a natural map

$$E_n(r) \times \coprod^{r+1} \mathbb{S}^{n-1} \longrightarrow \partial W_n(r) \tag{2.1}$$

of spaces over $E_n(r)$ identifying the first r copies of $\mathbb{S}^{n-1}$ with the ingoing boundary spheres and the last copy with the outgoing boundary spheres. This map is generally not a homeomorphism since some of the boundary spheres might be wedged together. By combining the restriction of the evaluation map (which is continuous by Lemma A.2)

$$\partial W_n(r) \times_{E_n(r)} W_n^T(r) \longrightarrow T$$

with (2.1) we obtain a map

$$\left(\coprod^{r+1} \mathbb{S}^{n-1}\right) \times W_n^T(r) \longrightarrow T$$

which, by adjunction, gives us a map

$$q : W_n^T(r) \longrightarrow \prod^{r+1} \mathrm{Map}(\mathbb{S}^{n-1}, T) \ . \tag{2.2}$$

Proposition 2.2 (Appendix, Proposition A.5). *The map $q : W_n^T(r) \longrightarrow \prod^{r+1} \mathrm{Map}(\mathbb{S}^{n-1}, T)$ is a Serre fibration.*

2.2   The operad E_n^T

From the map q we can construct a topological operad colored over the set of maps $\mathbb{S}^{n-1} \longrightarrow T$: Let $\underline{\varphi} = (\varphi_1, \ldots, \varphi_r)$ be an r-tuple of maps $\mathbb{S}^{n-1} \longrightarrow T$. Then for another map $\psi : \mathbb{S}^{n-1} \longrightarrow T$ we consider the fiber $E_n^T\left(\begin{smallmatrix}\psi\\\underline{\varphi}\end{smallmatrix}\right)$ of q over $(\underline{\varphi}, \psi)$, i.e. the pullback

$$
\begin{array}{ccc}
E_n^T\left(\begin{smallmatrix}\psi\\\underline{\varphi}\end{smallmatrix}\right) & \longrightarrow & W_n^T(r) \\
\downarrow & & \downarrow q \\
\star & \xrightarrow{(\underline{\varphi},\psi)} & \prod^{r+1} \mathrm{Map}(\mathbb{S}^{n-1}, T)
\end{array}
$$

which is also a homotopy pullback since q is a Serre fibration by Proposition 2.2. Informally, $E_n^T\left(\begin{smallmatrix}\psi\\\underline{\varphi}\end{smallmatrix}\right)$ consists of elements $f \in E_n(r)$ together with a map $\xi : \mathsf{C}(f) \longrightarrow T$ whose restriction to $\partial\mathsf{C}(f)$ is given by $(\underline{\varphi}, \psi)$. We will denote the point in $E_n^T\left(\begin{smallmatrix}\psi\\\underline{\varphi}\end{smallmatrix}\right)$ formed by f and ξ by $\langle f, \xi \rangle$.

The operad structure on E_n^T makes use of the operad structure of E_n for which we refer to [Fre17-I, Chapter 4]. The operadic identity $\star \longrightarrow E_n^T\left(\begin{smallmatrix}\varphi\\\varphi\end{smallmatrix}\right)$ is the operadic identity in $E_n(1)$, namely the identity embedding $I : \mathbb{D}^n \longrightarrow \mathbb{D}^n$, together with $\varphi : \mathsf{C}(I) = \mathbb{S}^{n-1} \longrightarrow T$. Moreover, the action of the symmetric group Σ_r on r letters on $E_n(r)$ turns E_n^T into a $\mathrm{Map}(\mathbb{S}^{n-1}, T)$-colored symmetric sequence.

The operadic composition consists of maps

$$\circ : E_n^T\begin{pmatrix}\psi\\\underline{\varphi}\end{pmatrix} \times \prod_{j=1}^r E_n^T\begin{pmatrix}\varphi_j\\\underline{\lambda}_j\end{pmatrix} \longrightarrow E_n^T\begin{pmatrix}\psi\\\otimes_{j=1}^r\underline{\lambda}_j\end{pmatrix} \ ,$$

where $\otimes$ denotes the juxtaposition of tuples. It sends

$$\left(\langle f, \xi\rangle, \prod_{j=1}^r \langle g_j, \mu_j\rangle\right) \in E_n^T\begin{pmatrix}\psi\\\underline{\varphi}\end{pmatrix} \times \prod_{j=1}^r E_n^T\begin{pmatrix}\varphi_j\\\underline{\lambda}_j\end{pmatrix}$$

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to

$$\left\langle f \circ \underline{g}, \xi \cup_{\coprod^r \mathbb{S}^{n-1}} \underline{\mu} \right\rangle \in E_n^T \begin{pmatrix} \psi \\ \otimes_{j=1}^r \underline{\lambda}_j \end{pmatrix} ,$$

where

- the composition of f with the r-tuple $\underline{g}$ of embeddings is formed via the composition in E_n,

- we use that for $1 \le j \le r$ the restriction of $\mu_j : \mathsf{C}(g_j) \longrightarrow T$ to the last copy of $\mathbb{S}^{n-1}$ (the outer sphere) is precisely φ_j in order to glue ξ and $\underline{\mu}$ along r copies of $\mathbb{S}^{n-1}$.

Proposition 2.3. *Let T be any space. With the above definitions E_n^T is a topological operad colored over $\mathrm{Map}(\mathbb{S}^{n-1}, T)$.*

Proof. The only non-trivial point is the continuity of the composition maps. It suffices to prove that the partial compositions

$$\circ_j : E_n^T \begin{pmatrix} \psi \\ \varphi_1, \ldots, \varphi_j, \ldots, \varphi_r \end{pmatrix} \times E_n^T \begin{pmatrix} \varphi_j \\ \underline{\lambda} \end{pmatrix} \longrightarrow E_n^T \begin{pmatrix} \psi \\ \varphi, \ldots, \underline{\lambda}, \ldots, \varphi_r \end{pmatrix} \tag{2.3}$$

are continuous.

To this end, set $r' := |\underline{\lambda}|$ and consider the restriction $W_n^T(r) \longrightarrow \mathrm{Map}(\mathbb{S}^{n-1}, T)$ to the outer boundary sphere and the restriction $W_n^T(r') \longrightarrow \mathrm{Map}(\mathbb{S}^{n-1}, T)$ to the j-th ingoing boundary sphere and observe that $E_n^T \begin{pmatrix} \psi \\ \varphi_1, \ldots, \varphi_j, \ldots, \varphi_r \end{pmatrix} \times E_n^T \begin{pmatrix} \varphi_j \\ \underline{\lambda} \end{pmatrix}$ is a subspace of the pullback $W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1}, T)} W_n^T(r')$ such that (2.3) is the restriction of

$$\widehat{\circ}_j : W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1}, T)} W_n^T(r') \longrightarrow W_n^T(r + r' - 1), \quad (\langle f, \xi \rangle, \langle f', \xi' \rangle) \longmapsto \langle f \circ f', \xi \cup_{\mathbb{S}^{n-1}} \xi' \rangle, \tag{2.4}$$

where $f \circ f'$ is the operadic composition in E_n and $\xi \cup_{\mathbb{S}^{n-1}} \xi'$ is the map obtained from gluing ξ and ξ' along $\mathbb{S}^{n-1}$. The map (2.4) is continuous by Lemma A.6 in the Appendix. $\square$

Remark 2.4. The model for E_n used in this article is Σ-cofibrant [Fre17-I, page 140]. For the same reasons, E_n^T is Σ-cofibrant.

3 The operad E_2^G of little G-bundles

Let us specialize the operad from Proposition 2.3 to aspherical spaces T to obtain what we will refer to as *little bundles operad*.

Recall that a space or simplicial set T called *aspherical* if $\pi_k(T) = 0$ for $k \ge 2$ and all choices of basepoints. Accordingly, we call an operad in spaces or simplicial sets *aspherical* if all its components are aspherical.

If T is an aspherical space (which we will assume to be connected without loss of generality), then, up to equivalence, T is the classifying space of its fundamental group G. Therefore, we set $E_n^G := E_n^{BG}$ for any group G.

For a space X the mapping space $\mathrm{Map}(X, BG)$ is the nerve $B\mathsf{PBun}_G(X)$ of the groupoid of principal G-bundles over X, i.e.

$$\mathrm{Map}(X, BG) \simeq B\mathsf{PBun}_G(X) . \tag{3.1}$$

In particular, $\mathrm{Map}(X, BG)$ is aspherical again with

$$\pi_0(\mathrm{Map}(X, BG)) \cong \pi_0(\mathsf{PBun}_G(X)) , \tag{3.2}$$

$$\pi_1(\mathrm{Map}(X, BG), \varphi) \cong \mathrm{Aut}(\varphi^* EG) , \tag{3.3}$$

where $\varphi^* EG$ is the pullback of the universal G-bundle $EG \longrightarrow BG$ along a map $\varphi : X \longrightarrow BG$ and where we denote by $\mathrm{Aut}(P)$ the group of automorphisms of a G-bundle P (the group of *gauge transformations*).

Recall that if X is connected, we find

$$\mathsf{PBun}_G(X) \simeq \mathrm{Hom}(\pi_1(X), G)//G \tag{3.4}$$

by the holonomy classification of G-bundles, i.e. after choice of a base point in X, the bundle groupoid $\mathsf{PBun}_G(X)$ is equivalent to the action groupoid associated to the action of G by conjugation on the set of group morphisms $\pi_1(X) \longrightarrow G$.

Note that for $n > 2$ the operad E_n^G is not really interesting since all G-bundles over $\mathbb{S}^{n-1}$ for $n > 2$ are trivializable. The case relevant to us is $n = 2$:

Definition 3.1. We call the topological operad E_2^G the *little bundles operad*.

In the remaining subsections of this section, we will show that E_2^G is aspherical (Proposition 3.8) and will explicitly describe its components as action groupoids (Proposition 3.9).

3.1 The space W_2^G as a Hurwitz space

In a first step we investigate the space $W_2^G := W_2^{BG}$ for a group G.

For this recall from [Fre17-I, Chapter 5] that $E_2(r)$ is the classifying space of the pure braid group P_r on r strands, i.e.

$$E_2(r) \simeq BP_r \ . \tag{3.5}$$

Alternatively (and for our applications more conveniently), we can describe the fundamental groupoid $\Pi E_2(r)$ as the action groupoid

$$\Pi E_2(r) \simeq \Sigma_r // B_r \ , \tag{3.6}$$

where the braid group B_r acts on Σ_r by $c.\sigma := \pi(c)\sigma$ for $c \in B_r$ and $\sigma \in \Sigma_r$, i.e. via the projection $\pi : B_r \longrightarrow \Sigma_r$ fitting into the short exact sequence

$$0 \longrightarrow P_r \longrightarrow B_r \xrightarrow{\pi} \Sigma_r \longrightarrow 0 \ .$$

If we consider the long exact sequence of homotopy groups for the Serre fibration from Proposition 2.1 whose fibers we computed in Lemma A.3 and take (3.2), (3.3) and (3.5) into account, we arrive at:

Lemma 3.2. *The space $W_2^G(r)$ is aspherical and for $f \in E_2(r)$ and $\varphi \in \mathrm{Map}(\mathsf{C}(f), BG)$ there is an exact sequence*

$$0 \longrightarrow \mathrm{Aut}(\varphi^* EG) \longrightarrow \pi_1(W_2^G(r), \langle f, \varphi \rangle) \longrightarrow P_r \longrightarrow \pi_0(\mathsf{PBun}_G(\mathsf{C}(f))) \longrightarrow \pi_0(W_2^G(r)) \longrightarrow 0 \ .$$

We will denote the homotopy fiber of a map $q : X \longrightarrow Y$ over $y \in Y$ by $q^{-1}[y]$. If X and Y are aspherical, we can make the following elementary observation:

Lemma 3.3. *Let $q : X \longrightarrow Y$ be a map between aspherical spaces, then for $y \in Y$ the natural map*

$$\Pi(q^{-1}[y]) \longrightarrow \Pi(q)^{-1}[y]$$

from the fundamental groupoid of the homotopy fiber $q^{-1}[y]$ to the homotopy fiber of $\Pi(q) : \Pi(X) \longrightarrow \Pi(Y)$ over $y \in Y$ is an equivalence.

The following Proposition will be the key for understanding the auxiliary spaces $W_2^G(r)$. It provides a link to a certain flavor of Hurwitz spaces, see also Remark 3.7 below.

Proposition 3.4. *There is an equivalence*

$$W_2^G(r) \simeq \operatorname*{hocolim}_{f \in \Pi E_2(r)} \operatorname{Map}(\mathsf{C}(f), BG) \ . \tag{3.7}$$

Here, by an equivalence we mean that there exists a zigzag of equivalences, i.e. the objects are isomorphic in the homotopy category.

Proof. Since $p : W_2^G(r) \longrightarrow E_2(r)$ is a Serre fibration (Lemma 2.1), $\Pi(p) : \Pi W_2^G(r) \longrightarrow \Pi E_2(r)$ is a categorical fibration. From this we easily deduce that it is also a category fibered in groupoids in the sense of [DM69]. Corresponding to this category fibered in groupoids we have by [Hol08, Section 3.3] a (pseudo-)functor $X : (\Pi E_2(r))^{\mathrm{opp}} \longrightarrow \mathsf{Grpd}$ (we can also see this as a $\Pi E_2(r)$-shaped diagram since a groupoid is equivalent to its opposite) such that for $f \in E_2(r)$ the groupoid $X(f)$ is equivalent to the fiber of $\Pi W_2^G(r) \longrightarrow \Pi E_2(r)$ over f, therefore to $\Pi p^{-1}(f)$ by Lemma 3.3 and finally to $\Pi \operatorname{Map}(\mathsf{C}(f), BG)$ by Lemma A.3. If we denote by $\int$ the Grothendieck construction, we conclude from [Hol08, Theorem 3.12] that there is a canonical fiberwise equivalence

$$\int X \longrightarrow \Pi W_2^G(r) \tag{3.8}$$

of groupoids over $\Pi E_2(r)$. It is then straightforward to verify that this is also an equivalence of groupoids.

By Thomason's Theorem [Th79, Theorem 1.2] we obtain a canonical equivalence

$$\operatorname*{hocolim}_{f \in \Pi E_2(r)} BX(f) \xrightarrow{\simeq} B \int X \ .$$

Combining this with the equivalence (3.8) yields the assertion if we additionally take into account that $\operatorname{Map}(\mathsf{C}(f), BG)$ and $W_2^G(r)$ are aspherical by (3.1) and Lemma 3.2, respectively. $\square$

Since $\mathsf{C}(f)$ is equivalent to a wedge $\bigvee_{j=1}^r \mathbb{S}^1$ of r circles, we conclude from (3.1) and (3.4)

$$\operatorname{Map}(\mathsf{C}(f), BG) \simeq B\left(\operatorname{Hom}(\mathbb{Z}^{*r}, G)//G\right) \simeq B(G^{\times r}//G) \ . \tag{3.9}$$

Lemma 3.5. *Under the identifications (3.6) and (3.9) the diagram from $\Pi E_2(r)$ to spaces underlying the homotopy colimit (3.7) is point-wise the nerve of the diagram*

$$\Sigma_r//B_r \longrightarrow \mathsf{Grpd}$$

sending $\sigma \in \Sigma_r$ to $(G^{\times r}//G)$. The generator $c_{j,j+1} \in B_r$ braiding strand j and $j + 1$ acts as the automorphism

$$G^{\times r}//G \longrightarrow G^{\times r}//G, \quad (g_1, \ldots, g_j, g_{j+1}, \ldots, g_r) \longmapsto (g_1, \ldots, g_j g_{j+1} g_j^{-1}, g_j, \ldots, g_r) \ . \tag{3.10}$$

Proof. We only have to observe that the transformation of holonomies under the braid group action are given by the formula (3.10) (sometimes called Hurwitz formula). For a detailed proof of this fact – given without loss of generality for two embedded disks – we refer to e.g. [MNS12, Lemma 3.25]. $\square$

By Proposition 3.4, $W_2^G(r)$ is the homotopy colimit of the nerve of the diagram presented in Lemma 3.5. For later purposes, we need to describe $W_2^G(r)$ explicitly as a groupoid. If we combine Lemma 3.5 with the homotopy colimit formula provided in Lemma A.7 in the Appendix, we obtain:

Lemma 3.6. *The groupoid $\Pi W_2^G(r)$ is equivalent to the groupoid with objects $\Sigma_r \times G^{\times r}$ and pairs $(c, h) \in B_r \times G$ as morphisms $(\sigma, g_1, \ldots, g_r) \longrightarrow (\pi(c)\sigma, c.(hg_1h^{-1}, \ldots, hg_rh^{-1}))$, where the action of the braid group on tuples of group elements is given by (3.10).*

Remark 3.7. The homotopy quotient of the space of G-bundles over a punctured plane by the braid group action (or its description in terms of holonomies) first appeared in [Cle72, Hur91] and is called a *Hurwitz space*, see [EVW16] for an overview.

3.2 Groupoid description of E_2^G

Our investigation of W_2^G is the key to the computation of the homotopy groups of the little bundles operad E_2^G. Using the long exact sequence for the Serre fibration $q : W_2^G(r) \longrightarrow \prod^{r+1} \mathrm{Map}(\mathbb{S}^1, BG)$ from Proposition 2.2 combined with Lemma 3.2 we obtain:

Proposition 3.8. *For any group G, the operad E_2^G is aspherical.*

Therefore, it suffices to compute the groupoid-valued operad ΠE_2^G. To this end, recall that the groupoid $\Pi \mathrm{Map}(\mathbb{S}^1, BG)$ is canonically equivalent to the groupoid of G-bundles over $\mathbb{S}^1$, hence for any fixed choice of basepoint we obtain an equivalence

$$\Pi \mathrm{Map}(\mathbb{S}^1, BG) \xrightarrow{\simeq} G//G \ .$$

In the sequel we choose a weak inverse

$$\widehat{-} : G//G \xrightarrow{\simeq} \Pi \mathrm{Map}(\mathbb{S}^1, BG) \ . \tag{3.11}$$

We make our choices such that the unit element e of the group G is mapped to the constant loop at the base point and $(0, 1) \in \mathbb{S}^1 \subset \mathbb{R}^2$ is mapped to the base point of BG. The object function $\widehat{-} : G \longrightarrow \mathrm{Map}(\mathbb{S}^1, BG)$ can be used to pull back E_2^G to a G-colored operad whose components are computed as follows:

Proposition 3.9. *For $\underline{g} \in G^r$ and $h \in G$ the groupoid $\Pi E_2^G\left(\genfrac{}{}{0pt}{}{\widehat{h}}{\widehat{g}}\right)$ is equivalent to the action groupoid of the B_r-action specified in Lemma 3.6 on the set*

$$\Sigma_r \times_h G^r := \left\{ (\sigma, \underline{b}) \in \Sigma_r \times G^r \ \middle| \ \prod_{j=1}^r b_{\sigma(j)} g_j b_{\sigma(j)} = h \right\} \ .$$

Proof. By Lemma 3.3 $\Pi E_2^G\left(\genfrac{}{}{0pt}{}{h}{g}\right)$ is equivalent to the homotopy fiber of

$$\Pi W_2^G(r) \longrightarrow \Pi \prod^{r+1} \mathrm{Map}(\mathbb{S}^1, BG) \simeq (G//G)^{r+1} \tag{3.12}$$

over $(\underline{g}, h)$. Using the presentation of $\Pi W_2^G(r)$ given in Lemma 3.6, the functor (3.12) sends $(\sigma, a_1, \ldots, a_r)$ to $(a_1, \ldots, a_r, a_{\sigma(1)} \ldots a_{\sigma(r)})$. Therefore, the homotopy fiber of (3.12) over $(\underline{g}, h)$ consists of all

$$(\sigma, \underline{a} = (a_1, \ldots, a_r)) \in \Sigma_r \times G^r, \quad \underline{b} = (b_1, \ldots, b_{r+1}) \in G^{r+1}$$
$$\text{such that} \quad b_j a_j b_j^{-1} = g_j, \quad 1 \leq j \leq r, \quad b_{r+1} a_{\sigma(1)} \ldots a_{\sigma(r)} b_{r+1}^{-1} = h \ .$$

From Lemma 3.6 it follows that, up to equivalence, we can concentrate on the full subgroupoid of the homotopy fiber spanned by those objects satisfying $b_{r+1} = 1$; and a morphism $(\sigma, \underline{a}, \underline{b}) \longrightarrow (\sigma', \underline{a}', \underline{b}')$ with $b_{r+1} = b'_{r+1} = 1$ is just an element of B_r. Of course, for an object $(\sigma, \underline{a}, \underline{b})$ the tuple $\underline{a}$ is redundant because $a_j = b_j^{-1} g_j b_j$. Also, we may work with the tuple $\underline{b}^{-1} = (b_1^{-1}, \ldots, b_r^{-1})$ instead of $\underline{b}$. $\square$

Remark 3.10. The above statement just gives the components of ΠE_2^G, but does not compute it as an operad. The latter problem will be addressed in Section 4.

In the sequel we will need a lifting condition for the functor $\Pi E_2^G\binom{\psi}{\underline{\varphi}} \longrightarrow \Pi E_2(r)$ whose proof we prepare with the following Lemma:

Lemma 3.11. *The forgetful functor* $(\Sigma_r \times_h G^r)\,//B_r \longrightarrow \Sigma_r//B_r$ *admits a unique solution to the lifting problem*

$$
\begin{array}{ccc}
0 & \xrightarrow{\ (\sigma,\underline{b})\ } & (\Sigma_r \times_h G^r)\,//B_r \\
\Big\downarrow & {\scriptstyle \exists!}\ \nearrow & \Big\downarrow \\
[1] & \xrightarrow[\ c:\sigma \longrightarrow \pi(c)\sigma\ ]{} & \Sigma_r//B_r
\end{array}
$$
.

Proof. The unique lift is $c : (\sigma, \underline{b}) \longrightarrow c.(\sigma, \underline{b})$. $\qquad\qquad\square$

Proposition 3.12. *For* $\underline{\varphi} \in \prod^r \mathrm{Map}(\mathbb{S}^1, BG)$ *and* $\psi \in \mathrm{Map}(\mathbb{S}^1, BG)$ *the forgetful functor*

$$
\Pi E_2^G\begin{pmatrix}\psi\\\underline{\varphi}\end{pmatrix} \longrightarrow \Pi E_2(r) \tag{3.13}
$$

admits lifts of the form

$$
\begin{array}{ccc}
0 & \xrightarrow{\ x_0\ } & \Pi E_2^G\big(\begin{smallmatrix}\psi\\\underline{\varphi}\end{smallmatrix}\big) \\
\Big\downarrow & \nearrow & \Big\downarrow \\
[1] & \xrightarrow{\ \ g\ \ } & \Pi E_2(r)
\end{array}
$$

as long as the end and starting point of g are points in $E_2(r)$ whose little disks have non-intersecting boundaries (however, in general (3.13) is not a fibration).

Additionally, we have the following uniqueness statement: For two lifts $\widetilde{g} : x_0 \longrightarrow x_1$ and $\widetilde{g}' : x_0 \longrightarrow x_1'$ there is a unique morphism $h : x_1 \longrightarrow x_1'$ such that $h\widetilde{g} = \widetilde{g}'$, i.e. the lift is completely determined by its start and end point. Furthermore, the set of allowed endpoints is the preimage of the endpoint of the lift from Lemma 3.11 under the equivalence $\Pi E_2^G\big(\begin{smallmatrix}\psi\\\underline{\varphi}\end{smallmatrix}\big) \longrightarrow (\Sigma_r \times_h G^r)\,//B_r$ intersected with the preimage of the endpoint of g under the projection $\Pi E_2^G\big(\begin{smallmatrix}\psi\\\underline{\varphi}\end{smallmatrix}\big) \longrightarrow \Pi E_2(r)$.

Proof. We start by showing that for $\underline{\varphi} \in \prod^r \mathrm{Map}(\mathbb{S}^{n-1}, BG)$ and $\psi \in \mathrm{Map}(\mathbb{S}^{n-1}, T)$ the composition $E_n^T\big(\begin{smallmatrix}\psi\\\underline{\varphi}\end{smallmatrix}\big) \longrightarrow W_n^T(r) \longrightarrow E_n(r)$ admits lifts for paths $I \longrightarrow E_n(r)$ whose little disks have non-intersecting boundaries. Indeed, the needed lifts

$$
\begin{array}{ccc}
0 & \xrightarrow{\qquad} & E_2^G\big(\begin{smallmatrix}\psi\\\underline{\varphi}\end{smallmatrix}\big) \\
\Big\downarrow & \nearrow & \Big\downarrow \\
I & \xrightarrow{\qquad} & E_2(r)
\end{array}
$$

can be constructed thanks to Lemma 2.1 if we allow the lift to take values in $W_2^G(r)$ without making sure that we hit the correct fiber. The parameter $t \in I$ will then describe a path in $W_2^G(r)$ whose restriction

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to the boundary circles will describe homotopies of $\underline{\varphi}$ and ψ. In order to remain in the fiber $E_n^T\left(\begin{smallmatrix}\psi\\\varphi\end{smallmatrix}\right)$, these restrictions would have to be constant. We can easily achieve that by fixing small non-intersecting collars around the boundary circles (this is possible since we assumed that the boundaries of the disks do not intersect) on which we place the inverses of the homotopies of $\underline{\varphi}$ and ψ mentioned above (this strategy is explained in more detail in the proof of Proposition A.5 in the Appendix).

More generally, a path in ΠE_2^G for which the start and end point do not contain intersecting disks admits a representative in E_2^G where the disks do not touch (by rescaling in the interior). This proves the existence.

For the proof of the uniqueness statement we consider the commutative diagram

$$
\begin{array}{ccccc}
0 & \xrightarrow{\ x_0\ } & \Pi E_2^G\left(\begin{smallmatrix}\psi\\\underline{\varphi}\end{smallmatrix}\right) & \xrightarrow{\ \simeq\ } & \left(\Sigma_r \times_h G^r\right)//B_r \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
[1] & \xrightarrow{\ g\ } & \Pi E_2(r) & \xrightarrow{\ \simeq\ } & \Sigma_r//B_r
\end{array}
\qquad,
$$

where $r = |\underline{\varphi}|$ and we have used a holonomy description of (φ, ψ) to apply Proposition 3.9. We deduce from Lemma 3.11 that the image of two lifts $\widetilde{g} : x_0 \longrightarrow x_1$ and $\widetilde{g}' : x_0 \longrightarrow x_1'$ under $\Pi E_2^G\left(\begin{smallmatrix}\psi\\\underline{\varphi}\end{smallmatrix}\right) \longrightarrow \left(\Sigma_r \times_h G^r\right)//B_r$ agree.

Now the assertion follows from the following elementary fact: Let $\Phi : \Gamma \longrightarrow \Omega$ be an equivalence of groupoids and $f : x \longrightarrow x'$ and $g : x \longrightarrow x''$ two morphisms with $\Phi(f) = \Phi(g)$. Then there is a unique morphism $h : x' \longrightarrow x''$ such that $hf = g$. Indeed, set $y := \Phi(x') = \Phi(x'')$, then $\Phi(g^{-1}f) = \mathrm{id}_y$ and h is the preimage of id_y under the bijection $\Gamma(x', x'') \longrightarrow \mathrm{Aut}(y)$. $\qquad\square$

4 Categorical algebras over the little bundle operad

Since any aspherical operad can be seen as an operad in groupoids, it is natural to consider its categorical algebras. For the little disks operad this leads to braided monoidal categories, see [Fre17-I, Chapter 5 and 6] for a detailed discussion that will also be briefly summarized below. For the little bundles operad, as we prove in this section, this leads to Turaev's crossed categories introduced in [Tur00].

4.1 Groupoid-valued operads in terms of generators and relations

For a fixed non-empty set $\mathfrak{C}$ of colors we denote by $U : \mathsf{Op}_{\mathfrak{C}}(\mathcal{M}) \longrightarrow \mathsf{Sym}_{\mathfrak{C}}(M)$ the forgetful functor from the category of $\mathfrak{C}$-colored operad valued in a symmetric monoidal category $\mathcal{M}$ to the category of symmetric sequences in $\mathcal{M}$. This functor admits a left adjoint $F : \mathsf{Sym}_{\mathfrak{C}}(M) \longrightarrow \mathsf{Op}_{\mathfrak{C}}(\mathcal{M})$, the free operad functor.
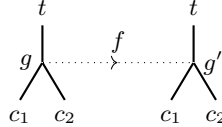
The free operad functor and the cocompleteness of the category of operads can be used to define an operad via generators and relations: Fix a collection of generators $G \in \mathsf{Sym}_{\mathfrak{C}}(\mathcal{M})$ and relations $R \in \mathsf{Sym}_{\mathfrak{C}}(\mathcal{M})$ together with two morphisms $r_1, r_2 : R \longrightarrow UF(G)$. This defines via the adjunction $F \dashv U$ two morphisms $F(R) \rightrightarrows F(G)$. The operad generated by G and R is the coequalizer of the parallel pair $F(R) \rightrightarrows F(G)$.

In the case that $\mathcal{M}$ is the category of groupoids, $\mathcal{M} = \mathsf{Grpd}$, we draw an object g of the groupoid $G\left(\begin{smallmatrix}t\\(c_1,\ldots,c_n)\end{smallmatrix}\right)$ as a planar graph with one vertex labelled by g, n ingoing legs labelled by $c_1,\ldots,c_n$ and one

outgoing edge labelled by t. For example, we depict an object $g \in G\left(\begin{smallmatrix} t \\ (c_1,c_2) \end{smallmatrix}\right)$ as
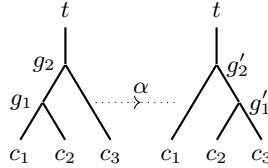
$$
\begin{array}{c}
t \\
\diagup\!\!\!\diagdown \\
c_1 \quad c_2
\end{array}
$$

We will draw morphisms as dotted lines between trees. For example we depict a morphism $f : g \longrightarrow g' \in G\left(\begin{smallmatrix} t \\ (c_1,c_2) \end{smallmatrix}\right)$ as

$$
\begin{array}{ccc}
t & \cdots\!\!\xrightarrow{f}\!\!\cdots & t \\
g\diagup\!\!\!\diagdown & & \diagup\!\!\!\diagdown g' \\
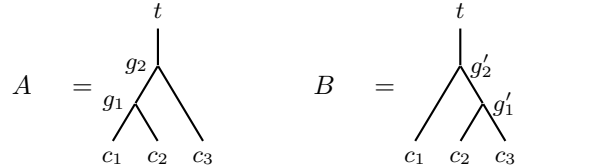c_1 \quad c_2 & & c_1 \quad c_2
\end{array}
$$

Furthermore, when we draw a list of generators we only draw 'elementary' generators and add elements corresponding to the action of permutations and the composition of morphisms. Put more formally, we only specify the groupoid as a directed graph, take the free groupoid generated by this graph and add elements corresponding to the action of the permutation group freely. Note that this automatically adds inverses for every morphism.

To simplify the notation later on, we draw diagrams like

$$
\begin{array}{ccc}
t & & t \\
g_2 \diagup & \xrightarrow{\alpha} & \diagdown g_2' \\
g_1\diagup\!\!\diagdown & & \diagup\!\!\diagdown g_1' \\
c_1 \quad c_2 \quad c_3 & & c_1 \quad c_2 \quad c_3
\end{array}
$$

with generators g_1, g_2, g_1', g_2' to describe the following: We formally add objects A and B in $G\left(\begin{smallmatrix} t \\ (c_1,c_2,c_3) \end{smallmatrix}\right)$, a morphism $\alpha : A \longrightarrow B$ between them and afterwards impose the relation

$$
A \;=\;
\begin{array}{c}
t \\
g_2 \diagup \\
g_1 \diagup\!\!\diagdown \\
c_1 \quad c_2 \quad c_3
\end{array}
\qquad\qquad
B \;=\;
\begin{array}{c}
t \\
\diagdown g_2' \\
\diagup\!\!\diagdown g_1' \\
c_1 \quad c_2 \quad c_3
\end{array}
\quad ,
$$

where the diagrams denote the operadic composition of generators.

Recall that the universal property of the coequalizer and the universal property of the free operad allow us to describe algebras over an operad defined in terms of generators and relations very concretely:

Proposition 4.1. *An algebra A in the category Cat of small categories over a $\mathfrak{C}$-colored operad in Grpd described by generators G and relations R consists of*

- *a category A_c for all $c \in \mathfrak{C}$,*

- *for every generating object $g \in G\left(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\right)$ a functor $A_g : A_{c_1} \times \cdots \times A_{c_n} \longrightarrow A_t$,*

- *for every generating morphism $f : g \longrightarrow g'$ a natural isomorphism $A_f : A_g \Longrightarrow A'_g$,*

such that all relations described by R are satisfied.

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4.2 E_2^G in terms of generators and relations

As a preparation for the description of the little bundle operad in terms of generators and relations we briefly recall the corresponding known description for the little disk operad [Fre17-I, Chapter 5 and 6]: One introduces the groupoid-valued operad PBr of *parenthesized braids* with the generators
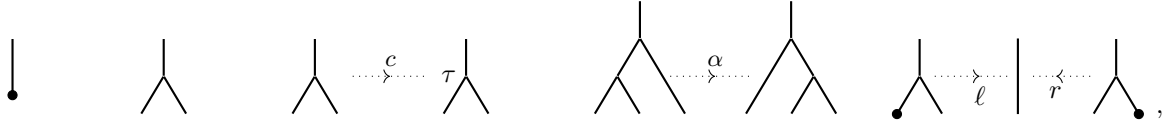
where τ denotes the application of the non-trivial permutation of two elements. As relations, we impose the pentagon identity for α, the hexagon identities for c and the triangle identity on ℓ and r. Proposition 4.1 implies that algebras over PBr are by construction braided tensor categories. Figure 1 indicates the definition of a morphism $F(G) \longrightarrow \Pi E_2$ of operads which by [Fre17-I, Theorem 6.2.4] descends to a morphism $PBr \longrightarrow E_2$.
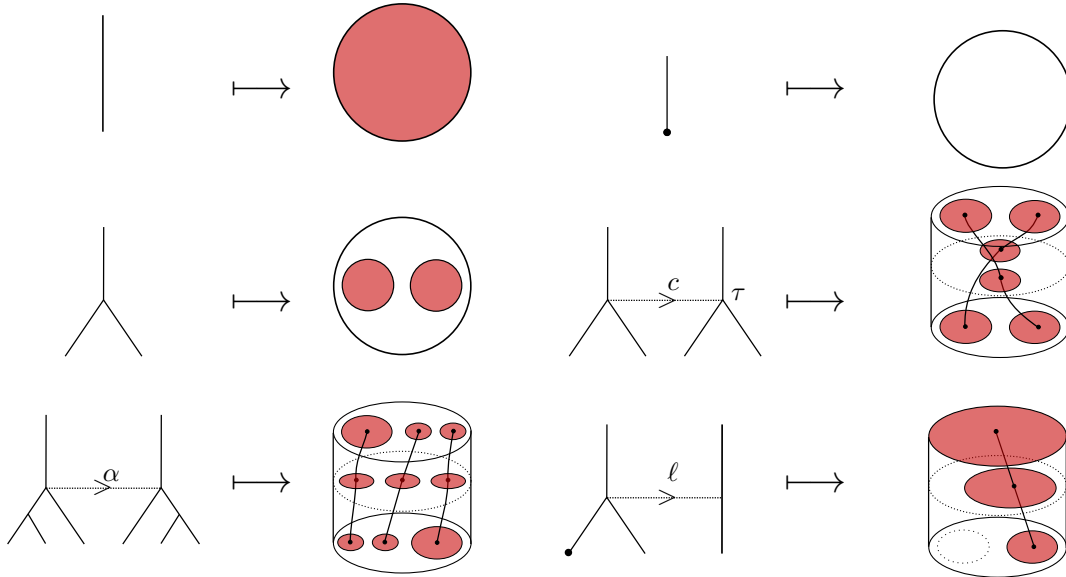

Figure 1: Definition of the morphism $PBr \longrightarrow \Pi E_2$ on generators. We only draw the center of the circles for paths in E_2. The definition for r is analogous to the definition for ℓ.

By [Fre17-I, Proposition 6.2.2] this morphism is a equivalence, thus giving us a presentation of ΠE_2 in terms of generators and relations.

After this short review of the non-equivariant case we generalize this description in terms of generators and relations to the little bundle operad E_2^G by introducing the G-colored operad PBr^G of *G-crossed parenthesized braids*. Its generating operations and isomorphisms are given by
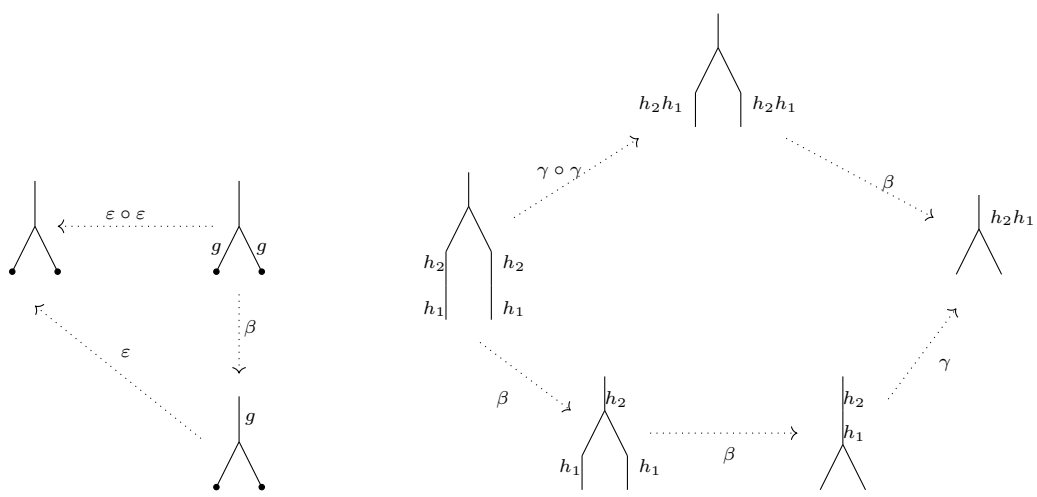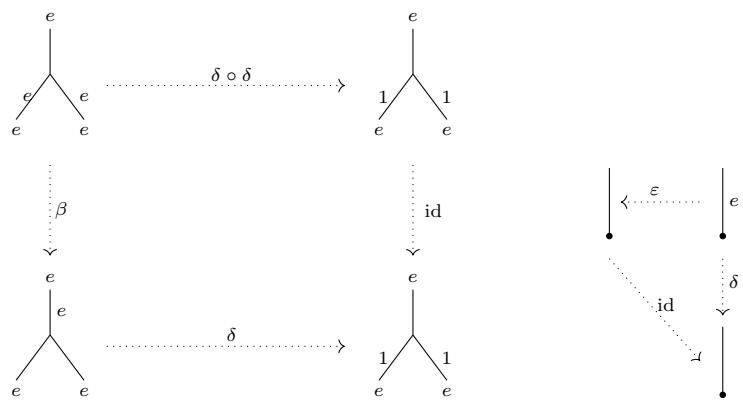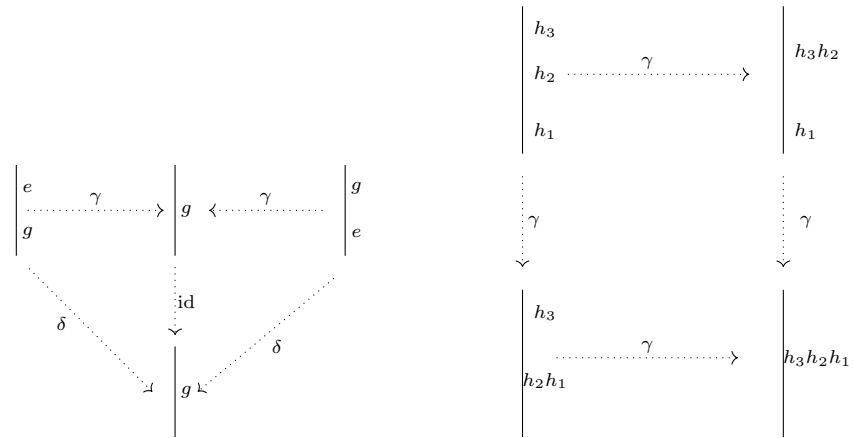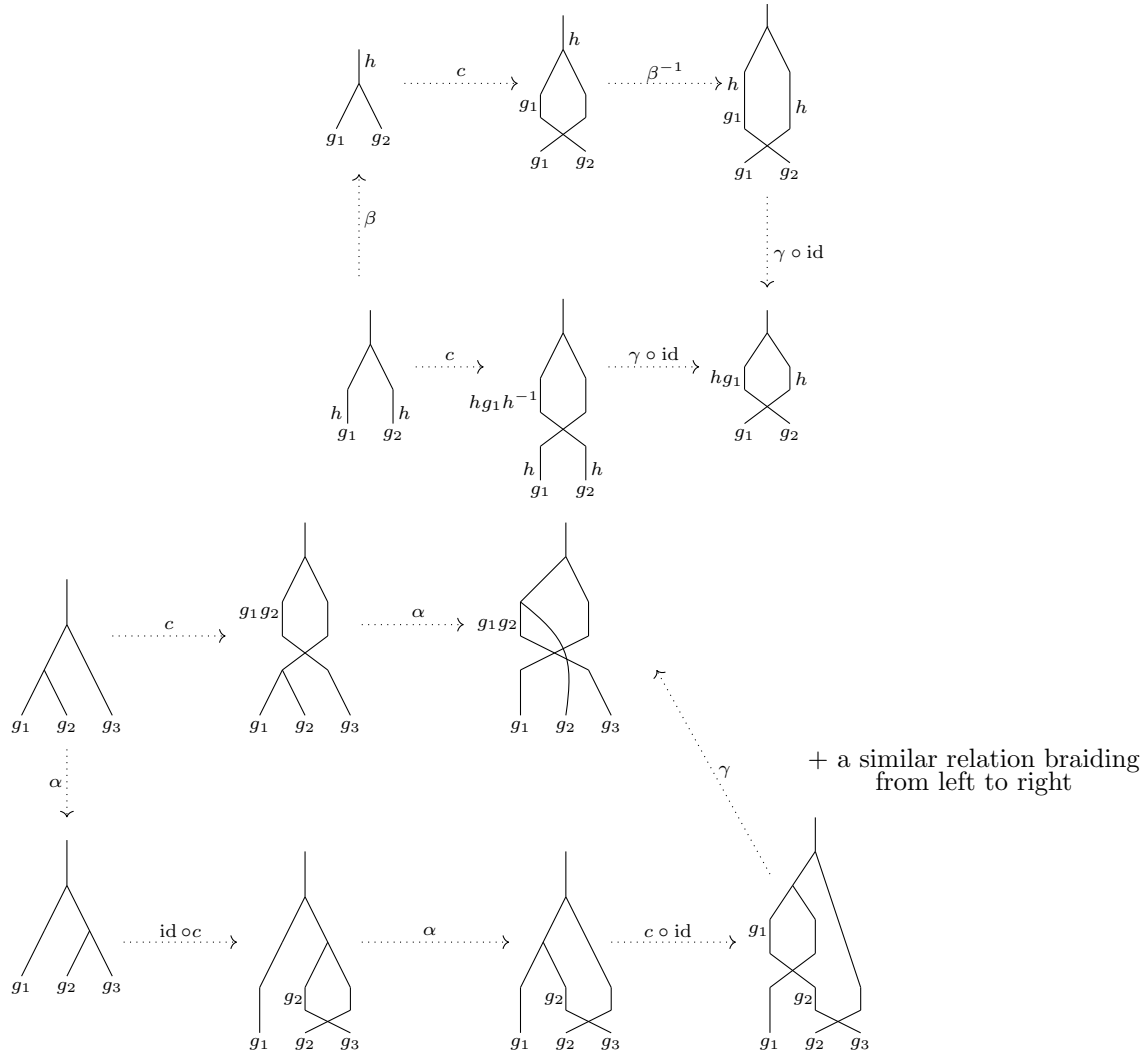
$$\text{(4.1)}$$

$$\text{(4.2)}$$

where 1 denotes the operadic unit.

As relations we introduce the pentagon equation for α, the triangle equations for ℓ and r and the following relations (the labels only indicate the corresponding morphisms and $\circ$ denotes the operadic composition):

$+$ a similar relation involving r instead of ℓ

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By construction and Proposition 4.1 we obtain

Proposition 4.2. *A categorical algebra over* PBr^G *is equivalent to the data of a G-crossed categories in the sense of [Tur10].*

We introduce the auxiliary operad P^G which differs from PBr^G by the omission of the isomorphism c in (4.1). Of course, this implies:

Corollary 4.3. *A categorical algebra over* P^G *is equivalent to the data of a G-equivariant monoidal category in the sense of [Tur10].*

Lemma 4.4. *The operad P^G is discrete and $\pi_0 \mathsf{P}^G \binom{h}{\underline{g}}$ is given by the set $\Sigma_r \times_h G^r$, i.e. by the set of all pairs $(\sigma, \underline{b}) \in \Sigma_r \times G^r$ with $r := |\underline{g}|$ such that*

$$\prod_{j=1}^{r} b_{\sigma(j)} g_j b_{\sigma(j)} = h \ . \tag{4.3}$$

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Proof. The Set-valued operad $\pi_0 \mathsf{P}^G$ has the same generators as P^G, but all the isomorphisms introduced in (4.1) and (4.2) have to be replaced by actual equalities. Recalling the generators and relations for the associative operad, we see that $\pi_0 \mathsf{P}^G$ is a colored version of the associative operad where all ingoing legs can be labeled by a group element. Group elements can be pushed through the multiplication (relation β) and be composed (relation γ) according to the group law. A label by the neutral element is treated as 'no label' (relation δ) and a label on a leg over the unit can be deleted (relation ε). These relations allow to bring each of the operations in $\pi_0 \mathsf{P}^G \binom{h}{\underline{g}}$ into a unique standard form where we can describe them as pair $(\sigma, \underline{b})$ and an r-tupel of group elements by arguments analogous to those in [BSW17, Section 4.3]. The prescription of ingoing and outgoing colors leads to the requirement (4.3) for $(\sigma, \underline{b})$.

We still have to prove that all fundamental groups of $\mathsf{P}^G \binom{h}{\underline{g}}$ are trivial: For this we have to make sure that the given coherence diagrams for $\alpha, \beta, \gamma, \delta$ and ε ensure that for each object in $\mathsf{P}^G \binom{h}{\underline{g}}$ there is only one morphism starting and ending at that object. The needed arguments amount precisely to the coherence theorem for equivariant monoidal categories [Gal17]. □

Adding the isomorphism c to P^G we get the following action groupoid description of G-crossed braids:

Proposition 4.5. *Color-wise there is an equivalence*

$$\mathsf{PBr}^G \binom{h}{\underline{g}} \simeq (\Sigma_r \times_h G^r)//B_r, \quad r = |\underline{g}|$$
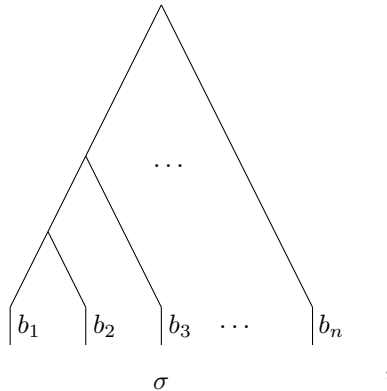
of groupoids.

Proof. We can describe PBr^G by adding to P^G the isomorphism c and the hexagon axiom that it has to satisfy.

For each $(\sigma, \underline{b}) \in \pi_0 \mathsf{P}^G \binom{h}{\underline{g}}$, see Lemma 4.4, and $1 \leq j \leq r-1$ the isomorphism c induces an isomorphism

$$(\sigma, \underline{b}) \xrightarrow{c_{j,j+1}} (\tau_{j,j+1}\sigma, (b_1, \ldots, b_j b_{j+1} b_j^{-1}, b_j, \ldots, b_r)) =: c_{j,j+1}.(\sigma, \underline{b}) , \qquad (4.4)$$

where $\tau_{j,j+1}$ is the transposition of j and $j+1$. Formally, this is achieved by choosing a standard representative for the classes, say

applying operations in P^G to bring it into a form such that the braiding can be applied to the legs j and $j+1$ and restoring the standard form by operations in P^G. The P^G-operations are always uniquely determined by starting point and endpoint thanks to discreteness of P^G (Lemma 4.4). Now $\mathsf{PBr}^G \binom{h}{\underline{g}}$ is equivalent to the groupoid whose objects are given by the set $\pi_0 \mathsf{P}^G \binom{h}{\underline{g}} \cong \Sigma_r \times_h G^r$ and whose morphisms are words in the $c_{j,j+1}$ modulo the (induced) hexagon relations which – as in the non-crossed case –

amount precisely to the braid group relations. This proves that $\mathsf{PBr}^G\binom{h}{g}$ is equivalent to the action groupoid of the B_r-action on $\Sigma_r \times_h G^r$ given by (4.4). □

Remark 4.6. We can read off from Proposition 4.5 that $\pi_0\mathsf{PBr}^G$-algebras with values in vector spaces are G-crossed algebras as considered in [Kau02, Tur10].

Next we construct an operad morphism $\Phi : \mathsf{PBr}^G \longrightarrow \Pi E_2^G$ generalizing the corresponding construction for E_2. As the underlying map of colors we use the object function $\widehat{-} : G \longrightarrow \mathrm{Map}(\mathbb{S}^1, BG)$ of the equivalence

$$\widehat{-} : G//G \xrightarrow{\simeq} \Pi\,\mathrm{Map}(\mathbb{S}^1, BG)$$

from (3.11), i.e. we need to exhibit a morphism $\mathsf{PBr}^G \longrightarrow \widehat{-}^* \Pi E_2^G$ of G-colored operads that by abuse of notation we will also denote by Φ.

To this end, we specify the images of the generators given in (4.1) and (4.2):

1. The generator

$$
e \\
\Big\downarrow \\
\bullet
$$

   is mapped by Φ to the embedding of an empty collection of disks (as in the non-equivariant case, see Figure 1) together with the constant map to BG.

2. The generator

$$
hgh^{-1} \\
\Big| h \\
g
$$

   is mapped by Φ to the embedding $\mathbb{D}^2 \longrightarrow \mathbb{D}^2, x \longmapsto x/2$ and an arbitrary choice

$$\mathbb{D}^2 \setminus \frac{\mathbb{D}^2}{2} \cong \mathbb{S}^1 \times \left[\frac{1}{2}, 1\right] \longrightarrow BG$$

   of a representative in the homotopy class $\widehat{h}$ corresponding to the morphism $h : g \longrightarrow hgh^{-1}$ in $G//G$.

3. The generator

$$
gh \\
\bigwedge \\
g \quad h
$$

   is mapped by Φ to the embedding (see also Figure 1)

$$f : \mathbb{D}^2 \sqcup \mathbb{D}^2 \longrightarrow \mathbb{D}^2$$
$$x_1 \longmapsto \frac{3}{8} \cdot x_1 - \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
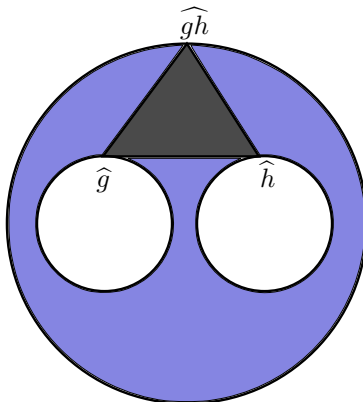$$x_2 \longmapsto \frac{3}{8} \cdot x_2 + \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad .$$

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Figure 2: A sketch for the definition of the map φ.

To equip $\mathsf{C}(f)$ with a continuous map φ into BG, we consider the decomposition of $\mathsf{C}(f)$ sketched in Figure 2. The value of φ on the boundary is given by $\widehat{g}, \widehat{h}$ and $\widehat{gh}$. On the black triangle we choose φ to be constant. Note that the blue area is homeomorphic to the standard 2-simplex. The 2-simplices of BG are described by pairs of group elements, and we equip the blue simplex with the BG-valued map corresponding to (g, h).

4. The path in $E_2(2)$ underlying the image under Φ of the braiding c in $\Pi E_2^G\left(\frac{\widehat{gh}}{(\widehat{g}, \widehat{h})}\right)$ is sketched in Figure 3a. By Proposition 3.12 there exists a lift to the fundamental groupoid of the little bundles operad which is unique once we specify starting point and endpoint. The starting point is the point defined in 3. The endpoint is determined by the images under Φ of the generators that the target of c in (4.1) is built from and their operadic composition. This describes the image of c in ΠE_2^G.

5. The image of the morphisms ℓ and r cannot be constructed by Proposition 3.12 since disks touch at the end point. However, we can use Proposition 3.12 to get a path from the start point of ℓ and r to the disk embedding $x \longmapsto x/2$ equipped with a map into BG which is constant in the radial direction. Now we can rescale the disk as in Figure 4b and leave the map constant. The composition of these two paths defines the image of ℓ and r.

6. To define the image of α, we first define the underlying path in $E_2(3)$ to agree with the corresponding path for PBr, see Figure 1. The corresponding morphism of $\Pi E_2^G\left(\frac{\widehat{g_1 g_2 g_3}}{(\widehat{g_1}, \widehat{g_2}, \widehat{g_3})}\right)$ is again the unique lift which exist by Proposition 3.12.

7. The image of β is again constructed by lifting a path in E_2 sketched in Figure 3b using Proposition 3.12; for another discussion see the proof of [SW18, Proposition 4.2].

8. The path in $E_2(1)$ underlying the image of γ is drawn in Figure 4a. We equip it with a representative for the unique homotopy class of maps into BG constructed from the unique homotopy relative boundary of maps into BG between the composition of the homotopies corresponding to $\widehat{h_1}$ and $\widehat{h_2}$ and the homotopy corresponding to $\widehat{h_1 h_2}$.

9. The image of δ is defined by a simple rescaling sketched in Figure 4b equipped with the constant map into BG.

10. To define the image of ε note that if we consider a disk such that any radial path from the origin to the boundary is labeled by g, the path in BG corresponding to the diameter of the disk is the composition of $\widehat{g}$ with $\widehat{g}^{-1}$ and hence homotopic to the constant map. We use such a homotopy to define the image of ε under Φ.
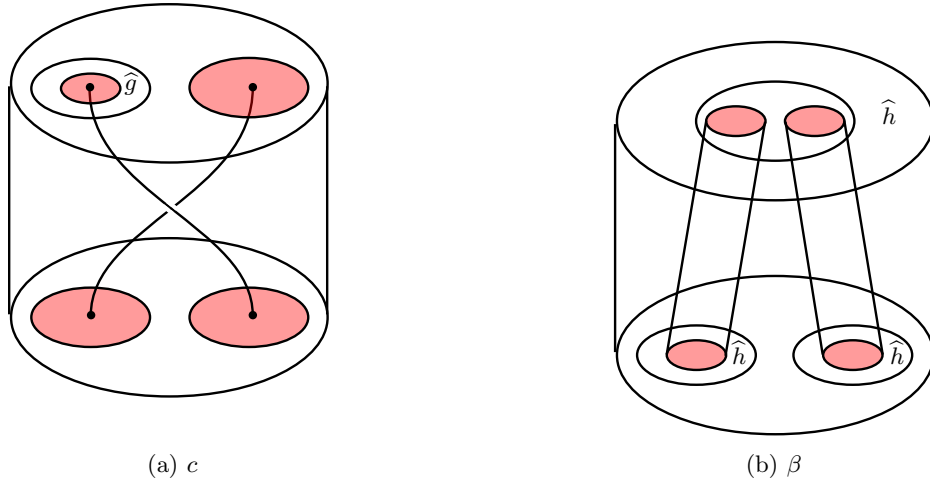
(a) c (b) β

Figure 3: Definition of the morphism $\mathsf{PBr}^G \longrightarrow \Pi E_2^G$ on c and β. In (a) the left circle at the bottom is labelled with $\widehat{g}$; the circle on the right with $\widehat{h}$. Again, we have only drawn the center of every disk when depicting paths.
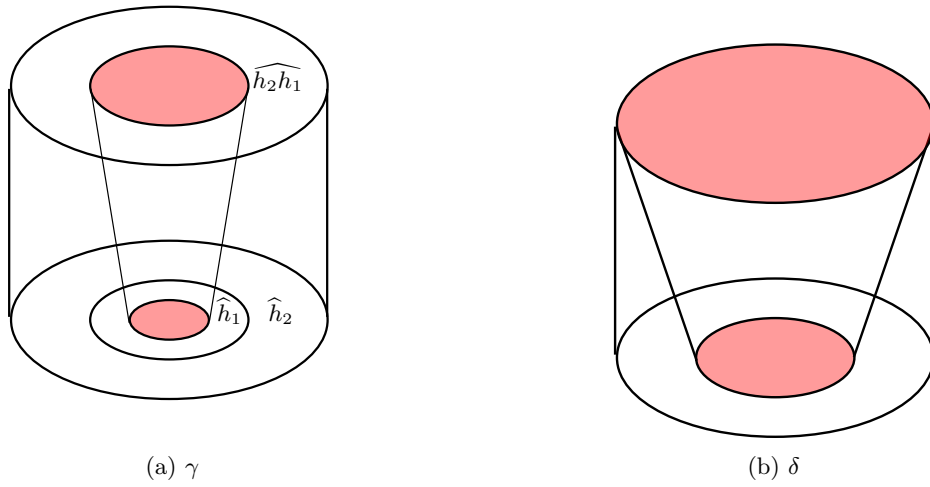

(a) γ (b) δ

Figure 4: Definition of the morphism $\mathsf{PBr}^G \longrightarrow \Pi E_2^G$ on γ and δ.

Theorem 4.7. *This assignment yields an equivalence* $\Phi : \mathsf{PBr}^G \longrightarrow \varPi E_2^G$ *of operads in groupoids.*

Proof.   1. To show that Φ is a map of operads we need to show that the assignments above are compatible with the relations listed on page 13 ff. Verifying a relation amounts to proving that two morphisms in components $\varPi E_2^G$ (namely those prescribed by the above assignments) are equal. This can be achieved by observing that they have the same source and target object and that they lift the same morphism in components of $\varPi E_2$. The latter follows by construction and the fact that in the non-crossed case $\mathsf{PBr} \longrightarrow \varPi E_2$ is a map of operads. Now we invoke to uniqueness statement of Proposition 3.12 to get the desired equality of morphisms. Note that the uniqueness statement of that Proposition can even be used in those cases where it does not grant existence of the lifts. Hence, we have shown that Φ descends to a morphism of $\Phi : \mathsf{PBr}^G \longrightarrow \varPi E_2^G$.

2. In the next step we prove that Φ is a equivalence. First observe that Φ induces an equivalence of the categories enriched in groupoids built from PBr^G and $\varPi E_2^G$ by discarding all non-unary operations. In fact, both these categories have discrete morphism spaces and the functor induced by Φ is actually the equivalence $\widehat{} : G//G \xrightarrow{\simeq} \varPi \operatorname{Map}(\mathbb{S}^1, BG)$ fixed in (3.11). Therefore, to conclude the proof that Φ is a equivalence, it is suffices to prove that its components

$$\Phi : \mathsf{PBr}^G\begin{pmatrix}h\\\underline{g}\end{pmatrix} \longrightarrow E_2^G\begin{pmatrix}\widehat{h}\\\widehat{\underline{g}}\end{pmatrix}$$

are equivalences of groupoids. This follows from the 2-out-of-3 property because these components fit into the weakly commutative triangle

$$
\begin{array}{ccc}
 & \Sigma_r \times_h G^r//B_r & \\
 \nearrow{\scriptstyle\simeq} & & \nwarrow{\scriptstyle\simeq} \\
\mathsf{PBr}^G\begin{pmatrix}h\\\underline{g}\end{pmatrix} \xrightarrow{\hspace{1.5cm}\Phi\hspace{1.5cm}} & & E_2^G\begin{pmatrix}\widehat{h}\\\widehat{\underline{g}}\end{pmatrix}
\end{array}
\quad,
$$

where $r = |\underline{g}|$. Here the equivalence $\mathsf{PBr}^G\begin{pmatrix}h\\\underline{g}\end{pmatrix} \simeq \Sigma_r \times_h G^r//B_r$ comes from Proposition 4.5 and the equivalence $E_2^G\begin{pmatrix}\widehat{h}\\\widehat{\underline{g}}\end{pmatrix} \simeq \Sigma_r \times_h G^r//B_r$ from Proposition 3.9.

$\square$

An operad valued in a model category is called admissible if its category of algebras inherits a model structure in which equivalences and fibrations are created by the forgetful functor to colored objects. From [BM07, Theorem 2.1] one can deduce that operads valued in Cat with its canonical model structure are admissible. Hence, via operadic left Kan extension Φ induces a Quillen adjunction

$$\Phi_! : \mathsf{Alg}(\mathsf{PBr}^G) \rightleftarrows \mathsf{Alg}(\varPi E_2^G) : \Phi^* \tag{4.5}$$

between the categories of algebras over PBr^G and $\varPi E_2^G$, respectively. It is even an equivalence:

Corollary 4.8. *The operad map $\Phi : \mathsf{PBr}^G \longrightarrow \varPi E_2^G$ induces a Quillen equivalence*

$$\Phi_! : \{\text{G-crossed categories}\} \;\underset{\longleftarrow}{\overset{\sim}{\longrightarrow}}\; \{\text{categorical little G-bundles algebras}\} : \Phi^* \,.$$

Proof. Taking into account Proposition 4.2, we need to show that (4.5) is a Quillen equivalence. By [Hin15, Theorem 2.4.5] this follows from Φ being a equivalence (Theorem 4.7) if PBr^G and $\varPi E_2^G$ are Σ-cofibrant.

To see the latter, observe $\varPi$ sends Σ-cofibrant topological operads to Σ-cofibrant categorical operads. Now $\varPi E_2^G$ is Σ-cofibrant thanks to Remark 2.4. It remains to prove that $B\mathsf{PBr}^G$ is Σ-cofibrant which easily follows from the fact that the permutation action is free. $\square$

5 Application to topological field theories

The little bundles operad describes the genus zero part of surfaces decorated with G-bundles, hence it is intimately related to equivariant topological field theories [Tur10, TuVi12, TuVi14] and or rather their homotopical analogues introduced in [MW18-II] using a Segal space model for the $(\infty, 1)$-category G-$\mathsf{Cob}(n)$ of G-cobordisms based on [CS15]:

Definition 5.1. For any group G we define an n-dimensional homotopical equivariant topological field theory with values in a symmetric monoidal $(\infty, 1)$-category $\mathcal{S}$ as a symmetric monoidal ∞-functor

$$Z : G\text{-}\mathsf{Cob}(n) \longrightarrow \mathcal{S}$$

We refer to [MW18-II] for the details and an example of such a field theory constructed using an equivariant version of higher derived Hochschild chains.

We will now make the relation between the little bundles operad and topological field theory precise by proving that the value of a two-dimensional homotopical equivariant topological field theory on the circle is a homotopy little bundles algebra, thereby generalizing [BZFN10, Proposition 6.3] or the earlier version of this result phrased in terms of Gerstenhaber algebras [Ge94].

Indeed, let $Z : G\text{-}\mathsf{Cob}(2) \longrightarrow \mathcal{S}$ be a homotopical two-dimensional G-equivariant topological field theory. Then any operation in $E_2^G \binom{\psi}{\underline{\varphi}}$ can be seen as bordism $(\mathbb{S}^1)^{\sqcup r} \longrightarrow \mathbb{S}^1$, where $r := |\underline{\varphi}|$, decorated with a map to BG whose restriction to the ingoing and outgoing boundary is $\underline{\varphi}$ and ψ, respectively. Strictly speaking, we can see only those operations as bordisms whose E_2-part consists of little disks with non-intersecting boundary. This leads us to considering an equivalent suboperad of E_2^G (which is not strictly unital any more, but only up to homotopy), but we will suppress this in the notation. In summary, we find maps

$$E_2^G \binom{\psi}{\underline{\varphi}} \longrightarrow [Z(\mathbb{S}^1, \varphi_1) \otimes \cdots \otimes Z(\mathbb{S}^1, \varphi_r), Z(\mathbb{S}^1, \psi)] , \tag{5.1}$$

where $[Z(\mathbb{S}^1, \varphi_1) \otimes \cdots \otimes Z(\mathbb{S}^1, \varphi_r), Z(\mathbb{S}^1, \psi)]$ denotes the space of maps from $Z(\mathbb{S}^1, \varphi_1) \otimes \cdots \otimes Z(\mathbb{S}^1, \varphi_r)$ to $Z(\mathbb{S}^1, \psi)$. Since Z respects the composition of bordisms up to coherent homotopy, the maps (5.1) endow the $\mathrm{Map}(\mathbb{S}^1, BG)$-colored object $\big(Z(\mathbb{S}^1, \varphi)\big)_{\varphi \in \mathrm{Map}(\mathbb{S}^1, BG)}$ with the structure of a homotopy E_2^G-algebra, i.e. an algebra over the Boardmann-Vogt resolution of E_2^G. Hence, we have proven:

Theorem 5.2. *For any homotopical two-dimensional G-equivariant topological field theory Z, the values of Z on the circle with varying G-bundle decoration combine into a homotopy algebra over the little bundles operad E_2^G.*

We can use the constructions in this paper also to obtain results about ordinary (non-homotopical) 3-2-1-dimensional topological field theories with non-aspherical target space [Tur10, SW18]. Statements about the non-aspherical case are scarce in the literature, and the following Proposition is supposed to indicate that we can make at least a statement about the value of such theories on the circle:

Proposition 5.3. *Let T be a space such that $\pi_k(T) = 0$ for $k \geq 3$ and $Z : T\text{-}\mathsf{Cob}(3, 2, 1) \longrightarrow 2\mathsf{Vect}_\mathbb{C}$ a 3-2-1-dimensional topological field theory with target T valued in the symmetric monoidal bicategory of complex 2-vector spaces. Then the operad E_2^T takes values in 2-groupoids and the values of Z on the circle combine into a homotopy E_2^T-algebra in 2-vector spaces.*

Proof. Similar arguments as those for Proposition 3.8 show that E_2^T takes values in 2-groupoids. Now we restrict Z to a two-dimensional non-extended $(\infty, 1)$ topological field theory with target T and proceed as in the proof of Theorem 5.2. $\qquad\square$

We will, however, not spell out the data of a (homotopy) E_2^T-algebra; a presentation of E_2^T in terms of generators and relations is beyond the scope of this article. A first approach to E_2^T-algebras might be through the examples that we can produce from a cohomology class in $H^3(T; U(1))$ using Proposition 5.3 and [MW18-I, Theorem 3.19].

A   Appendix

A.1   Properties of the auxiliary spaces $W_n^T(r)$

In order to investigate the spaces $W_n^T(r)$, we will need the following construction for a pair $\langle f, \varphi \rangle \in W_n^T(r)$, i.e. for $f \in E_n(r)$ and a map $\varphi : \mathsf{C}(f) \longrightarrow T$: First note that $\mathsf{C}(f)$ arises from $\mathbb{D}^n$ by cutting out r open disks specified by their radii and centers. We now reduce each of these radii by half. Additionally, we can double the radius of the outer disk. The resulting manifold with boundary is a 'fattening' of $\mathsf{C}(f)$ and denoted by $\widehat{\mathsf{C}}(f)$, see Figure 5. One can use the value of φ on the boundary of $\mathsf{C}(f)$ to extend it to a map $\widehat{\varphi} : \widehat{\mathsf{C}}(f) \longrightarrow T$. This extension will be referred to as *radial extension*.

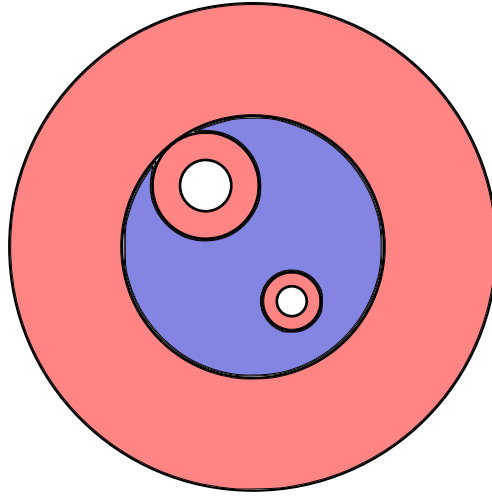

Figure 5: The fattening $\widehat{\mathsf{C}}(f)$ for an element f in $E_2(2)$. The complement $\mathsf{C}(f)$ of f is indicated in blue. To obtain $\widehat{\mathsf{C}}(f)$ we have to take the union with the red area.

We define the subspace

$$\widehat{W}_n(r) := \{(f, x) \in E_n(r) \times \mathbb{D}_2^n \mid x \in \widehat{\mathsf{C}}(f)\} \subset E_n(r) \times \mathbb{D}_2^n \ ,$$

where we denote by $\mathbb{D}_2^n$ the n-dimensional disk of radius 2.

Lemma A.1 (Continuity of radial extension)**.** *For a topological space Y let $g : Y \longrightarrow W_n^T(r)$ be a map of sets satisfying condition (I) and (II) on page 3. Then the radial extension*

$$\widehat{W}_n(r) \times_{E_n(r)} Y \longrightarrow T, \quad (f, x, y) \longmapsto \widehat{g(y)}(x) \tag{A.1}$$

is continuous.

Proof. Let $U \subset T$ be open and let (f_0, x_0, y_0) be in its preimage under (A.1). We need to exhibit a neighbourhood of (f_0, x_0, y_0) in $\widehat{W}_n(r) \times_{E_n(r)} Y$ whose image under (A.1) is contained in U.

If $x_0 \in \mathsf{C}(\mathring{f_0})$, then (f_0, x_0, y_0) is contained in the subspace $W_n(r) \times_{E_n(r)} Y \subset \widehat{W}_n(r) \times_{E_n(r)} Y$. By assumption, the map $W_n(r) \times_{E_n(r)} Y \longrightarrow T$ is continuous, thus we find open neighbourhoods V of f_0 in $E_n(r)$, V'' of y_0 in Y and additionally for some $\varepsilon > 0$ an open ball $\mathbb{B}_\varepsilon(x_0)$ of radius $\varepsilon > 0$ around x_0 such that the image of the open neighbourhood $(V' \times \mathbb{B}_\varepsilon(x_0) \times V'') \cap W_n(r) \times_{E_n(r)} Y$ of (f_0, x_0, y_0) under $W_n(r) \times_{E_n(r)} Y \longrightarrow T$ is contained in U. But since

$$(V' \times \mathbb{B}_\varepsilon(x_0) \times V'') \cap W_n(r) \times_{E_n(r)} Y = (V' \times \mathbb{B}_\varepsilon(x_0) \times V'') \cap \widehat{W}_n(r) \times_{E_n(r)} Y$$

the set $(V' \times \mathbb{B}_\varepsilon(x_0) \times V'') \cap W_n(r) \times_{E_n(r)} Y$ is also an open neighbourhood of (f_0, x_0, y_0) in $\widehat{W}_n(r) \times_{E_n(r)} Y$ being mapped to U under (A.1).

If $x_0 \in \partial\mathsf{C}(f)$, then, as in the case $x_0 \in \mathsf{C}(\mathring{f_0})$, we find suitable neighbourhoods V', V'' and $\mathbb{B}_\varepsilon(x_0)$ such that $(V' \times \mathbb{B}_\varepsilon(x_0) \times V'') \cap W_n(r) \times_{E_n(r)} Y$ is mapped to U (again by continuity of $W_n(r) \times_{E_n(r)} Y \longrightarrow T$). By construction of the radial extension, $(V' \times \mathbb{B}_\varepsilon(x_0) \times V'') \cap \widehat{W}_n(r) \times_{E_n(r)} Y$ is also mapped to U, which gives us the desired neighbourhood is this case.

Now assume that x_0 is not in $\mathsf{C}(f_0)$. Then the corresponding point on the boundary of $\mathsf{C}(f)$ obtained by following a straight line in radial direction is also in the preimage of U, again by construction of the radial extension. For this point there exists the desired open neighbourhood as argued above. We can translate this neighbourhood to x_0, rescale it to get the desired neighbourhood for x_0 and proceed as above. $\qquad\square$

Lemma A.2. *The evaluation map* $\mathrm{ev} : W_n(r) \times_{E_n(r)} W_n^T(r) \longrightarrow T$ *is continuous.*

Proof. Let $U \subset T$ be open and $(f_0, x_0, \varphi_0) \in \mathrm{ev}^{-1}(U)$. We need to show that there exist an open neighbourhood of (f_0, x_0, φ_0) in $W_n(r) \times_{E_n(r)} W_n^T(r)$ such that its image under the evaluation is contained in U.

In a first step consider the radial extension $\widehat{\varphi}_0 : \widehat{\mathsf{C}}(f_0) \longrightarrow T$ of φ_0 to the fattening of $\mathsf{C}(f)$ as discussed above. By continuity of $\widehat{\varphi}_0$ there is an $\varepsilon > 0$ such that

(a) $\mathbb{B}_\varepsilon(x_0) \subset \widehat{\mathsf{C}}(f_0)$,

(b) $\overline{\mathbb{B}_\varepsilon(x_0)} \cap \partial\widehat{\mathsf{C}}(f_0) = \emptyset$

(c) and $\widehat{\varphi}_0\left(\overline{\mathbb{B}_\varepsilon(x_0)}\right) \subset U$.

Now define the subset $U_W \subset W_n^T(r)$ of those $(f, \varphi) \in W_n^T(r)$ for which

(a') $\mathbb{B}_\varepsilon(x_0) \subset \widehat{\mathsf{C}}(f)$,

(b') $\overline{\mathbb{B}_\varepsilon(x_0)} \cap \partial\widehat{\mathsf{C}}(f) = \emptyset$

(c') and $\widehat{\varphi}(\overline{\mathbb{B}_\varepsilon(x_0)}) \subset U$.

Next recall that $W_n(r) \times_{E_n(r)} W_n^T(r)$ is a subspace of $E_n(r) \times \mathbb{D}_2^n \times W_n^T(r)$, where $\mathbb{D}_2^n$ is the closed n-disk of radius 2. The intersection of $E_n(r) \times \mathbb{B}_\varepsilon(x_0) \times U_W \subset E_n(r) \times \mathbb{D}_2^n \times W_n^T(r)$ with $W_n(r) \times_{E_n(r)} W_n^T(r)$ contains (f_0, x_0, φ_0) and is mapped to U under the evaluation. Hence, it remains to show that $E_n(r) \times \mathbb{B}_\varepsilon(x_0) \times U_W$ is open in $W_n(r) \times_{E_n(r)} W_n^T(r)$. For this, it suffices to prove that U_W is open in $W_n^T(r)$.

By definition of the final topology, a subset $V \subset W_n^T(r)$ is open if and only if for all maps $g : Y \longrightarrow W_n^T(r)$ of sets from a topological space Y satisfying the conditions (I) and (II) on page 3 the preimage $g^{-1}(V)$ of V is open. For the proof that U_W meets this requirement let $g : Y \longrightarrow W_n^T(r)$ be a map satisfying conditions (I) and (II). First we remark that it follows from conditions (a') and (b') above that the image $p(U_W)$ of U_W under the projection $p : W_n^T(r) \longrightarrow E_n(r)$ is open in $E_n(r)$. Hence, by (I) the set

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$Y' := (p \circ g)^{-1}(p(U_W))$ is also open. This implies that a subset of Y' is open if and only if it is open seen as a subset of Y. By (II) the map $W_n(r) \times_{E_n(r)} Y \longrightarrow T$ is continuous. This implies that its restriction $W_n(r) \times_{E_n(r)} Y' \longrightarrow T$ is continuous as well. It naturally gives rise to a map

$$\widehat{W}_n(r) \times_{E_n(r)} Y' \longrightarrow T, \quad (f, x, y) \longmapsto \widehat{g(y)}(x)$$

which is continuous by Lemma A.1.

There is a natural embedding

$$\mathbb{B}_\varepsilon(x_0) \times Y' \longrightarrow W_n(r) \times_{E_n(r)} Y', \quad (x, y) \longmapsto (p \circ g(y), x, y)$$

which is continuous by the universal property of the subspace and product topology. This implies that the composition

$$\mathbb{B}_\varepsilon(x_0) \times Y' \longrightarrow T$$
$$(x, y) \longmapsto \widehat{g(y)}(x)$$

is continuous and hence, by adjunction, gives rise to a continuous map

$$Y' \longrightarrow \mathrm{Map}(\overline{\mathbb{B}_\varepsilon(x_0)}, T) \ . \tag{A.2}$$

By definition the subspace $M(\overline{\mathbb{B}_\varepsilon(x_0)}, U) \subset \mathrm{Map}(\overline{\mathbb{B}_\varepsilon(x_0)}, T)$ of all maps $\overline{\mathbb{B}_\varepsilon(x_0)} \longrightarrow T$ sending $\overline{\mathbb{B}_\varepsilon(x_0)}$ to U is open in the compact-open topology. This implies that the preimage of $M(\overline{\mathbb{B}_\varepsilon(x_0)}, U)$ under (A.2) is open. But this preimage is just $g^{-1}(U_W)$ showing that $g^{-1}(U_W)$ is open and finishing the proof. $\square$

Lemma A.3. *For $f \in E_n(r)$ denote by $p^{-1}(f)$ the fiber of p over f endowed with the subspace topology induced from $W_n^T(r)$. Then the evaluation $\mathsf{C}(f) \times p^{-1}(f) \longrightarrow T$ is continuous and the topology on $p^{-1}(f)$ agrees with the compact-open topology, i.e. $p^{-1}(f) = \mathrm{Map}(\mathsf{C}(f), T)$.*

Proof. The map $\mathsf{C}(f) \times p^{-1}(f) \longrightarrow T$ is the restriction of the map $\mathrm{ev} : W_n(r) \times_{E_n(r)} W_n^T(r) \longrightarrow T$ from Lemma A.2 to the fiber of $W_n(r) \times_{E_n(r)} W_n^T(r) \longrightarrow E_n(r)$ over f and hence continuous.

Moreover, $p^{-1}(f) = \mathrm{Map}(\mathsf{C}(f), T)$ as sets, so it remains to show that the identity map is continuous in both directions.

The identity is continuous as a map $\mathrm{Map}(\mathsf{C}(f), T) \longrightarrow p^{-1}(f)$: For this it suffices to show that the composition $\mathrm{Map}(\mathsf{C}(f), T) \longrightarrow W_n^T(r)$ with the inclusion $p^{-1}(f) \longrightarrow W_n^T(r)$ is continuous. The composition of $\mathrm{Map}(\mathsf{C}(f), T) \longrightarrow W_n^T(r) \longrightarrow E_n(r)$ factors through $\{f\} \longrightarrow E_n(r)$ and is therefore continuous, so condition (I) is fulfilled. For condition (II) to be fulfilled we need the evaluation map $\mathsf{C}(f) \times \mathrm{Map}(\mathsf{C}(f), T) \longrightarrow T$ to be continuous. But this is the case because $\mathsf{C}(f)$ is locally compact.

The identity is continuous as a map $p^{-1}(f) \longrightarrow \mathrm{Map}(\mathsf{C}(f), T)$: By adjunction (and since $\mathsf{C}(f)$ is locally compact), continuity of $p^{-1}(f) \longrightarrow \mathrm{Map}(\mathsf{C}(f), T)$ is equivalent to continuity of $\mathsf{C}(f) \times p^{-1}(f) \longrightarrow T$, which we have already established. $\square$

Proposition A.4. *The map $p : W_n^T(r) \longrightarrow E_n(r)$ is a Serre fibration.*

Proof. Let us fix $f_0 \in E_n(r)$. We define an open neighbourhood U of f_0 in $E_n(r)$ consisting of those $f \in E_n(r)$ satisfying for sufficiently small fixed $\varepsilon > 0$ the following requirements (illustrated in Figure 6):

- A ball of radius ε around every center of f_0 does not intersect with $\mathsf{C}(f)$,

- The center of any disk corresponding to f has a distance of less than ε from a center of f_0 (by the first requirement this center is uniquely determined).
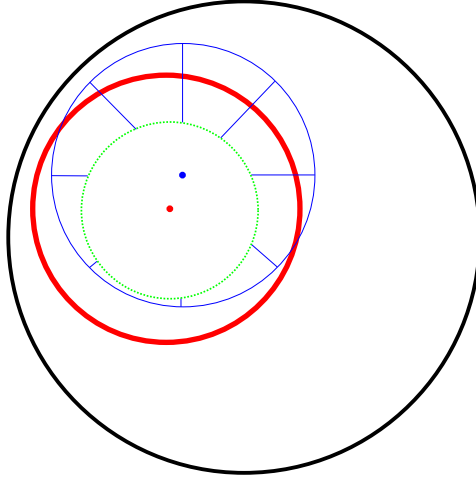
Figure 6: For the proof of Proposition A.4. The red circle corresponds to f_0, the blue circle to f and the green circle to a circle of radius ε around the center of f_0. The blue lines indicate the direction of the radial extension.

We denote the complement of the ε-balls around the centers of f_0 by $\mathsf{C}(\varepsilon, f_0)$. Note that $\mathsf{C}(f) \subset \mathsf{C}(\varepsilon, f_0)$.

Since being a Serre fibration is a local property [Bre93, Theorem VII.6.11], it suffices to prove that for $m \geq 0$ the lifting problem

$$
\begin{array}{ccc}
\mathbb{D}^m \times 0 & \xrightarrow{\ \ K\ \ } & p^{-1}(U) \\
\big\downarrow & \overset{\widetilde{L}}{\nearrow} & \big\downarrow{\scriptstyle p} \\
\mathbb{D}^m \times I & \xrightarrow{\ \ L\ \ } & U
\end{array}
$$

can be solved. For every $x \in \mathbb{D}^m$ we write $K(x) = (K'(x), K''(x))$, where $K'(x) \in E_n(r)$ and $K''(x) : \mathsf{C}(K'(x)) \longrightarrow T$. We can extend $K''(x)$ radially in the direction of the center of $K'(x)$ to a map $\widetilde{K}''(x) : \mathsf{C}(\varepsilon, f_0) \longrightarrow T$ (constantly along the blue radial lines in Figure 6). Since $\mathsf{C}(L(x,t)) \subset \mathsf{C}(\varepsilon, f_0)$, we can set $\widetilde{L}(x,t) := (L(x,t), \widetilde{K}''(x)|_{\mathsf{C}(L(x,t))})$ for $x \in \mathbb{D}^m$ and $t \in I$. This is obviously a p-lift of L as a map of sets. It remains to show that $\widetilde{L}$ is continuous: Indeed, condition (I) is satisfied by definition. For (II) we investigate the map

$$
W_n(r) \times_{E_n(r)} (\mathbb{D}^m \times I) \longrightarrow T \tag{A.3}
$$

and realize that its domain is a subspace

$$
W_n(r) \times_{E_n(r)} (\mathbb{D}^m \times I) \subset U \times \mathsf{C}(\varepsilon, f_0) \times \mathbb{D}^m \times I
$$

and that (A.3) is the restriction of

$$
U \times \mathsf{C}(\varepsilon, f_0) \times \mathbb{D}^m \times I \xrightarrow{\ \mathrm{pr}\ } U \times \mathsf{C}(\varepsilon, f_0) \times \mathbb{D}^m \xrightarrow{\ F\ } T \ ,
$$

where

$$
\begin{aligned}
F : U \times \mathsf{C}(\varepsilon, f_0) \times \mathbb{D}^m &\longrightarrow T \\
(f, x, x') &\longmapsto (\widetilde{K}''(x'))(x)
\end{aligned}
$$

Hence, it suffices to prove that F is continuous. For this, we slightly modify the proof of Lemma A.1: Note that for all radii r in U we have $\varepsilon \leq r \leq 1$. We can then repeat the argument of the proof of Lemma A.1 with 2 replaced by ε and $Y = \mathbb{D}^m$ to see that F is continuous. $\qquad\square$

We also need to prove the following statement about the map q from (2.2):

Proposition A.5. *The map $q : W_n^T(r) \longrightarrow \prod^{r+1} \mathrm{Map}(\mathbb{S}^{n-1}, T)$ obtained by restriction to the boundary is a Serre fibration.*

Proof. We need to prove that for $m \geq 0$ the lifting problem

$$
\begin{array}{ccc}
\mathbb{D}^m \times 0 & \xrightarrow{\;\;K\;\;} & W_n^T(r) \\
\downarrow & \overset{\widetilde{L}}{\nearrow} & \downarrow q \\
\mathbb{D}^m \times I & \xrightarrow{\;\;L\;\;} & \prod^{r+1} \mathrm{Map}(\mathbb{S}^{n-1}, T)
\end{array}
$$

can be solved. For this we write $K(x) = (K'(x), K''(x))$ for $x \in \mathbb{D}^m$, where $K'(x) \in E_n(r)$ and $K''(x) \in \mathrm{Map}(\mathsf{C}(K'(x)), T)$. Next note that L gives us for each $x \in \mathbb{D}^m$ paths $h_1^x, \ldots, h_{r+1}^x$ in $\mathrm{Map}(\mathbb{S}^{n-1}, T)$, and for $1 \leq j \leq r+1$ the path h_j^x is a homotopy of maps $\mathbb{S}^{n-1} \longrightarrow T$ starting at the j-th component $q_j(K''(x))$ of the restriction of $K''(x) : \mathsf{C}(K'(x)) \longrightarrow T$ to the boundary of $\mathsf{C}(K'(x))$.

The desired lift $\widetilde{L} : \mathbb{D}^m \times I \longrightarrow W_n^T(r)$ can now be described as follows: For $(x, t) \in \mathbb{D}^m \times I$ the $E_n(r)$-part of $\widetilde{L}(x, t)$ is obtained from $K'(x)$ by enhancing the radius of the outer disk by t and reducing the radii of the inner disks by multiplying them by $1 - t/2$. Afterwards, we rescale by the factor $1/(1+t)$ to really obtain a point in $E_n(r)$. The needed map from the complement of this point in $E_n(r)$ to T is obtained by gluing together $K''(x)$ and the restriction of the homotopies $h_1^x, \ldots, h_{r+1}^x$ to $[0, t]$. $\qquad\square$

The spaces W_n^T allow for a gluing map that we need to prove that the composition of the little bundles operad is continuous:

Lemma A.6. *The gluing map*

$$
\widehat{\circ}_j : W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1}, T)} W_n^T(r') \longrightarrow W_n^T(r + r' - 1), \; (\langle f, \xi \rangle, \langle f', \xi' \rangle) \longmapsto \langle f \circ f', \xi \cup_{\mathbb{S}^{n-1}} \xi' \rangle \,,
$$

where $f \circ f'$ is the operadic composition in E_n and $\xi \cup_{\mathbb{S}^{n-1}} \xi'$ is the map obtained from gluing ξ and ξ' along $\mathbb{S}^{n-1}$, is continuous.

Proof. By definition of the topology of the spaces W_n^T, continuity of the gluing map amounts to proving that the composition with with $W_n^T(r + r' - 1) \longrightarrow E_n(r + r' - 1)$ is continuous (which is obvious) and that the evaluation

$$
W_n(r + r' - 1) \times_{E_n(r+r'-1)} \left(W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1}, T)} W_n^T(r') \right) \longrightarrow T \tag{A.4}
$$

is continuous.

The latter can be seen by factorizing (A.4) into continuous maps. As a first observation, we describe

the left hand side of (A.4) as the pushout

$$\mathbb{S}^{n-1} \times E_n(r) \times E_n(r') \times W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r') \longrightarrow W_n(r') \times_{E_n(r')} W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r')$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$W_n(r) \times_{E_n(r)} W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r') \longrightarrow \begin{array}{l} \left(W_n(r) \times_{E_n(r)} \left(W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r') \right) \right) \\ \cup \left(W_n(r') \times_{E_n(r)} \left(W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r') \right) \right) \end{array}$$

where the map

$$\mathbb{S}^{n-1} \times E_n(r) \times E_n(r') \times W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r') \longrightarrow W_n(r) \times_{E_n(r)} W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r')$$

first projects to $\mathbb{S}^{n-1} \times E_n(r) \times W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r')$ and then identifies $\mathbb{S}^{n-1}$ with the outer boundary sphere and the map

$$\mathbb{S}^{n-1} \times E_n(r) \times E_n(r') \times W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r') \longrightarrow W_n(r') \times_{E_n(r')} W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r')$$

first projects to $\mathbb{S}^{n-1} \times E_n(r') \times W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r')$ and identifies $\mathbb{S}^{n-1}$ with the j-th ingoing boundary sphere. Indeed, there is a homeomorphism

$$W_n(r + r' - 1) \times_{E_n(r+r'-1)} \left(W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r') \right)$$
$$\xrightarrow{\cong} \left(W_n(r) \times_{E_n(r)} W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r') \right) \cup \left(W_n(r') \times_{E_n(r')} W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r') \right)$$

given by

$$((f_1 \circ f_2, x), \langle f_1, \xi_1 \rangle, \langle f_2, \xi_2 \rangle) \longmapsto \begin{cases} ((f_1, x), \langle f_1, \xi_1 \rangle, \langle f_2, \xi_2 \rangle) \ , & \text{if } x \in \mathsf{C}(f_1) \\ ((f_2, x), \langle f_1, \xi_1 \rangle, \langle f_2, \xi_2 \rangle) \ , & \text{if } x \in \mathsf{C}(f_2) \end{cases}$$

Now (A.4) is the composition of continuous maps

$$W_n(r + r' - 1) \times_{E_n(r+r'-1)} \left(W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r') \right)$$

$$\downarrow \cong$$

$$\left(W_n(r) \times_{E_n(r)} W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r') \right) \cup \left(W_n(r') \times_{E_n(r')} W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r') \right)$$

$$\downarrow \text{projection}$$

$$\left(W_n(r) \times_{E_n(r)} W_n^T(r) \right) \cup \left(W_n(r') \times_{E_n(r')} W_n^T(r') \right)$$

$$\downarrow \text{evaluation}$$

$$T \ .$$

$$\square$$

A.2 Computation of (some) homotopy colimits

In this Appendix we present the computation of a homotopy colimit needed for the description of a groupoid model of the little bundles operad.

First recall that for any diagram X from a groupoid Ω to spaces, the homotopy colimit is given by the realization of the simplicial space with level n given by

$$\coprod_{\vec{y}:[n] \longrightarrow \Omega} X(y_0) \ ,$$

where the coproduct runs over all strings $\vec{y} : [n] \longrightarrow \Omega$ of length $n \geq 0$, see e.g. [Rie14, Corollary 5.1.3] for the definition of the face and degeneracy maps.

Lemma A.7. *Let Γ be a diagram from a groupoid Ω to groupoids. The homotopy colimit of $B\Gamma$ is the nerve of a groupoid admitting the following description:*

- *The objects are given by pairs (y_0, x_0), where $y_0 \in \Omega$ and $x_0 \in \Gamma_0(y_0)$.*

- *For every pair (g_0, f_0), where $g_0 : y_0 \longrightarrow y_1$ is a morphism in Ω and $f_0 : x_0 \longrightarrow x_1$ a morphism in $\Gamma(y_0)$, we get a morphism $(y_0, x_0) \longrightarrow (y_1, g_0.x_1)$.*

- *The composition of morphisms is given by*

$$\left(y_1 \xrightarrow{g_1} y_2, g_0.x_1 \xrightarrow{f_1} x_2 \right) \circ \left(y_0 \xrightarrow{g_0} y_1, x_0 \xrightarrow{f_0} x_1 \right) := \left(y_0 \xrightarrow{g_1 g_0} y_1, x_0 \xrightarrow{(g_0^{-1}.f_1)f_0} g_0^{-1} x_2 \right)$$

Proof. As just explained, the desired homotopy colimit is the realization of the simplicial space with

$$\coprod_{\vec{y}:[n]\longrightarrow\Omega} B\Gamma(y_0)$$

in level n. Since the realization can be computed as the diagonal, we find

$$\left(\operatorname*{hocolim}_{\Omega} B\Gamma \right)_n = \coprod_{\vec{y}:[n]\longrightarrow\Omega} B_n\Gamma(y_0) \ .$$

Carefully writing out the low degree face and degeneracy maps yields the claim. $\qquad\square$

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