EXTREMAL PROBLEMS IN UNIFORMLY DENSE HYPERGRAPHS

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Abstract. For a k-uniform hypergraph F let $\mathrm{ex}(n, F)$ be the maximum number of edges of a k-uniform n-vertex hypergraph H which contains no copy of F. Determining or estimating $\mathrm{ex}(n, F)$ is a classical and central problem in extremal combinatorics. While for graphs ($k = 2$) this problem is well understood, due to the work of Mantel, Turán, Erdős, Stone, Simonovits and many others, only very little is known for k-uniform hypergraphs for $k > 2$. Already the case when F is a k-uniform hypergraph with three edges on $k+1$ vertices is still wide open even for $k = 3$.

We consider variants of such problems where the large hypergraph H enjoys additional hereditary density conditions. Questions of this type were suggested by Erdős and Sós about 30 years ago. In recent work with Rödl and Schacht it turned out that the *regularity method for hypergraphs*, established by Gowers and by Rödl et al. about a decade ago, is a suitable tool for extremal problems of this type and we shall discuss some of those recent results and some interesting open problems in this area.

§1. Introduction

1.1. **Turán's extremal problem.** Extremal graph theory is known to have been initiated by Turán's seminal article [34], in which he proved that for $n \geqslant r \geqslant 2$ there is, among all graphs on n vertices not containing a clique of order r, exactly one whose number of edges is maximal, namely the balanced complete $(r-1)$-partite graph. Turán then asked for similar results, where instead of a clique one intends to find the 1-skeleton of a given platonic solid in the host graph. Moreover, he proposed to study analogous questions in the context of hypergraphs.

Fixing some terminology, we say for a nonnegative integer k that a pair $H = (V, E)$ is a *k-uniform hypergraph*, if V is a finite set of *vertices* and $E \subseteq V^{(k)} = \{e \subseteq V : |e| = k\}$ is a set of k-element subsets of V, whose members are called the *edges* of H. As usual 2-uniform hypergraphs are simply called *graphs*. Associated with every given k-uniform hypergraph F one has *Turán's extremal function* $\mathrm{ex}(\cdot, F)$ mapping every positive integer n to

$$\mathrm{ex}(n, F) = \max\left\{|E| : H = (V, E) \text{ is an } F\text{-free, } k\text{-uniform hypergraph with } |V| = n\right\},$$

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i.e., to the largest number of edges that a k-uniform hypergraph on n vertices without containing F as a (not necessarily induced) subhypergraph can have. In its strictest sense, *Turán's hypergraph problem* asks to determine this function for every hypergraph F.

Using an averaging argument, Katona, Nemetz, and Simonovits [18] have shown that for every k-uniform hypergraph F the sequence $n \longmapsto \mathrm{ex}(n, F)/\binom{n}{k}$ is nonincreasing. Therefore the limit

$$\pi(F) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, F)}{\binom{n}{k}}\,,$$

known as the *Turán density of* F, exists. The problem to determine the Turán densities of all hypergraphs is likewise called *Turán's hypergraph problem*.

It may be observed that these problems are trivial for $k \in \{0, 1\}$, while the case $k = 2$ is fairly well understood. Turán himself [34] determined $\mathrm{ex}(n, K_r)$ for all integers n and r, thus proving $\pi(K_r) = \frac{r-2}{r-1}$ for every integer $r \geqslant 2$. This was further generalised by Erdős and Stone [12], and their result can be shown to yield the full answer to the Turán density problem in the case of graphs. Explicitly, we have

$$\pi(F) = \frac{\chi(F) - 2}{\chi(F) - 1} \tag{1.1}$$

for every graph F with at least one edge, where $\chi(F)$ denotes the *chromatic number* of F, i.e., the least integer r for which there exists a graph homomorphism from F to K_r (see also [10], where the connection with the chromatic number appeared first).

Already for $k = 3$, however, our knowledge is very limited and there are only very few 3-uniform hypergraphs F for which the function $\mathrm{ex}(\cdot, F)$ is completely known. A notable example occurs when F denotes the Fano plane. Sós conjectured in the 1970s that for $n \geqslant 7$ the balanced, complete, bipartite hypergraph is extremal for this problem. The first result in this direction is due to de Caen and Füredi [6], who proved that at least the consequence $\pi(F) = \frac{3}{4}$ of Sós's conjecture holds. By combining their work with Simonovits's stability method [33] it was shown in [15, 20] that the conjecture holds for all sufficiently large hypergraphs. A full proof applying to all $n \geqslant 7$ was recently obtained in [4].

On the other hand, even concerning the 3-uniform hypergraphs on four vertices with three and four edges, denoted by $K_4^{(3)-}$ and $K_4^{(3)}$ respectively, it is only known that

$$\frac{2}{7} \leqslant \pi(K_4^{(3)-}) \leqslant 0.2871 \qquad \text{and} \qquad \frac{5}{9} \leqslant \pi(K_4^{(3)}) \leqslant 0.5616\,.$$

The lower bounds are derived from explicit constructions due to Frankl and Füredi [13] and to Turán (see, e.g., [7]), and in both cases they are universally believed to be optimal. The upper bounds were obtained by computer assisted calculations based on Razborov's *flag algebra method* introduced in [23]. They are due to Baber and Talbot [3], and to Razborov himself [24]. For a more detailed discussion of our current knowledge about Turan's hypergraph problem we refer to Keevash's survey [19].

1.2. **Uniformly dense hypergraphs.** Let us now restrict our attention to 3-uniform hypergraphs. Accordingly, the word *hypergraph* will henceforth always mean 3-uniform hypergraph. Concerning the extremal problem for $K_4^{(3)-}$ it was thought for a while that its Turán density might be $\frac{1}{4}$.

This notion was based on the following construction, which goes back to the work of Erdős and Hajnal [9]. Take a random tournament T on a large set V of vertices. Evidently any three vertices in V induce either a transitive subtournament of T or a cyclic triangle. Furthermore, the former happens with a probability of $\frac{3}{4}$ and the latter with a probability of $\frac{1}{4}$. Define, depending on T, a random hypergraph $H(T)$ on V whose edges correspond to the cyclic triangles in T. One checks easily that $H(T)$ can never contain a $K_4^{(3)-}$ and the random choice of T causes $H(T)$ to have, with high probability, an edge density close to $\frac{1}{4}$.

While the construction of Frankl and Füredi [13] mentioned earlier shows that the hypergraphs $H(T)$ cannot be *optimal among all $K_4^{(3)-}$-free hypergraphs*, it was suggested by Erdős and Sós (see e.g., [8, 11]) that there might still be a natural sense in which they are optimal $K_4^{(3)-}$-free hypergraphs. Specifically, they suggested to focus only on *uniformly dense* host hypergraphs defined as follows.

Definition 1.1. For real numbers $d \in [0, 1]$ and $\eta > 0$ we say that a 3-uniform hypergraph $H = (V, E)$ is *uniformly (d, η)-dense* if for all $U \subseteq V$ the estimate

$$|U^{(3)} \cap E| \geqslant d\binom{|U|}{3} - \eta\,|V|^3$$

holds.

Using standard probabilistic estimates one checks easily that for every accuracy parameter $\eta > 0$ the probability that $H(T)$ is uniformly $(\frac{1}{4}, \eta)$-dense tends to 1 as the number of vertices tends to infinity. The Turán theoretic question about the optimal density of uniformly dense hypergraphs not containing a given hypergraph F (such as $K_4^{(3)-}$) can be made precise by introducing the quantities

$$\begin{aligned}\pi_{\therefore}(F) = \sup\big\{d \in [0, 1] \colon &\text{ for every } \eta > 0 \text{ and } n \in \mathbb{N} \text{ there exists an } F\text{-free,} \\ &\text{uniformly } (d, \eta)\text{-dense hypergraph } H \text{ with } |V(H)| \geqslant n\big\}\,, \end{aligned} \tag{1.2}$$

which are to be regarded as modified versions of the usual Turán densities for uniformly dense hypergraphs. With this notation at hand, the tournament construction shows that $\pi_{\therefore}(K_4^{(3)-}) \geqslant \frac{1}{4}$ and the aforementioned conjecture of Erdős and Sós states that, actually, this holds with equality. Recently this has been shown independently in [16] and in [25].

Theorem 1.2. *We have $\pi_{\therefore}(K_4^{(3)-}) = \frac{1}{4}$.*

One of the two proofs referred to above consists of a computer-generated argument based on Razborov's flag algebra method, while the other one uses the hypergraph regularity method. The subsequent progress in this area (see [28, 29]) has followed the latter approach. Moreover, continuing the collaboration with Rödl and Schacht, we have shown that there is a large number of further variants of the classical Turán density that can likewise be studied by means of the hypergraph regularity method (see [26, 27]). The goal of this article is to survey these recent developments.

Before we proceed any further, however, we would like to draw the reader's attention to perhaps the most urgent problem in the area, the determination of $\pi_{\therefore}(K_4^{(3)})$. The following construction, due to Rödl [30], shows that this number has to be at least $\frac{1}{2}$. Consider, for a sufficiently large natural number n, the elements of $[n] = \{1, 2, \ldots, n\}$ as vertices. Assign to every pair ij of vertices uniformly at random one of the colours *red* or *green*. Declare a triple ijk with $1 \leqslant i < j < k \leqslant n$ to be an edge of the hypergraph H we are about to exhibit, if the colours of ij and ik disagree. Of course this happens with a probability of $\frac{1}{2}$ and, again, standard probabilistic arguments show that for every $\eta > 0$ it happens asymptotically almost surely that H is uniformly $(\frac{1}{2}, \eta)$-dense. Moreover, it is impossible that H contains a tetrahedron. This is because for any four vertices $i < j < k < \ell$ it must be the case that two of the three pairs ij, ik, and $i\ell$ receive the same colour, meaning that the three triples ijk, $ij\ell$, and $ik\ell$ cannot be present in H at the same time.

Conjecture 1.3. Rödl's construction is optimal, i.e., we have $\pi_{\therefore}(K_4^{(3)}) = \frac{1}{2}$.

A partial result in this direction is given by Theorem 1.4 below.

1.3. **Further Turán densities.** For proving results about $\pi_{\therefore}(\cdot)$ one typically works with a property of hypergraphs that turns out to be more useful than the uniform density condition introduced in Definition 1.1. Rather than knowing something about the edge densities *within single* sets of vertices, it is more helpful to have comparable knowledge about the edge densities *between any three* sets of vertices. Explicitly, if $H = (V, E)$ denotes a hypergraph and $A, B, C \subseteq V(H)$, we set

$$E_{\therefore}(A, B, C) = \{(a, b, c) \in A \times B \times C \colon abc \in E\}\,.$$

Moreover, for two real numbers $d \in [0, 1]$ and $\eta > 0$ we say that H is $(d, \eta, \therefore)$-*dense* if

$$|E_{\therefore}(A, B, C)| \geqslant d|A||B||C| - \eta|V|^3$$

holds for all $A, B, C \subseteq V$. One checks immediately by setting $U = A = B = C$ that every $(d, \eta, \therefore)$-dense hypergraph is also uniformly $(d, \eta/6)$-dense. In the converse direction one can only show that large uniformly dense hypergraphs contain linear sized subhypergraphs that are still dense in this new sense with almost the same density, and that this is enough for proving

$$\pi_{\therefore}(F) = \sup\big\{d \in [0,1]\colon \text{for every } \eta > 0 \text{ and } n \in \mathbb{N} \text{ there exists}$$
$$\text{an } F\text{-free, } (d,\eta,\therefore)\text{-dense hypergraph } H \text{ with } |V(H)| \geqslant n\big\}\,. \tag{1.3}$$

A proof of this equality can be found in [28, Proposition 2.5], where one has to set $k = 3$ and $j = 1$. Alternatively, the reader may prefer to regard (1.3) as the "official definition" of $\pi_{\therefore}(\cdot)$ and treat (1.2) just like an additional piece of information that is not going to be used throughout the rest of this article. As a matter of fact, this may even be the more natural approach to this subject, and the three dots occurring in the symbol $\pi_{\therefore}(\cdot)$ are intended to remind us of the three sets A, B, and C mentioned in the definition of being $(d,\eta,\therefore)$-dense.

We proceed with a more restrictive property of hypergraphs shared by both the random tournament construction and by Rödl's hypergraph introduced in the previous subsection. Given a hypergraph $H = (V,E)$, a set $A \subseteq V$, and a set of ordered pairs $P \subseteq V^2$ we set

$$E_{\overset{\cdot}{\smile}}(A,P) = \{(a,b,c) \in V^3\colon a \in A, (b,c) \in P, \text{ and } abc \in E\}\,.$$

So for instance $E_{\overset{\cdot}{\smile}}(A, B \times C) = E_{\therefore}(A,B,C)$ holds for all $A,B,C \subseteq V$. Next, for two real numbers $d \in [0,1]$ and $\eta > 0$ we say that H is $(d,\eta,\overset{\cdot}{\smile})$-*dense* provided that

$$|E_{\overset{\cdot}{\smile}}(A,P)| \geqslant d|A||P| - \eta|V|^3$$

holds for all $A \subseteq V$ and $P \subseteq V^2$. Finally we define

$$\pi_{\overset{\cdot}{\smile}}(F) = \sup\big\{d \in [0,1]\colon \text{for every } \eta > 0 \text{ and } n \in \mathbb{N} \text{ there exists}$$
$$\text{an } F\text{-free, } (d,\eta,\overset{\cdot}{\smile})\text{-dense hypergraph } H \text{ with } |V(H)| \geqslant n\big\}$$

for every hypergraph F. Since every $(d,\eta,\overset{\cdot}{\smile})$-dense hypergraph is, in particular, also $(d,\eta,\therefore)$-dense, we have

$$\pi_{\overset{\cdot}{\smile}}(F) \leqslant \pi_{\therefore}(F)$$

for every hypergraph F. Let us remark at this point that due to the fact that Rödl's hypergraph is $(\frac{1}{2},\eta,\overset{\cdot}{\smile})$-dense we have $\pi_{\overset{\cdot}{\smile}}(K_4^{(3)}) \geqslant \frac{1}{2}$. Thus the following result from [26] shows that a considerably weaker version of Conjecture 1.3 is true.

Theorem 1.4. *We have* $\pi_{\overset{\cdot}{\smile}}(K_4^{(3)}) = \frac{1}{2}$.

The process of replacing a pair of sets by a set of pairs may be repeated once more. For a hypergraph $H = (V,E)$ and two sets of ordered pairs of vertices $P, Q \subseteq V^2$ one defines

$$\mathcal{K}_{\wedge}(P,Q) = \{(a,b,c) \in V^3\colon (a,b) \in P \text{ and } (b,c) \in Q\}$$

as well as

$$E_{\wedge}(P,Q) = \{(a,b,c) \in \mathcal{K}_{\wedge}(P,Q) \colon abc \in E\}\,.$$

Notice that for all $A \subseteq V$ and $P \subseteq V^2$ we have

$$|\mathcal{K}_{\wedge}(A \times V, P)| = |A||P| \quad \text{and} \quad E_{\wedge}(A \times V, P) = E_{\dot{-}}(A,P)\,.$$

Next, we declare H to be $(d,\eta,\wedge)$-*dense* for two real numbers $d \in [0,1]$ and $\eta > 0$ if

$$|E_{\wedge}(P,Q)| \geqslant d|\mathcal{K}_{\wedge}(P,Q)| - \eta|V|^3$$

holds for all $P, Q \subseteq V^2$. If this is the case, then H is $(d,\eta,\dot{-})$-dense as well. The generalised Turán densities appropriate for this concept are defined by

$$\pi_{\wedge}(F) = \sup\bigl\{d \in [0,1] \colon \text{for every } \eta > 0 \text{ and } n \in \mathbb{N} \text{ there exists}$$
$$\text{an } F\text{-free, } (d,\eta,\wedge)\text{-dense hypergraph } H \text{ with } |V(H)| \geqslant n\bigr\}$$

for every hypergraph F, and as before we may observe that

$$\pi_{\wedge}(F) \leqslant \pi_{\dot{-}}(F)\,.$$

The investigation of these quantities was initiated in [27], where the case that F is a clique received particular attention. This led to the curious situation that while the value of $\pi_{\wedge}(K_5^{(3)})$ is still unknown, it has been be shown that $\pi_{\wedge}(K_{11}^{(3)}) = \frac{2}{3}$ holds (see Theorem 2.9). We would like to mention that $\wedge$-dense hypergraphs have recently also been studied by Aigner-Horev and Levy [1] in the context of hypergraph Hamiltonicity problems.

It is natural to expect at this moment some definitions of sets like $\mathcal{K}_{\triangle}(P,Q,R)$ and $E_{\triangle}(P,Q,R)$ involving three sets of ordered pairs, but it can be shown that the corresponding generalised Turán densities $\pi_{\triangle}(F)$ vanish for all hypergraphs F (see [21]).

Still, there are some further variations on this theme. We refer to the concluding remarks in [26] for a complete enumeration of all uniform density notions in the context of 3-uniform hypergraphs*. A more systematic account applying to k-uniform hypergraphs for all $k \geqslant 2$ has been given in [28, Section 2]. In this survey, however, we shall mainly focus on the most concrete cases $\therefore$, $\dot{-}$, and $\wedge$.

*Strictly speaking, that article deals with quasirandomness notions instead of density notions, the difference being that in [26] there are also *upper bounds* imposed on the numbers $|E_{\therefore}(P,Q)|$, etc. It seems, however, that the present version demanding only lower bounds on these numbers is more natural from the perspective of hypergraph Turán problems.

§2. Examples

All known lower bounds on quantities of the form $\pi_\star(F)$ with $\star \in \{\therefore, \overset{\cdot}{\cdot\!-\!\cdot}, \boldsymbol{\wedge}\}$ are derived from probabilistic constructions that can be viewed as appropriate modifications of Rödl's hypergraph introduced at the end of Subsection 1.2. Basically, these constructions combine an ordering of the vertices, a colouring of the pairs of vertices, and certain rules telling us which colour patterns on triples of vertices are to be translated into edges of the envisioned hypergraph.

As a matter of fact, even the Erdős-Hajnal tournament hypergraph can be presented in this manner, even though prima facie it depends on an orientation rather than on a colouring of the pairs. Once its vertices receive an arbitrary ordering, however, there will be "forward" and "backward" arcs between the vertices, and this state of affairs can alternatively be encoded by using two colours. Moreover, one can decide the presence or absence of an edge abc in the hypergraph as soon as one knows the three "colours" received by the pairs ab, ac, and bc (as well as the ordering of $\{a, b, c\}$).

For all these reasons, we shall now describe an abstract framework for presenting such constructions. Given a nonempty finite set Φ of *colours* we call a set $\mathscr{P} \subseteq \Phi^3$ a *palette* (over Φ). So the elements of palettes are ordered triples of colours, called *colour patterns*. Such a palette is said to be $(d, \therefore)$*-dense* for a real number $d \in [0, 1]$ if $|\mathscr{P}| \geqslant d|\Phi|^3$ holds. Given a vertex set V equipped with a linear ordering $<$ and a colouring $\varphi\colon V^{(2)} \longrightarrow \Phi$ we define a hypergraph $H^{\mathscr{P}}_{\varphi} = (V, E)$ by

$$E = \left\{\{x, y, z\} \in V^{(3)}\colon x < y < z \text{ and } \big(\varphi(x, y), \varphi(x, z), \varphi(y, z)\big) \in \mathscr{P}\right\}. \tag{2.1}$$

In practice, one usually takes $V = [n]$ for a sufficiently large integer n and adopts the standard ordering on this set as $<$. This causes no loss of generality in the sense that one still considers the same isomorphism types of hypergraphs as in the general case.

Now the important observation is that if the underlying palette $\mathscr{P}$ is $(d, \therefore)$-dense for some real $d \in [0, 1]$, and if the colouring φ gets chosen uniformly at random (among all $|\Phi|^{\binom{n}{2}}$ possibilities), then for every $\eta > 0$ the hypergraph $H^{\mathscr{P}}_{\varphi}$ is asymptotically almost surely $(d, \eta, \therefore)$-dense. Furthermore, given a hypergraph F and a palette $\mathscr{P}$ it can be decided in finite time whether there exists a hypergraph of the form $H^{\mathscr{P}}_{\varphi}$ containing F. Specifically, this happens if and only if there exists an ordering $<$ of $V(F)$ as well as a colouring $\varphi\colon \partial F \longrightarrow \Phi$ of the set of pairs covered by edges of F such that every edge xyz of F with $x < y < z$ satisfies

$$\big(\varphi(x, y), \varphi(x, z), \varphi(y, z)\big) \in \mathscr{P}.$$

Thus, whenever F fails to admit such a pair $(<, \varphi)$, one knows that $\pi_{\therefore}(F) \geqslant d$.

Example 2.1. The simplest (nontrivial) palettes that can be imagined just consist of a single colour pattern. Owing to potential repetitions of colours in such a pattern, there arise several distinct possibilities, the most restrictive of which is given by three distinct colours. So let us consider the case that $\Phi = \{\text{red}, \text{blue}, \text{green}\}$ and $\mathscr{P} = \{(\text{red}, \text{blue}, \text{green})\}$.

Clearly, $\mathscr{P}$ is $(\frac{1}{27}, \therefore)$-dense. Therefore, the previous discussion shows that if a hypergraph F does not have property *(b)* in Theorem 2.2 below, then $\pi_{\therefore}(F) \geqslant \frac{1}{27}$. In other words, if $\pi_{\therefore}(F) < \frac{1}{27}$, then F needs to admit such an ordering of its vertices together with such a colouring of its shadow. The main result of [29] informs us that under this condition one actually has $\pi_{\therefore}(F) = 0$. This implies that $\pi_{\therefore}(F) \notin (0, \frac{1}{27})$ holds for every hypergraph F.

Theorem 2.2. *For a 3-uniform hypergraph F, the following are equivalent:*

(a) $\pi_{\therefore}(F) = 0$.
(b) *There is an enumeration of the vertex set $V(F) = \{v_1, \dots, v_f\}$ and there is a three-colouring $\varphi\colon \partial F \to \{\text{red}, \text{blue}, \text{green}\}$ of the pairs of vertices covered by hyperedges of F such that every hyperedge $\{v_i, v_j, v_k\} \in E(F)$ with $i < j < k$ satisfies*

$$\varphi(v_i, v_j) = \text{red}, \quad \varphi(v_i, v_k) = \text{blue}, \quad \textit{and} \quad \varphi(v_j, v_k) = \text{green}.$$

Example 2.3. As indicated by the discussion in the second paragraph of this section, the tournament hypergraph can be defined by the $(\frac{1}{4}, \therefore)$-dense palette

$$\mathscr{P} = \{(\rightarrow, \leftarrow, \rightarrow), (\leftarrow, \rightarrow, \leftarrow)\}$$

over $\Phi = \{\rightarrow, \leftarrow\}$. The proof of Theorem 1.2 presented in [25] proceeds by showing that for $n^{-1} \ll \eta \ll \varepsilon$ every $(\frac{1}{4} + \varepsilon, \eta, \therefore)$-dense hypergraph on n vertices possesses a vertex whose link graph contains a triangle. It thus seems natural to wonder whether similar ideas can be used to settle the value of $\pi_{\therefore}(F)$ for all hypergraphs F having a special vertex contained in every edge. Given a graph G, let us call the hypergraph obtained from G by adding a new vertex ∞ having all triples ∞vw with $vw \in E(G)$ as edges the *cone over* G, denoted by CG. So $K_4^{(3)-} = CK_3$ and the question is what one can say about $\pi_{\therefore}(CG)$ in general. This problem is already very interesting when G is a clique. Concerning *stars* $S_k = CK_k$ the proof in [25] shows more generally that

$$\pi_{\therefore}(S_k) \leqslant \left(\frac{k-2}{k-1}\right)^2$$

holds for all $k \geqslant 2$, but it remains unclear at this moment whether this is sharp for any $k \geqslant 4$. The $(\frac{1}{3}, \therefore)$-dense palette

$$\mathscr{P} = \{(1,2,1), (1,3,1), (2,1,2), (2,3,2), (3,1,3), (3,2,3), (1,2,3), (2,3,1), (3,1,2)\}$$

over $\Phi = \{1, 2, 3\}$ establishes the lower bound $\pi_{\therefore}(S_4) \geqslant \frac{1}{3}$ and a generalisation of this idea leads to

$$\pi_{\therefore}(S_k) \geqslant \frac{k^2 - 5k + 7}{(k-1)^2}$$

for all $k \geqslant 3$ (see [25, Section 5.3.1]).

Example 2.4. Rödl's hypergraph, let us recall, is defined by the $(\frac{1}{2}, \therefore)$-dense palette

$$\mathscr{P} = \{(\text{red}, \text{green}, \text{red}), (\text{red}, \text{green}, \text{green}), (\text{green}, \text{red}, \text{red}), (\text{green}, \text{red}, \text{green})\}$$

over $\Phi = \{\text{red}, \text{green}\}$ and establishes $\pi_{\therefore}(K_4^{(3)}) \geqslant \frac{1}{2}$. More generally, given a set Φ consisting of $r \geqslant 2$ colours one may use the palette

$$\mathscr{P} = \{(\alpha, \beta, \gamma) \in \Phi^3 \colon \alpha \neq \beta\}$$

for showing

$$\pi_{\therefore}(K_{r+2}^{(3)}) \geqslant \frac{r-1}{r}\,.$$

It would be exciting if equality turned out to hold here for all $r \geqslant 2$. It should be pointed out, however, that if this is true it might be much more difficult to prove than Conjecture 1.3, as for $r = 4$ there is a second, apparently sporadic, construction that yields the lower bound $\pi_{\therefore}(K_6^{(3)}) \geqslant \frac{3}{4}$ as well. Namely, one takes the palette over $\{\text{red}, \text{green}\}$ containing all six colour patterns involving both colours (see [25, Section 5.1]). This construction works because of $6 \longrightarrow (3)_2^2$. However, we are probably just exploiting a numerical coincidence here and it seems unlikely that similar Ramsey theoretic constructions are relevant to the problem of determining $\pi_{\therefore}(K_{r+2}^{(3)})$ (but see also Example 2.8).

Example 2.5. Finally, we briefly discuss the case where $F = C_5^{(3)}$ is a cycle of length five, i.e., $V(C_5^{(3)}) = \mathbb{Z}/5\mathbb{Z}$ and $E(C_5^{(3)}) = \big\{\{i, i+1, i+2\} \colon i \in \mathbb{Z}/5\mathbb{Z}\big\}$. The lower bound $\pi_{\therefore}(C_5^{(3)}) \geqslant \frac{4}{27}$ can be shown by using the set of colours $\Phi = \{\text{dark red}, \text{light red}, \text{green}\}$ and the palette consisting of all four colour patterns of the form (red, red, green), where "red" means either "dark red" or "light red". As far as we know, no interesting upper bound on $\pi_{\therefore}(C_5^{(3)})$ has ever been obtained.

The last example suggests that occasionally it may be more convenient to work with a weighted version of the concepts introduced so far. Let us say that a *weighted set of colours* is a pair (Φ, w) consisting of a finite nonempty set of colours Φ and a weight function $w\colon \Phi \to [0, 1]$ with the property $\sum_{\gamma \in \Phi} w(\gamma) = 1$. If no weight function has been specified, we imagine that $w(\gamma) = |\Phi|^{-1}$ for all $\gamma \in \Phi$ is implicitly understood. Now when we have a palette $\mathscr{P} \subseteq \Phi^3$ over such a weighted set of colours (Φ, w) we say that $\mathscr{P}$ is $(d, \therefore)$-*dense* if $\sum_{(\alpha,\beta,\gamma)\in\mathscr{P}} w(\alpha)w(\beta)w(\gamma) \geqslant d$. In an obvious sense, this extends the meaning of being $(d, \therefore)$-dense introduced earlier. Now instead of artificially talking about dark and light red in Example 2.5 we could have just said that we consider the weighted set $\Phi = \{\text{red}, \text{green}\}$ with $w(\text{red}) = \frac{2}{3}$ and $w(\text{green}) = \frac{1}{3}$, as well as the palette $\mathscr{P} = \{(\text{red}, \text{red}, \text{green})\}$.

As long as the values attained by our weight function w are rational numbers, it remains, of course, purely a matter of taste whether one prefers weighted sets of colours or whether one rather wants to speak about different shades of colours that are somewhat immaterial to the definition of the palette. It is an interesting open question, however, whether allowing irrational weights of the colours can ever give rise to an optimal lower bound on $\pi_{\therefore}(F)$ for any hypergraph F.

This roughly exhausts the lower bound constructions for $\pi_{\therefore}(\cdot)$ that have been used so far, and we proceed with a discussion of $\pi_{\underline{\therefore}}(\cdot)$. Returning for simplicity to the unweighted setting, we say that a palette $\mathscr{P}$ over a set of colours Φ is $(d, \underline{\therefore})$-*dense* for a real number $d \in [0, 1]$ provided that

for every	we have
$\alpha \in \Phi$	$\big\lvert\{(\beta,\gamma) \in \Phi^2 \colon (\alpha,\beta,\gamma) \in \mathscr{P}\}\big\rvert \geqslant d\lvert\Phi\rvert^2\,,$
$\beta \in \Phi$	$\big\lvert\{(\alpha,\gamma) \in \Phi^2 \colon (\alpha,\beta,\gamma) \in \mathscr{P}\}\big\rvert \geqslant d\lvert\Phi\rvert^2\,,$
$\gamma \in \Phi$	$\big\lvert\{(\alpha,\beta) \in \Phi^2 \colon (\alpha,\beta,\gamma) \in \mathscr{P}\}\big\rvert \geqslant d\lvert\Phi\rvert^2\,.$

Again easy probabilistic arguments show that whenever a palette $\mathscr{P}$ is $(d, \underline{\therefore})$-dense, and a colouring φ gets chosen uniformly at random, then for every $\eta > 0$ the hypergraph $H^{\mathscr{P}}_{\varphi}$ defined in (2.1) is asymptotically almost surely $(d, \eta, \underline{\therefore})$-dense. Thus lower bounds on $\pi_{\underline{\therefore}}(F)$ can be established almost in the same way as for $\pi_{\therefore}(F)$, the only additional thing that needs to be checked being whether the palette one uses satisfies the three conditions in the above table.

For instance, the palettes we referred to in the Examples 2.3 and 2.4 are easily verified to be $\underline{\therefore}$-dense for the expected values of d. Hence the lower bounds on $\pi_{\therefore}(\cdot)$ obtained there apply to $\pi_{\underline{\therefore}}(\cdot)$ as well. In particular, we learn

$$\pi_{\underline{\therefore}}(S_k) \geqslant \frac{k^2 - 5k + 7}{(k-1)^2}$$

for every $k \geqslant 3$ and

$$\pi_{\underline{\therefore}}(K^{(3)}_{r+2}) \geqslant \frac{r-1}{r}$$

for every $r \geqslant 2$. But with the exception of Theorem 1.4 (and Theorem 1.2) it is not known whether equality holds here either. The reader might briefly wonder at this point whether $\pi_{\therefore}(F)$ and $\pi_{\underline{\therefore}}(F)$ agree for all hypergraphs F. But in unpublished work with Rödl and Schacht it was shown that $\pi_{\therefore}(F) > \pi_{\underline{\therefore}}(F) = 0$ holds for some hypergraph F. Moreover, we obtained an explicit description of the class $\{F \colon \pi_{\underline{\therefore}}(F) = 0\}$.

The story of $\pi_{\wedge}(\cdot)$ starts similarly, but the few results that have been obtained so far seem to suggest that this generalised Turán density behaves quite differently. To begin with, given a real number $d \in [0,1]$ and a palette $\mathscr{P}$ over a set of colours Φ, we say that $\mathscr{P}$ is $(d, \boldsymbol{\wedge})$-*dense* if

for all	we have
$\alpha, \beta \in \Phi$	$\big\lvert\{\gamma \in \Phi \colon (\alpha, \beta, \gamma) \in \mathscr{P}\}\big\rvert \geqslant d\lvert\Phi\rvert\,,$
$\alpha, \gamma \in \Phi$	$\big\lvert\{\beta \in \Phi \colon (\alpha, \beta, \gamma) \in \mathscr{P}\}\big\rvert \geqslant d\lvert\Phi\rvert\,,$
$\beta, \gamma \in \Phi$	$\big\lvert\{\alpha \in \Phi \colon (\alpha, \beta, \gamma) \in \mathscr{P}\}\big\rvert \geqslant d\lvert\Phi\rvert\,.$

For clarity we emphasise that the two colours, α, β, etc. mentioned in the left column of this table are allowed to be identical. Now again standard probabilistic arguments show that if $\mathscr{P}$ is $(d, \boldsymbol{\wedge})$-dense, then for every $\eta > 0$ the hypergraph $H^{\mathscr{P}}_{\varphi}$ is asymptotically almost surely $(d, \eta, \boldsymbol{\wedge})$-dense and this principle can be used in the standard way for producing lower bounds on $\pi_{\boldsymbol{\wedge}}(F)$ for many hypergraphs F.

All palettes $\mathscr{P}$ used in this connection so far are *symmetrical* in the sense that for every pattern $(\gamma_1, \gamma_2, \gamma_3) \in \mathscr{P}$ and every permutation $\sigma \in S_3$ one has $(\gamma_{\sigma(1)}, \gamma_{\sigma(2)}, \gamma_{\sigma(3)}) \in \mathscr{P}$. In other words, this means that permuting the entries of a triple does not affect its membership in the palette. For symmetrical palettes any two of our three conditions are equivalent to each other, which reduces the amount of work one needs for checking them by a factor of three.*

When specifying a symmetrical palette, it is convenient to enumerate only a small proportion of its colour patterns from which the remaining ones can be deduced owing to the symmetry condition. More precisely, given an arbitrary palette $\mathscr{P} \subseteq \Phi^3$ we call the inclusion-wise minimal symmetrical palette containing $\mathscr{P}$ the *symmetrical palette generated by* $\mathscr{P}$. One may observe that the three symmetrical palettes in the examples that follow possess some further symmetries induced by permutations of colours.

Example 2.6. The symmetrical palette over $\{1, 2, 3\}$ generated by

$$\{(1,1,2), (2,2,3), (3,3,1)\}$$

is $(\frac{1}{3}, \boldsymbol{\wedge})$-dense and shows $\pi_{\boldsymbol{\wedge}}(K_5^{(3)}) \geqslant \frac{1}{3}$ (see [27, Section 13.1.3]).

Example 2.7. Similarly, the symmetrical palette over $\{1, 2\}$ generated by $\{(1,1,2), (1,2,2)\}$ is $(\frac{1}{2}, \boldsymbol{\wedge})$-dense and, due to the well-known Ramsey theoretic fact $6 \longrightarrow (3)^2_2$, this proves that $\pi_{\boldsymbol{\wedge}}(K_6^{(3)}) \geqslant \frac{1}{2}$.

Example 2.8. Finally, the symmetrical palette over $\{1, 2, 3\}$ generated by

$$\{(1,1,2), (1,1,3), (2,2,1), (2,2,3), (3,3,1), (3,3,2)\}$$

*It is for this reason that in [27, Section 13.1.3] only symmetrical palettes were introduced. Therefore, when writing [27], it seemed more convenient to define palettes as collections of *multisets* of colours instead of ordered triples, but it is unlikely that this will cause any confusion.

is $(\frac{2}{3}, \curlywedge)$-dense and because of a Ramsey theoretic result due to Chung and Graham [5] this proves $\pi_{\curlywedge}(K_{11}^{(3)}) \geqslant \frac{2}{3}$.

The main result of [27] provides an upper bound on the $\curlywedge$-Turán-densities of cliques that turns out to be sharp in surprisingly many small cases.

Theorem 2.9. *For every integer $r \geqslant 2$ one has $\pi_{\curlywedge}(K_{2^r}) \leqslant \frac{r-2}{r-1}$.*

Together with the Examples 2.6–2.8 this yields

$$\begin{array}{rcl} \pi_{\curlywedge}(K_4^{(3)}) = & 0, & \\ & \frac{1}{3} & \leqslant \pi_{\curlywedge}(K_5^{(3)}), \\ \pi_{\curlywedge}(K_6^{(3)}) = \pi_{\curlywedge}(K_7^{(3)}) = \pi_{\curlywedge}(K_8^{(3)}) = & \frac{1}{2} & \leqslant \pi_{\curlywedge}(K_9^{(3)}) \leqslant \pi_{\curlywedge}(K_{10}^{(3)}), \\ \pi_{\curlywedge}(K_{11}^{(3)}) = \cdots = \pi_{\curlywedge}(K_{16}^{(3)}) = & \frac{2}{3} & , \end{array}$$

i.e., the exact value of $\pi_{\curlywedge}(K_t^{(3)})$ for all $t \leqslant 16$ with the exception of $t = 5, 9, 10$. It seems likely that if $\pi_{\curlywedge}(K_5^{(3)}) = \frac{1}{3}$ turned out to be true, then the methods of [27] would allow to prove $\pi_{\curlywedge}(K_{10}^{(3)}) \leqslant \frac{3}{5}$ as well. More generally, there are some good reasons to believe that $\pi_{\curlywedge}(K_\ell^{(3)}) = \alpha$ implies $\pi_{\curlywedge}(K_{2\ell}^{(3)}) \leqslant \frac{1}{2-\alpha}$.

§3. Reduced hypergraphs

It is currently open whether all extremal hypergraphs for $\pi_{\therefore}$, $\pi_{\underline{\therefore}}$, and $\pi_{\curlywedge}$ can be derived from palettes, i.e., whether they are of the form $H_\varphi^{\mathscr{P}}$. There is, however, a slightly more general method to construct $(d, \eta, \star)$-dense hypergraphs with $\star \in \{\therefore, \underline{\therefore}, \curlywedge\}$, for which such a result can be proved. This construction relies on so-called *reduced hypergraphs* that are going to be introduced next.

The main new idea is that when we have an ordered vertex set $(V, <)$ as well as a colouring φ of the pairs in $V^{(2)}$, then in hypergraphs of the form $H_\varphi^{\mathscr{P}} = (V, E)$ the presence or absence of a triple xyz with $x < y < z$ in E depends entirely on the colours received by the pairs xy, xz, and yz without taking the relative positions of x, y, and z in the linear ordering $<$ into account. But one could imagine, for instance, hypergraphs with vertex set $[2n]$ for some huge $n \in \mathbb{N}$, where for converting colour patterns observed on pairs into edges there is one rule applying to triples with two vertices in $[n]$ and a completely different rule for triples with two vertices in $[n+1, 2n]$.

Reduced hypergraphs can be thought of as a framework for capturing the combinatorial core of all such constructions. Let us consider a finite set I of *indices*. Suppose that to any pair of distinct indices $i, j \in I$ there has been assigned a finite nonempty set of vertices $\mathcal{P}^{ij} = \mathcal{P}^{ji}$, and that for distinct pairs of indices the corresponding vertex sets are disjoint. Finally, assume that for every triple of indices $ijk \in I^{(3)}$ there has been specified a 3-uniform tripartite hypergraph $\mathscr{A}^{ijk}$ with vertex classes $\mathcal{P}^{ij}$, $\mathcal{P}^{ik}$, and $\mathcal{P}^{jk}$. In such situations we call the $\binom{|I|}{2}$-partite 3-uniform hypergraph $\mathscr{A}$ with

$$V(\mathscr{A}) = \dot{\bigcup_{ij\in I^{(2)}}} \mathcal{P}^{ij} \quad \text{and} \quad E(\mathscr{A}) = \dot{\bigcup_{ijk\in I^{(3)}}} E(\mathscr{A}^{ijk})$$

a *reduced hypergraph* with *index set* I, *vertex classes* $\mathcal{P}^{ij}$, and *constituents* $\mathscr{A}^{ijk}$.

When translating such a reduced hypergraph into a hypergraph $H = (V, E)$ of Turán theoretic significance one starts with a huge vertex set V having an equipartition $V = \dot\bigcup_{i\in I} V_i$ and takes a "colouring" of pairs of vertices such that for $x \in V_i$ and $y \in V_j$ with $i \neq j$ the pair xy receives uniformly at random some element of $\mathcal{P}^{ij}$ as its "colour" $\varphi(xy)$. Then for any three vertices from distinct partition classes $x \in V_i$, $y \in V_j$, and $z \in V_k$ one decides whether $xyz \in E$ should be the case depending on the colours $\varphi(xy)$, $\varphi(xz)$, and $\varphi(yz)$ by using the constituent $\mathscr{A}^{ijk}$ as if it were a palette; so explicitly one demands

$$xyz \in E \iff \{\varphi(xy), \varphi(xz), \varphi(yz)\} \in E(\mathscr{A}^{ijk})\,.$$

Next we need to express our density conditions in terms of reduced hypergraphs. The definition that follows is easy to remember. Intuitively it just tells us that $\therefore$, $\underline{\therefore}$, and $\curlywedge$ correspond to ordinary density, a minimum vertex degree condition, and a minimum pair degree condition for the constituents of $\mathscr{A}$, respectively.*

Definition 3.1. Let $\mathscr{A}$ denote a reduced hypergraph with index set I, vertex classes $\mathcal{P}^{ij}$, and constituents $\mathscr{A}^{ijk}$, and let $d \in [0, 1]$ be a real number.

(*i*) If $e(\mathscr{A}^{ijk}) \geqslant d|\mathcal{P}^{ij}||\mathcal{P}^{ik}||\mathcal{P}^{jk}|$ holds for any three distinct indices $i, j, k \in I$ we say that $\mathscr{A}$ is $(d, \therefore)$-*dense.*

(*ii*) Moreover, if for any three distinct indices $i, j, k \in I$ and every vertex $P^{ij} \in \mathcal{P}^{ij}$ we have

$$\left|\left\{(P^{ik}, P^{jk}) \in \mathcal{P}^{ik} \times \mathcal{P}^{jk} \colon (P^{ij}, P^{ik}, P^{jk}) \in E(\mathscr{A}^{ijk})\right\}\right| \geqslant d|\mathcal{P}^{ik}||\mathcal{P}^{jk}|\,,$$

then $\mathscr{A}$ is called $(d, \underline{\therefore})$-*dense.*

(*iii*) Finally, if for any three distinct indices $i, j, k \in I$ and all vertices $P^{ij} \in \mathcal{P}^{ij}$, $P^{ik} \in \mathcal{P}^{ik}$ we have

$$\left|\left\{P^{jk} \in \mathcal{P}^{jk} \colon (P^{ij}, P^{ik}, P^{jk}) \in E(\mathscr{A}^{ijk})\right\}\right| \geqslant d|\mathcal{P}^{jk}|\,,$$

then $\mathscr{A}$ is called $(d, \curlywedge)$-*dense.*

*In the same way, the case $\triangle$ dismissed at the the end of Section 1.3 would correspond to a minimum triple degree condition or, in other words, to the condition that all constituents be complete tripartite hypergraphs (if $d > 0$). This is, of course, related to the fact that $\triangle$-dense hypergraphs of positive density contain everything, i.e., that $\pi_{\triangle}(F) = 0$ holds for every hypergraph F.

Whether the hypergraphs described by a given reduced hypergraph $\mathscr{A}$ are capable of containing a given hypergraph F can be expressed in terms of the existence of so-called reduced maps, that are going to be introduced next.

Definition 3.2. A *reduced map* from a hypergraph F to a reduced hypergraph $\mathscr{A}$ with index set I, vertex classes $\mathcal{P}^{ij}$, and constituents $\mathscr{A}^{ijk}$ is a pair (λ, φ) such that

(*i*) $\lambda\colon V(F) \longrightarrow I$ and $\varphi\colon \partial F \longrightarrow V(\mathscr{A})$, where ∂F denotes the set of all pairs of vertices covered by an edge of F;

(*ii*) if $uv \in \partial F$, then $\lambda(u) \neq \lambda(v)$ and $\varphi(uv) \in \mathcal{P}^{\lambda(u)\lambda(v)}$;

(*iii*) if $uvw \in E(F)$, then $\{\varphi(uv), \varphi(uw), \varphi(vw)\} \in E(\mathscr{A}^{\lambda(u)\lambda(v)\lambda(w)})$.

If some such reduced map exists, we say that $\mathscr{A}$ *contains a reduced image of* F, and otherwise $\mathscr{A}$ is called *F-free.*

Now the main result about the reduced hypergraph construction asserts the following.

Theorem 3.3. *If F is a hypergraph and $\star \in \{\therefore, \dot{-}, \wedge\}$, then*

$$\pi_\star(F) = \sup\big\{d \in [0,1]\colon \textit{For every } m \in \mathbb{N} \textit{ there is a } (d,\star)\textit{-dense,} \\ F\textit{-free, reduced hypergraph with an index set of size } m\big\}\,. \tag{3.1}$$

Large parts of the proof of this result are implicit in [25–27, 29]. Still, we believe it to be useful to gather the argument in its entirety in the remainder of this section and the two subsequent sections. To this end, we shall temporarily denote the right side of (3.1) by $\pi_\star^{\mathrm{rd}}(F)$, where the superscript "rd" means "reduced".

The inequality $\pi_\star^{\mathrm{rd}}(F) \leqslant \pi_\star(F)$, proved in Proposition 3.4 below, simply expresses the fact that the narrative of this section does indeed indicate a valid strategy for establishing lower bounds on $\pi_\star(F)$ by means of reduced hypergraphs. The proof of the other direction, $\pi_\star^{\mathrm{rd}}(F) \geqslant \pi_\star(F)$, requires more involved reasoning based on the hypergraph regularity method.

Proposition 3.4. *For every hypergraph F and every symbol $\star \in \{\therefore, \dot{-}, \wedge\}$ we have*

$$\pi_\star^{\mathrm{rd}}(F) \leqslant \pi_\star(F)\,.$$

Let us recall the following standard concepts and facts required in the proof. A bipartite graph $G = (X \cupdot Y, E)$ is called (δ, d)-*quasirandom* for two real numbers $\delta > 0$ and $d \in [0,1]$ if for all $A \subseteq X$ and $B \subseteq Y$ the estimate $\big|e(A,B) - d|A||B|\big| \leqslant \delta|X||Y|$ holds. Suppose now that for two nonempty disjoint sets X and Y we create a random bipartite graph G with vertex set $X \cupdot Y$ by declaring each pair in $K(X,Y) = \{\{x,y\}\colon x \in X \text{ and } y \in Y\}$ uniformly at random to be an edge of G with probability d. Then for any fixed pair of sets $A \subseteq X$ and $B \subseteq Y$ Chernoff's inequality (see e.g. [2, Theorem A.1.4]) implies

$$\mathbb{P}\Big(\big|e(A,B) - d|A||B|\big| > \delta|X||Y|\Big) \leqslant 2\exp(-2\delta^2|X||Y|),$$

whence

$$\mathbb{P}(G \text{ fails to be } (\delta, d)\text{-quasirandom}) \leqslant 2^{|X|+|Y|+1} \exp(-2\delta^2 |X||Y|)\,.$$

In particular, if $|X| = |Y|$ tends to infinity, then G is asymptotically almost surely (δ, d)-quasirandom.

An important result about quasirandomness, utilised below, is the so-called *triangle counting lemma*. It informs us that if a tripartite graph $P = (X \cupdot Y \cupdot Z, E)$ has the property that its naturally induced bipartite subgraphs on $X \cupdot Y$, $X \cupdot Z$, and $Y \cupdot Z$ are (δ, d_{XY})-, (δ, d_{XZ})-, and (δ, d_{YZ})-quasirandom, respectively, then the size of the set $\mathcal{K}_3(P) = \{\{x, y, z\} \colon xy, xz, yz \in E\}$ of triangles it contains obeys the estimate

$$\big||\mathcal{K}_3(P)| - d_{XY} d_{XZ} d_{YZ} |X||Y||Z|\big| \leqslant 3\delta |X||Y||Z|\,.$$

Proof of Proposition 3.4. Let a real number $d \in [0, 1]$ be given which has the property that for every $m \in \mathbb{N}$ there exists a $(d, \star)$-dense, F-free reduced hypergraph with m indices. We need to show that $d \leqslant \pi_\star(F)$. So consider an arbitrary real $\eta > 0$ as well as some $n \in \mathbb{N}$. Now we need to produce a $(d, \eta, \star)$-dense, F-free hypergraph $H = (V, E)$ with $|V| \geqslant n$. For this purpose, we set

$$m = \left\lceil \frac{6}{\eta} \right\rceil \tag{3.2}$$

and appeal to our hypothesis on d. It yields an F-free, $(d, \star)$-dense reduced hypergraph $\mathscr{A}$, say with index set I, vertex classes $\mathcal{P}^{ij}$, and constituents $\mathscr{A}^{ijk}$, where $|I| = m$. Now set

$$P = \max\big\{|\mathcal{P}^{ij}| \colon ij \in I^{(2)}\big\}, \quad \delta = \frac{\eta}{6P^3}$$

and let $h \gg P, m, n, \eta^{-1}$ be sufficiently large.

We shall construct the desired hypergraph H on a set of vertices $V = \dot\bigcup_{i \in I} V_i$ with $|V_i| = h$ for every $i \in I$. Owing to the probabilistic argument discussed before this proof we may assume that there is a family $\{\varphi_{ij} \colon ij \in I^{(2)}\}$ of colourings $\varphi_{ij} \colon K(V_i, V_j) \longrightarrow \mathcal{P}^{ij}$ such that for every pair of indices $ij \in I^{(2)}$ and every $P^{ij} \in \mathcal{P}^{ij}$ the bipartite graph $G(P^{ij})$ between V_i and V_j whose set of edges is $\varphi_{ij}^{-1}(P^{ij})$ happens to be $(\delta, |\mathcal{P}^{ij}|^{-1})$-quasirandom. Depending on such colourings φ_{ij} we complete the definition of H in the expected way by setting

$$\begin{aligned} E(H) = \big\{xyz \in V^{(3)} \colon &\text{There are distinct } i, j, k \in I \text{ with } x \in V_i,\ y \in V_j,\ z \in V_k, \\ &\text{and } \{\varphi_{ij}(xy), \varphi_{ik}(xz), \varphi_{jk}(yz)\} \in E(\mathscr{A}^{ijk})\big\}\,. \end{aligned}$$

Let us remark that all edges of H are *crossing* in the sense of intersecting each of the vertex classes V_i with $i \in I$ at most once. The rationale behind our choice of m in (3.2) is that it allows us to bound the number of non-crossing triples $(x, y, z) \in V^3$ in a useful way. Clearly, this number is h^3 times the number of triples $(i, j, k) \in I^3$ for which $i = j$, $i = k$, or $j = k$ holds. As this number is in turn at most $3m^2$, we conclude that the number of non-crossing ordered triples is at most $3m^2h^3 = 3m^{-1}|V|^3$, which by (3.2) is at most $\frac{\eta}{2}|V|^3$.

Now our choice of h clearly guarantees $|V(H)| = hm \geqslant n$. Next we would like to check that H is indeed F-free. Otherwise there would exist an embedding $\psi\colon F \longrightarrow H$. For each $u \in V(F)$ let $\lambda(u) \in I$ denote the index for which $\psi(u) \in V_{\lambda(u)}$ is true. For every pair $uv \in \partial F$ we know that $\lambda(u) \neq \lambda(v)$, because the edges of H are crossing. Thus we may define $\varphi\colon \partial F \to V(\mathscr{A})$ by

$$\varphi(uv) = \varphi_{\lambda(u)\lambda(v)}\big(\psi(u)\psi(v)\big)$$

for every pair $uv \in \partial F$. Evidently λ and φ satisfy the first two clauses of Definition 3.2. As ψ maps edges of F to edges of H, they satisfy *(iii)* as well. Thus (λ, φ) is a reduced map from F to $\mathscr{A}$, contrary to the choice of $\mathscr{A}$ as being F-free.

It remains to check that H is $(d, \eta, \star)$-dense and for this purpose we consider the three possibilities for $\star$ separately.

First Case. $\star = \therefore$

Given arbitrary $A, B, C \subseteq V$ we need to prove that $|E_{\therefore}(A, B, C)| \geqslant d|A||B||C| - \eta|V|^3$. Whenever $i, j, k \in I$ are distinct and $\{P^{ij}, P^{ik}, P^{jk}\} \in E(\mathscr{A}^{ijk})$, the triangle counting lemma entails that the tripartite subgraph of $G(P^{ij}) \cup G(P^{ik}) \cup G(P^{jk})$ induced by $A \cap V_i$, $B \cap V_j$, and $C \cap V_k$ contains at least

$$\frac{|A \cap V_i||B \cap V_j||C \cap V_k|}{|\mathcal{P}^{ij}||\mathcal{P}^{ik}||\mathcal{P}^{jk}|} - 3\delta h^3$$

triangles each of which gives rise to an edge of H. Thus for distinct $i, j, k \in I$ we have

$$E_{\therefore}(A \cap V_i, B \cap V_j, C \cap V_k) \geqslant \frac{|E(\mathscr{A}^{ijk})||A \cap V_i||B \cap V_j||C \cap V_k|}{|\mathcal{P}^{ij}||\mathcal{P}^{ik}||\mathcal{P}^{jk}|} - 3P^3\delta h^3\,,$$

which by our assumption that $\mathscr{A}$ be $(d, \therefore)$-dense and by our choice of δ yields

$$E_{\therefore}(A \cap V_i, B \cap V_j, C \cap V_k) \geqslant d|A \cap V_i||B \cap V_j||C \cap V_k| - \tfrac{\eta}{2}h^3\,.$$

Summing over all ordered triples (i, j, k) of distinct indices we infer that, up to an additive error of at most $\frac{\eta}{2}|V|^3$, the size of $E_{\therefore}(A, B, C)$ is at least d times the number of crossing triples in $A \times B \times C$. As there at most $\frac{\eta}{2}|V|^3$ non-crossing triples altogether, it follows that we have indeed $|E_{\therefore}(A, B, C)| \geqslant d|A||B||C| - \eta|V|^3$.

Second Case. $\star = \overset{\cdot}{\underline{\cdot\,\cdot}}$

Given $A \subseteq V$ and $Q \subseteq V^2$ we need to prove that $|E_{\overset{\cdot}{\underline{\cdot\,\cdot}}}(A, Q)| \geqslant d|A||Q| - \eta|V|^3$. Getting rid of non-crossing triples as in the previous case, it suffices to this end if we show for any three distinct indices $i, j, k \in I$ that

$$|E_{\overset{\cdot}{\underline{\cdot\,\cdot}}}(A \cap V_i, Q \cap (V_j \times V_k)| \geqslant d|A \cap V_i||Q \cap (V_j \times V_k)| - \tfrac{\eta}{2}h^3\,.$$

For this in turn it is enough to establish that for every $P^{jk} \in \mathcal{P}^{jk}$ the sets

$$\widetilde{K}(P^{jk}) = \big\{(x,y,z) \in V_i \times V_j \times V_k \colon x \in A, (y,z) \in Q, \text{ and } \varphi_{jk}(yz) = P^{jk}\big\}$$

and

$$\widetilde{E}(P^{jk}) = \big\{(x,y,z) \in \widetilde{K}(P^{jk}) \colon xyz \in E(H)\big\}$$

satisfy

$$\big|\widetilde{E}(P^{jk})\big| \geqslant d\,\big|\widetilde{K}(P^{jk})\big| - \tfrac{\eta}{2P} h^3 \,.$$

Now we distinguish the triples (x,y,z) contributing to the left side according to the values of $\varphi_{ij}(xy)$ and $\varphi_{ik}(xz)$. By the assumed $(d, \therefore)$-denseness of $\mathscr{A}$ we know

$$\big|\big\{(P^{ik}, P^{jk}) \in \mathcal{P}^{ik} \times \mathcal{P}^{jk} \colon (P^{ij}, P^{ik}, P^{jk}) \in E(\mathscr{A}^{ijk})\big\}\big| \geqslant d|\mathcal{P}^{ik}||\mathcal{P}^{jk}| \,,$$

and thus it remains to show that for every edge $\{P^{ij}, P^{ik}, P^{jk}\}$ of $\mathscr{A}^{ijk}$ we have

$$\big|\big\{(x,y,z) \in \widetilde{K}(P^{jk}) \colon \varphi_{ij}(xy) = P^{ij} \text{ and } \varphi_{ik}(xz) = P^{ik}\big\}\big| \geqslant \frac{\big|\widetilde{K}(P^{jk})\big|}{|\mathcal{P}^{ij}||\mathcal{P}^{ik}|} - 3\delta h^3 \,. \tag{3.3}$$

Now appealing for $y \in V_j$ to the $(\delta, |\mathcal{P}^{ik}|^{-1})$-quasirandomness of $G(P^{ik})$ we learn that the sets

$$A_y = \big\{x \in A \cap V_i \colon \varphi_{ij}(xy) = P^{ij}\big\}$$

and

$$C_y = \big\{z \in V_k \colon (y,z) \in Q \text{ and } \varphi_{jk}(yz) = P^{jk}\big\}$$

satisfy

$$\big|\big\{(x,z) \in A_y \times C_y \colon \varphi_{ik}(xz) = P^{ik}\big\}\big| \geqslant \frac{|A_y||C_y|}{|\mathcal{P}^{ik}|} - \delta h^2 \,.$$

Summing over all $y \in Y$ we deduce

$$\begin{aligned} &\big|\big\{(x,y,z) \in \widetilde{K}(P^{jk}) \colon \varphi_{ij}(xy) = P^{ij} \text{ and } \varphi_{ik}(xz) = P^{ik}\big\}\big| \\ &\qquad\qquad \geqslant |\mathcal{P}^{ik}|^{-1}\big|\big\{(x,y,z) \in \widetilde{K}(P^{jk}) \colon \varphi_{ij}(xy) = P^{ij}\big\}\big| - \delta h^3 \,. \end{aligned} \tag{3.4}$$

Thus (3.3) will follow if can prove additionally that

$$\big|\big\{(x,y,z) \in \widetilde{K}(P^{jk}) \colon \varphi_{ij}(xy) = P^{ij}\big\}\big| \geqslant |\mathcal{P}^{ij}|^{-1}\big|\widetilde{K}(P^{jk})\big| - \delta h^3 \,.$$

This estimate can be verified, however, in the same way as (3.4), the only difference being that this time one works with a sum over all $z \in V_k$ and exploits the $(\delta, |\mathcal{P}^{ij}|^{-1})$-quasirandomness of $G(P^{ij})$.

Third Case. $\star = \curlywedge$

Proceeding almost exactly as in the previous case we consider two given sets of pairs $Q, R \in V^2$ and aiming at $|E_{\curlywedge}(Q,R)| \geqslant d|\mathcal{K}_{\curlywedge}(Q,R)| - \eta|V|^3$ we begin again by eliminating the noncrossing triples from our consideration, this time by reducing our claim to the statement that for any three distinct indices $i, j, k \in I$ the inequality

$$\big|E_{\wedge}\big(Q \cap (V_i \times V_j), R \cap (V_j \times V_k)\big)\big| \geqslant d\big|\mathcal{K}_{\wedge}\big(Q \cap (V_i \times V_j), R \cap (V_j \times V_k)\big)\big| - \tfrac{\eta}{2}h^3$$

holds. This will be clear once we know that for all $P^{ij} \in \mathcal{P}^{ij}$ and $P^{jk} \in \mathcal{P}^{jk}$ the sets

$$\widetilde{K}(P^{ij}, P^{jk}) = \big\{(x, y, z) \in V_i \times V_j \times V_k \cap \mathcal{K}_{\wedge}(Q, R) \colon \varphi_{ij}(xy) = P^{ij} \text{ and } \varphi_{jk}(yz) = P^{jk}\big\}$$

and

$$\widetilde{E}(P^{ij}, P^{jk}) = \big\{(x, y, z) \in \widetilde{K}(P^{ij}, P^{jk}) \colon xyz \in E(H)\big\}$$

satisfy

$$\big|\widetilde{E}(P^{ij}, P^{jk})\big| \geqslant d\big|\widetilde{K}(P^{ij}, P^{jk})\big| - 3\delta P h^3 \,.$$

As $\mathscr{A}$ is $(d, \wedge)$-dense, we have

$$\big|\big\{P^{ik} \in \mathcal{P}^{ik} \colon (P^{ij}, P^{ik}, P^{jk}) \in E(\mathscr{A}^{ijk})\big\}\big| \geqslant d|\mathcal{P}^{ik}|\,,$$

and, hence, it is enough to check that to every edge $\{P^{ij}, P^{ik}, P^{jk}\}$ of the constituent $\mathscr{A}^{ijk}$ there corresponds an inequality

$$\big|\big\{(x, y, z) \in \widetilde{K}(P^{ij}, P^{jk}) \colon \varphi_{ik}(xz) = P^{ik}\big\}\big| \geqslant \frac{\big|\widetilde{K}(P^{ij}, P^{jk})\big|}{|\mathcal{P}^{ik}|} - \delta h^3 \,. \tag{3.5}$$

Now indeed for every $y \in V_j$ the $(\delta, |\mathcal{P}^{ik}|^{-1})$-quasirandomness of $G(P^{ik})$ tells us that for the sets

$$A_y = \big\{x \in V_i \colon (x, y) \in Q \text{ and } \varphi_{ij}(xy) = P^{ij}\big\}$$

and

$$C_y = \big\{z \in V_k \colon (y, z) \in R \text{ and } \varphi_{jk}(yz) = P^{jk}\big\}$$

one has

$$\big|\big\{(x, z) \in A_y \times C_y \colon \varphi_{ik}(xz) = P^{ik}\big\}\big| \geqslant \frac{|A_y||C_y|}{|\mathcal{P}^{ik}|} - \delta h^2 \,.$$

By summing this over all $y \in Y$ one arrives at (3.5). □

§4. Irregular triads

The definition of $\pi_\star^{\mathrm{rd}}(F)$ assures us that for $\varepsilon > 0$ every $(\pi_\star^{\mathrm{rd}}(F) + \varepsilon, \star)$-dense reduced hypergraph $\mathscr{A}$ with sufficiently many indices contains a reduced image of F. For our intended application of this fact, however, we need to know that it remains true if one allows the deletion of a small number of edges from $\mathscr{A}$ (see Proposition 4.4 below).

For $\star = \therefore$ this turns out to be somewhat easier to prove than in the other two cases. The additional argument we want to put forth if $\star = \therefore$ is the following.

Suppose that a reduced hypergraph $\mathscr{A}$ with index set I, vertex classes $\mathcal{P}^{ij}$, and constituents $\mathscr{A}^{ijk}$ as well as two real numbers $d > 0$ and $\eta \geqslant 0$ are given. We shall say that $\mathscr{A}$

is $(d, \eta, \therefore)$-*dense* if for any three distinct indices $i, j, k \in I$ the exceptional set $\mathcal{X}_k^{ij} \subseteq \mathcal{P}^{ij}$ consisting of all $P^{ij} \in \mathcal{P}^{ij}$ with

$$\left|\left\{(P^{ik}, P^{jk}) \in \mathcal{P}^{ik} \times \mathcal{P}^{jk} \colon (P^{ij}, P^{ik}, P^{jk}) \in E(\mathscr{A}^{ijk})\right\}\right| < d|\mathcal{P}^{ik}||\mathcal{P}^{jk}|$$

satisfies $|\mathcal{X}_k^{ij}| \leqslant \eta|\mathcal{P}^{ij}|$. So for reduced hypergraphs being $(d, 0, \therefore)$-dense means the same as being $(d, \therefore)$-dense.

Lemma 4.1. *For every hypergraph F and every $\varepsilon > 0$ there exist $m \in \mathbb{N}$ and $\eta > 0$ such that every $(\pi_{\therefore}^{\mathrm{rd}}(F) + \varepsilon, \eta, \therefore)$-dense reduced hypergraph with m indices contains a reduced image of F.*

Proof. Choose m in such a way that every $\left(\pi_{\therefore}^{\mathrm{rd}}(F) + \frac{\varepsilon}{2}, \therefore\right)$-dense reduced hypergraph with m indices contains a reduced image of F, set

$$\eta = \frac{\varepsilon}{4m}\,,$$

and consider an arbitrary $(\pi_{\therefore}^{\mathrm{rd}}(F) + \varepsilon, \eta, \therefore)$-dense reduced hypergraph $\mathscr{A}$ with m indices. As usual we denote the index set, vertex classes, and constituents of $\mathscr{A}$ by I, $\mathcal{P}^{ij}$, and $\mathscr{A}^{ijk}$ respectively. Let the exceptional sets $\mathcal{X}_k^{ij}$ be defined as above with $\pi_{\therefore}^{\mathrm{rd}}(F) + \varepsilon$ here in place of d there.

Now the plan is to show that if one deletes all exceptional vertices from $\mathscr{A}$ one gets a $\left(\pi_{\therefore}^{\mathrm{rd}}(F) + \frac{\varepsilon}{2}, \therefore\right)$-dense reduced hypergraph, which, therefore, contains a reduced image of F. Thus we define

$$\mathcal{Q}^{ij} = \mathcal{P}^{ij} \smallsetminus \bigcup_{k \in I \smallsetminus \{i,j\}} \mathcal{X}_k^{ij} \quad \text{for every pair } ij \in I^{(2)}\,,$$

and notice that our assumption on $\mathscr{A}$ implies

$$|\mathcal{Q}^{ij}| \geqslant (1 - m\eta)\,, \tag{4.1}$$

whence, in particular, $\mathcal{Q}^{ij} \neq \varnothing$.

For this reason there exists a reduced hypergraph $\mathscr{B}$ with index set I and vertex classes $\mathcal{Q}^{ij}$ whose constituents $\mathscr{B}^{ijk}$ are the restrictions of $\mathscr{A}^{ijk}$ to $\mathcal{Q}^{ij} \,\dot\cup\, \mathcal{Q}^{ik} \,\dot\cup\, \mathcal{Q}^{jk}$.

Consider any three distinct indices $i, j, k \in I$ as well as an arbitrary vertex $P^{ij} \in \mathcal{Q}^{ij}$. From $P^{ij} \notin \mathcal{X}_k^{ij}$ we conclude

$$\left|\left\{(P^{ik}, P^{jk}) \in \mathcal{P}^{ik} \times \mathcal{P}^{jk} \colon (P^{ij}, P^{ik}, P^{jk}) \in E(\mathscr{A}^{ijk})\right\}\right| \geqslant (\pi_{\therefore}^{\mathrm{rd}}(F) + \varepsilon)|\mathcal{P}^{ik}||\mathcal{P}^{jk}|\,. \tag{4.2}$$

By (4.1) we have $|\mathcal{Q}^{ik}| \geqslant (1 - m\eta)|\mathcal{P}^{ik}|$ and $|\mathcal{Q}^{jk}| \geqslant (1 - m\eta)|\mathcal{P}^{jk}|$, wherefore

$$|(\mathcal{P}^{ik} \times \mathcal{P}^{jk}) \smallsetminus (\mathcal{Q}^{ik} \times \mathcal{Q}^{jk})| \leqslant 2m\eta|\mathcal{P}^{ik}||\mathcal{P}^{jk}| = \tfrac{\varepsilon}{2}|\mathcal{P}^{ik}||\mathcal{P}^{jk}|\,.$$

Combined with (4.2) this yields

$$\begin{aligned}\big|\big\{(P^{ik},P^{jk})\in\mathcal{Q}^{ik}\times\mathcal{Q}^{jk}\colon (P^{ij},P^{ik},P^{jk})\in E(\mathscr{A}^{ijk})\big\}\big| &\geqslant \big(\pi^{\mathrm{rd}}_{\therefore}(F)+\varepsilon-\tfrac{\varepsilon}{2}\big)|\mathcal{P}^{ik}||\mathcal{P}^{jk}|\\ &\geqslant \big(\pi^{\mathrm{rd}}_{\therefore}(F)+\tfrac{\varepsilon}{2}\big)|\mathcal{Q}^{ik}||\mathcal{Q}^{jk}|\,,\end{aligned}$$

as desired. □

Similar considerations can be undertaken with respect to $\boldsymbol{\wedge}$. As expected, a reduced hypergraph $\mathscr{A}$ with standard notation is called $(d,\eta,\boldsymbol{\wedge})$-*dense* for two real numbers $d\in[0,1]$ and $\eta>0$ provided that for any three distinct indices $i,j,k\in I$ the set $\mathcal{Y}_i^{jk}$ of all exceptional pairs $(P^{ij},P^{ik})\in\mathcal{P}^{ij}\times\mathcal{P}^{ik}$ with

$$\big|\big\{P^{jk}\in\mathcal{P}^{jk}\colon (P^{ij},P^{ik},P^{jk})\in E(\mathscr{A}^{ijk})\big\}\big|<d|\mathcal{P}^{jk}|\,,$$

satisfies $|\mathcal{Y}_i^{jk}|\leqslant\eta|\mathcal{P}^{ij}||\mathcal{P}^{ik}|$.

Lemma 4.2. *Given $\varepsilon>0$ and a hypergraph F, there are $m\in\mathbb{N}$ and $\eta>0$ such that every $(\pi^{\mathrm{rd}}_{\boldsymbol{\wedge}}(F)+\varepsilon,\eta,\boldsymbol{\wedge})$-dense reduced hypergraph with m indices contains a reduced image of F.*

Proof. Take m so large that every $\big(\pi^{\mathrm{rd}}_{\boldsymbol{\wedge}}(F)+\frac{\varepsilon}{2},\boldsymbol{\wedge}\big)$-dense reduced hypergraph with m indices contains a reduced image of F. Take $\ell\in\mathbb{N}$ and $\eta>0$ fitting into the hierarchy

$$\eta\ll\ell^{-1}\ll m^{-1},\varepsilon\,,$$

and let $\mathscr{A}$ be a $(\pi^{\mathrm{rd}}_{\boldsymbol{\wedge}}(F)+\varepsilon,\eta,\boldsymbol{\wedge})$-dense reduced hypergraph with index set I of size m, vertex classes $\mathcal{P}^{ij}$, constituents $\mathscr{A}^{ijk}$, and exceptional sets $\mathcal{Y}_i^{jk}$ (defined with $\pi^{\mathrm{rd}}_{\boldsymbol{\wedge}}(F)+\varepsilon$ in place of d). We want to prove that there is a reduced map from F to $\mathscr{A}$.

To this end we consider a random reduced hypergraph $\mathscr{B}$ with index set I whose vertex sets $\mathcal{Q}^{ij}$ are any $\binom{|I|}{2}$ disjoint sets of size ℓ. The intended randomness is induced by a family $\psi=\{\psi^{ij}\colon ij\in I^{(2)}\}$ of maps $\psi^{ij}\colon\mathcal{Q}^{ij}\longrightarrow\mathcal{P}^{ij}$. Depending on ψ the constituents of $\mathscr{B}$ are defined so as to satisfy

$$\{Q^{ij},Q^{ik},Q^{jk}\}\in E(\mathscr{B}^{ijk})\iff\{\psi^{ij}(Q^{ij}),\psi^{ik}(Q^{ik}),\psi^{jk}(Q^{jk})\}\in E(\mathscr{A}^{ijk})$$

for all $ijk\in I^{(3)}$ and all $Q^{ij}\in\mathcal{Q}^{ij}$, $Q^{ik}\in\mathcal{Q}^{ik}$, and $Q^{jk}\in\mathcal{Q}^{jk}$.

Let us observe first that if for some choice of ψ it happens that $\mathscr{B}$ contains a reduced image of F, then we are done. This is because if (λ,φ) is a reduced map from F to $\mathscr{B}$, then $(\lambda,\psi\circ\varphi)$ is a reduced map from F to $\mathscr{A}$, where by $(\psi\circ\varphi)\colon\partial F\longrightarrow V(\mathscr{A})$ we mean the map defined by $(\psi\circ\varphi)(uv)=\psi^{\lambda(u)\lambda(v)}\big(\varphi(uv)\big)$ for all $uv\in\partial F$.

In the remainder of the proof we shall show that if ψ gets chosen uniformly at random, then with positive probability $\mathscr{B}$ is $\big(\pi^{\mathrm{rd}}_{\boldsymbol{\wedge}}(F)+\frac{\varepsilon}{2},\boldsymbol{\wedge}\big)$-dense, which will conclude the proof due to our choice of m.

So let us study for fixed distinct indices $i,j,k\in I$ and fixed vertices $Q^{ij}\in\mathcal{Q}^{ij}$, $Q^{ik}\in\mathcal{Q}^{jk}$ the unpleasant event $\mathscr{E}$ that the pair degree D of Q^{ij} and Q^{ik} in $\mathscr{B}^{ijk}$ is smaller than

$\big(\pi^{\mathrm{rd}}_{\text{\Lambda}}(F) + \frac{\varepsilon}{2}\big)\ell$. If also the vertices $P^{ij} = \psi^{ij}(Q^{ij})$ and $P^{ik} = \psi^{ik}(Q^{ik})$ are given, then this pair degree D depends only on ψ^{jk} and not on the remaining maps comprising ψ. Moreover, the distribution of D is the same as if one draws ℓ random elements from $\mathcal{P}^{jk}$ and keeps track of how many of them belong to the common neighbourhood of P^{ij} and P^{ik} in $\mathscr{A}^{ijk}$. Thus if $(P^{ij}, P^{ik}) \notin \mathcal{Y}^{jk}_i$ the expected value of D is at least $(\pi^{\mathrm{rd}}_{\text{\Lambda}}(F) + \varepsilon)\ell$ and Chernoff's inequality (see e.g. [2, Theorem A.1.4]) yields

$$\mathbb{P}\big(\mathscr{E}\,\big|\psi^{ij}(Q^{ij}) = P^{ij} \text{ and } \psi^{ik}(Q^{ik}) = P^{ik}\big) \leqslant \exp(-\varepsilon^2\ell/2)\,.$$

Owing to $|\mathcal{Y}^{jk}_i| \leqslant \eta|\mathcal{P}^{ij}||\mathcal{P}^{ik}|$ we infer

$$\begin{aligned}\mathbb{P}(\mathscr{E}) &= |\mathcal{P}^{ij}|^{-1}|\mathcal{P}^{ik}|^{-1} \sum_{(P^{ij},P^{ik})\in\mathcal{P}^{ij}\times\mathcal{P}^{ik}} \mathbb{P}\big(\mathscr{E}\,\big|\psi^{ij}(Q^{ij}) = P^{ij}, \psi^{ik}(Q^{ik}) = P^{ik}\big)\\ &\leqslant \eta + \exp(-\varepsilon^2\ell/2)\,.\end{aligned}$$

As there are altogether no more than $m^3\ell^2$ possibilities to choose i, j, k, Q^{ij}, and Q^{ik}, this proves

$$\mathbb{P}\big(\mathscr{B} \text{ fails to be } \big(\pi^{\mathrm{rd}}_{\text{\Lambda}}(F) + \tfrac{\varepsilon}{2}, \text{\Lambda}\big)\text{-dense}\big) \leqslant m^3\ell^2\big(\eta + \exp(-\varepsilon^2\ell/2)\big)\,,$$

and by an appropriate choice of ℓ and η the right side can be pushed below 1. □

Remark 4.3. It should be clear that the same construction could have been used for establishing Lemma 4.1. In fact, it generalises much further and applies to the study of the Turán densities $\pi_{\mathscr{A}}(F)$ initiated in [28] as well.

Proposition 4.4. *Given a hypergraph F, a positive real number ε, and a symbol $\star \in \{\therefore, \therefore, \text{\Lambda}\}$ there exist $m \in \mathbb{N}$ and $\delta > 0$ such that the following holds. If two reduced hypergraphs $\mathscr{A}$ and $\mathscr{B}$ with the same set of indices I of size at least m and with the same vertex classes $\mathcal{P}^{ij}$ have the properties that $\mathscr{A}$ is $(\pi^{\mathrm{rd}}_{\star}(F) + \varepsilon, \star)$-dense and*

$$\sum_{ijk\in I^{(3)}} \frac{|E(\mathscr{A}^{ijk}) \smallsetminus E(\mathscr{B}^{ijk})|}{|\mathcal{P}^{ij}||\mathcal{P}^{ik}||\mathcal{P}^{jk}|} \leqslant \delta|I|^3\,, \tag{4.3}$$

then $\mathscr{B}$ contains a reduced image of F.

Proof. We work with the hierarchy

$$\delta \ll \xi, m^{-1} \ll \varepsilon\,.$$

Call a triple $ijk \in I^{(3)}$ *useless* if

$$E(\mathscr{A}^{ijk}) \smallsetminus E(\mathscr{B}^{ijk})| > \xi|\mathcal{P}^{ij}||\mathcal{P}^{ik}||\mathcal{P}^{jk}|\,.$$

As a consequence of (4.3) the number of such useless triples is at most $\delta\xi^{-1}|I|^3$. Since $|I| \geqslant m$ is sufficiently large, we have $\binom{|I|}{3} > \frac{1}{7}|I|^3$ and thus a proportion of no more than $7\delta\xi^{-1}$ among all triples is useless. Therefore, if one draws a set $J \subseteq I$ with $|J| = m$ uniformly at random, the expected number of useless triples in J is at most $7\delta\xi^{-1}\binom{m}{3}$, which by an appropriate choice of δ can be made less than 1. For this reason, there exists a set $J \subseteq I$ with $|J| = m$ spanning no useless triple. We shall now prove that the restriction of $\mathscr{B}$ to J, denoted by $\mathscr{B}'$ in the sequel, contains a reduced image of F. To this end we treat the three cases $\star = \therefore, \underline{\therefore}, \wedge$ separately.

First Case. $\star = \therefore$

Since $\xi \leqslant \frac{\varepsilon}{2}$, the reduced hypergraph $\mathscr{B}'$ is $\bigl(\pi^{\mathrm{rd}}_{\therefore}(F) + \frac{\varepsilon}{2}, \therefore\bigr)$-dense and thus it does indeed contain a reduced image of F.

Second Case. $\star = \underline{\therefore}$

Owing to Lemma 4.1 it suffices to check that $\mathscr{B}'$ is $\bigl(\pi^{\mathrm{rd}}_{\therefore}(F) + \frac{\varepsilon}{2}, 2\xi\varepsilon^{-1}, \underline{\therefore}\bigr)$-dense. So let any three distinct indices $i, j, k \in J$ be given and let $\mathcal{X}^{ij}_k \subseteq \mathcal{P}^{ij}$ denote the exceptional set of all vertices $P^{ij} \in \mathcal{P}^{ij}$ with

$$\bigl|\bigl\{(P^{ik}, P^{jk}) \in \mathcal{P}^{ik} \times \mathcal{P}^{jk} \colon (P^{ij}, P^{ik}, P^{jk}) \in E(\mathscr{B}^{ijk})\bigr\}\bigr| < \bigl(\pi^{\mathrm{rd}}_{\therefore}(F) + \tfrac{\varepsilon}{2}\bigr)|\mathcal{P}^{ik}||\mathcal{P}^{jk}|\,.$$

Since $\mathscr{A}$ is $(\pi^{\mathrm{rd}}_{\therefore}(F) + \varepsilon, \underline{\therefore})$-dense and ijk is useful, we have

$$|\mathcal{X}^{ij}_k| \cdot \tfrac{\varepsilon}{2}|\mathcal{P}^{ik}||\mathcal{P}^{jk}| \leqslant |E(\mathscr{A}^{ijk}) \smallsetminus E(\mathscr{B}^{ijk})| \leqslant \xi|\mathcal{P}^{ij}||\mathcal{P}^{ik}||\mathcal{P}^{jk}|\,,$$

which yields indeed $|\mathcal{X}^{ij}_k| \leqslant 2\xi\varepsilon^{-1}|\mathcal{P}^{ij}|$.

Third Case. $\star = \wedge$

Arguing as the previous case one proves that $\mathscr{B}'$ is $\bigl(\pi^{\mathrm{rd}}_{\therefore}(F) + \frac{\varepsilon}{2}, 2\xi\varepsilon^{-1}, \wedge\bigr)$-dense, which yields the desired conclusion in view of Lemma 4.2. □

§5. Hypergraph regularity

The proof of Theorem 3.3 can now be completed by means of the hypergraph regularity method, which for 3-uniform hypergraphs is due to Frankl and Rödl [14]. Our presentation below also takes the later works [17, 22, 31, 32] into account.

A central notion in this area is that of a hypergraph H being regular with respect to a tripartite graph P, which roughly speaking means that the triangles in P behave in an important way as if a random subset of them would correspond to edges of H.

Definition 5.1. A 3-uniform hypergraph $H = (V, E_H)$ is (δ_3, d_3)*-regular with respect to a tripartite graph* $P = (X \cupdot Y \cupdot Z, E_P)$ with $V \supseteq X \cup Y \cup Z$ if for every tripartite subgraph $Q \subseteq P$ we have

$$\bigl||E_H \cap \mathcal{K}_3(Q)| - d_3|\mathcal{K}_3(Q)|\bigr| \leqslant \delta_3|\mathcal{K}_3(P)|\,.$$

Moreover, we simply say that H *is* δ_3*-regular with respect to* P, if it is (δ_3, d_3)-regular for some $d_3 \geqslant 0$. We also define the *relative density* of H with respect to P by

$$d(H|P) = \frac{|E_H \cap \mathcal{K}_3(P)|}{|\mathcal{K}_3(P)|}\,,$$

where we use the convention $d(H|P) = 0$ if $\mathcal{K}_3(P) = \varnothing$.

Now the hypergraph regularity lemma tells us that large hypergraphs can in the following approximate sense be decomposed into regular parts.

Theorem 5.2 (Regularity Lemma). *For every $\delta_3 > 0$, every $\delta_2 \colon \mathbb{N} \to (0,1]$, and every $t_0 \in \mathbb{N}$ there exists an integer T_0 such that for every $n \geqslant t_0$ and every n-vertex 3-uniform hypergraph $H = (V, E_H)$ the following holds.*

There are integers $t \in [t_0, T_0]$ and $\ell \leqslant T_0$, a vertex partition $V_0 \charfusion[\mathbin]{\cup}{\cdot} V_1 \charfusion[\mathbin]{\cup}{\cdot} \dots \charfusion[\mathbin]{\cup}{\cdot} V_t = V$, and for all $1 \leqslant i < j \leqslant t$ there exists a partition

$$\mathcal{P}^{ij} = \{P^{ij}_\alpha = (V_i \charfusion[\mathbin]{\cup}{\cdot} V_j, E^{ij}_\alpha) \colon 1 \leqslant \alpha \leqslant \ell\}$$

of the edge set of the complete bipartite graph $K(V_i, V_j)$ satisfying the following properties.

- (i) *$|V_0| \leqslant \delta_3 n$ and $|V_1| = \dots = |V_t|$;*
- (ii) *for every $1 \leqslant i < j \leqslant t$ and $\alpha \in [\ell]$ the bipartite graph P^{ij}_α is $(\delta_2(\ell), 1/\ell)$-regular;*
- (iii) *and H is δ_3-regular with respect to $P^{ijk}_{\alpha\beta\gamma}$ for all but at most $\delta_3 t^3 \ell^3$ tripartite graphs*

$$P^{ijk}_{\alpha\beta\gamma} = P^{ij}_\alpha \charfusion[\mathbin]{\cup}{\cdot} P^{ik}_\beta \charfusion[\mathbin]{\cup}{\cdot} P^{jk}_\gamma = (V_i \charfusion[\mathbin]{\cup}{\cdot} V_j \charfusion[\mathbin]{\cup}{\cdot} V_k, E^{ij}_\alpha \charfusion[\mathbin]{\cup}{\cdot} E^{ik}_\beta \charfusion[\mathbin]{\cup}{\cdot} E^{jk}_\gamma) \tag{5.1}$$

  with $1 \leqslant i < j < k \leqslant t$ and α, β, $\gamma \in [\ell]$.

The tripartite graphs occurring in (5.1) are called *triads*. In order to get a better feeling as to why (in our context) such a decomposition of a given hypergraph H is a useful thing to have, it may be helpful to imagine the following special outcome.

- (a) $V_0 = \varnothing$, i.e., the entire vertex set gets partitioned;
- (b) every edge of H intersects each partition class V_i at most once;
- (c) there are no irregular triads, i.e., (iii) holds without any exceptions;
- (d) moreover, all triads are either "full" in the sense that all their triangles correspond to edges of H, or "empty" in the sense that none of their triangles correspond to edges of H.

It is not hard to see that if these four things happen at the same time, then H is essentially of the form constructed in the proof of Proposition 3.4. The underlying reduced hypergraph $\mathscr{A}$ on which such a construction would be based has index set $[t]$, vertex classes $\mathcal{P}^{ij}$, and the possible edges $\{P^{ij}_\alpha, P^{ik}_\beta, P^{jk}_\gamma\}$ in its constituents $\mathscr{A}^{ijk}$ would indicate which triads $P^{ijk}_{\alpha\beta\gamma}$ are "full".

So in a vague sense what remains to be done for completing the proof of Theorem 3.3 is that we need to address how to deal with the possible failures of (a)–(d) when the regularity lemma gets applied. There will be no difficulties with (a) or (b), for the concepts we study are sufficiently robust, so that deleting the small set V_0 for (a) and ignoring the small proportion of noncrossing edges for (b) has essentially no effect. We are prepared for (c) in the light of Proposition 4.4.

Finally, regarding (*d*) we will treat triads with respect to which the relative density H is not too small as if they were full. That is, for some appropriate constant $d_3 > 0$ we will put an edge $\{P^{ij}_\alpha, P^{ik}_\beta, P^{jk}_\gamma\}$ into $\mathscr{A}^{ijk}$ if and only if $d(H|P^{ijk}_{\alpha\beta\gamma}) \geqslant d_3$. This will allow us to rather easily transfer denseness properties from H to $\mathscr{A}$, but we will need an argument as to why a reduced map from F to $\mathscr{A}$ does still give rise to a copy of F in H, even though the triads we want to use are not known to be full. This is, however, a standard situation in hypergraph regularity theory, for which the *counting lemma* has been developed. Below we shall require the following consequence of this result.

Theorem 5.3 (Embedding Lemma)**.** *For every 3-uniform hypergraph F and every $d_3 > 0$ there exist $\delta_3 > 0$, and functions $\delta_2\colon \mathbb{N} \longrightarrow (0,1]$ and $N\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that the following holds for every $\ell \in \mathbb{N}$.*

Suppose that

- *$\lambda\colon V(F) \longrightarrow I$ is a map from $V(F)$ to some set I with $\lambda(u) \neq \lambda(v)$ for all $uv \in \partial F$,*
- *that $\{V_i\colon i \in I\}$ is a family of mutually disjoint sets of the same size $N_* \geqslant N(\ell)$,*
- *and that for every $uv \in \partial F$ one has a $(\delta_2(\ell), \frac{1}{\ell})$-quasirandom bipartite graph P_{uv} between $V_{\lambda(u)}$ and $V_{\lambda(v)}$.*

Then a hypergraph H with $V(H) \supseteq \bigcup_{i\in I} V_i$ posseses a subhypergraph isomorphic to F provided that for every edge $uvw \in E(F)$

- *one has $d(H|P_{uv} \cup P_{uw} \cup P_{vw}) \geqslant d_3$*
- *and H is δ_3-regular with respect to the tripartite graph $P_{uv} \cup P_{uw} \cup P_{vw}$.*

For completeness we shall briefly discuss how this statement relates to the standard reference [22, Corollary 2.3]. First of all, a more conventional setup for the counting lemma would be the case that $V(F) = I = [f]$ holds for some natural number f and that λ is the identity. Secondly, in this special case the full counting lemma allows to estimate the number of homomorphisms ψ from F to H with $\psi(u) \in V_{\lambda(u)}$ for every $u \in V(F)$ in a satisfactory way. In particular, a suitable choice of δ_3, $\delta_2(\cdot)$, and $N(\cdot)$ entails that this number is at least $\frac{1}{2} d_3^{e(F)} \ell^{-|\partial F|} N_*^{|V(F)|}$. Thirdly, this assertion generalises immediately to the case of general F, I, and λ, even if λ should fail to be injective. Finally, by increasing $N(\ell)$ if necessary, one can achieve that this lower bound on the number of homomorphisms from F to H exceeds the number of non-injective maps ψ from $V(F)$ to $V(H)$ with $\psi(u) \in V_{\lambda(u)}$ for every $u \in V(F)$. Therefore, [22, Corollary 2.3] does indeed imply Theorem 5.3.

We may now proceed to the second half of Theorem 3.3.

Proposition 5.4. *If F is a hypergraph and $\star \in \{\therefore, \overset{\bullet}{\underline{\bullet\,\bullet}}, \wedge\}$, then $\pi_\star(F) \leqslant \pi^{\mathrm{rd}}_\star(F)$.*

Proof. We may suppose $\pi^{\mathrm{rd}}_\star(F) < 1$, since otherwise the result is clear. Let an arbitrary $\varepsilon \in (0, 1 - \pi^{\mathrm{rd}}_\star(F)]$ be given. By plugging F and $d_3 = \frac{1}{7}\varepsilon$ into Theorem 5.3 we obtain $\delta_3 > 0$

and functions $\delta_2\colon \mathbb{N} \longrightarrow (0,1]$ as well as $N\colon \mathbb{N} \longrightarrow \mathbb{N}$. Without loss of generality, we may suppose that $\delta_3 < \frac{1}{2}$ is sufficiently small, that $\delta_2(\ell) \leqslant \frac{1}{21}\varepsilon\ell^{-3}$ holds for every $\ell \in \mathbb{N}$ and that N is increasing. By Proposition 4.4 and our flexibility to decrease δ_3 we may assume that there exists $t_0 \in \mathbb{N}$ such that if for arbitrary $t \geqslant t_0$ and $\ell \geqslant 1$ one deletes deletes at most $\delta_3 t^3 \ell^3$ edges from a $\big(\pi^{\mathrm{rd}}_\star(F) + \frac{1}{7}\varepsilon, \star\big)$-dense reduced hypergraph with index set $[t]$ whose vertex classes have size ℓ, then the resulting reduced hypergraph contains a reduced image of F. With this choice of δ_3, $\delta_2(\cdot)$, and t_0 we appeal to the regularity lemma, thus getting an integer T_0. Finally, we set

$$n_0 = 2T_0 N(T_0) \quad \text{and} \quad \eta = \frac{\varepsilon}{56T_0^5}\,.$$

Now we contend that every $(\pi^{\mathrm{rd}}_\star(F) + \varepsilon, \eta, \star)$-dense hypergraph H on $n \geqslant n_0$ vertices has a subhypergraph isomorphic to F, which clearly implies the desired result.

Suppose that the regularity lemma applied to H yields the integers $t \in [t_0, T_0]$ and $\ell \leqslant T_0$, the vertex partition $V(H) = V_0 \cup V_1 \cup \ldots \cup V_t$ and for $1 \leqslant i < j \leqslant t$ the pair partition

$$\mathcal{P}^{ij} = \{P^{ij}_\alpha = (V_i \cup V_j, E^{ij}_\alpha)\colon\ 1 \leqslant \alpha \leqslant \ell\}$$

of $K(V_i, V_j)$ such that *(i)*, *(ii)*, and *(iii)* hold.

This situation gives rise to two reduced hypergraphs $\mathscr{A}$ and $\mathscr{B}$ with index set $[t]$ and vertex classes $\mathcal{P}^{ij}$ for $ij \in [t]^{(2)}$ defined as follows. A triple $\{P^{ij}_\alpha, P^{ik}_\beta, P^{jk}_\gamma\}$ is declared to form an edge of the constituent $\mathscr{A}^{ijk}$ if the corresponding triad $P^{ijk}_{\alpha\beta\gamma}$ satisfies $d(H|P^{ijk}_{\alpha\beta\gamma}) \geqslant d_3$. If in addition H is δ_3-regular with respect to this triad, then we put this edge into $\mathscr{B}^{ijk}$ as well. We shall verify later that

$$\mathscr{A} \text{ is } \big(\pi^{\mathrm{rd}}_\star(F) + \tfrac{1}{7}\varepsilon, \star\big)\text{-dense.} \tag{5.2}$$

Based on this fact, the argument can be completed as follows. By Theorem 5.2*(iii)* we have

$$|E(\mathscr{A}) \smallsetminus E(\mathscr{B})| \leqslant \delta_3 t^3 \ell^3\,,$$

so due to our choice of t_0 according to Proposition 4.4 there is a reduced map (λ, φ) from F to $\mathscr{B}$. Now the embedding lemma applies to $I = [t]$, λ, the sets V_i for $i \in I$, and the bipartite graphs called $\varphi(uv) \in \mathcal{P}^{\lambda(u)\lambda(v)}$ here playing the rôles of P_{uv} there. The lower bound imposed there on the sets V_i follows from

$$|V_i| = \frac{|V| - |V_0|}{t} \geqslant \frac{(1-\delta_3)n}{T_0} \geqslant \frac{n_0}{2T_0} = N(T_0) \geqslant N(\ell)\,,$$

for every $i \in [t]$. Moreover, H satisfies the last two bullets of Theorem 5.3 by Definition 3.2*(iii)* and the construction of $\mathscr{B}$. So altogether we obtain indeed $F \subseteq H$ and it remains to establish (5.2).

A key observation towards this goal is that for $M = |V_1| = \ldots = |V_t|$ every triad spans at most $\big(\ell^{-3} + 3\delta_2(\ell)\big)M^3$ triangles due to the triangle counting lemma, and because of

$\delta_2(\ell) \leqslant \frac{1}{21}\varepsilon\ell^{-3}$ this is turn at most $(1+\frac{1}{7}\varepsilon)\ell^{-3}M^3$. So by our choice of d_3 a triad that does not correspond to an edge of $\mathscr{A}$ can accomodate at most $\frac{1}{7}\varepsilon(1+\frac{1}{7}\varepsilon)\ell^{-3}M^3$ edges of H.

Furthermore, it will be helpful to be aware that our choice of η guarantees

$$\eta n^3 = \frac{\varepsilon}{7T_0^2}\left(\frac{n}{2T_0}\right)^3 \leqslant \frac{\varepsilon M^3}{7\ell^2}\,.$$

From now on we treat the three possibilities for $\star$ separately.

First Case. $\star = \therefore$

Given any three distinct indices $i, j, k \in [t]$ we need to prove $|E(\mathscr{A}^{ijk})| \geqslant \big(\pi^{\mathrm{rd}}_{\therefore}(F) + \frac{1}{7}\varepsilon\big)\ell^3$. Applying the assumption that H is $(\pi^{\mathrm{rd}}_{\therefore}(F)+\varepsilon, \eta, \therefore)$-dense to V_i, V_j, and V_k we obtain

$$|E_{\therefore}(V_i, V_j, V_k)| \geqslant \big(\pi^{\mathrm{rd}}_{\therefore}(F)+\varepsilon\big)M^3 - \eta n^3 \geqslant \big(\pi^{\mathrm{rd}}_{\therefore}(F)+\tfrac{6}{7}\varepsilon\big)M^3\,.$$

Counting the edges of the left side according to the triad to which they belong we obtain

$$\big(\pi^{\mathrm{rd}}_{\therefore}(F)+\tfrac{6}{7}\varepsilon\big)M^3 \leqslant \big(|E(\mathscr{A}^{ijk})| + \tfrac{1}{7}\varepsilon\ell^3\big)\big(1+\tfrac{1}{7}\varepsilon\big)\ell^{-3}M^3\,.$$

Owing to

$$\big(\pi^{\mathrm{rd}}_{\therefore}(F)+\tfrac{4}{7}\varepsilon\big)\big(1+\tfrac{1}{7}\varepsilon\big) \leqslant \pi^{\mathrm{rd}}_{\therefore}(F)+\tfrac{6}{7}\varepsilon$$

this yields

$$\big(\pi^{\mathrm{rd}}_{\therefore}(F)+\tfrac{3}{7}\varepsilon\big)\ell^3 \leqslant |E(\mathscr{A}^{ijk})|\,,$$

which is more than required.

Second Case. $\star = \overset{\bullet}{\bullet\!\!-\!\!\bullet}$

Consider three distinct indices $i, j, k \in [t]$, a bipartite graph $P^{jk}_\gamma \in \mathcal{P}^{jk}$, and its neighbourhood

$$N = \big\{(P^{ij}_\alpha, P^{ik}_\beta) \in \mathcal{P}^{ij}\times\mathcal{P}^{ik} \colon \{P^{ij}_\alpha, P^{ik}_\beta, P^{jk}_\gamma\} \in E(\mathscr{A}^{ijk})\big\}$$

in the constituent $\mathscr{A}^{ijk}$. Observe that

$$|E(P^{jk}_\gamma)| \geqslant \big(\ell^{-1} - \delta_2(\ell)\big)M^2 \geqslant \big(1-\tfrac{1}{7}\varepsilon\big)\ell^{-1}M^2\,.$$

Since H is $(\pi^{\mathrm{rd}}_{\overset{\bullet}{\bullet\!\!-\!\!\bullet}}(F)+\varepsilon, \eta, \overset{\bullet}{\bullet\!\!-\!\!\bullet})$-dense, this yields

$$\begin{aligned}|E_{\overset{\bullet}{\bullet\!\!-\!\!\bullet}}(V_i, P^{jk}_\gamma)| &\geqslant \big(\pi^{\mathrm{rd}}_{\overset{\bullet}{\bullet\!\!-\!\!\bullet}}(F)+\varepsilon\big)M|E(P^{jk}_\gamma)| - \eta n^3\\ &\geqslant \big(\pi^{\mathrm{rd}}_{\overset{\bullet}{\bullet\!\!-\!\!\bullet}}(F)+\varepsilon\big)(1-\tfrac{1}{7}\varepsilon)\ell^{-1}M^3 - \tfrac{1}{7}\varepsilon\ell^{-1}M^3\\ &\geqslant \big(\pi^{\mathrm{rd}}_{\overset{\bullet}{\bullet\!\!-\!\!\bullet}}(F)+\tfrac{4}{7}\varepsilon\big)\ell^{-1}M^3\,,\end{aligned}$$

where we have identified P^{jk}_γ in the natural way with a subset of $V_j \times V_k$. As in the previous case this leads to

$$\big(\pi^{\mathrm{rd}}_{\overset{\bullet}{\bullet\!\!-\!\!\bullet}}(F)+\tfrac{4}{7}\varepsilon\big)\ell^{-1}M^3 \leqslant \big(N + \tfrac{1}{7}\varepsilon\ell^2\big)\big(1+\tfrac{1}{7}\varepsilon\big)\ell^{-3}M^3\,,$$

which in turn implies

$$\big(\pi^{\mathrm{rd}}_{\therefore}(F) + \tfrac{1}{7}\varepsilon\big)\ell^2 \leqslant N\,.$$

Thus $\mathscr{A}$ is indeed $\big(\pi^{\mathrm{rd}}_{\therefore}(F) + \frac{1}{7}\varepsilon, \therefore\big)$-dense.

Third Case. $\star = \curlywedge$

This time let three distinct indices $i, j, k \in [t]$ as well as two bipartite graphs $P^{ij}_{\alpha} \in \mathcal{P}^{ij}$ and $P^{jk}_{\gamma} \in \mathcal{P}^{jk}$ be given, which we identify with the corresponding subsets of $V_i \times V_j$ and $V_j \times V_k$, respectively. The graph counting lemma implies

$$\mathcal{K}_{\curlywedge}(P^{ij}_{\alpha}, P^{jk}_{\gamma}) \geqslant \big(\ell^{-2} - 2\delta_2(\ell)\big)M^3 \geqslant \big(1 - \tfrac{1}{7}\varepsilon\big)\ell^{-2}M^3$$

and it follows from H being $(\pi^{\mathrm{rd}}_{\curlywedge}(F) + \varepsilon, \eta, \curlywedge)$-dense that

$$\begin{aligned}
\big|E_{\curlywedge}(P^{ij}_{\alpha}, P^{jk}_{\gamma})\big| &\geqslant \big(\pi^{\mathrm{rd}}_{\curlywedge}(F) + \varepsilon\big)\big|\mathcal{K}_{\curlywedge}(P^{ij}_{\alpha}, P^{jk}_{\gamma})\big| - \eta n^3 \\
&\geqslant \big(\pi^{\mathrm{rd}}_{\curlywedge}(F) + \varepsilon\big)\big(1 - \tfrac{1}{7}\varepsilon\big)\ell^{-2}M^3 - \tfrac{1}{7}\varepsilon\ell^{-2}M^3 \\
&\geqslant \big(\pi^{\mathrm{rd}}_{\curlywedge}(F) + \tfrac{4}{7}\varepsilon\big)\ell^{-2}M^3
\end{aligned}$$

Regarding the common neighbourhood

$$J = \big\{P^{ik}_{\beta} \in \mathcal{P}^{ik} \colon (P^{ij}_{\alpha}, P^{ik}_{\beta}, P^{jk}_{\gamma}) \in E(\mathscr{A}^{ijk})\big\}$$

this tells us

$$\big(\pi^{\mathrm{rd}}_{\curlywedge}(F) + \tfrac{4}{7}\varepsilon\big)\ell^{-2}M^3 \leqslant \big(|J| + \tfrac{1}{7}\ell\varepsilon\big)\big(1 + \tfrac{1}{7}\varepsilon\big)\ell^{-3}M^3\,,$$

which yields

$$\big(\pi^{\mathrm{rd}}_{\curlywedge}(F) + \tfrac{1}{7}\varepsilon\big)\ell \leqslant |J|\,,$$

as desired. $\square$

§6. More on tetrahedra

In order to illustrate how Theorem 3.3 can be applied we conclude this article by sketching a proof of $\pi_{\curlywedge}(K^{(3)}_4) = 0$. This result forms the first interesting case of Theorem 2.9 and the reader seeking further information or more details is referred to [27].

Given $\varepsilon > 0$ we want to show that every $(\varepsilon, \curlywedge)$-dense reduced hypergraph with sufficiently many indices contains the reduced image of a tetrahedron. Let $\mathscr{A}$ be such a reduced hypergraph with index set I, vertex classes $\mathcal{P}^{ij}$, and constituents $\mathscr{A}^{ijk}$. Write I as a disjoint union $I = X \cup Y$, where $|X| > \frac{1}{\varepsilon}$ and Y is much larger then X.

The first step is to assign to every pair $(x, y) \in X \times Y$ an arbitrary vertex $P^{xy} \in \mathcal{P}^{xy}$.

Next we look at two distinct vertices $x, x' \in X$. For every $y \in Y$ the common neighbourhood of P^{xy} and $P^{x'y}$ in the constituent $\mathscr{A}^{xx'y}$ contains, by our hypothesis on $\mathscr{A}$, at least $\varepsilon|\mathcal{P}^{xx'}|$ vertices. Thus, by double counting, we may fix a vertex $P^{xx'} \in \mathcal{P}^{xx'}$ belonging to this neighbourhood for at least $\varepsilon|Y|$ many choices of $y \in Y$. In other words, we may shrink Y by a factor of no more than ε to a subset Y' such that $P^{xx'}P^{xy}P^{x'y}$ is edge of $\mathscr{A}$ for every $y \in Y'$.

This argument can be applied iteratively to all pairs of vertices in X. That is, we enumerate all pairs in $X^{(2)}$ and when processing a pair in the list we select a vertex from the corresponding vertex class and shrink the subset of Y under current consideration by a further factor of ε. When this procedure ends, we have chosen for every pair $xx' \in X^{(2)}$ a vertex $P^{xx'} \in \mathcal{P}^{xx'}$. Moreover, if Y^* denotes the subset of Y that has survived through all stages, then $\{P^{xx'}, P^{xy}, P^{x'y}\} \in E(\mathscr{A}^{xx'y})$ holds for all distinct $x, x' \in X$ and all $y \in Y^*$.

By starting with a sufficiently large set Y we can ensure that $|Y^*| \geqslant 2$. Pick once and for all two distinct indices $y, y' \in Y^*$. Reversing the rôles of X and Y we may now select a suitable vertex $P^{yy'}$ in $\mathcal{P}^{yy'}$ and shrink X in the same way as above to a set X^* with $|X^*| \geqslant \varepsilon|X|$ such that $P^{xy}P^{xy'}P^{yy'} \in E(\mathscr{A}^{xyy'})$ holds for all $x \in X^*$. Due to $|X| > \frac{1}{\varepsilon}$ there will be at least two survivors x and x' in X^*.

Now the four indices x, x', y, and y' form together with the six vertices $P^{xx'}$, P^{xy}, $P^{xy'}$, $P^{x'y}$, $P^{x'y'}$, and $P^{yy'}$ the desired reduced image of a tetrahedron in $\mathscr{A}$.

It should be clear that the same argument also establishes $\pi_{\wedge}(B) = 0$ for every bipartite hypergraph B. There are, however, many further hypergraphs whose $\wedge$-Turán-density vanishes. For instance, as a consequence of Theorem 2.2 the Fano plane $\mathscr{F}$ satisfies $\pi_{\therefore}(\mathscr{F}) = 0$ and, hence, also $\pi_{\wedge}(\mathscr{F}) = 0$. We shall return to the rather subtle problem of characterising the set $\{F\colon \pi_{\wedge}(F) = 0\}$ at another occasion.

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References

[1] E. Aigner-Horev and G. Levy, *Tight Hamilton cycles in cherry quasirandom 3-uniform hypergraphs*, available at arXiv:1712.00186. ↑1.3

[2] N. Alon and J. H. Spencer, *The probabilistic method*, 3rd ed., Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., Hoboken, NJ, 2008. With an appendix on the life and work of Paul Erdős. MR2437651 ↑3, 4

[3] R. Baber and J. Talbot, *Hypergraphs do jump*, Combin. Probab. Comput. **20** (2011), no. 2, 161–171, DOI 10.1017/S0963548310000222. MR2769186 (2012g:05166) ↑1.1

[4] L. Bellmann and Chr. Reiher, *Turán's Theorem for the Fano plane*, available at arXiv:1804.07673. Submitted. ↑1.1

[5] F. R. K. Chung and R. L. Graham, *Edge-colored complete graphs with precisely colored subgraphs*, Combinatorica **3** (1983), no. 3-4, 315–324, DOI 10.1007/BF02579187. MR729784 (85g:05107) ↑2.8

[6] D. De Caen and Z. Füredi, *The maximum size of 3-uniform hypergraphs not containing a Fano plane*, J. Combin. Theory Ser. B **78** (2000), no. 2, 274–276, DOI 10.1006/jctb.1999.1938. MR1750899 ↑1.1

[7] P. Erdős, *Paul Turán, 1910–1976: his work in graph theory*, J. Graph Theory **1** (1977), no. 2, 97–101. MR0441657 (56 #61) ↑1.1

[8] ______, *Problems and results on graphs and hypergraphs: similarities and differences*, Mathematics of Ramsey theory, Algorithms Combin., vol. 5, Springer, Berlin, 1990, pp. 12–28. MR1083590 ↑1.2

[9] P. Erdős and A. Hajnal, *On Ramsey like theorems. Problems and results*, Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972), Inst. Math. Appl., Southend-on-Sea, 1972, pp. 123–140. MR0337636 (49 #2405) ↑1.2

[10] P. Erdős and M. Simonovits, *A limit theorem in graph theory*, Studia Sci. Math. Hungar **1** (1966), 51–57. MR0205876 (34 #5702) ↑1.1

[11] P. Erdős and V. T. Sós, *On Ramsey-Turán type theorems for hypergraphs*, Combinatorica **2** (1982), no. 3, 289–295, DOI 10.1007/BF02579235. MR698654 (85d:05185) ↑1.2

[12] P. Erdős and A. H. Stone, *On the structure of linear graphs*, Bull. Amer. Math. Soc. **52** (1946), 1087–1091. MR0018807 (8,333b) ↑1.1

[13] P. Frankl and Z. Füredi, *An exact result for 3-graphs*, Discrete Math. **50** (1984), no. 2-3, 323–328, DOI 10.1016/0012-365X(84)90058-X. MR753720 (85k:05063) ↑1.1, 1.2

[14] P. Frankl and V. Rödl, *Extremal problems on set systems*, Random Structures Algorithms **20** (2002), no. 2, 131–164, DOI 10.1002/rsa.10017.abs. MR1884430 (2002m:05192) ↑5

[15] Z. Füredi and M. Simonovits, *Triple systems not containing a Fano configuration*, Combin. Probab. Comput. **14** (2005), no. 4, 467–484, DOI 10.1017/S0963548305006784. MR2160414 ↑1.1

[16] R. Glebov, D. Kráľ, and J. Volec, *A problem of Erdős and Sós on 3-graphs*, Israel J. Math. **211** (2016), no. 1, 349–366, DOI 10.1007/s11856-015-1267-4. MR3474967 ↑1.2

[17] W. T. Gowers, *Quasirandomness, counting and regularity for 3-uniform hypergraphs*, Combin. Probab. Comput. **15** (2006), no. 1-2, 143–184, DOI 10.1017/S0963548305007236. MR2195580 (2008b:05175) ↑5

[18] G. Katona, T. Nemetz, and M. Simonovits, *On a problem of Turán in the theory of graphs*, Mat. Lapok **15** (1964), 228–238 (Hungarian, with Russian and English summaries). MR0172263 ↑1.1

[19] P. Keevash, *Hypergraph Turán problems*, Surveys in combinatorics 2011, London Math. Soc. Lecture Note Ser., vol. 392, Cambridge Univ. Press, Cambridge, 2011, pp. 83–139. MR2866732 ↑1.1

[20] P. Keevash and B. Sudakov, *The Turán number of the Fano plane*, Combinatorica **25** (2005), no. 5, 561–574, DOI 10.1007/s00493-005-0034-2. MR2176425 ↑1.1

[21] Y. Kohayakawa, V. Rödl, and J. Skokan, *Hypergraphs, quasi-randomness, and conditions for regularity*, J. Combin. Theory Ser. A **97** (2002), no. 2, 307–352, DOI 10.1006/jcta.2001.3217. MR1883869 (2003b:05112) ↑1.3

[22] B. Nagle, A. Poerschke, V. Rödl, and M. Schacht, *Hypergraph regularity and quasi-randomness*, Proceedings of the Twentieth Annual ACM-SIAM Symposium on Discrete Algorithms, SIAM, Philadelphia, PA, 2009, pp. 227–235. MR2809322 ↑5, 5

[23] A. A. Razborov, *Flag algebras*, J. Symbolic Logic **72** (2007), no. 4, 1239–1282, DOI 10.2178/jsl/1203350785. MR2371204 (2008j:03040) ↑1.1

[24] ______, *On 3-hypergraphs with forbidden 4-vertex configurations*, SIAM J. Discrete Math. **24** (2010), no. 3, 946–963, DOI 10.1137/090747476. MR2680226 (2011k:05171) ↑1.1

[25] Chr. Reiher, V. Rödl, and M. Schacht, *On a Turán problem in weakly quasirandom 3-uniform hypergraphs*, Journal of the European Mathematical Society (JEMS) **20** (2018), no. 5, 1139–1159, DOI 10.4171/JEMS/784. MR3790065 ↑1.2, 2.3, 2.4, 3, 6

[26] ______, *Embedding tetrahedra into quasirandom hypergraphs*, J. Combin. Theory Ser. B **121** (2016), 229–247, DOI 10.1016/j.jctb.2016.06.008. MR3548293 ↑1.2, 1.3, 1.3, $*$, 3, 6

[27] ———, *Some remarks on $\pi_{\text{⋀}}$*, Connections in discrete mathematics, Cambridge Univ. Press, Cambridge, 2018, pp. 214–239. MR3821841 ↑1.2, 1.3, 2.6, $*$, $*$, 2, 2, 3, 6, 6

[28] ———, *On a generalisation of Mantel's Theorem to Uniformly Dense Hypergraphs*, Int. Math. Res. Not. IMRN **16** (2018), 4899–4941, DOI 10.1093/imrn/rnx017. MR3848224 ↑1.2, 1.3, 1.3, 4.3, 6

[29] ———, *Hypergraphs with vanishing Turán density in uniformly dense hypergraphs*, Journal of the London Mathematical Society (2) **97** (2018), no. 1, 77–97, DOI 10.1112/jlms.12095. MR3764068 ↑1.2, 2.1, 3, 6

[30] V. Rödl, *On universality of graphs with uniformly distributed edges*, Discrete Math. **59** (1986), no. 1-2, 125–134, DOI 10.1016/0012-365X(86)90076-2. MR837962 (88b:05098) ↑1.2

[31] V. Rödl and M. Schacht, *Regular partitions of hypergraphs: regularity lemmas*, Combin. Probab. Comput. **16** (2007), no. 6, 833–885. MR2351688 (2008h:05083) ↑5

[32] ———, *Regular partitions of hypergraphs: counting lemmas*, Combin. Probab. Comput. **16** (2007), no. 6, 887–901. MR2351689 (2008j:05238) ↑5

[33] M. Simonovits, *A method for solving extremal problems in graph theory, stability problems*, Theory of Graphs (Proc. Colloq., Tihany, 1966), Academic Press, New York, 1968, pp. 279–319. MR0233735 ↑1.1

[34] P. Turán, *Eine Extremalaufgabe aus der Graphentheorie*, Mat. Fiz. Lapok **48** (1941), 436–452 (Hungarian, with German summary). MR0018405 (8,284j) ↑1.1

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