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GEOMETRY AND HOLONOMY OF INDECOMPOSABLE CONES

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Abstract. We study the geometry and holonomy of semi-Riemannian, time-like metric cones that are indecomposable, i.e., which do not admit a local decomposition into a semi-Riemannian product. This includes irreducible cones, for which the holonomy can be classified, as well as non irreducible cones. The latter admit a parallel distribution of null k-planes, and we study the cases $k = 1$ and $k = 2$ in detail. In these cases, i.e., when the cone admits a distribution of parallel null tangent lines or planes, we give structure theorems about the base manifold. Moreover, in the case $k = 1$ and when the base manifold is Lorentzian, we derive a description of the cone holonomy. This result is obtained by a computation of certain cocycles of indecomposable subalgebras in $\mathfrak{so}(1, n-1)$.

Contents

1. Introduction

1.1. **Background.** Cone constructions are a valuable tool in differential geometry to study overdetermined PDEs on manifolds. They are applied in conformal [12, 13] and projective geometry [21, 2], but the most striking example is Bär's classification of Riemannian manifolds with real Killing spinors [3]. Bär's observation that real Killing spinors on a Riemannian manifold (M, g) correspond to parallel spinors on the cone

$$(\widehat{M} = \mathbb{R}^{>0} \times M, \breve{g} = dr^2 + r^2 g),$$

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allows to relate and apply several fundamental results in differential geometry: Berger's list of irreducible Riemannian holonomy groups [7] and the classification of those that belong to manifolds with parallel spinors by Wang [22], the understanding of the geometric structures that correspond to these holonomy groups, and finally Gallot's Theorem [15] that the cone $(\breve{M}, \breve{g})$ over a complete manifold (M, g) is either flat or irreducible. This result allows to determine the geometry of (M, g): if the cone $(\breve{M}, \breve{g})$ is flat, then (M, g) has constant sectional curvature 1, and if the cone is irreducible, the geometry of (M, g) is determined by the special holonomy of the cone (Ricci-flat Kähler, hyper-Kähler, or exceptional).

One of the motivations to study semi-Riemannian cones is the Killing spinor equation on semi-Riemannian manifolds, but indefinite cones already become relevant in the Riemannian context. Indeed, *imaginary* Killing spinors on a Riemannian manifold (M, g) correspond to parallel spinors on the time-like cone

$$(\widehat{M} = \mathbb{R}^{>0} \times M, \widehat{g} = -\mathrm{d}r^2 + r^2 g). \tag{1.1}$$

Riemannian manifolds with imaginary Killing spinors were classified by Baum in [5, 4] without using the cone construction, but our results about Lorentzian cones in [1] allow to reprove Baum's classification.

Another motivation stems from supergravity (and string theory), where semi-Riemannian cones play a two-fold role. One the one hand, they appear as scalar geometries (of arbitrary dimension) in the superconformal formulation of supergravity theories, on the other hand, they can be used to study space-times which are part of supersymmetric solutions of the equations of motion of theories of (Poincaré) supergravity or of string theories. In the latter case, the supersymmetry equations can be analysed by passing to the time-like cone over the Lorentzian space-time manifold, which is a semi-Riemannian cone of index 2.

A generalisation of Bär's method to *indefinite* semi-Riemannian manifolds has two aspects: a holonomy classification of indefinite semi-Riemannian cones and the description of the corresponding geometry of the base. Both tasks face several difficulties in the semi-Riemannian context. The fundamental difficulty is that for metrics of arbitrary signature the holonomy group may not act completely reducibly: there are semi-Riemannian manifolds whose holonomy group admits an invariant subspace that is degenerate for the metric. As a consequence, those manifolds cannot be decomposed into a product of manifolds with irreducible holonomy, as it is the case for Riemannian manifolds. Hence, in an indefinite semi-Riemannian context, irreducibility has to be replaced by indecomposability. A semi-Riemannian manifold is *indecomposable* if its holonomy representation (i.e., the representation of the holonomy algebra on the tangent space) does not admit an invariant subspace that is non-degenerate for the metric. By the splitting theorems of de Rham [9] and Wu [23], such metrics do not have a local decomposition into product metrics, hence the term *indecomposable*. Therefore, a generalisation of Bär's method to semi-Riemannian geometry requires two steps:

(A) Generalise Gallot's Theorem to the case of semi-Riemannian cones.
(B) For indecomposable semi-Riemannian cones, describe the holonomy of the cone and the local geometry of the base.

The problem in (A) was solved in [1], where we studied decomposable indefinite semi-Riemannian cones and obtained a generalisation of Gallot's result. In fact we showed that a cone over a complete and compact semi-Riemannian manifold is either flat or indecomposable. Further results about decomposable cones have been obtained in [11, Theorems 5 and 6].

1.2. **Results.** In this article we deal with problem (B), i.e., we study the local geometry of the base and the holonomy of the cone in the case when the cone is *indecomposable*. This setting naturally splits into two different scenarios: the holonomy of the cone is *irreducible*, or it admits an *invariant subspace that is totally null but no non-degenerate invariant subspace*. The irreducible case is well understood as there is Berger's classification of irreducible holonomy groups [7], which we describe in Section 2.2 with the following result:

Theorem 1.1. *If $(\widehat{M},\widehat{g})$ is a time-like cone with irreducible holonomy algebra $\mathfrak{g}$, then $\mathfrak{g}$ is isomorphic to one of the following Lie algebras*

(1.2)
$$\begin{array}{ccllcllcl} & & \mathfrak{so}(t,s), & \mathfrak{u}(p,q),\ \mathfrak{su}(p,q) & \subset & \mathfrak{so}(2p,2q), & \mathfrak{sp}(p,q) & \subset & \mathfrak{so}(4p,4q),\\ \mathfrak{so}(n,\mathbb{C}) & \subset & \mathfrak{so}(n,n), & \mathfrak{g}_2^{\mathbb{C}} & \subset & \mathfrak{so}(7,7), & \mathfrak{spin}(7,\mathbb{C}) & \subset & \mathfrak{so}(8,8),\\ & & & \mathfrak{g}_2 & \subset & \mathfrak{so}(7), & \mathfrak{spin}(7) & \subset & \mathfrak{so}(8),\\ & & & \mathfrak{g}_{2(2)} & \subset & \mathfrak{so}(3,4), & \mathfrak{spin}(3,4) & \subset & \mathfrak{so}(4,4). \end{array}$$

More interesting is the non-irreducible indecomposable case. Here the cone admits a totally null vector distribution of rank $k>0$ that is invariant under parallel transport, or equivalently, its space of sections is invariant under differentiation with respect to the Levi-Civita connection. In general, this case is rather difficult and no general holonomy classification is known. However, the parallel vector distribution determines the local structure of the base. This became obvious in [1] where we studied the case of *Lorentzian* indecomposable cones. As mentioned, some of our motivation comes from the equations of motion of supersymmetric theories of gravity, where the space-time metric is Lorentzian (that is of index 1). Hence we will focus on cones that have index 2, that is signature $(2,n-2)$. For these the totally null parallel vector distribution is of rank 1, i.e., a *null line*, or of rank 2, i.e., a *null plane*. Many of our results will however hold for cones in *arbitrary signature* but with an invariant null line or null plane.

In Section 3 we will study the case of a *parallel null line*, and describe the local structure of the base as well as of the cone:

Theorem 1.2. *Let $(\widehat{M},\widehat{g})$ be the time-like cone over a semi-Riemannian manifold (M,g). If the cone admits a parallel null line field $\mathbf{L}$, then locally there is a parallel trivializing section of $\mathbf{L}$. Moreover, on a dense open subset $\widehat{M}_{\mathrm{reg}}\subset\widehat{M}$, the metric $\widehat{g}$ is locally isometric to a warped product of the form*

$$\widetilde{g}=2\,\mathrm{d}u\,\mathrm{d}v+u^2g_0, \tag{1.3}$$

with a semi-Riemannian metric g_0, and the metric g is locally of the form

$$g=\mathrm{d}s^2+\mathrm{e}^{2s}g_0.$$

In the case when the above decompositions hold globally, the situation can be summarised in the commutative diagram:

$$\begin{tikzcd}
\text{signature } (t+1,s) & & (\widehat{M},\widehat{g}) \arrow[r, "\text{isometry } \psi"] & (\widetilde{M},\widetilde{g}) \\
\text{signature } (t,s) & (M,g) \arrow[ur, "\widehat{g}=-\mathrm{d}r^2+r^2g", "\text{cone}"'] & & \\
\text{signature } (t,s-1) & & (M_0,g_0) \arrow[ul, "\text{warp}", "g=\mathrm{d}s^2+\mathrm{e}^{2s}g_0"'] \arrow[uur, "\text{double warp}", "\widetilde{g}=2\mathrm{d}u\mathrm{d}v+u^2g_0"'] &
\end{tikzcd} \tag{1.4}$$

Here $\widetilde{M}=\mathbb{R}^+\times\mathbb{R}^-\times M_0$, see (3.1) for the definition of ψ. This result motivates the study of metrics of the form (1.3) in Section 4. Such metrics have a parallel null vector field ∂_v and it was shown in [18] that their holonomy algebra $\widetilde{\mathfrak{g}}=\mathfrak{hol}(\widetilde{g})$ is contained in $\mathfrak{hol}(g_0)\ltimes\mathbb{R}^{t,s}$, where (t,s) is the signature of the metric g_0, and moreover that $\mathrm{pr}_{\mathfrak{so}(n)}(\widetilde{\mathfrak{g}})=\mathfrak{hol}(g_0)$. For a Lorentzian metric $\widetilde{g}$, i.e., when g_0 is Riemannian, it was shown in [18, 1] that we have in fact

$$\widetilde{\mathfrak{g}}=\mathfrak{hol}(g_0)\ltimes\mathbb{R}^n,$$

which means that the holonomy of the cone is determined solely by the holonomy of the metric g_0. In higher signatures, i.e., when g_0 is not Riemannian, this is no longer true, as examples will show. Our approach is to consider the ideal of translations in $\mathfrak{hol}(\widetilde{g})$,

$$T:=\mathfrak{hol}(\widetilde{g})\cap\mathbb{R}^{t,s},$$

and use this for a first, purely algebraic study of indecomposable subalgebras in the stabiliser of a null vector. This will be carried out in Section 5.1, which is the most technical section of the paper. The key observation is that

$$\widetilde{\mathfrak{g}}/T=\{(X,\varphi(X))\mid X\in\mathfrak{hol}(g_0)\},\qquad\text{with } \varphi\in Z^1(\mathfrak{hol}(g_0),\mathbb{R}^{t,s}/T),$$

where $Z^1(\mathfrak{hol}(g_0),\mathbb{R}^{t,s}/T)$ denotes the cocycles of $\mathfrak{hol}(g_0)$ with values in $\mathbb{R}^{t,s}/T$. For example, in order to obtain results for time-like cones over Lorentzian manifolds, we will compute $Z^1(\mathfrak{g},\mathbb{R}^{1,n-1}/T)$, for indecomposable subalgebras $\mathfrak{g}$ of $\mathfrak{so}(1,n-1)$ (these belong to one of four types according to [6]).

In Section 6 we apply these algebraic results to obtain the following result.

Theorem 1.3. *Let g_0 be a Lorentzian metric in dimension n and $\widetilde{g}$ the metric of signature $(2,n)$ defined in (1.3). If the holonomy of $\widetilde{g}$ acts indecomposably and with invariant null line, then*

$$\mathfrak{hol}(\widetilde{g})=\mathfrak{hol}(g_0)\ltimes\mathbb{R}^{1,n-1},$$

or g_0 admits a parallel null vector field and $\widetilde{g}$ admits two linearly independent parallel null vector fields that are orthogonal to each other.

This theorem shows that if the holonomy of $\widetilde{g}$ is not equal to the semi-direct product $\mathfrak{hol}(g_0)\ltimes\mathbb{R}^{1,n-1}$, then $\widetilde{g}$ and hence the cone admits a parallel null plane (which in addition is spanned by two parallel null vector fields). We study the case of cones admitting a totally null parallel 2-plane in the remainder of the article. In Section 7 we show:

Theorem 1.4. *The timelike cone* $(\widehat{M}, \widehat{g})$ *over a semi-Riemannian manifold* (M, g) *admits a parallel, totally null* 2*-plane field if and only if, locally over an open dense subset, the base* (M, g) *admits two vector fields* V *and* Z *satisfying*

$$g(V, V) = 0, \quad g(Z, Z) = 1, \quad g(V, Z) = 0, \tag{1.5}$$

and such that

$$\nabla_X V = \alpha(X)V + g(X, V)Z, \tag{1.6}$$

$$\nabla_X Z = -X + \beta(X)V + g(X, Z)Z, \tag{1.7}$$

with 1*-forms* α *and* β *on* M*. In particular, the base* (M, g) *admits a geodesic, shearfree null congruence defined by* V*.*

Note that equation (1.6) implies that $V^\perp$ is integrable. This allows us to determine the local form of the metrics with vector field V and Z satisfying equations (1.5–1.7):

Theorem 1.5. *A semi-Riemannian metric* (M, g) *admits vector fields* V *and* Z *with (1.5–1.7) if and only if* (M, g) *is locally of the form* $M = M_0 \times \mathbb{R}^3$ *and*

$$g = \mathrm{d}s^2 + e^{-2s} g_0(u) + 2\,\mathrm{d}u\,\eta,$$

for a family of metrics $g_0(u)$ *on* M_0 *depending on* u *and a* 1*-form* η *on* M *such that* $\eta(\partial_t)$ *is nowhere vanishing satisfying the following system of first order PDEs:*

$$\begin{array}{rcl} \partial_t \eta_t = \partial_s \eta_t &=& X\eta_t \;=\; 0, \\ \partial_t \eta_s &=& 2\eta_t, \\ \partial_t(\eta(X)) &=& 0, \\ \partial_s\, \eta(X) - X\ \eta_s &=& -2\eta(X) \end{array} \tag{1.8}$$

for all $X \in \mathfrak{X}(M_0)$ *and where* $\eta_t := \eta(\partial_t)$ *and* $\eta_s := \eta(\partial_s)$*.*

Finally we give explicitly the general solution for the system (1.8), providing us with a construction method of metrics whose cone admits a totally null two plane.

2. Preliminaries

2.1. **Fundamental properties of time-like cones.** Let (M, g) be a semi-Riemannnian manifold and $\widehat{M} := \mathbb{R}^+ \times M$ with the metric

$$\widehat{g} := -\mathrm{d}r^2 + r^2 g \tag{2.1}$$

be the *time-like cone* or just the *cone over* (M, g). We denote by

$$\xi = r\frac{\partial}{\partial r}$$

the *Euler vector field*. The Levi-Civita connection $\widehat{\nabla}$ of $\widehat{g}$ reduces to the Levi-Civita connection ∇ of g in the following way

$$\begin{array}{rcl} \widehat{\nabla}\xi &=& \mathrm{Id} \\ \widehat{\nabla}_X Y &=& \nabla_X Y + g(X, Y)\xi, \end{array} \tag{2.2}$$

where here and in the following formulas $X, Y, Z \in \mathfrak{X}(M)$, and the curvature is given as

$$\begin{array}{rcl} \xi \lrcorner \widehat{R} &=& 0, \\ \widehat{R}(X, Y)Z &=& R(X, Y)Z + g(Y, Z)X - g(X, Z)Y. \end{array} \tag{2.3}$$

Hence, for the Ricci tensor we obtain that

$$\begin{aligned} \xi \lrcorner \widehat{Ric} &= 0, \\ \widehat{Ric}(X,Y) &= Ric(X,Y) + (n-1)g(X,Y). \end{aligned} \tag{2.4}$$

This leads to the following observations:

Proposition 2.1. *Let $(\widehat{M}, \widehat{g})$ be the cone over (M,g).*

1. *(M,g) has constant curvature -1 if and only if the cone $(\widehat{M}, \widehat{g})$ is flat.*
2. *If $(\widehat{M}, \widehat{g})$ is Einstein, then it is Ricci-flat.*
3. *If (M,g) is Einstein with $Ric = (1-n)g$, then $(\widehat{M}, \widehat{g})$ is Ricci-flat.*

Finally we observe that the existence of a time-like vector field ξ with $\nabla \xi = \mathrm{Id}$ characterises cones locally.

Proposition 2.2. *Let $(\widehat{M}, \widehat{g})$ be a semi-Riemannian manifold of dimension $n+1$ that admits a time-like vector field ξ such that $\hat{\nabla}\xi = \mathrm{Id}$. Then there are local coordinates $(r, x^1, \dots x^n)$ such that $\widehat{g}$ is of the form*

$$\widehat{g} = -\mathrm{d}r^2 + r^2 g_{ij}(x^1, \dots, x^n)\mathrm{d}x^i \mathrm{d}x^j,$$

where i, j run from 1 to n, we use the Einstein summation convention, and $g_{ij} = g_{ij}(x^1, \dots, x^n)$ are functions of the x^k coordinates only.

Proof. The vector field ξ defines a positive function r via

$$\widehat{g}(\xi, \xi) = -r^2.$$

Differentiating this relation gives

$$2r\mathrm{d}r = \mathrm{d}(r^2) = -\mathrm{d}(\widehat{g}(\xi,\xi)) = -2g(\xi, \cdot) = -2\xi^\flat,$$

where the musical isomorphism $\flat$ denotes the metric dual with respect to $\widehat{g}$. Hence

$$\xi^\flat = -\mathrm{d}\left(\frac{r^2}{2}\right),$$

is exact and therefore $\xi = -\hat{\nabla}\frac{r^2}{2}$ is a gradient vector field. The level sets of the function r are orthogonal to ξ and we can fix coordinates $(x^1, \dots, x^n)$ on the level sets such that $(r, x^1, \dots, x^n)$ are local coordinates on $\widehat{M}$. In these coordinates the metric has the form

$$g = -\mathrm{d}r^2 + \widehat{g}_{ij}(r, x^1, \dots, x^n)\mathrm{d}x^i \mathrm{d}x^j,$$

and it holds $\xi = r\partial_r$. Since $\hat{\nabla}\xi = \mathrm{Id}$, the vector field ξ is a homothety,

$$\mathcal{L}_\xi \widehat{g} = 2\widehat{g},$$

which implies that

$$\widehat{g}_{ij}(r, x^1, \dots, x^n) = r^2 g_{ij}(x^1, \dots, x^n)$$

for some functions $g_{ij}(x^1, \dots, x^n)$ of the x^i coordinates. $\square$

2.2. **The holonomy of irreducible cones.** For irreducible cones the possible holonomy groups are known from the Berger list [7], which comprises the orthogonal algebra and the three lists (2.5–2.7) below. In the following let $\mathfrak{h} \subset \mathfrak{so}(t+1,q)$ the irreducible holonomy algebra of a semi-Riemannian manifold $(\widehat{M},\widehat{g})$, i.e., one of the entries in Berger's list. For each possible $\mathfrak{h}$ we will now determine if it can be the holonomy algebra of a cone.

(1) $\mathfrak{h} = \mathfrak{so}(t+1,s)$: This is the holonomy algebra of a generic semi-Riemannian manifold of signature $(t+1,s)$.

Proposition 2.3. *Let (M,g) be a semi-Riemannian manifold of signature (t,s) and of constant curvature $\kappa \neq -1$ and let $(\widehat{M},\widehat{g})$ be the time-like cone over (M,g). Then $\mathfrak{hol}(\widehat{M},\widehat{g}) = \mathfrak{so}(t+1,s)$.*

Proof. The curvature endomorphisms of (M,g) are of the form

$$R(X,Y) = \kappa\big(g(Y,\cdot)X - g(X,\cdot)Y\big).$$

Since the holonomy algebra contains all curvature endomorphisms, equation (2.3) shows that

$$\mathfrak{so}(t,s) \subset \mathfrak{hol}(\widehat{M},\widehat{g}),$$

where $\mathfrak{so}(t,s)$ is embedded as the stabiliser of the vector ξ. Moreover, equations (2.2–2.3) show that

$$(\widehat{\nabla}_X\widehat{R})(Y,Z)\xi = -\widehat{R}(X,Y)Z = -2\left(g(Y,Z)X - g(X,Z)Y\right).$$

This establishes $\mathfrak{hol}(\widehat{M},\widehat{g}) = \mathfrak{so}(t+1,s)$. $\square$

(2) $\mathfrak{h}$ is the holonomy of an irreducible symmetric space or one of the following algebras:

$$\begin{array}{rcl} \mathfrak{sp}(1)\oplus\mathfrak{sp}(p,q) & \subset & \mathfrak{so}(2p,2q),\\ \mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sp}(m,\mathbb{R}) & \subset & \mathfrak{so}(2m,2m),\\ \mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sp}(m,\mathbb{C}) & \subset & \mathfrak{so}(4m,4m), \end{array} \tag{2.5}$$

where $p+q$ and m are >1. In the first case the metric is quaternionic Kähler of signature $(4p,4q)$ and in the second it is quaternionic para-Kähler. Examples of the third type are obtained by complexifying manifolds with holonomy of the first two types, as discussed below. In these examples $(\widehat{M},\widehat{g})$ is Einstein with *nonzero* Einstein constant. Hence, these cases can be excluded as holonomy of cones by Proposition 2.1.

(3) $\mathfrak{h}$ is one of the following:

$$\begin{array}{rclrcl} \mathfrak{u}(p,q),\ \mathfrak{su}(p,q) & \subset & \mathfrak{so}(2p,2q), & \mathfrak{sp}(p,q) & \subset & \mathfrak{so}(4p,4q),\\ \mathfrak{g}_2 & \subset & \mathfrak{so}(7), & \mathfrak{spin}(7) & \subset & \mathfrak{so}(8),\\ \mathfrak{g}_{2(2)} & \subset & \mathfrak{so}(3,4), & \mathfrak{spin}(3,4) & \subset & \mathfrak{so}(4,4). \end{array} \tag{2.6}$$

The geometric structures corresponding to these algebras do exist on cones over semi-Riemannian manifolds with certain structures. In fact, the following relations between structure on the base (M,g) and on the cone are well known:

(i) The cone over a (semi-Riemannian) Sasaki, Einstein-Sasaki or 3-Sasaki manifold is, respectively, a Kähler, Ricci-flat Kähler or hyper-Kähler manifold and hence has holonomy contained in $\mathfrak{u}(p,q)$, $\mathfrak{su}(p,q)$ or $\mathfrak{sp}(p,q)$.

(ii) The cone over a strict nearly-Kähler manifold of dimension 6, Riemannian or of signature $(2,4)$, has a parallel $\mathbf{G}_2$- or $\mathbf{G}_{2(2)}$-structure and hence has holonomy contained in $\mathfrak{g}_2$ or $\mathfrak{g}_{2(2)}$. Similarly, the cone over a nearly para-Kähler manifold with $|\nabla J|^2 \neq 0$ has holonomy contained in $\mathfrak{g}_{2(2)}$, see [8, Prop. 3.1].

(iii) The cone over a 7-manifold with a nearly-parallel $\mathbf{G}_2$-structure, Riemannian or of signature $(3,4)$, has a parallel $\mathbf{Spin}(7)$- or $\mathbf{Spin}(3,4)$-structure and hence has holonomy contained in $\mathfrak{spin}(7)$ or $\mathfrak{spin}(3,4)$.

The question remains, whether the holonomy of the cone is not only contained but actually *equal* to one of the algebras in the list (2.6). In the Riemannian setting (which corresponds to the case where the base of the time-like cone is negative definite) this can be established by using Gallot's Theorem that the (space-like) cone over a *complete* Riemannian manifold (M,g) is either flat or irreducible and then by constructing a complete (M,g) with the corresponding structure. For indefinite metrics several gaps open up in this argument: our generalisation of Gallot's Theorem in [1] assumes that (M,g) to be compact and complete and implies that the cones is flat or *indecomposable*, but not necessarily irreducible. Hence, even if one constructed compact and complete indefinite semi-Riemannian manifolds with the above structures, the cone would not have to be irreducible and hence its holonomy could be an indecomposable, non irreducible subalgebra of the algebras in (2.6). We suspect however, that for a "generic" semi-Riemannian manifold with one of the above structures, the cone has holonomy equal to the algebras in (2.6).

(4) $\mathfrak{h}$ is one of the following algebras:

$$\begin{array}{rclrcl} \mathfrak{so}(n,\mathbb{C}) & \subset & \mathfrak{so}(n,n) & \mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sp}(m,\mathbb{C}) & \subset & \mathfrak{so}(4m,4m) \\ \mathfrak{g}_2^{\mathbb{C}} & \subset & \mathfrak{so}(7,7), & \mathfrak{spin}(7,\mathbb{C}) & \subset & \mathfrak{so}(8,8). \end{array} \tag{2.7}$$

Examples can be obtained by complexification as we will explain now. In the case of $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sp}(m,\mathbb{C})$ the metric is then Einstein of nonzero scalar curvature (incompatible with a cone), whereas in the two exceptional cases it is Ricci-flat.

Realisation of complex holonomy algebras. Let (M,g) be a connected real analytic manifold endowed with a real analytic semi-Riemannian metric. Then it is easy to see that M can be embedded into a connected complex manifold $M^{\mathbb{C}}$ with the following properties.

(1) There exists an atlas of $M^{\mathbb{C}}$ such that each of its charts $\varphi : U \to \mathbb{C}^n$ is real-valued on $U\cap M$ and the restrictions $\varphi|_{U\cap M} : U\cap M \to \mathbb{R}^n$, $U\cap M \neq \varnothing$, form an atlas of M.
(2) The metric coefficients $g_{ij}(x)$ with respect to the real coordinates $x=(x^1,\ldots,x^n)=\varphi|_{U\cap M}$ are given by real power series converging in $U\cap M$.
(3) The power series $g_{ij}(z)$ in the holomorphic coordinates $z=(z^1,\ldots,z^n)=\varphi$ converges in U for all i,j.

It follows that we can define a holomorphic symmetric tensor field $g^{\mathbb{C}}$ on $M^{\mathbb{C}}$ by

$$g^{\mathbb{C}}|_U = \sum g_{ij}(z)\mathrm{d}z^i\mathrm{d}z^j.$$

The tensor field is non-degenerate on a neighborhood of M and by restriction we can always assume that it is non-degenerate on $M^{\mathbb{C}}$. Then it defines what is called a *holomorphic Riemannian metric* on $M^{\mathbb{C}}$. We will call $(M^{\mathbb{C}},g^{\mathbb{C}})$ a *complexification* of (M,g). Recall that a pair consisting of a complex manifold and a holomorphic Riemannian metric on that manifold is called a *holomorphic Riemannian manifold*. Note that $(M^{\mathbb{C}}, g^{\mathbb{C}})$ is unique as a germ of holomorphic Riemannian manifold along M.

We define the *holonomy algebra* of a holomorphic Riemannian manifold $(M^{\mathbb{C}}, g^{\mathbb{C}})$ at $p \in M^{\mathbb{C}}$ as the Lie algebra spanned by all the skew-symmetric endomorphisms

$$((\nabla^{\mathbb{C}})^k_{v_1,\ldots,v_k} R^{\mathbb{C}})(v_{k+1}, v_{k+2}) \in \mathfrak{so}(T^{1,0}_p M^{\mathbb{C}}) \cong \mathfrak{so}(T_pM)^{\mathbb{C}},$$

where $v_1, \ldots, v_{k+2} \in T^{1,0}_p M^{\mathbb{C}}$ and $k \geqslant 0$. Here $\nabla^{\mathbb{C}}$ denotes the (holomorphic) Levi-Civita connection of $g^{\mathbb{C}}$ and $R^{\mathbb{C}}$ its curvature tensor.

Proposition 2.4. *Let $(M^{\mathbb{C}}, g^{\mathbb{C}})$ be a complexification of a connected semi-Riemannian manifold (M, g). Then the holonomy algebra of $(M^{\mathbb{C}}, g^{\mathbb{C}})$ is given by the complexification $\mathfrak{h}^{\mathbb{C}}$ of the holonomy algebra $\mathfrak{h}$ of (M, g).*

Proof. By the Ambrose-Singer theorem for real analytic semi-Riemannian manifolds we know that $\mathfrak{h}$ is spanned by all the endomorphisms $(\nabla^k_{v_1,\ldots,v_k} R)(v_{k+1}, v_{k+2}) \in \mathfrak{so}(T_pM)$, where $v_1, \ldots, v_{k+2} \in T_pM$ and $k \geqslant 0$. From the definition of $g^{\mathbb{C}}$ as complex-analytic extension of g it is clear that the Levi-Civita connection $\nabla^{\mathbb{C}}$ of $g^{\mathbb{C}}$ coincides with the complex-analytic extension of the Levi-Civita connection ∇ of g. The same relation holds for the curvature tensors and their covariant derivatives. This implies the proposition. □

Next we consider the real analytic manifold N of dimension $2n$ underlying the complex manifold $M^{\mathbb{C}}$. It carries a corresponding integrable complex structure J and we can identify (N, J) with $M^{\mathbb{C}}$. We endow N with the real analytic semi-Riemannian metric

$$g_N := 2\,\mathrm{Re}\, g^{\mathbb{C}}. \tag{2.8}$$

Note that g_N can be considered as a (fibrewise) real bilinear form on TN by means of the canonical identification

$$TN \cong T^{1,0}N, \quad X \mapsto X^{1,0} = \frac{1}{2}(X - iJX).$$

The factor 2 in (2.8) is chosen such that $g^{\mathbb{C}}$ is obtained by restricting (the complex bilinear extension of) g_N to $T^{1,0}N$.

We observe that the metric g_N can be defined on the real analytic manifold N underlying any holomorphic Riemannian manifold $(M^{\mathbb{C}}, g^{\mathbb{C}})$ irrespective of whether $(M^{\mathbb{C}}, g^{\mathbb{C}})$ is a complexification of a semi-Riemannian manifold (M, g).

Theorem 2.5. *Let $(M^{\mathbb{C}}, g^{\mathbb{C}})$ be a connected holomorphic Riemannian manifold and (N, g_N) the corresponding semi-Riemannian manifold. Then (N, g_N) has neutral signature and its holonomy algebra is isomorphic to the holonomy algebra of $(M^{\mathbb{C}}, g^{\mathbb{C}})$.*

Proof. Note first that $g_N(J\cdot, J\cdot) = -g_N$, since g_N is of type $(2,0) + (0,2)$ with respect to J. This implies that g_N has neutral signature, since J it maps a maximal definite subspace of T_pN to a maximal definite subspace of the same dimension and of opposite signature.

We consider first the Lie algebra $\mathfrak{so}(T_pN)$, $p \in N$, with respect to g_N and its subalgebra

$$\mathfrak{so}(T_pN)^J := \{A \in \mathfrak{so}(T_pN) \mid [A, J] = 0\}.$$

The latter can be considered as a complex Lie algebra with the complex structure $A \mapsto JA$. Indeed, JA is g_N-skew-symmetric as the product of a symmetric with a commuting skew-symmetric endomorphism. The symmetry of J follows from the fact that J is an anti-isometry squaring to minus one.

We claim that $\mathfrak{so}(T_pN)^J$ is canonically isomorphic to the complex Lie algebra $\mathfrak{so}(T_p^{1,0}N)$ with respect to $g^{\mathbb{C}}$. Using the metric g_N, we can identify $\mathfrak{so}(T_pN)^J$ with the set of real points in $\bigwedge^{2,0} T_pN \oplus \bigwedge^{0,2} T_pN$ and the latter can be identified with $\bigwedge^{2,0} T_pN \cong \bigwedge^2 T^{1,0}M$ by projecting to the $(2,0)$-component. Finally, using the metric $g^{\mathbb{C}}$, we can identify $\bigwedge^2 T^{1,0}M$ with $\mathfrak{so}(T_p^{1,0}N)$. This yields a canonical isomorphism

$$\Phi : \mathfrak{so}(T_pN)^J \to \mathfrak{so}(T_p^{1,0}N) \tag{2.9}$$

of complex vector spaces. It simply maps $A \in \mathfrak{so}(T_pN)^J$ to its restriction to $T^{1,0}N$. Therefore it is even an isomorphism of Lie algebras.

Next we show, for all $v_1, \dots v_{k+2} \in T_pN$, that under the canonical isomorphism (2.9) the tensor $(\nabla^N)^k_{v_1,\dots,v_k} R^N(v_{k+1}, v_{k+2})$ is mapped to $(\nabla^{\mathbb{C}})^k_{w_1,\dots,w_k} R^{\mathbb{C}}(w_{k+1}, w_{k+2})$, where $w_j = v_j^{1,0}$, ∇^N denotes the Levi-Civita connection of g_N and R^N its curvature. This implies the theorem, in virtue of the Ambrose-Singer theorem. First we show that ∇^N can be constructed from the holomorphic connection $\nabla^{\mathbb{C}}$. Let ∇' be the unique connection in $(TN)^{\mathbb{C}}$ with the following properties:

(1) $\nabla'_Z W = \nabla^{\mathbb{C}}_Z W$ for all holomorphic vector fields Z, W on $M^{\mathbb{C}}$.
(2) $\nabla'_{\bar{Z}} W = 0$ for all holomorphic vector fields Z, W on $M^{\mathbb{C}}$.
(3) ∇' is real, that is restricts to a connection in TN.

Notice that the above properties imply that the subbundles $T^{1,0}N$ and $T^{0,1}N$ are ∇'-parallel and, hence, that $\nabla' J = 0$. Moreover, using these properties, it is straightforward to check that ∇' is metric torsion-free, since $\nabla^{\mathbb{C}}$ is. This implies that ∇' (when considered as a connection in TN) coincides with the Levi-Civita connection ∇^N. As a consequence, we see that $\nabla^N J = 0$ and thus $(\nabla^N)^k_{v_1,\dots,v_k} R^N(v_{k+1}, v_{k+2}) \in \mathfrak{so}(T_pN)^J$. Now let X, Y be real vector fields on an open set $U \subset N$ which are infinitesimal automorphisms of J. Then we have the formula

$$(\nabla^N_X Y)^{1,0} = \nabla^{\mathbb{C}}_{X^{1,0}} Y^{1,0}. \tag{2.10}$$

This follows immediately from the defining properties of $\nabla' = \nabla^N$ by decomposing $X = Z + \bar{Z}$ and $Y = W + \bar{W}$, where $Z = X^{1,0}, W = Y^{1,0}$ are holomorphic. From (2.10) we deduce that

$$\left(\left((\nabla^N)^k_{v_1,\dots,v_k} R^N(v_{k+1}, v_{k+2}) \right) v_{k+3} \right)^{1,0} = \left((\nabla^{\mathbb{C}})^k_{w_1,\dots,w_k} R^{\mathbb{C}}(w_{k+1}, w_{k+2}) \right) w_{k+3},$$

for all $v_1, \dots, v_{k+3} \in T_pN$, where we recall that $w_j = v_j^{1,0}$. Since the left-hand side is precisely

$$\Phi \left((\nabla^N)^k_{v_1,\dots,v_k} R^N(v_{k+1}, v_{k+2}) \right) w_{k+3},$$

we can conclude that

$$\Phi \left((\nabla^N)^k_{v_1,\dots,v_k} R^N(v_{k+1}, v_{k+2}) \right) = (\nabla^{\mathbb{C}})^k_{w_1,\dots,w_k} R^{\mathbb{C}}(w_{k+1}, w_{k+2}),$$

finishing the proof. □

This leads to the following consequence:

Corollary 2.6. *The complex holonomies*

$$\mathfrak{so}(n, \mathbb{C}) \subset \mathfrak{so}(n, n), \quad \mathfrak{g}_2^{\mathbb{C}} \subset \mathfrak{so}(7, 7), \quad \mathfrak{spin}(7, \mathbb{C}) \subset \mathfrak{so}(8, 8), \tag{2.11}$$

are holonomy algebras of time-like cones.

Proof. This follows from the above considerations and from the fact that real forms of the complex holonomy algebras in (2.11) can be realised by cones. Indeed, if $(\widehat{M}, \widehat{g})$ is a time-like cone with holonomy $\mathfrak{so}(p,q)$, $\mathfrak{g}_{2(2)} \subset \mathfrak{so}(3,4)$, or $\mathfrak{spin}(3,4) \subset \mathfrak{so}(4,4)$, then there is the Euler vector field $\xi \in \Gamma(T\widehat{M})$. Hence the real analytic metric $\widehat{g}^{\mathbb{C}}$ on $\widehat{M}^{\mathbb{C}}$ has the holomorphic Euler vector field $\xi^{\mathbb{C}}$ with $\widehat{\nabla}^{\mathbb{C}}\xi^{\mathbb{C}} = \mathrm{Id}$. On the real manifold $N = \widehat{M}^{\mathbb{C}}$ we then have that $\eta = 2\mathrm{Re}\,\xi^{\mathbb{C}}$ satisfies $\nabla^N\eta = \mathrm{Id}$, as a consequence of equation (2.10) applied here to $N = \widehat{M}^{\mathbb{C}}$ instead of $M^{\mathbb{C}}$. By Proposition 2.2 we then get that N is locally a cone, which by Theorem 2.5 has one of the complex holonomies in (2.11) as holonomy algebra. □

This proof can be made more explicit in local coordinates. Locally the metric $\widehat{g}$ is of the form

$$\widehat{g} = -\mathrm{d}r^2 + r^2 g_{ij}(x^k)\mathrm{d}x^i\mathrm{d}x^j$$

with Euler vector field $\xi = r\partial_r \in \Gamma(T\widehat{M})$. The analytic metric $\widehat{g}^{\mathbb{C}}$ on $\widehat{M}^{\mathbb{C}}$ then is of the form

$$\widehat{g}^{\mathbb{C}} = -\mathrm{d}u^2 + u^2 g_{ij}(z^k)\mathrm{d}z^i\mathrm{d}z^j$$

with coordinates $(u = r + is, z^1, \dots, z^n)$ with $z^k = x^k + \mathrm{i}y^k$ and holomorphic Euler vector field $\xi^{\mathbb{C}} = u\partial_u$ with $\widehat{\nabla}^{\mathbb{C}}\xi^{\mathbb{C}} = \mathrm{Id}$. Then the metric $\widehat{h} = \frac{1}{2}g_N$ on $N = \widehat{M}^{\mathbb{C}}$ is given by

$$\widehat{h} = -\mathrm{d}r^2 + \mathrm{d}s^2 + (r^2 - s^2)\mathrm{Re}(g_{ij}(z^k)\mathrm{d}z^i\mathrm{d}z^j) - 2rs\,\mathrm{Im}(g_{ij}(z^k)\mathrm{d}z^i\mathrm{d}z^j).$$

One can directly check that $\eta = r\partial_r + s\partial_s$ satisfies $\nabla^N\eta = \mathrm{Id}$. Moreover, the cone coordinate with respect to $\widehat{h}$ is given by $\rho = \sqrt{r^2 - s^2}$, which satisfies $\widehat{h}(\eta, \cdot) = -\rho\mathrm{d}\rho$.

2.3. **Manifolds with parallel null line bundle.** In the following manifolds with a parallel null line bundle will be crucial. In this section we will collect some facts about them.

Let (M, g) be a semi-Riemannian manifold with a *parallel null line bundle* $\mathbf{L}$, i.e., $\mathbf{L}$ is a rank 1 subbundle of TM the fibres of which are null with respect to the metric g and invariant under parallel transport with respect to the Levi-Civita connection ∇ of g. This implies that every non-vanishing section $\chi \in \Gamma(\mathbf{L})$ satisfies

$$\nabla\chi = \alpha \otimes \chi, \tag{2.12}$$

for a uniquely determined 1-form α. Any vector field that satisfies equation (2.12) for some 1-form α is called a *recurrent vector field*.

Proposition 2.7. *Let χ be a recurrent vector field on a connected semi-Riemannian manifold (M, g). Then the function $f = g(\chi, \chi)$ is either everywhere positive, negative or zero. In particular, χ can only have zeros if $f \equiv 0$.*

Proof. The equation (2.12) yields the ODE $X(f) = 2\alpha(X)f$ for every vector field X. These ODEs imply that if f vanishes at a point, then f vanishes in a neighbourhood of this point. Due to the continuity of f this shows that M is a disjoint union of the three open sets $\{f > 0\}$, $\{f < 0\}$ and $\{f = 0\}$. Now, since M is connected, the proposition follows. □

Hence, locally the existence of a parallel null line bundle is equivalent to the existence of a *recurrent null vector field*, where we recall that a vector field χ is null if $g(\chi, \chi) = 0$ and χ does not vanish [19, Definition 3 in Chapter 3]. Moreover, a nowhere vanishing recurrent vector field χ can be rescaled to parallel vector field $h \cdot \chi$, for a non-vanishing function h, if and only if the 1-form α is exact. Indeed, if $\alpha = df$, then $h = \mathrm{e}^{-f}$ as

$$\nabla(\mathrm{e}^{-f}\chi) = 0.$$

Conversely, if $h \cdot \chi$ is parallel, then

$$0 = R(X,Y)\chi = \mathrm{d}\alpha(X,Y)\chi$$

for all $X, Y \in TM$.

Hence, on simply connected manifolds (M,g), nowhere vanishing recurrent vector fields can be rescaled to parallel ones if and only if α is closed[1]. The choice we have when locally choosing a recurrent vector field that spans a null line bundle $\mathbf{L}$ can be used to find special recurrent sections of $\mathbf{L}$.

Lemma 2.8. *Let $\mathbf{L}$ be a parallel null line bundle. Then locally there is a recurrent gradient vector field χ which spans $\mathbf{L}$. This vector field satisfies that $\nabla\chi = h\,\chi^\flat \otimes \chi$ for a function h.*

Proof. Since $\mathbf{L}$ is parallel, the hyperplane distribution $\mathbf{L}^\perp = \{X \in TM \mid g(X,\mathbf{L}) = 0$ is parallel and hence involutive. Hence by Frobenius' Theorem $\mathbf{L}^\perp$ is integrable and the integral manifolds are given as $f \equiv$ *constant* for some local function f. Hence $\mathbf{L}^\perp = \ker(df)$ and the gradient $\chi := \mathrm{grad}(f)$ of f spans $\mathbf{L}$. Then χ is recurrent, i.e., $\nabla\chi = \alpha \otimes \chi$. But then $\chi = \mathrm{grad}(f)$ implies that

$$0 = \mathrm{d}\chi^\flat = \alpha \wedge \chi^\flat,$$

which shows that $\alpha = h\,\chi^\flat$ for a local function h. □

3. Cones with parallel null lines

In this section we assume that the cone (2.1) over a semi-Riemannian manifold (M,g) admits a null line that is invariant under parallel transport. We will show that locally this implies that the cone admits a parallel null vector field and that the base (M,g) is locally an exponential extension of a semi-Riemannian manifold (M_0, g_0), see Definition 3.2. The total space of the cone will then be shown to be locally isometric to a double warped extension $(\widetilde{M}, \widetilde{g})$ of (M_0, g_0), see Definition 3.2. This will generalise our results for Lorentzian cones in [1, Section 9].

Proposition 3.1. *Let $(\widehat{M}, \widehat{g})$ be a timelike cone and assume that $(\widehat{M}, \widehat{g})$ admits a parallel null line $\mathbf{L}$. Then the following holds:*

- *(i) The set $\widehat{M}_{\mathrm{reg}}$ where $\mathbf{L}$ is not perpendicular to the Euler vector field ξ is open and dense and invariant under the flow of ξ. So, in particular, $\widehat{M}_{\mathrm{reg}} = \widehat{M_{\mathrm{reg}}}$, where $M_{\mathrm{reg}} := \widehat{M}_{\mathrm{reg}} \cap M$.*
- *(ii) $\mathbf{L}$ is flat and, hence, locally (and globally if M is simply connected) there is a parallel null vector field that spans $\mathbf{L}$.*

Proof. By passing to the universal cover of $(\widehat{M}, \widehat{g})$, that is to the cone over the universal cover of M, we can assume that M and $\widehat{M}$ are simply connected. Hence, we can assume

[1] This also shows that non-null recurrent vector fields can always be rescaled locally to a parallel vector field, as either $\chi = 0$ or $0 = \widehat{R}(X,Y,\chi,\chi) = \mathrm{d}\alpha(X,Y)\,\widehat{g}(\chi,\chi)$ shows that α is closed.

that the parallel null line $\mathbf{L}$ is spanned by a nowhere vanishing recurrent vector field χ on $(\widehat{M}, \widehat{g})$. Then we decompose

$$\chi = f\xi + Z,$$

where Z is tangent to M and nowhere vanishing. We claim that the function f cannot vanish on a nonempty open set. If it did, formulae (2.2) would give

$$\alpha(X)Z = \widehat{\nabla}_X \chi = \nabla_X Z + g(X, Z)\xi,$$

on the open set, and hence $g(X, Z) = 0$ for all $X \in TM$, which is a contradiction. This proves that the open set $\widehat{M}_{\mathrm{reg}} = \{p \in \widehat{M} \mid f(p) \neq 0\}$ is dense. The invariance of $\widehat{M}_{\mathrm{reg}}$ under the homothetic flow of ξ follows from the invariance of $\mathbf{L}$ under the flow. The latter is obtained by writing the Lie derivative as $\mathcal{L}_\xi = \widehat{\nabla}_\xi - \mathrm{Id}$ and using that $\mathbf{L}$ is parallel.

On $\widehat{M}_{\mathrm{reg}}$ we have

$$\mathrm{d}\alpha(X, \xi)\chi = \widehat{R}(X, \xi)\chi = 0$$

and

$$\mathrm{d}\alpha(X, Y)\chi = \widehat{R}(X, Y)\chi = \widehat{R}(X, Y)Z \in TM,$$

for $X, Y \in TM$. This implies $\mathrm{d}\alpha = 0$, since $\widehat{M}_{\mathrm{reg}}$ is dense, proving that $\mathbf{L}$ is flat. □

In the next proposition we describe an example of a cone with a parallel null line before showing that every simply connected example is of this form.

Definition 3.2. Let (M_0, g_0) be a semi-Riemannian manifold. Then the warped products $(M = \mathbb{R} \times M_0, g = ds^2 + \mathrm{e}^{2s} g_0)$ and $(\widetilde{M} = \mathbb{R}^+ \times \mathbb{R}^- \times M_0, \widetilde{g} = 2\,\mathrm{d}u\,\mathrm{d}v + u^2 g_0)$ will be called the *exponential extension* and the *double warped extension* of (M_0, g_0), respectively.

Proposition 3.3. *Let (M_0, g_0) be a semi-Riemannian manifold. The time-like cone $(\widehat{M}, \widehat{g})$ over the exponential extension (M, g) of (M_0, g_0) is globally isometric to the double warped extension $(\widetilde{M}, \widetilde{g})$ of (M_0, g_0). In particular, the cone admits the parallel null vector field ∂_v.*

Proof. The cone metric over (M, g) is given by

$$\widehat{g} = -\mathrm{d}r^2 + r^2 \mathrm{d}s^2 + r^2 \mathrm{e}^{2s} g_0,$$

with $r \in \mathbb{R}^+$ and $s \in \mathbb{R}$. For the diffeomorphism

$$\psi : \widehat{M} = \mathbb{R}^+ \times \mathbb{R} \times M_0 \ni (r, s, p) \longmapsto \left(u = r\mathrm{e}^s, v = -\tfrac{1}{2} r\mathrm{e}^{-s}, p\right) \in \widetilde{M} = \mathbb{R}^+ \times \mathbb{R}^- \times M_0 \tag{3.1}$$

one checks that

$$(\psi^{-1})^* \widehat{g} = 2\,\mathrm{d}u\,\mathrm{d}v + u^2 g_0.$$

This proves the statement. □

Theorem 3.4. *Let $(\widehat{M}, \widehat{g})$ be a time-like cone over a semi-Riemannian manifold (M, g). Assume that $(\widehat{M}, \widehat{g})$ admits a parallel null line $\mathbf{L}$. Then the open dense subset $\widehat{M}_{\mathrm{reg}} \subset (\widehat{M}, \widehat{g})$, cf. Proposition 3.1, is locally isometric to the double warped extension $(\widetilde{M}, \widetilde{g})$ of a semi-Riemannian manifold (M_0, g_0) and the open dense subset $M_{\mathrm{reg}} \subset (M, g)$ is locally isometric to the exponential extension of (M_0, g_0).*

Proof. By Proposition 3.1, $\mathbf{L}$ admits a parallel trivializing section χ. We write the parallel null vector field χ on $\widehat{M}$ as

$$\chi = \hat{f}\xi + \hat{Z}$$

with $\hat{Z}$ a nowhere vanishing vector field tangent to M and $\hat{f}$ a smooth function on $\widehat{M}$. We will show that $\hat{Z}$ defines vector field Z on M. From

$$[\xi, \hat{Z}] = -d\hat{f}(\xi)\xi + [\xi, \chi] = -d\hat{f}(\xi)\xi - \widehat{\nabla}_\chi \xi = -d\hat{f}(\xi)\xi - \chi = -(d\hat{f}(\xi) + \hat{f})\xi - \hat{Z},$$

with $[\xi, \hat{Z}]$ being tangent to M, we get on the one hand that

$$\mathrm{d}\hat{f}(\xi) + \hat{f} = r\partial_r(\hat{f}) + \hat{f} = 0,$$

and on the other that

$$[\xi, \hat{Z}] + \hat{Z} = 0$$

The first equation shows that

$$\hat{f} = \frac{1}{r} f$$

with f a function on M and the second that

$$\hat{Z} = \frac{1}{r} Z,$$

with $Z = r\hat{Z}$ a vector field on M, i.e., $[\xi, Z] = 0$. Hence we have

$$\chi = \frac{1}{r}(f\xi + Z).$$

Differentiating in direction of $X \in TM$, by (2.2) we get

$$0 = r\widehat{\nabla}_X \chi = (df(X) + g(X, Z))\xi + fX + \nabla_X Z.$$

which shows that

$$Z = -\mathrm{grad}(f),$$

where grad denotes the gradient with respect to g, and

$$\nabla Z = -f\,\mathrm{Id}. \tag{3.2}$$

Hence, the distribution $Z^\perp$ on M is integrable and its leafs are given by the level sets of f. The vector field Z is not only a gradient but also a conformal vector field, since from (3.2) we compute

$$\mathcal{L}_Z g = -2fg.$$

Note also that on $M_{\mathrm{reg}} = \widehat{M}_{\mathrm{reg}} \cap M$, the vector field Z is transversal to the level sets of f. This follows from $df(Z) = g(\mathrm{grad}(f), Z) = -g(Z, Z) = -f^2$. Hence, locally on M_{reg} the metric g is given as

$$g = \frac{(\mathrm{d}f)^2}{f^2} + f^2 g_0$$

where $c^2 g_0$ is the metric g restricted to a level set $\{f = c\}$. Setting $s = \log|f|$ and using Proposition 3.3 this proves the statement in the Theorem. □

The statement of this theorem is summarised in diagram (1.4) in the introduction. In the following we will study metrics of the form $\widetilde{g} = 2\mathrm{d}u\mathrm{d}v + u^2 g_0$. For brevity we will drop the index 0 at g_0.

4. Metrics of the form $\widetilde{g} = 2\mathrm{d}u\,\mathrm{d}v + u^2 g$

4.1. **Levi-Civita connection, curvature and holonomy.** Let g be a semi-Riemannian metric (of signature (t,s)) on a manifold M of dimension n. We want to study the geometry and the holonomy of metrics of signature $(t+1, s+1)$ of the form

$$\widetilde{g} = 2\mathrm{d}u\,\mathrm{d}v + u^2 g, \tag{4.1}$$

from now on to be considered on the maximal domain $\widetilde{M} := \mathbb{R}^+ \times \mathbb{R} \times M \supset \mathbb{R}^+ \times \mathbb{R}^- \times M$. Such metrics admit a 2-dimensional solvable group of homotheties given by $(u,v,p) \mapsto (\mathrm{e}^r u, \mathrm{e}^r v + s, p)$. Its infinitesimal generators are the parallel null vector field ∂_v and the homothetic vector field $U = u\partial_u + v\partial_v$.

There are obvious inclusions of $M = \{1\} \times \{0\} \times M \subset \widetilde{M}$, $TM \subset T\widetilde{M}$ and $\Gamma(TM) \subset \Gamma(T\widetilde{M})$. Using these identifications, the Levi-Civita connection of $\widetilde{g}$ can be expressed easily as follows

$$\begin{array}{rcl} \widetilde{\nabla}_X Y &=& \nabla_X Y - u\, g(X,Y)\,\partial_v, \quad \text{with } Y \in \Gamma(TM) \\ \widetilde{\nabla}_X \partial_u &=& \dfrac{1}{u}\, X \end{array} \tag{4.2}$$

with $X \in TM$, ∇ the Levi-Civita connection of g, and all other derivatives that are not determined by the vanishing of the torsion of ∇. Note that the homothetic vector field $U = u\partial_u + v\partial_v$ satisfies $\widetilde{\nabla} U = \mathrm{Id}$. Moreover, for the curvature of $\widetilde{g}$ one computes that

$$\begin{array}{rcl} \partial_v \lrcorner \widetilde{R} &=& \partial_u \lrcorner \widetilde{R} \;=\; 0, \\ \widetilde{R}(X,Y)Z &=& R(X,Y)Z, \quad \text{for all } X,Y,Z \in TM, \end{array} \tag{4.3}$$

where R is the curvature tensor of (M,g). Note that this implies for an arbitrary tensor field Q that

$$\widetilde{\nabla}_{\partial_u} \widetilde{\nabla}_X Q = \widetilde{\nabla}_X \widetilde{\nabla}_{\partial_u} Q \tag{4.4}$$

for all $X \in \Gamma(TM)$.

For the derivatives of $\widetilde{R}$ we get the following formulae, which determine all possible derivatives. First we observe that

$$(\widetilde{\nabla}_{\partial_u} \widetilde{R})(\partial_u, X) = 0 \tag{4.5}$$

for all $X \in TM$. For the q-th derivative in ∂_u-direction we compute

$$(\widetilde{\nabla}_{\partial_u} \cdots \widetilde{\nabla}_{\partial_u} \widetilde{R})(X,Y)Z = \frac{(-1)^q (q+1)!}{u^q}\, R(X,Y)Z. \tag{4.6}$$

Moreover, a simple induction shows

$$\begin{array}{rcl} (\widetilde{\nabla}_{X_1} \cdots \widetilde{\nabla}_{X_p} \widetilde{R})(X,Y)Z &=& (\nabla_{X_1} \cdots \nabla_{X_p} R)(X,Y)Z \\ && - u \sum_{i=1}^p (\nabla_{X_1} \overset{\overset{i}{\uparrow}}{\cdots} \nabla_{X_p} R)(X,Y,Z,X_i)\partial_v, \end{array} \tag{4.7}$$

for all $X_i, Y, Z, W \in TM$ and where the symbol $\overset{\overset{i}{\uparrow}}{\cdot\,\cdot}$ indicates the omission of the ith term. In general, a straightforward computations shows

Proposition 4.1. *The $(p+q)$th derivative of $\widetilde{R}$ is determined by the relations*

$$\begin{array}{rcl} \partial_v \lrcorner \widetilde{\nabla}^k \widetilde{R} &=& 0, \\ (\widetilde{\nabla}_{\partial_u} \widetilde{\nabla}_{X_1} \cdots \widetilde{\nabla}_{X_p} \widetilde{R})(Y,Z)(W) &=& (\widetilde{\nabla}_{X_1} \widetilde{\nabla}_{\partial_u} \widetilde{\nabla}_{X_2} \cdots \widetilde{\nabla}_{X_p} \widetilde{R})(Y,Z)(W), \end{array}$$

and the formula

$$
\begin{aligned}
&(\widetilde{\nabla}_{\partial_u} \cdots \widetilde{\nabla}_{\partial_u} \widetilde{\nabla}_{X_1} \cdots \widetilde{\nabla}_{X_p} \widetilde{R})(X,Y)Z = \\
&\quad = \frac{c(p,q)}{u^q} \left((\nabla_{X_1} \cdots \nabla_{X_p} R)(X,Y)Z - u \sum_{i=1}^{p} (\nabla_{X_1} \overset{\overset{i}{\uparrow}}{\cdots} \nabla_{X_p} R)(X,Y,Z,X_i)\partial_v \right),
\end{aligned}
$$

where $c(p,0) = 1$ *and* $c(p,q) = (-1)^q (p+2) \cdot \ldots \cdot (p+q+1)$ *when* $q \geqslant 1$.

Our aim is to study the holonomy of metrics $\widetilde{g} = 2\mathrm{d}u\mathrm{d}v + u^2 g$. Since ∂_v is a parallel vector field on $(\widetilde{M}, \widetilde{g})$, the holonomy of $(\widetilde{M}, \widetilde{g})$ is contained in the stabiliser of the vector ∂_v at a point. By splitting $T\widetilde{M} = \mathbb{R}\partial_v \oplus TM \oplus \mathbb{R}\partial_u$, where $\mathrm{span}\{\partial_v, \partial_u\} = TM^\perp$, and fixing an orthonormal basis in T_pM we can identify $\mathfrak{so}(T_pM, g) \simeq \mathfrak{so}(t,s)$ and have $\mathfrak{hol}(M,g) \subset \mathfrak{so}(t,s)$. Hence, we can identify the stabiliser of ∂_v in $\mathfrak{so}(t+1, s+1)$ with $\mathfrak{so}(t+1,s+1)_{\partial_v} = \mathfrak{so}(t,s) \ltimes \mathbb{R}^{t,s}$ and we get that

$$
\mathfrak{hol}(\widetilde{M}, \widetilde{g}) \subset \mathfrak{so}(t,s) \ltimes \mathbb{R}^{t,s} = \left\{ \begin{pmatrix} 0 & g(w,.) & 0 \\ 0 & A & -w \\ 0 & 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{so}(t,s), w \in \mathbb{R}^{t,s} \right\}, \tag{4.8}
$$

where the matrices are with respect to the splitting $T\widetilde{M} = \mathbb{R}\partial_v \oplus TM \oplus \mathbb{R}\partial_u$ and the identification $T_pM = \mathbb{R}^{t,s}$. With these identifications, there are two projections

$$
\mathrm{pr}_{\mathfrak{so}(t,s)} : \mathfrak{hol}(\widetilde{M}, \widetilde{g}) \to \mathfrak{so}(t,s), \qquad \mathrm{pr}_{\mathbb{R}^{t,s}} : \mathfrak{hol}(\widetilde{M}, \widetilde{g}) \to \mathbb{R}^{t,s},
$$

to the linear part A and the translational part w in (4.8) of $\mathfrak{so}(t+1,s+1)_{\partial_v} = \mathfrak{so}(t,s) \ltimes \mathbb{R}^{t,s}$. Since derivatives of the curvature are contained in the holonomy algebra, Proposition 4.1 implies that

$$
\begin{aligned}
\mathrm{pr}_{\mathfrak{so}(t,s)} \left(\widetilde{\nabla}^q_{\partial_u} \widetilde{\nabla}_{X_1} \cdots \widetilde{\nabla}_{X_p} \widetilde{R})(Y,Z) \right) &= \tfrac{c}{u^q} (\nabla_{X_1} \cdots \nabla_{X_p} R)(Y,Z) \\
\mathrm{pr}_{\mathbb{R}^{t,s}} \left(\widetilde{\nabla}^q_{\partial_u} \widetilde{\nabla}_{X_1} \cdots \widetilde{\nabla}_{X_p} \widetilde{R})(Y,Z) \right) &= \tfrac{c}{u^{q-1}} \textstyle\sum_{i=1}^{p} (\nabla_{X_1} \overset{\overset{i}{\uparrow}}{\cdots} \nabla_{X_p} R)(Y,Z)X_i,
\end{aligned} \tag{4.9}
$$

where $X_i, Y, Z \in T_pM$ and c is a nonzero constant.

A first description of the holonomy of $(\widetilde{M}, \widetilde{g})$ was obtained in [18]. This description is the first part of the following proposition.

Proposition 4.2 ([18, Theorem 4.2]). *Let g be a semi-Riemannian metric on M with holonomy algebra $\mathfrak{hol}(g)$ and $\widetilde{g}$ the metric $\widetilde{g} = 2\,\mathrm{d}u\,\mathrm{d}v + u^2 g$ on $\mathbb{R}^+ \times \mathbb{R} \times M$. Then*

$$
\mathfrak{hol}(\widetilde{g}) \;\subset\; \mathfrak{hol}(g) \ltimes \mathbb{R}^{t,s} \;\subset\; \mathfrak{so}(t,s) \ltimes \mathbb{R}^{t,s} \;=\; \mathfrak{so}(t+1,s+1)_{\partial_v},
$$

and

$$
\mathrm{pr}_{\mathfrak{so}(t,s)}(\mathfrak{hol}(\widetilde{g})) \;=\; \mathfrak{hol}(g).
$$

Moreover, if (M,g) admits a nonzero parallel vector field X, then

$$
\mathfrak{hol}(\widetilde{g}) \;\subset\; \mathfrak{hol}(g) \ltimes X^\perp,
$$

where $X^\perp \subset T_pM$ denotes the subspace orthogonal to X_p with respect to g.

Proof. The proof of the first part of the proposition was given in [18] and uses equations (4.2) to compute explicitly the parallel transport in $(\widetilde{M}, \widetilde{g})$. Indeed, let $\widetilde{\gamma} : [t_0, t_1] \to \widetilde{M}$ be a piecewise smooth curve given by $\widetilde{\gamma}(t) = (u(t), v(t), \gamma(t))$ with a curve γ in M. Let $Y(t)$ be a parallel vector field along γ with respect to ∇ and tangential to M. Then one checks that the vector field

$$\widetilde{Y}(t) = \frac{1}{u(t)} Y(t) + f(t) \cdot \partial_v$$

is parallel with respect to $\widetilde{\nabla}$ along $\widetilde{\gamma}$, where $f(t) = \int_{t_0}^{t} g_{\gamma(s)}\left(\dot\gamma(s), Y(s)\right) \mathrm{d}s$. In particular, the parallel transport of $Z \in T_{(u(t_0),v(t_0),\gamma(t_0))}M$ along $\widetilde{\gamma}$ is given by

$$\widetilde{\mathcal{P}}_{\widetilde{\gamma}}(Z) = \tfrac{1}{u(t_1)} \mathcal{P}_\gamma(Z) + \left(\int_{t_0}^{t_1} g_{\gamma(t)}\left(\dot\gamma(t), \mathcal{P}_\gamma|_{[t_0,t]}(Z)\right) \mathrm{d}t \right) \partial_v|_{\widetilde{\gamma}(t_1)}.$$

This implies that for a loop $\widetilde{\gamma}$ starting and ending at $(u, v, p) \in \widetilde{M}$ we have that

$$\mathrm{pr}_{\mathfrak{so}(t,s)}\left(\widetilde{\mathcal{P}}_{\widetilde{\gamma}}\right) = \tfrac{1}{u}\,\mathcal{P}_\gamma,$$

which shows that $\mathrm{pr}_{\mathfrak{so}(t,s)}(\mathfrak{hol}(\widetilde{g})) = \mathfrak{hol}(g)$.

For the second part, in the case where (M, g) admits a parallel vector field X, the statement follows from the the Ambrose-Singer Holonomy Theorem and the second equation in (4.9) as $(\nabla_{X_1} \cdots \nabla_{X_p} R)(Y, Z, X_i, X) = 0$ for all $X_i \in TM$ if X is parallel. □

Note that this does *not* establish the inclusion $\mathfrak{hol}(g) \subset \mathfrak{hol}(\widetilde{g})$. Hence, for a metric of the form $\widetilde{g} = 2\mathrm{d}u\mathrm{d}v + u^2 g$ this result allows for the possibility that $\mathfrak{hol}(\widetilde{g})$ is not completely determined by $\mathfrak{hol}(g)$. Indeed, for the space of translations in $\mathfrak{hol}(\widetilde{g})$,

$$T := \mathfrak{hol}(\widetilde{g}) \cap \mathbb{R}^{t,s}$$

we have the following possibilities:

(1) $T = \mathbb{R}^{t,s}$: In this case the holonomy of $\widetilde{g}$ is completely determined by the holonomy of g and we have $\mathfrak{hol}(\widetilde{g}) = \mathfrak{hol}(g) \ltimes \mathbb{R}^{t,s}$.
(2) $T \neq \mathbb{R}^{t,s}$: In this case we can distinguish two situations:
  (a) $\mathfrak{hol}(g) \subset \mathfrak{hol}(\widetilde{g})$: In this case there is a subspace of translations $T \subsetneq \mathbb{R}^{t,s}$ such that $\mathfrak{hol}(\widetilde{g}) = \mathfrak{hol}(g) \ltimes T$.
  (b) $\mathfrak{hol}(g) \not\subset \mathfrak{hol}(\widetilde{g})$.

In both cases in (2) it seems as if $\mathfrak{hol}(g)$ does not determine $\mathfrak{hol}(\widetilde{g})$ completely and that further knowledge about the geometry of g is needed in order to decide whether (a) or (b) occur, to determine T, etc. In Sections 5 and 6 we will study these questions further, first purely algebraically and then using geometric properties of $\widetilde{g}$. But first we will give some examples.

4.2. **Locally symmetric spaces and other examples.**

4.2.1. *Locally symmetric spaces.* Here we consider manifolds $(\widetilde{M}, \widetilde{g})$ that arise via the construction (4.1) from semi-Riemannian locally symmetric spaces (M, g).

Theorem 4.3. *Let (M, g) be a semi-Riemannian locally symmetric space, i.e., a semi-Riemannian manifold with $\nabla R = 0$. For (M, g) we consider the metric $\widetilde{g} = 2\mathrm{d}u\,\mathrm{d}v + u^2 g$ on $\widetilde{M} = \mathbb{R}^+ \times \mathbb{R} \times M$. Then*

$$\mathfrak{hol}_{\widetilde{p}}(\widetilde{M}, \widetilde{g}) = \mathfrak{hol}_p(M, g) \ltimes T,$$

where $T = \mathfrak{hol}_p(M, g)V$ with $V = T_pM$ and $\widetilde{p} = (1, 0, p) \in \widetilde{M}$.

Proof. As a consequence of the Ambrose-Singer theorem and $\nabla R = 0$ we have that

$$\mathfrak{hol}_p(M,g) = \operatorname{span}\{R(X,Y)|_p \mid X,Y \in T_pM\}. \tag{4.10}$$

The curvature $\widetilde{R}$ of $(\widetilde{M},\widetilde{g})$ satisfies equation (4.3), which, together with equation (4.10), shows that $\mathfrak{g} = \mathfrak{hol}(M,g)$ is contained in $\widetilde{\mathfrak{g}} = \mathfrak{hol}_p(\widetilde{M},\widetilde{g})$. Moreover, by Proposition 4.1 we have that

$$(\widetilde{\nabla}_{\partial_u} \cdots \widetilde{\nabla}_{\partial_u} \widetilde{\nabla}_{X_1} \widetilde{R})(X,Y)Z = \frac{c}{u^{q-1}} R(X,Y,Z,X_1)\partial_v,$$

for nonzero constant c and $X,Y,Z,X_1 \in TM$, and all other derivatives of $\widetilde{R}$ are zero. This implies the claim. □

Corollary 4.4. *Let (M,g) be a semi-Riemannian locally symmetric space, which is locally the product of (non-flat) irreducible symmetric spaces. Then*

$$\mathfrak{hol}_{\widetilde{p}}(\widetilde{M},\widetilde{g}) = \mathfrak{hol}_p(M,g) \ltimes T_pM.$$

Example 4.5. The following example shows that Corollary 4.4 does not extend to indecomposable symmetric spaces such as the Cahen-Wallach space of dimension $n = m+2$,

$$(M,g) = \left(\mathbb{R}^n, g_{CW} = 2\mathrm{d}x\mathrm{d}z + \sum_{i,j=1}^m \lambda_{ij} y^i y^j \,\mathrm{d}z^2 + \sum_{i=1}^m (\mathrm{d}y^i)^2\right),$$

where $(x, y^1, \ldots, y^m, z)$ are global coordinates on $\mathbb{R}^{m+2}$ and where $S = (\lambda_{ij})$ is a constant symmetric matrix with $\det(S) \neq 0$. In this case we have $\mathfrak{hol}(M,g) = \mathbb{R}^m \subset \mathfrak{so}(1,m+1)_{\partial_x} = \mathfrak{so}(m) \ltimes \mathbb{R}^m$ and $T = \operatorname{span}(\partial_x, \partial_1 \ldots, \partial_m)$ where $\partial_i = \frac{\partial}{\partial y^i}$. We will explain these Lie algebras in more detail later on.

4.2.2. *pp-waves and plane waves.* The pp-waves are Lorentzian manifolds that are generalisations of Cahen-Wallach spaces. Again we consider $M = \mathbb{R}^n = \mathbb{R}^{m+2}$ with global coordinates $(x, y^1, \ldots, y^m, z)$ and f a function $f = f(y^1, \ldots, y^m, z)$ of $y^1, \ldots, y^m$ and z but not of x. Then a general pp-wave metric on $\mathbb{R}^{m+2}$ is given by

$$g = 2\mathrm{d}x\mathrm{d}z + 2f(y^1, \ldots, y^m, z)\mathrm{d}z^2 + \sum_{i=1}^m (\mathrm{d}y^i)^2. \tag{4.11}$$

The Levi-Civita connection and the curvature are determined by

$$\nabla \partial_x = 0, \quad \nabla_{\partial_i}\partial_j = 0, \quad \nabla_{\partial_z}\partial_i = \partial_i f \partial_x, \quad \nabla_{\partial_z}\partial_z = \partial_z f \partial_x - \sum_{i=1}^m \partial_i f\, \partial_i,$$

and

$$\partial_x \lrcorner R = 0, \quad R(\partial_i, \partial_j) = 0, \quad R(\partial_i, \partial_z, \partial_z, \partial_j) = -\partial_i \partial_j f.$$

In the basis $(\partial_x, \partial_1, \ldots, \partial_m, \partial_z - f\partial_x)$ the metric is given by

$$\eta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbf{1}_m & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and we can write the curvature and its derivatives as endomorphisms in $\mathfrak{so}(\eta)$ as

$$(\nabla_{X_1}\dots\nabla_{X_p}R)(\partial_i,\partial_z)=\begin{pmatrix}0 & (X_1\dots X_p\partial_i\partial_j(f))_{j=1}^m & 0\\ 0 & 0 & -(X_1\dots X_p\partial_i\partial_j(f))_{j=1}^m\\ 0 & 0 & 0\end{pmatrix}, \tag{4.12}$$

where the X_i are constant vector fields on $M=\mathbb{R}^n$. As for Cahen-Wallach spaces, their holonomy algebra contained in (and equal to, if the Hessian $\partial_i\partial_j f$ of f is invertible) $\mathbb{R}^m\subset\mathfrak{so}(1,m+1)_{\partial_x}$ and hence abelian.

Now we consider the semi-Riemannian manifold $(\widetilde{M},\widetilde{g})$ of signature $(2,m+2)$ for a given pp-wave (M,g) of dimension $n=m+2$. Then, by setting

$$A_{qrk_1\dots k_si}:=(\widetilde{\nabla}^q_{\partial_u}\widetilde{\nabla}^r_{\partial_z}\widetilde{\nabla}_{\partial_{k_1}}\cdots\widetilde{\nabla}_{\partial_{k_s}}\widetilde{R})(\partial_i,\partial_z),$$

equations (4.9) in this case are

(4.13)

$$\begin{aligned}\mathrm{pr}_{\mathfrak{so}(1,n-1)}\left(A_{qrk_1\dots k_si}\right) &= \tfrac{c}{u^q}\begin{pmatrix}0 & \left(\partial_{k_1}\dots\partial_{k_s}\partial_i\partial_j\ f^{(r)}\right)_{j=1}^m & 0\\ 0 & 0 & \vdots\\ 0 & 0 & 0\end{pmatrix},\\ \mathrm{pr}_{\mathbb{R}^{1,n-1}}\left(A_{qrk_1\dots k_si}\right) &= \tfrac{c}{u^{q-1}}\Big(s\partial_{k_1}\cdots\partial_{k_s}\partial_i f^{(r)}\partial_x+\sum_{j=1}^m\partial_{k_1}\cdots\partial_{k_s}\partial_i\partial_j f^{(r-1)}\partial_j\Big).\end{aligned}$$

where $f^{(r)}$ denotes the r-th derivative of f with respect to the coordinate z. This shows that $\mathfrak{hol}(\widetilde{M},\widetilde{g})\subset\mathfrak{hol}(M,g)\ltimes\partial_x^\perp$, with $\partial^\perp=\mathrm{span}(\partial_x,\partial_1,\dots,\partial_m)$, as claimed in Proposition 4.2. In general these projections could be coupled to each other, but for a special case we can say more:

Proposition 4.6. *Let (M,g) be a pp-wave as in (4.11) but with the condition that f does not depend on z, i.e., $f=f(y^1,\dots,y^n)$ and such that $\det(\partial_i\partial_j f)\neq 0$ at one point (or, more generally, such that at one point*

$$\mathrm{span}\{d(\partial_{k_1}..\partial_{k_p}f)\mid p\geqslant 1,\ k_1,\dots,k_p\in\underline{m}\}=(\mathbb{R}^m)^*, \tag{4.14}$$

where $\underline{m}=\{1,\dots,m\}$). Then

$$\mathfrak{hol}(\widetilde{M},\widetilde{g})=\mathfrak{hol}(M,g)\ltimes\partial_x^\perp.$$

Proof. We evaluate formulae (4.9) for $r=1$: since f is independent of z, we have $f'=0$ and hence

$$\mathrm{pr}_{\mathfrak{so}(1,m+1)}\Big((\widetilde{\nabla}_{\partial_z}\widetilde{\nabla}_{\partial_{k_1}}\cdots\widetilde{\nabla}_{\partial_{k_p}}\widetilde{R})(\partial_i,\partial_z)\Big)=0,$$

and

$$\mathrm{pr}_{\mathbb{R}^{1,m+1}}\Big((\widetilde{\nabla}_{\partial_z}\widetilde{\nabla}_{\partial_{k_1}}\cdots\widetilde{\nabla}_{\partial_{k_p}}\widetilde{R})(\partial_i,\partial_z)\Big)=\sum_{j=1}^m\partial_{k_1}\cdots\partial_{k_p}\partial_i\partial_j f\partial_j.$$

If $\det(\partial_i\partial_j f)\neq 0$ (or if (4.14) holds at one point), this shows that $\mathrm{span}(\partial_1\dots,\partial_m)\subset\mathfrak{hol}(\widetilde{M},\widetilde{g})\cap\mathbb{R}^{1,m+1}$. This space however is not invariant under $\mathfrak{hol}(M,g)$ and is mapped under the adjoint representation in $\mathfrak{hol}(\widetilde{M},\widetilde{g})$ to $\mathbb{R}\partial_x$, which shows that $\mathfrak{hol}(\widetilde{M},\widetilde{g})=\mathfrak{hol}(M,g)\ltimes\partial_x^\perp$. ∎

This proposition can be clearly generalised to functions f that are polynomial, say of degree d, in z (and have arbitrary dependence on the y^i). It suffices to replace $r = 1$ in the proof with $r = d+1$ and the condition on f by the corresponding condition on $f^{(d)}$. It does not hold however for general f as the following example shows:

Example 4.7. Let $f(y,z) = \mathrm{e}^z y^2$ and g a plane wave metric[2] on $\mathbb{R}^3$ defined by f,

$$g = 2\mathrm{d}x\mathrm{d}z + \mathrm{e}^z y^2 \mathrm{d}z^2 + \mathrm{d}y^2.$$

Its curvature and derivatives thereof are given by equation (4.12) as follows

$$\nabla_{\partial_y} R = 0, \qquad (\nabla^{(r)}_{\partial_z} R)(\partial_y, \partial_z) = 2\begin{pmatrix} 0 & \mathrm{e}^z & 0 \\ 0 & 0 & -\mathrm{e}^z \\ 0 & 0 & 0 \end{pmatrix} =: A(z),$$

for all $r \geqslant 0$. Its holonomy algebra is one-dimensional. When we now consider the metric $\widetilde{g}$, formula (4.13) shows that

$$\begin{aligned} (\widetilde{\nabla}^q_{\partial_u} \widetilde{\nabla}^r_{\partial_z} \widetilde{R})(\partial_y, \partial_z) &= \tfrac{c}{u^q} \begin{pmatrix} 0 & 2u\,\mathrm{e}^z dy & 0 \\ 0 & A(z) & -2u\,\mathrm{e}^z \partial_y \\ 0 & 0 & 0 \end{pmatrix}, \\ (\widetilde{\nabla}^q_{\partial_u} \widetilde{\nabla}^r_{\partial_z} \widetilde{\nabla}_{\partial_y} \widetilde{R})(\partial_y, \partial_z) &= \tfrac{2c}{u^{q-1}} \mathrm{e}^z \partial_x, \end{aligned}$$

with all other derivatives of the curvature being zero. Since $\widetilde{g}$ is analytic, its holonomy is determined by the derivatives of the curvature at a point, say at $v = x = y = z = 0$ and $u = 1$, and is spanned by the two matrices arising from $(\widetilde{\nabla}^q_{\partial_u} \widetilde{\nabla}^r_{\partial_z} \widetilde{R})(\partial_y, \partial_z)$ and $(\widetilde{\nabla}^q_{\partial_u} \widetilde{\nabla}^r_{\partial_z} \widetilde{\nabla}_{\partial_y} \widetilde{R})(\partial_y, \partial_z)$,

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This shows that the holonomy of $\widetilde{g}$ is abelian and is neither purely translational nor a semidirect sum of $\mathfrak{hol}(g)$ with a Lie algebra of translations.

4.3. **Lift of parallel objects.** In this section we analyse how parallel objects on (M, g), such as vector fields and vector distributions, lift to $(\widetilde{M}, \widetilde{g})$. First we analyse how certain vector fields on M lift to $\widetilde{M}$.

Lemma 4.8. *Let ξ be a homothetic gradient vector field on (M, g), i.e., a vector field with*

$$\nabla \xi = a \,\mathrm{Id} \tag{4.15}$$

for a constant $a \in \mathbb{R}$ and such that $\xi^\flat$ is not only closed but exact, $\xi^\flat = df$ for a smooth function f. Then the vector field $\widetilde{\xi}$ defined by

$$\widetilde{\xi} = f\,\partial_v + \frac{1}{u}\xi - a\partial_u,$$

[2] Plane waves are pp-waves for which the function f is a quadratic polynomial in the y^i's with z-dependent coefficients, i.e., $f(y^1, \dots y^m, z) = \sum_{i,j=1}^m f_{ij}(z) y^i y^j$, with $S(z) = (f_{ij}(z))$ a symmetric matrix of functions of $z \in \mathbb{R}$.

is parallel for $\widetilde{\nabla}$. In particular, if ξ is parallel for (M,g), then $\widetilde{\xi} = f\,\partial_v + \frac{1}{u}\xi$ is parallel for $(\widetilde{M},\widetilde{g})$.

Proof. First note that the condition (4.15) implies that $\xi^\flat$ is closed, i.e., locally we can always find a function f such that $\xi^\flat = df$. Then we compute

$$\widetilde{\nabla}_{\partial_u}\widetilde{\xi} = -\frac{1}{u^2}\xi + \frac{1}{u}\widetilde{\nabla}_{\partial_u}\xi = 0,$$

because of (4.2). Moreover, we have

$$\widetilde{\nabla}_X\widetilde{\xi} = df(X)\partial_v + \frac{a}{u}X - g(\xi,X)\partial_v - a\widetilde{\nabla}_X\partial_u = 0,$$

again by (4.2) and $df = \xi^\flat$. □

In a similar way we can prove:

Lemma 4.9. *Let* $\mathbf{L}$ *be a parallel null line bundle on (M,g). Then the totally null 2-plane bundle* $\mathbf{P}$ *on $(\widetilde{M},\widetilde{g})$ spanned by ∂_v and* $\mathbf{L}$ *is parallel for $\widetilde{\nabla}$.*

Proof. This follows from applying equation (4.2) to a recurrent null vector field ξ spanning $\mathbf{L}$ and ∂_v being parallel for $\widetilde{\nabla}$. □

The following proposition will be used in Section 6 for the proof of Theorem 1.3:

Proposition 4.10. *Let (M,g) be a manifold with parallel null line bundle* $\mathbf{L}$. *Assume that the metric $\widetilde{g} = 2\mathrm{d}u\mathrm{d}v + u^2 g$ admits a recurrent vector field in the span of ∂_v and* $\mathbf{L}$ *that is not a multiple of ∂_v. Then locally g admits a parallel null vector field in* $\mathbf{L}$.

Proof. By Lemma 2.8 we can assume that $\mathbf{L}$ is spanned by a recurrent *gradient* vector field $\xi = \mathrm{grad}(f)$, i.e., with $\xi^\flat = df$ and $\nabla\xi = \theta\otimes\xi$ with θ a multiple of $\xi^\flat$. Then the vector field

$$\widetilde{\xi} = f\partial_v + \frac{1}{u}\xi$$

satisfies

$$\widetilde{\nabla}_{\partial_u}\widetilde{\xi} \;=\; 0, \tag{4.16}$$

$$\widetilde{\nabla}_X\widetilde{\xi} \;=\; \frac{1}{u}\theta(X)\xi \;=\; \theta(X)(\widetilde{\xi} - f\partial_v), \qquad \text{for } X \in TM. \tag{4.17}$$

Without loss of generality the assumption implies that $\widetilde{g}$ admits a recurrent vector field of the form $\zeta = \widetilde{\xi} + h\partial_v$ for a function h defining a one-form α by $\widetilde{\nabla}\zeta = \alpha\otimes\zeta$. Then the fact that ∂_v is parallel and equation (4.16) immediately show that

$$\partial_u h = \alpha(\partial_u) = \partial_v h = \alpha(\partial_v) = 0.$$

Equation (4.17) implies that

$$\widetilde{\nabla}_X\zeta = \theta(X)\widetilde{\xi} + (dh(X) - f\theta(X))\partial_v.$$

Hence the equation $\widetilde{\nabla}\zeta = \alpha\otimes\zeta$ implies that $\alpha = \theta$ and

$$\mathrm{d}h = (f+h)\theta.$$

Differentiating this and taking into account that $\mathrm{d}f\wedge\theta = \mathrm{d}h\wedge\theta = 0$ gives

$$0 = (f+h)\mathrm{d}\theta$$

If $\mathrm{d}\theta \neq 0$ this implies $h = -f$. This contradicts the above $\mathrm{d}h = (f+h)\theta$, as it would imply that h and hence f are constant. So we must have $\mathrm{d}\theta = 0$. This however implies that one can rescale ξ to a parallel null vector field. □

Finally, for parallel distributions of (M, g) we get

Lemma 4.11. *Let $W \subset TM$ be a parallel distribution on (M, g). Then the distribution $\mathbb{R}\partial_v \oplus W \subset T\widetilde{M}$ is parallel.*

Proof. The distribution W is locally spanned by vector fields $W_1, \ldots, W_k$. Then one checks that for $\widetilde{W}_i := \partial_v + \frac{1}{u} W_i$ we have $\tilde{\nabla}_{\partial_u} \widetilde{W}_i = 0$ and

$$\tilde{\nabla}_X \widetilde{W}_i = -g(X, W_i)\partial_v + \frac{1}{u} \nabla_X W_i \in \mathbb{R}\partial_v \oplus W,$$

for all $X \in TM$. □

5. Results about indecomposable subalgebras of $\mathfrak{so}(t+1, s+1)$

In this section we will prove several algebraic results about indecomposable subalgebras of $\mathfrak{so}(t+1, s+1)$ stabilising a null line or a null vector. We will use these results in the next section when studying further the holonomy of metrics of the form $\tilde{g} = 2\mathrm{d}u\mathrm{d}v + u^2 g$.

5.1. **Indecomposable subalgebras stabilising a null vector.** In this section we will fix some notations and observe some fundamental facts about indecomposable subalgebras of $\mathfrak{so}(t+1, s+1)$ stabilising a null vector. In particular, in this section we will see why the vector space $Z^1(\mathfrak{g}, V)$ of 1-cocycles of a Lie algebra $\mathfrak{g}$ with values in a $\mathfrak{g}$-module V comes into play. Recall that

$$Z^1(\mathfrak{g}, V) := \{\varphi : \mathfrak{g}^* \otimes V \mid \varphi([X, Y]) = X\varphi(Y) - Y\varphi(X) \text{ for all } X, Y \in \mathfrak{g}\} \tag{5.1}$$

and

$$H^1(\mathfrak{g}, V) := \frac{Z^1(\mathfrak{g}, V)}{\mathrm{d}V},$$

where

$$\mathrm{d} : V \to Z^1(\mathfrak{g}, V), \quad \mathrm{d}v(X) := Xv, \quad v \in V, \quad X \in \mathfrak{g}.$$

Let $\widetilde{V}$ be a semi-Euclidean vector space of signature $(t+1, s+1)$ with metric $\tilde{g}$ and let $\mathbf{e}_\pm$ be two null vectors such that $\tilde{g}(\mathbf{e}_-, \mathbf{e}_+) = 1$. We split $\widetilde{V} = L_- \oplus V \oplus L_+$ with $L_\pm = \mathbb{R} \cdot \mathbf{e}_\pm$ and $V = (L_- \oplus L_+)^\perp$ which is equipped with the metric $g = \tilde{g}|_{V \times V}$. With respect to this splitting the stabiliser of L_- in $\mathfrak{so}(\widetilde{V})$, denoted by $\mathfrak{so}(\widetilde{V})_{L_-}$ is given as

$$\begin{aligned} \mathfrak{so}(\widetilde{V})_{L_-} &= (\mathbb{R} \oplus \mathfrak{so}(V)) \ltimes V \\ &:= \left\{ (a, X, v) := \begin{pmatrix} a & -v^\flat & 0 \\ 0 & X & v \\ 0 & 0 & -a \end{pmatrix} \mid a \in \mathbb{R}, X \in \mathfrak{so}(V), v \in V \right\}. \end{aligned}$$

The action of $\mathfrak{so}(\widetilde{V})_{L_-}$ on $\widetilde{V} = L_- \oplus V \oplus L_+ \cong \mathbb{R} \oplus V \oplus \mathbb{R}$ is given by

$$(a, X, v) \cdot \begin{pmatrix} r \\ u \\ s \end{pmatrix} = \begin{pmatrix} ar - g(v, u) \\ Xu + sv \\ -as \end{pmatrix}. \tag{5.2}$$

Furthermore we record the formula for the Lie bracket in $\mathfrak{so}(\widetilde{V})_{L_-}$:

$$[(a, X, v), (b, Y, w)] = (0, [X, Y], (X + a)w - (Y + b)v)\,. \tag{5.3}$$

The stabiliser of the *vector* $\mathbf{e}_-$ is given as $\mathfrak{so}(\widetilde{V})_{\mathbf{e}_-} = \mathfrak{so}(V) \ltimes V$, i.e., is obtained by requiring a to be zero in the above formulae. Note that, the adjoint of action of $\mathfrak{so}(\widetilde{V})_{\mathbf{e}_-} = \mathfrak{so}(V) \ltimes V$ preserves the ideal V, whereas the linear action on $\widetilde{V}$ does not preserve the subspace $V \subset \widetilde{V}$.

Furthermore note that there are natural projections pr_V and $\mathrm{pr}_{\mathfrak{so}(V)}$ on V and $\mathfrak{so}(V)$. For a subalgebra $\widetilde{\mathfrak{g}} \subset \mathfrak{so}(V) \ltimes V$ we call $\mathfrak{g} := \mathrm{pr}_{\mathfrak{so}(V)}(\widetilde{\mathfrak{g}})$ the *linear part of* $\widetilde{\mathfrak{g}}$ and $T := \widetilde{\mathfrak{g}} \cap V$ the *translations in* $\widetilde{\mathfrak{g}}$. Note that $\widetilde{\mathfrak{g}} \subset \mathfrak{g} \ltimes V$ but in general $\mathfrak{g} \not\subset \widetilde{\mathfrak{g}}$.

Proposition 5.1. *Let $\widetilde{\mathfrak{g}} \subset \mathfrak{so}(\widetilde{V})_{\mathbf{e}_-} = \mathfrak{so}(V) \ltimes V$ be a subalgebra, $\mathfrak{g}$ its linear part and T the translations in $\widetilde{\mathfrak{g}}$. Then*

(1) T is an ideal in $\widetilde{\mathfrak{g}}$.
(2) $T \subset V$ is invariant under $\mathfrak{g}$, and consequently $\mathfrak{g}$ acts on V/T.
(3) We have an inclusion of Lie algebras $\widetilde{\mathfrak{g}}/T \subset \mathfrak{g} \ltimes V/T$.
(4) There is a $\varphi \in Z^1(\mathfrak{g}, V/T)$ such that $\widetilde{\mathfrak{g}}/T = \{(X, \varphi(X)) \mid X \in \mathfrak{g}\}$.
(5) If T has a $\mathfrak{g}$-invariant complement T', then there is a $\varphi \in Z^1(\mathfrak{g}, T')$, such that

$$\widetilde{\mathfrak{g}} = \mathfrak{h}_\varphi \ltimes T, \qquad \textit{where } \mathfrak{h}_\varphi = \{(X, \varphi(X)) \in \widetilde{\mathfrak{g}} \mid X \in \mathfrak{g}\}$$

Proof. Items 1 to 3 are obvious from the definitions. For Item 4 we define $\varphi(X) = v \mod T$ if $(X, v) \in \widetilde{\mathfrak{g}}$. Since $(X, v) \in \widetilde{\mathfrak{g}}$ and $(X, w) \in \widetilde{\mathfrak{g}}$ implies that $v - w \in T$, this map is well defined. From equation (5.3) we see that φ is an element in $Z^1(\mathfrak{g}, V/T)$. Finally, Item 5 follows easily from Item 4 using the identification $V/T = T'$ as $\mathfrak{g}$-modules. □

Theorem 5.2. *Let $\widetilde{\mathfrak{g}} \subset \mathfrak{so}(\widetilde{V})_{\mathbf{e}_-} = \mathfrak{so}(V) \ltimes V$ be a subalgebra acting indecomposably on $\widetilde{V}$. Let $\mathfrak{g} \subset \mathfrak{so}(V)$ and $T \subset V$ be respectively the linear part and translational ideal of $\widetilde{\mathfrak{g}}$.*

(1) If T has a $\mathfrak{g}$-invariant complement T' and $H^1(\mathfrak{g}, V) = 0$, then, up to conjugation in $\mathfrak{so}(V) \ltimes V$, $\widetilde{\mathfrak{g}} = \mathfrak{g} \ltimes T$ and $T^\perp$ is degenerate or zero. In particular, if T is nondegenerate and $H^1(\mathfrak{g}, V) = 0$, then $T = V$.
(2) If T is degenerate such that $L = T \cap T^\perp$ is a null line (this is the case for example when T is degenerate and g Lorentzian) and if the representation of $\mathfrak{g}$ on $V/L^\perp$ satisfies that $H^1(\mathfrak{g}, V/L^\perp) = 0$, then $\mathfrak{g}$ acts trivially on L or, up conjugation in $\mathfrak{so}(V) \ltimes V$, $\widetilde{\mathfrak{g}}$ preserves L.

Proof. (1) First assume $V = T \oplus T'$ is a $\mathfrak{g}$-invariant decomposition. In virtue of Proposition 5.1, $\widetilde{\mathfrak{g}} = \mathfrak{h}_\varphi \ltimes T$, for some $\varphi \in Z^1(\mathfrak{g}, T')$. Since $Z^1(\mathfrak{g}, V) = \mathrm{d}V$ and $Z^1(\mathfrak{g}, T') \subset Z^1(\mathfrak{g}, V)$, we find a $v \in V$ such that

$$\varphi(X) = Xv,$$

for all $X \in \mathfrak{g}$. Then every element $(X, \varphi(X)) = (X, Xv) \in \mathfrak{h}_\varphi$ can be conjugated to X by a conjugation with the translation given by v, i.e., with

$$A_v = \begin{pmatrix} 1 & -v^\flat & -\frac{1}{2}g(v,v) \\ 0 & \mathbf{1} & v \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.4}$$

Indeed, for each $X \in \mathfrak{g}$ we get

$$A_v \begin{pmatrix} 0 & -(Xv)^\flat & 0 \\ 0 & X & Xv \\ 0 & 0 & 0 \end{pmatrix} A_v^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & X & Xv \\ 0 & 0 & 0 \end{pmatrix} A_{-v} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

using that $X \in \mathfrak{so}(V)$. This shows that after conjugation with a translation, we have that $\mathfrak{g} \subset \widetilde{\mathfrak{g}}$. Hence $\widetilde{\mathfrak{g}} = \mathfrak{g} \ltimes T$, where $T = \widetilde{\mathfrak{g}} \cap V$. Note that this already implies that T is nonzero, because otherwise $\widetilde{\mathfrak{g}} = \mathfrak{g} \subset \mathfrak{so}(V)$, which contradicts indecomposability. Since T is $\mathfrak{g}$ invariant, also the orthogonal complement $T^\perp$ of T in V is $\mathfrak{g}$ invariant. Then equation (5.2) shows that $T^\perp \subset \widetilde{V}$ is also invariant under the action of $T \subset \mathfrak{so}(\widetilde{V})$ on $\widetilde{V}$ and therefore $T^\perp$ is $\widetilde{\mathfrak{g}}$-invariant. Hence, by indecomposability of $\widetilde{\mathfrak{g}}$, $T^\perp$ has to be degenerate or zero.

(2) Assume that T is degenerate such that $L := T \cap T^\perp$ is a null line. By Item 2 of Proposition 5.1, L is invariant under $\mathfrak{g}$. Moreover, by Item 5 of Proposition 5.1 we have that there is a $\varphi \in Z^1(\mathfrak{g}, V/T)$ such that $\widetilde{\mathfrak{g}}/T = \{(X, \varphi(X)) \mid X \in \mathfrak{g}\} \subset \mathfrak{g} \ltimes V/T$. Hence, if $\widetilde{\varphi} : \mathfrak{g} \to V$ is a lift of φ we can write $\widetilde{\mathfrak{g}} = \mathfrak{h}_{\widetilde{\varphi}} + T$, where $\mathfrak{h}_{\widetilde{\varphi}} = \{(X, \widetilde{\varphi}(X)) \in \mathfrak{g} \ltimes V \mid X \in \mathfrak{g}\}$. Note that, since T may not have an invariant complement, in general we do not have that $\widetilde{\varphi} \in Z^1(\mathfrak{g}, V)$ and neither that $\mathfrak{h}_{\widetilde{\varphi}}$ is a subalgebra.

Let $L^\perp$ be the hyperplane in V that is orthogonal to L. It is $L \subset T \subset L^\perp$ and hence, by formula (5.2), L is annihilated by the translations T in $\widetilde{\mathfrak{g}} = \mathfrak{h}_{\widetilde{\varphi}} + T$. It remains to show that L is invariant under $\mathfrak{h}_{\widetilde{\varphi}}$, unless $\mathfrak{g}$ acts trivially on L. For this we consider the projection $\pi : V/T \twoheadrightarrow V/L^\perp$ and distinguish two cases:

Case 1: $\pi \circ \varphi : \mathfrak{g} \to V/L^\perp$ is zero. This means that the image of the lift $\widetilde{\varphi}$ is contained in $L^\perp$. This however implies that L is not only invariant under $\mathfrak{g}$ but also under $\widetilde{\mathfrak{g}} = \mathfrak{h}_{\widetilde{\varphi}} + T$. Indeed, from formula (5.2) it follows for an element $(X, \widetilde{\varphi}(X)) \in \mathfrak{h}_{\widetilde{\varphi}}$ and $\ell \in L$, that $(X, \widetilde{\varphi}(X)) \cdot \ell = X \cdot \ell - g(\widetilde{\varphi}(X), \ell)\mathrm{e}_- = X \cdot \ell \in L$, since $\widetilde{\varphi}(X) \in L^\perp$ and $\mathfrak{g}$ leaves L invariant. Hence, in this case L is $\widetilde{\mathfrak{g}}$-invariant.

Case 2: $\pi \circ \varphi : \mathfrak{g} \to V/L^\perp$ is not zero, i.e., the image of $\widetilde{\varphi}$ is not contained in $L^\perp$. In this case, similarly to (1), we try to find a conjugation with a translation that shows that L is invariant under $\mathfrak{h}_{\widetilde{\varphi}}$ (after conjugation). For $v \in V$ to be determined, we consider the associated translation A_v as in equation (5.4). Then, as in (1), for an element

$$(X, \widetilde{\varphi}(X)) = \begin{pmatrix} 0 & -(\widetilde{\varphi}(X))^\flat & 0 \\ 0 & X & \widetilde{\varphi}(X) \\ 0 & 0 & 0 \end{pmatrix}$$

we get that

$$A_v(X, \widetilde{\varphi}(X))A_v^{-1} = \begin{pmatrix} 0 & -(\widetilde{\varphi}(X) - Xv)^\flat & 0 \\ 0 & X & \widetilde{\varphi}(X) - Xv \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.5}$$

Fix $\ell \in L$ and $\hat{\ell} \in V$ such that $g(\ell, \hat{\ell}) = 1$. Then define $0 \neq \lambda \in \mathfrak{g}^*$ and $\rho \in \mathfrak{g}^*$ by $\widetilde{\varphi}(X) = \lambda(X)\hat{\ell} \mod L^\perp$ and $X\ell = -\rho(X)\ell$, for $X \in \mathfrak{g}$. This is summarised in $(X, \widetilde{\varphi}(X)) \cdot \ell = -\lambda(X)\mathrm{e}_- - \rho(X)\ell$. It also implies that $X\hat{\ell} = \rho(X)\hat{\ell} \mod L^\perp$, i.e., $\rho : \mathfrak{g} \to \mathfrak{gl}(V/L^\perp)$ is the induced representation of $\mathfrak{g}$ on $V/L^\perp$. If we assume that $\mathfrak{g}$ does not act trivially on L, ρ is not zero. The key observation now is that $H^1(\mathfrak{g}, V/L^\perp) = 0$ implies that $\lambda = c\rho$ for a constant c. Indeed, $\varphi \in Z^1(\mathfrak{g}, V/T)$ induces an element $\overline{\varphi} \in Z^1(\mathfrak{g}, V/L^\perp)$. So $H^1(\mathfrak{g}, V/L^\perp) = 0$ implies that $\overline{\varphi}(X) = X(c\,\hat{\ell} \mod L^\perp) = cX\hat{\ell} \mod L^\perp = c\rho(X)\hat{\ell} \mod L^\perp$ and thus $\widetilde{\varphi}(X) = c\rho(X)\hat{\ell} \mod L^\perp$.

Now, in equation (5.4) we set $v := c\,\hat{\ell}$. Taking into account that $g(\hat{\ell}, \ell) = 1$, formula (5.5) shows that

$$A_v(X, \widetilde{\varphi}(X))A_v^{-1} \cdot \ell = -(\lambda(X) - c\rho(X))\mathbf{e}_- - \rho(X)\ell = -\rho(X)\ell.$$

This shows that after conjugation with a translation the null line L is invariant under $\mathfrak{h}_{\widetilde{\varphi}}$ and hence under $\widetilde{\mathfrak{g}}$. □

Example 5.3. Consider $\mathfrak{g} = \mathbb{R}^n \subset \mathfrak{so}(n) \ltimes \mathbb{R}^n = \mathfrak{so}(1, n+1)_{\mathrm{e}_0}$, where $\mathrm{e}_0 \in \mathbb{R}^{1,n+1}$ is a null vector. Then for $T = \mathbb{R} \cdot \mathrm{e}_0$ one can check that $\widetilde{\mathfrak{g}} = \mathfrak{g} \ltimes T \subset \mathfrak{so}(2, n+2)_{\mathrm{e}_-}$ is indecomposable. Similarly, for $T = \operatorname{span}(\mathrm{e}_0, \dots \mathrm{e}_n)$, $\widetilde{\mathfrak{g}} = \mathfrak{g} \ltimes T$ is indecomposable. Note that the latter is the holonomy algebra of a $(\widetilde{M}, \widetilde{g})$ for a Cahen-Wallach space (M, g_{CW}) of dimension $n+2$ presented in Example 4.5.

5.2. **Indecomposable subalgebras with completely reducible linear part.** The main result of this section is the following theorem, which is a generalisation to arbitrary signature of a result in [6] for an indecomposable stabiliser in $\mathfrak{so}(1, n+1)$ of a null *vector*[3]. It gives a description of all indecomposable subalgebras $\widetilde{\mathfrak{g}} \subset \mathfrak{so}(\widetilde{V})_{\mathbf{e}_-} = \mathfrak{so}(V) \ltimes V$ with completely reducible linear part and non-degenerate translational part. We use the same conventions as in Section 5.1.

Theorem 5.4. *Let $\widetilde{\mathfrak{g}} \subset \mathfrak{so}(\widetilde{V})_{\mathbf{e}_-} = \mathfrak{so}(V) \ltimes V$ an indecomposable subalgebra which satisfies the following properties*

(1) $\mathfrak{g} = \mathrm{pr}_{\mathfrak{so}(V)}(\widetilde{\mathfrak{g}})$ acts completely reducibly on V, and

(2) the translational ideal $T = \widetilde{\mathfrak{g}} \cap V$ is non-degenerate.

Under these assumptions, let $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$ be the decomposition of $\mathfrak{g}$ into its centre and the semisimple derived Lie algebra. Then, $\mathfrak{g}$ acts trivially on $T^\perp$ and $T \neq 0$. Moreover, there is a linear map $\varphi : \mathfrak{g} \to T^\perp$ with $\varphi|_{\mathfrak{g}'} = 0$ such that after conjugation in $\mathfrak{so}(V) \ltimes V$, $\widetilde{\mathfrak{g}}$ is of the form $\widetilde{\mathfrak{g}} = \mathfrak{h}_\varphi \ltimes T$, where

$$\mathfrak{h}_\varphi = \{(X, \varphi(X)) \in \mathfrak{so}(V) \ltimes V \mid X \in \mathfrak{g}\}, \tag{5.6}$$

and the image of φ is co-null in $T^\perp$, i.e., $(\operatorname{im}\varphi)^\perp \subset T^\perp$ is totally null.

The proof of this theorem requires several lemmas which we are going to prove now.

Lemma 5.5. *Let V be a semi-Euclidean vector space and $\mathfrak{g} \subset \mathfrak{so}(V)$ be a Lie subalgebra which acts completely reducibly on V. Then there exists a canonical inclusion*

$$\iota : Z^1(\mathfrak{z}, \ker \mathfrak{g}') \longrightarrow Z^1(\mathfrak{g}, \ker \mathfrak{g}') \subset Z^1(\mathfrak{g}, V),$$

such that

$$Z^1(\mathfrak{g}, V) = \mathrm{d}V + \iota(Z^1(\mathfrak{z}, \ker \mathfrak{g}')), \quad \mathrm{d}V \cap \iota(Z^1(\mathfrak{z}, \ker \mathfrak{g}')) = \mathrm{d} \ker \mathfrak{g}',$$

where $\ker \mathfrak{g}' = \{v \in V | \mathfrak{g}' v = 0\} \subset V$ *denotes the maximal $\mathfrak{g}$-submodule on which $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ acts trivially and $\mathfrak{z}$ the center of $\mathfrak{g}$.*

[3] We should point out that in [6] a similar result for the stabiliser in $\mathfrak{so}(1, n+1)$ of a null *line* is given.

Proof. The complete reducibility of $\mathfrak{g} \subset \mathfrak{so}(V)$ implies that $\mathfrak{g}$ is reductive, that is

$$\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}',$$

where $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is the maximal semi-simple factor of $\mathfrak{g}$, see for example [16]. Let us denote by

$$\tilde{\varphi} := \varphi \circ \pi_{\mathfrak{z}} : \mathfrak{g} \longrightarrow V$$

the canonical extension of a linear map $\varphi : \mathfrak{z} \longrightarrow V$, where

$$\pi_{\mathfrak{z}} : \mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}' \longrightarrow \mathfrak{z}, \quad \pi' : \mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}' \longrightarrow \mathfrak{g}' \tag{5.7}$$

are the canonical projections.

One can easily check that $\tilde{\varphi} \in Z^1(\mathfrak{g}, V)$ if and only if $\varphi \in Z^1(\mathfrak{z}, V)$ and $\operatorname{im} \varphi \subset \ker \mathfrak{g}'$. This defines the inclusion $\iota : Z^1(\mathfrak{z}, \ker \mathfrak{g}') \longrightarrow Z^1(\mathfrak{g}, V)$. Next we check $\mathrm{d}V \cap \iota(Z^1(\mathfrak{z}, \ker \mathfrak{g}')) = \mathrm{d} \ker \mathfrak{g}'$. Let $v \in V$ such that $\mathrm{d}v \in \iota(Z^1(\mathfrak{z}, \ker \mathfrak{g}'))$ and, hence, $Xv = \mathrm{d}v(X) = 0$ for all $X \in \mathfrak{g}'$. This shows that $v \in \ker \mathfrak{g}'$.

In order to prove the remaining inclusion $Z^1(\mathfrak{g}, V) \subset \mathrm{d}V + \iota(Z^1(\mathfrak{z}, \ker \mathfrak{g}'))$, we consider for every linear map $\varphi : \mathfrak{g} \longrightarrow V$ the linear maps

$$\varphi_{\mathfrak{z}} := \varphi \circ \pi_{\mathfrak{z}} : \mathfrak{g} \longrightarrow V \quad \text{and} \quad \varphi' := \varphi \circ \pi' : \mathfrak{g} \longrightarrow V,$$

cf. (5.7), such that

$$\varphi = \varphi_{\mathfrak{z}} + \varphi'.$$

If $\varphi \in Z^1(\mathfrak{g}, V)$ then $\varphi|_{\mathfrak{g}'} \in Z^1(\mathfrak{g}', V)$ and, by the Whitehead lemma (see for example [17, p. 77]),

$$Z^1(\mathfrak{g}', V) = \mathrm{d}'V,$$

where d' denotes the differential of the Lie algebra cohomology of $\mathfrak{g}'$. This shows that there exists $v \in V$ such that $\varphi|_{\mathfrak{g}'} = \mathrm{d}'v$, which implies

$$\varphi' = \varphi \circ \pi' = \varphi|_{\mathfrak{g}'} \circ \pi' = \mathrm{d}'v \circ \pi' = \mathrm{d}v \circ \pi' = (\mathrm{d}v)'$$

and

$$\psi := \varphi - \mathrm{d}v = \varphi_{\mathfrak{z}} + (dv)' - \mathrm{d}v = (\varphi - \mathrm{d}v)_{\mathfrak{z}}.$$

So $\psi \in Z^1(\mathfrak{g}, V)$ vanishes on $\mathfrak{g}'$, which implies $\operatorname{im} \psi \subset \ker \mathfrak{g}'$. This proves that $\psi \in \iota(Z^1(\mathfrak{z}, \ker \mathfrak{g}'))$. □

In order to refine Lemma 5.5, we need to study $Z^1(\mathfrak{z}, \ker \mathfrak{g}')$. Since $\mathfrak{g}'$ is semisimple, and therefore V is completely reducible, we have a direct decomposition

$$V = \ker \mathfrak{g}' \oplus \mathfrak{g}'V$$

into $\mathfrak{g}'$ submodules. As $\mathfrak{g}' \subset \mathfrak{so}(V)$, this is an orthogonal decomposition, which implies that the subspace $\ker \mathfrak{g}' \subset V$ is non-degenerate.

Lemma 5.6. *Let U be a semi-Euclidean vector space and $\mathfrak{z} \subset \mathfrak{so}(U)$ be an Abelian Lie subalgebra which acts completely reducibly on U. Then*

$$Z^1(\mathfrak{z}, U) = \mathrm{d}U \oplus Z^1(\mathfrak{z}, \ker \mathfrak{z}), \quad Z^1(\mathfrak{z}, \ker \mathfrak{z}) = \mathfrak{z}^* \otimes \ker \mathfrak{z}.$$

Proof. It is clear that $Z^1(\mathfrak{z}, \ker \mathfrak{z}) = \mathfrak{z}^* \otimes \ker \mathfrak{z}$. We consider the decomposition

$$U = \ker \mathfrak{z} \overset{\perp}{\oplus} W, \quad W := \mathfrak{z}U.$$

Notice that $\mathrm{d}U = \mathrm{d}W \subset Z^1(\mathfrak{z}, W)$ and $Z^1(\mathfrak{z}, U) = Z^1(\mathfrak{z}, W) \oplus Z^1(\mathfrak{z}, \ker \mathfrak{z})$. Therefore it suffices to show that $Z^1(\mathfrak{z}, W) = \mathrm{d}W$. The $\mathfrak{z}$-module W is an orthogonal sum of 2-dimensional indecomposable modules W_i and $Z^1(\mathfrak{z}, W) = \oplus_i Z^1(\mathfrak{z}, W_i)$. Therefore we can assume without loss of generality that $W = W_1$ is 2-dimensional. Let us denote by I a generator of the 1-dimensional Lie algebra $\mathfrak{so}(W)$ such that $I^2 = \epsilon \mathrm{Id}$, $\epsilon = \pm 1$. Then there exists $0 \neq \lambda \in \mathfrak{z}^*$ such that $Xv = \lambda(X) Iv$ for all $X \in \mathfrak{z}$ and $v \in W$. Given $\varphi \in Z^1(\mathfrak{z}, W)$, we have

$$0 = X\varphi(Y) - Y\varphi(X) = \lambda(X) I\varphi(Y) - \lambda(Y) I\varphi(X),$$

for all $X, Y \in \mathfrak{z}$. The latter equation implies that there exists a vector $v \in W$ such that

$$I\varphi(X) = \lambda(X)v,$$

for all $X \in \mathfrak{z}$. This shows that $\varphi = \epsilon\lambda \otimes Iv = \epsilon \mathrm{d}v \in \mathrm{d}W$ and finishes the proof of the lemma. □

Lemma 5.7. *Let V be a semi-Euclidean vector space and $\mathfrak{g} \subset \mathfrak{so}(V)$ be a Lie subalgebra which acts completely reducibly on V. Then*

$$Z^1(\mathfrak{g}, V) = \mathrm{d}V \oplus \iota(Z^1(\mathfrak{z}, \ker \mathfrak{g})),$$

where $\ker \mathfrak{g} = \{v \in V | \mathfrak{g}v = 0\} \subset V$ *denotes the maximal $\mathfrak{g}$-submodule on which $\mathfrak{g}$ acts trivially and $\mathfrak{z}$ the center of $\mathfrak{g}$.*

Proof. Since the $\mathfrak{g}$-submodule $U = \ker \mathfrak{g}' \subset V$ is a sum of irreducible $\mathfrak{g}$-submodules, the center $\mathfrak{z}$ acts on U completely reducibly. Applying Lemma 5.6 to $U = \ker \mathfrak{g}'$ (occurring in Lemma 5.5) we obtain that

$$\iota(Z^1(\mathfrak{z}, U)) = \iota(\mathrm{d}_{\mathfrak{z}} U) \oplus \iota(Z^1(\mathfrak{z}, \ker \mathfrak{g})) = \mathrm{d}U \oplus \iota(Z^1(\mathfrak{z}, \ker \mathfrak{g})),$$

where $d_{\mathfrak{z}}$ denotes the differential for the Lie algebra cohomology of $\mathfrak{z}$. Next we claim

$$\mathrm{d}V \cap Z^1(\mathfrak{g}, \ker \mathfrak{g}) = 0,$$

which implies the lemma by Lemma 5.5. To prove the claim let $v \in V$ be a vector such that $dv \in Z^1(\mathfrak{g}, \ker \mathfrak{g})$. Then $Xv = \mathrm{d}v(X) \in \ker \mathfrak{g}$ for all $X \in \mathfrak{g}$ and the $\mathfrak{g}$-invariant decomposition

$$V = \ker \mathfrak{g} \oplus \mathfrak{g}V,$$

where $\mathfrak{g}V$ is a sum of irreducible $\mathfrak{g}$-modules, shows that $v \in \ker \mathfrak{g}$ and, hence, $dv = 0$. □

Now we are in a position to prove Theorem 5.4:

Proof of Theorem 5.4. From Proposition 5.1 we have that $\widetilde{\mathfrak{g}} = \mathfrak{h}_\varphi \ltimes T$, where $\mathfrak{h}_\varphi$ is given by equation (5.6) with $\varphi \in Z^1(\mathfrak{g}, T^\perp)$. It remains to verify that $\varphi|_{\mathfrak{g}'} = 0$. Lemma 5.7 shows that, up to conjugation of $\widetilde{\mathfrak{g}}$ in $\mathfrak{so}(V) \ltimes V$ by a translation in $T^\perp$, $\varphi \in \iota(Z^1(\mathfrak{z}, T^\perp \cap \ker \mathfrak{g}))$. This shows that φ vanishes on $\mathfrak{g}'$ and takes values in $T^\perp \cap \ker \mathfrak{g}$. The $\mathfrak{g}$-invariant decomposition

$$T^\perp = (T^\perp \cap \ker \mathfrak{g}) \overset{\perp}{\oplus} \mathfrak{g}T^\perp$$

shows that the subspace $\mathfrak{g}T^\perp \subset V$ is non-degenerate. Let us check that it is not only invariant under $\mathfrak{g}$ but also under $\widetilde{\mathfrak{g}}$. For this is suffices to observe that, by our description of $\widetilde{\mathfrak{g}}$ and the fact that $\operatorname{im} \varphi \subset T^\perp \cap \ker \mathfrak{g}$, the translational part of any element of $\widetilde{\mathfrak{g}}$ is contained in $(T^\perp \cap \ker \mathfrak{g}) \oplus T$. Therefore it is perpendicular to $\mathfrak{g}T^\perp$, which shows that $\mathfrak{g}T^\perp \subset V \subset \widetilde{V}$ is $\widetilde{\mathfrak{g}}$-invariant. Since $\widetilde{\mathfrak{g}}$ is indecomposable this proves that $\mathfrak{g}T^\perp = 0$.

Note that this implies that $T \neq 0$, because otherwise $T^\perp = V$ and hence $\mathfrak{g} = 0$ and $\widetilde{\mathfrak{g}} = T = 0$, which contradicts the indecomposability of $\widetilde{\mathfrak{g}}$.

Finally, let $(\operatorname{im}\varphi)^\perp$ be the orthogonal space of $\operatorname{im}\varphi$ in $T^\perp$ and W be a $\mathfrak{g}$-invariant complement of $\operatorname{im}\varphi \cap (\operatorname{im}\varphi)^\perp$ in $(\operatorname{im}\varphi)^\perp$. Then W is non-degenerate. Again it is not only $\mathfrak{g}$-invariant but also $\widetilde{\mathfrak{g}}$-invariant because the translational part of any element in $\widetilde{\mathfrak{g}}$ is contained in $(\operatorname{im}\varphi) \oplus T$ and $W \subset (\operatorname{im}\varphi)^\perp \subset T^\perp$. Since $\widetilde{\mathfrak{g}}$ is indecomposable this shows that $W = 0$ and, hence, that $(\operatorname{im}\varphi)^\perp \subset \operatorname{im}\varphi$. □

5.3. **Cocycles for indecomposable subalgebras in $\mathfrak{so}(1, n+1)$.** In this section we compute the 1-cocycles for subalgebras $\mathfrak{g}$ of $\mathfrak{so}(1, n+1)$ that act indecomposably on $V = \mathbb{R}^{1,n+1}$. Such a subalgebra is either irreducible, in which case it is equal to $\mathfrak{so}(1, n+1)$ [10] and hence $H^1(\mathfrak{g}, V) = 0$, or admits a parallel null-line $L = L_- = \mathbb{R}\mathrm{e}_-$. That such a subalgebra belongs to one of the four types discussed in the proof of Theorem 5.8 below, was proven in [6].

In the following we will use equations (5.2) and (5.3) and the identifications in Section 5.1 with $(\widetilde{V}, \widetilde{g}, V, g)$ replaced by (V, g, V_0, g_0). Note that $g_0 = g|_{V_0 \times V_0}$ is the standard Euclidean scalar product on $V_0 = \mathbb{R}^n$. We will use the standard decomposition $V = \mathbb{R}\cdot\mathrm{e}_- \oplus V_0 \oplus \mathbb{R}\cdot\mathrm{e}_+$ and the notation $\mathfrak{g}_0 = \mathrm{pr}_{\mathfrak{so}(V_0)}(\mathfrak{g})$, $\mathfrak{g}_0' = [\mathfrak{g}_0, \mathfrak{g}_0]$, $\mathfrak{z} = \mathfrak{z}(\mathfrak{g}_0)$ for a subalgebra $\mathfrak{g} \subset \mathfrak{so}(V)_L$.

Theorem 5.8. *Let $V = \mathbb{R}\cdot\mathrm{e}_- \oplus V_0 \oplus \mathbb{R}\cdot\mathrm{e}_+$ be the Minkowski space with null vectors $\mathrm{e}_\pm$ and Euclidean vector space V_0, and let $\mathfrak{g} \subset \mathfrak{so}(V)_L \subset \mathfrak{so}(V)$ be an indecomposable subalgebra. Then*

$$H^1(\mathfrak{g}, V) = 0,$$

or $\mathfrak{g}$ leaves $\mathbf{e}_-$ invariant.

Proof. We will use the following lemma, which does not yet require the classification of indecomposable Lorentzian Lie algebras.

Lemma 5.9. *For $\mathfrak{g}$ as in the theorem we set $\mathfrak{k} = \mathfrak{so}(V_0) \cap \mathfrak{g}$ and by $T_0 = \mathfrak{g} \cap V_0$. We write $\varphi \in Z^1(\mathfrak{g}, V)$ as $\varphi = (\varphi^-, \varphi^0, \varphi^+)$ according to the decomposition $V = \mathbb{R}\cdot\mathrm{e}_- \oplus V_0 \oplus \mathbb{R}\cdot\mathrm{e}_+$. Then we have the following:*

(i) $\varphi^-|_{[\mathfrak{k},\mathfrak{k}]} = 0$,
(ii) *if $T_0 \neq 0$, then $\varphi^+|_{\mathfrak{k}} = 0$, and if $\dim T_0 \geqslant 2$, $\varphi^+|_{T_0} = 0$,*
(iii) *the endomorphism $S := \mathrm{pr}_{T_0} \circ \varphi^0|_{T_0} : T_0 \to T_0$ is symmetric and commutes with $\mathfrak{k}$, $\varphi^0|_{T_0} : T_0 \to V_0$ is $\mathfrak{k}$-equivariant and*
(iv) *there exists $u \in V_0$ such that*

$$\varphi^0|_{\mathfrak{k}} = \mathrm{d}u \in \mathrm{d}V_0 \subset Z^1(\mathfrak{k}, V_0), \quad \varphi^-|_{\mathfrak{k}T_0} = -g(u, \cdot).$$

Proof. The cocycle condition (5.1) for φ becomes

$$\begin{aligned}
(5.8)\quad \varphi^-(0, [X, \hat{X}], (X+a)\hat{v} - (\hat{X}+\hat{a})v) &= a\varphi^-(\hat{a}, \hat{X}, \hat{v}) - \hat{a}\varphi^-(a, X, v) \\
&\quad - g(v, \varphi^0(\hat{a}, \hat{X}, \hat{v})) + g(\hat{v}, \varphi^0(a, X, v)), \\
(5.9)\quad \varphi^0(0, [X, \hat{X}], (X+a)\hat{v} - (\hat{X}+\hat{a})v) &= X\varphi^0(\hat{a}, \hat{X}, \hat{v}) - \hat{X}\varphi^0(a, X, v) \\
&\quad + \varphi^+(\hat{a}, \hat{X}, \hat{v})v - \varphi^+(a, X, v)\hat{v}, \\
(5.10)\quad \varphi^+(0, [X, \hat{X}], (X+a)\hat{v} - (\hat{X}+\hat{a})v) &= -a\varphi^+(\hat{a}, \hat{X}, \hat{v}) + \hat{a}\varphi^+(a, X, v),
\end{aligned}$$

where $(a, X, v), (\hat{a}, \hat{X}, \hat{v}) \in \mathfrak{g} \subset \mathfrak{so}(V)_L \cong \mathbb{R} \oplus \mathfrak{so}(V_0) \ltimes V_0$. The equations (5.8), (5.9), and (5.10) evaluated only on elements from $\mathfrak{k}$ yield

$$\varphi^{\pm}|_{[\mathfrak{k},\mathfrak{k}]} = 0 \tag{5.11}$$

as well as

$$\varphi^0|_{\mathfrak{k}} \in Z^1(\mathfrak{k}, V_0). \tag{5.12}$$

Since $\mathfrak{k}$ is reductive with possible centre $\mathfrak{z}$, by Lemma 5.7 in Section 5.2 we know that

$$\varphi^0|_{\mathfrak{k}} = \mathrm{d}u + f, \tag{5.13}$$

where $u \in V_0$ and $f \in \mathfrak{z}^* \otimes \ker(\mathfrak{k})$ and $\ker(\mathfrak{k}) \subset V_0$ is the maximal subspace on which $\mathfrak{k}$ acts trivially.

Next evaluating (5.8) and (5.9) on elements of T_0 we obtain the equations

$$\begin{aligned} 0 &= g(v, \varphi^0(0,0,\hat{v})) - g(\hat{v}, \varphi^0(0,0,v)), && (5.14)\\ 0 &= \varphi^+(0,0,\hat{v})v - \varphi^+(0,0,v)\hat{v}, && (5.15)\end{aligned}$$

for all $v, \hat{v} \in T_0$. The first one is equivalent to the symmetry of the endomorphism S of T_0 defined in (iii). The second implies that

$$\varphi^+|_{T_0} = 0, \tag{5.16}$$

whenever $\dim(T_0) \geqslant 2$. Finally, we pair $\mathfrak{k}$ with T_0 and obtain

$$\varphi^-(0,0,Xv) \quad = \quad g(v, \varphi^0(0,X,0)) \tag{5.17}$$

and

$$0 = \varphi^0(0,0,Xv) - X\varphi^0(0,0,v) + \varphi^+(0,X,0)v. \tag{5.18}$$

Taking the T_0-component of this equation, we get

$$0 = [S, X]v + \varphi^+(0,X,0)v,$$

which is nothing else than

$$[S, X]|_{T_0} = -\varphi^+(0,X,0)\,\mathrm{Id}_{T_0}.$$

Taking the trace shows that

$$\varphi^+|_{\mathfrak{k}} = 0 \tag{5.19}$$

if $T_0 \neq 0$ and that S commutes with $\mathfrak{k}$.

Taking the $T_0^{\perp}$ component of (5.18), we get

$$\mathrm{pr}_{T_0^{\perp}}(\varphi^0(0,0,Xv)) = X\mathrm{pr}_{T_0^{\perp}}(\varphi^0(0,0,v)), \tag{5.20}$$

proving (iii). In particular, $\varphi^0 : \mathfrak{g} \to V_0$ takes values in $\mathfrak{k}V_0 \subset V_0 = \mathfrak{k}V_0 \oplus \ker \mathfrak{k}$ and, hence, $f = 0$ in equation (5.13). Thus

$$\varphi^0|_{\mathfrak{k}} = \mathrm{d}u, \quad u \in V_0, \tag{5.21}$$

and from (5.17) we obtain

$$\varphi^-(0,0,Xv) = g(v, \varphi^0(0,X,0)) = -g(Xv, u) \tag{5.22}$$

for all $v \in T_0$, $X \in \mathfrak{k}$. This proves (iv). $\square$

According to [6], an indecomposable subalgebra of $\mathfrak{so}(V)_L$, which does not annihilate e_- belongs to one of the following two types.

Type $\mathfrak{g} = (\mathbb{R} \oplus \mathfrak{g}_0) \ltimes V_0$: For this type we have $\mathfrak{k} = \mathfrak{g}_0$ and $T_0 = V_0$. First note that if $V_0 = 0$, then

$$\mathfrak{g} = \mathfrak{so}(1,1) = \mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $H^1(\mathfrak{g}, V)$ clearly is trivial. Otherwise, by Lemma 5.9 we have $\varphi^+|_{\mathfrak{g}_0} = 0$ and there exists $u \in V_0$ such that

$$\varphi^0|_{\mathfrak{g}_0} = \mathrm{d}u, \quad \varphi^-|_{\mathfrak{g}_0 V_0} = -g(u, \cdot). \tag{5.23}$$

Next we further evaluate the equations (5.8), (5.9), and (5.10) by pairing $\mathbb{R}$ with $\mathfrak{g}_0$ and V_0. Pairing V_0 and $\mathbb{R}$ in equation (5.10) shows that

$$\varphi^+|_{V_0} = 0,$$

independent of the dimension of V_0. Pairing $\mathbb{R}$ with $\mathfrak{g}_0$, equation (5.8) becomes

$$\varphi^-|_{\mathfrak{g}_0} = 0.$$

Pairing $\mathbb{R}$ with V_0, equation (5.8) and the linearity of φ^- imply

$$\varphi^0|_{\mathbb{R}} = 0.$$

Moreover, (5.9) gives

$$\varphi^0|_{V_0} = s\ \mathrm{Id}, \qquad \text{with } s = -\varphi^+(1,0,0).$$

Now, if we write $\varphi^-(0,0,v) = -g(u', v)$, with $u' \in V_0$, and split $V_0 = \ker(\mathfrak{g}_0) \oplus \mathfrak{g}_0 V_0$ then equation (5.23) shows that the $\mathfrak{g}_0 V_0$-component of u and u' are the same, so up to adding an element of $\ker \mathfrak{g}_0$ to u, we can assume that $u = u'$. Hence, with $r := \varphi^-(1,0,0)$, we obtain that

$$\varphi(a, X, v) = \begin{pmatrix} ar - g(v,u) \\ Xu + sv \\ -sa \end{pmatrix} = \begin{pmatrix} a & -v^\flat & 0 \\ 0 & X & v \\ 0 & 0 & -a \end{pmatrix}\begin{pmatrix} r \\ u \\ s \end{pmatrix} = d\begin{pmatrix} r \\ u \\ s \end{pmatrix} \cdot (a, X, v),$$

i.e., that $Z^1(\mathfrak{g}, V) = \mathrm{d}V$.

Type $\mathfrak{g} = (\mathfrak{h}_f \oplus \mathfrak{g}_0') \ltimes V_0$, with $0 \neq f \in \mathfrak{z}^*$ and $\mathfrak{h}_f = \{(f(Z), Z) \mid Z \in \mathfrak{z}\} \subset \mathbb{R} \oplus \mathfrak{z}$: Here we have $T_0 = V_0$, $\mathfrak{k} = \ker f \oplus \mathfrak{g}_0' \subset \mathfrak{g}_0 = \mathfrak{z} \oplus \mathfrak{g}_0'$ and

$$\mathfrak{g} = (\mathbb{R}\zeta_0 \oplus \mathfrak{k}) \ltimes V_0,$$

where $\zeta_0 = (1, Z_0)$ and $Z_0 \in \mathfrak{z}$ is a vector such that $f(Z_0) = 1$. In particular $\dim(V_0) \geqslant 2$. Thus by Lemma 5.9, in particular, we have $\varphi^+|_{\mathfrak{k} \ltimes V_0} = 0$, $S = \varphi^0|_{V_0} : V_0 \to V_0$ is symmetric and there exists $u \in V_0$ such that

$$\varphi^0|_{\mathfrak{k}} = \mathrm{d}u, \quad \varphi^-|_{\mathfrak{k}V_0} = -g(u, \cdot). \tag{5.24}$$

Notice that u is determined only up to addition of an element of $\ker \mathfrak{k}$, a freedom to be used below. The one-form $\varphi^-|_{V_0}$ is given by

$$\varphi^-|_{V_0} = -g(u_1, \cdot), \tag{5.25}$$

where $u_1 \in V_0$ satisfies $\mathfrak{k}(u - u_1) = 0$.

Next we further evaluate the equations (5.8), (5.9), and (5.10) by pairing ζ_0 with $\mathfrak{g}_0$ and V_0. First, pairing ζ_0 with V_0, eq. (5.8) and the linearity of φ^- gives

$$\varphi^-(0,0,Z_0v) = g(v,\varphi^0(\zeta_0)).$$

This implies that the vector $\varphi^0(\zeta_0)$ can be written as $\varphi^0(\zeta_0) = Z_0 w$ for some $w \in V_0$.

Moreover eq. (5.9) implies

$$[S,Z_0] = -S - \varphi^+(\zeta_0)\,\mathrm{Id}. \tag{5.26}$$

Taking the trace yields

$$s := \varphi^+(\zeta_0) = -\frac{\mathrm{tr}(S)}{n}.$$

Taking the trace-free part of equation (5.26) we see that the trace-free part S_0 of S satisfies

$$[S_0,Z_0] = -S_0.$$

If K is the trace form on matrices, we have

$$0 = K([Z_0,S_0],S_0) = K(S_0,S_0),$$

which shows $S_0 = 0$, since the trace-form is positive definite on symmetric matrices. This shows that

$$S = -s\,\mathrm{Id}_{V_0}.$$

We set $r := \varphi^-(\zeta_0)$. Pairing ζ_0 with $\mathfrak{k}$ in (5.8), (5.9) yields respectively $\varphi^-|_{\mathfrak{k}} = 0$ and $w_0 := Z_0(u-w) \in \ker\mathfrak{k}$. Now we redefine u by $u' := u - \mathrm{pr}_{\ker\mathfrak{k}}(u-w)$. Using the decomposition $u - w = \mathrm{pr}_{\ker\mathfrak{k}}(u-w) + \mathrm{pr}_{\mathfrak{k}V_0}(u-w) \in \ker\mathfrak{k} \oplus \mathfrak{k}V_0$, we see that

$$w_0 = Z_0\mathrm{pr}_{\ker\mathfrak{k}}(u-w),$$

which implies $Z_0u' = Z_0u - Z_0(u-w) = Z_0w$. This shows that we can assume that $w_0 = 0$ and denote u' again by u.

Altogether this implies for $Z \in \mathbb{R}Z_0$, $X \in \mathfrak{k}$ and $v \in V_0$

$$\begin{aligned}
\varphi^-(f(Z),Z+X,v) &= rf(Z) - g(v,u_1),\\
\varphi^0(f(Z),Z+X,v) &= (Z+X)u_1 - sv,\\
\varphi^+(f(Z),X+Z,v) &= sf(Z),
\end{aligned}$$

compare equation (5.25). Hence we obtain for all $(a,X,v) \in \mathfrak{g} \subset (\mathbb{R}\oplus\mathfrak{so}(V_0)) \ltimes V_0$:

$$\varphi(a,X,v) = \begin{pmatrix} a & -v^\flat & 0\\ 0 & X & v\\ 0 & 0 & -a\end{pmatrix}\begin{pmatrix} r\\ u_1\\ -s\end{pmatrix},$$

i.e. $\varphi = d(r,u_1,-s)$ and the first cohomology is trivial. □

Remark 5.10. We want to remark that Lemma 5.9 can be used to determine $H^1(\mathfrak{g},V)$ for the other two types of indecomposable subalgebras of $\mathfrak{so}(V)_L$, those that leave invariant the null vector e_- (notations as in Theorem 5.8, for details about these subalgebras see [6]). One of them is of the form $\mathfrak{g} = \mathfrak{g}_0 \ltimes V_0$ and one can show that

$$H^1(\mathfrak{g},V) = S_0(\mathfrak{g}_0) \oplus \mathfrak{z}(\mathfrak{g}_0)^* \oplus \ker(\mathfrak{g}_0)^*,$$

where $S(\mathfrak{g}_0)$ denotes the trace-free, symmetric matrices that commute with $\mathfrak{g}_0$, $\mathfrak{z}(\mathfrak{g}_0)$ is the centre of $\mathfrak{g}_0$ and $\ker(\mathfrak{g}_0) \subset V_0$ are all vectors in V_0 annihilated by $\mathfrak{g}_0$.

A similar statement holds for the remaining type where $\mathfrak{g} = (\mathfrak{h}_f \oplus \mathfrak{g}_0') \ltimes T_0$, with $0 \neq T_0 \subsetneq V_0$ invariant under $\mathfrak{g}_0$ such that $T_0^\perp \subset \ker(\mathfrak{g}_0)^\perp$ and $\mathfrak{h}_f = \{(0, Z, f(Z)) \mid Z \in \mathfrak{z}$, with $f : \mathfrak{z} \to T_0^\perp$ surjective.

Finally we study the two types of indecomposable subalgebras of $\mathfrak{so}(1, n+1)$ that stabilise the null line L but act non trivially on L, i.e., the types considered in the previous theorem.

Proposition 5.11. *Let $V = \mathbb{R}\cdot e_- \oplus V_0 \oplus \mathbb{R}\cdot e_+$ be the Minkowski space with null vectors $e_\pm$, and let $\mathfrak{g} \subset \mathfrak{so}(V)_L \subset \mathfrak{so}(V)$ be an indecomposable subalgebra stabilising a null line $L = \mathbb{R}e_-$ but acting non trivially on L. Let $\rho \in \mathfrak{g}^*$ be defined by the representation of $\mathfrak{g}$ on $V/L^\perp$, i.e.,*

$$(a, X, v)[u] = \rho(a, X, v)[u], \quad i.e., \ \rho(a, X, v) = -a,$$

(according to formula (5.2)). Then, every $\varphi \in Z^1(\mathfrak{g}, V/L^\perp) \subset \mathfrak{g}^*$ is a multiple of ρ, or equivalently, $Z^1(\mathfrak{g}, V/L^\perp) = d(V/L^\perp))$.

Proof. First we consider the type $\mathfrak{g} = (\mathbb{R} \oplus \mathfrak{g}_0) \ltimes V_0$. Note that we not exclude the case $V_0 = 0$, for which $\mathfrak{g} = \mathfrak{so}(1, 1)$. For $a \neq 0$, every $\varphi \in Z^1(\mathfrak{g}, V/L^\perp)$ satisfies

$$0 = \varphi([(a, 0, 0), (0, X, 0)]) = -a\varphi(0, X, 0),$$

for all $X \in \mathfrak{g}_0$. Hence $\varphi|_{\mathfrak{g}_0} = 0$. Similarly, we get

$$a\varphi(0, 0, v) = \varphi([(a, 0, 0), (0, 0, v)]) = -a\varphi(0, 0, v),$$

for all $v \in \mathbb{R}^n$. Hence $\varphi|_{V_0} = 0$. This implies that φ is a multiple of ρ.

Now we assume that $\mathfrak{g} = (\mathbb{R}\zeta_0 \oplus \mathfrak{k}) \ltimes V_0$, where $\mathfrak{k} = \ker f \oplus \mathfrak{g}_0' \subset \mathfrak{z} \oplus \mathfrak{g}_0' = \mathfrak{g}_0 = \mathrm{pr}_{\mathfrak{so}(n)}\mathfrak{g}$, $f \in \mathfrak{z}^*$, $\zeta_0 = (1, Z_0)$ and $Z_0 \in \mathfrak{z}$ is a vector in the centre $\mathfrak{z}$ of $\mathfrak{g}_0$ such that $f(Z_0) = 1$. In particular, $\dim(V_0) \geqslant 2$. For $X \in \mathfrak{k}$ we obtain

$$0 = \varphi([\zeta_0, (0, X, 0)]) = -\varphi(0, X, 0),$$

i.e., $\varphi|_{\mathfrak{k}} = 0$. Moreover, for all $v \in \mathbb{R}^n$ from the cocycle condition we get

$$\begin{aligned} -\varphi(0, 0, v) &= \varphi([\zeta_0, (0, 0, v)]) \\ &= \varphi(0, 0, (1 + Z_0)v) \ = \ \varphi(0, 0, v) + \varphi(0, 0, Z_0 v), \end{aligned}$$

i.e., that

$$\varphi(0, 0, Z_0 v) = -2\varphi(0, 0, v). \tag{5.27}$$

Applying equation (5.27) twice one obtains

$$\varphi(0, 0, Z_0^2 v) = -2\varphi(0, 0, Z_0 v) = 4\varphi(0, 0, v).$$

Since $Z_0 \in \mathfrak{so}(n)$, its square Z_0^2 is diagonalisable with only nonpositive eigenvalues. Hence we get that $\varphi|_{V_0} = 0$. This implies that φ is a multiple of ρ. $\square$

6. Holonomy of metrics $\widetilde{g} = 2\,\mathrm{d}u\,\mathrm{d}v + u^2 g$

In this section we will use the geometric lifting properties of metrics of the form $\widetilde{g} = 2\,\mathrm{d}u\,\mathrm{d}v + u^2 g$ derived in Section 4 and the algebraic results of Section 5 in order study the holonomy of $\widetilde{g}$. For cones over manifolds (M, g) of arbitrary signature but with completely reducible holonomy, Theorem 5.4 has the following consequences.

Corollary 6.1. *Let g be a semi-Riemannian metric of signature (t,s) on a manifold M the holonomy algebra $\mathfrak{hol}(g)$ of which acts completely reducibly. Consider the metric*

$$\widetilde{g} = 2\,\mathrm{d}u\,\mathrm{d}v + u^2 g$$

on $\widetilde{M} = \mathbb{R}^+ \times \mathbb{R} \times M$ and assume that the holonomy $\widetilde{\mathfrak{g}} := \mathfrak{hol}(\widetilde{g})$ of $\widetilde{g}$ acts indecomposably, i.e. without a proper non-degenerate invariant subspace, and that the translational ideal $T := \widetilde{\mathfrak{g}} \cap V$ is non-degenerate. Then

$$\mathfrak{hol}(\widetilde{g}) = \mathfrak{hol}(g) \ltimes V.$$

Proof. First Proposition 4.2 gives that $\mathfrak{g} = \mathrm{pr}_{\mathfrak{so}(t,s)}(\widetilde{\mathfrak{g}}) = \mathfrak{hol}(g)$. Then Theorem 5.4 applied to $\widetilde{\mathfrak{g}}$ shows that $\mathfrak{g}T^\perp = 0$. If $T^\perp \neq \{0\}$, then g admits a non-degenerate parallel vector field which, according to Lemma 4.8, would lift to a non-degenerate parallel vector field for $\widetilde{g}$. This is excluded by the assumption of indecomposability of $\widetilde{g}$. □

As an aside, let us record the consequence of Theorem 5.4 for Lorentzian metrics of the form $\widetilde{g} = 2\mathrm{d}u\mathrm{d}v + u^2 g$. We have obtained this result in [1, Section 9].

Corollary 6.2. *Let g be a Riemannian metric in dimension n and $\widetilde{g} = 2\,\mathrm{d}u\,\mathrm{d}v + u^2 g$ a Lorentzian metric. If the holonomy of $\widetilde{g}$ acts indecomposably, then*

$$\mathfrak{hol}(\widetilde{g}) = \mathfrak{hol}(g) \ltimes \mathbb{R}^n.$$

In the main result of this section we deal with metrics $\widetilde{g}$ over *Lorentzian* metrics g.

Theorem 6.3. *Let g be a Lorentzian metric in dimension n and $\widetilde{g} = 2\,\mathrm{d}u\,\mathrm{d}v + u^2 g$ of signature $(2,n)$. If the holonomy of $\widetilde{g}$ acts indecomposably, then*

$$\mathfrak{hol}(\widetilde{g}) = \mathfrak{hol}(g) \ltimes \mathbb{R}^{1,n-1},$$

or g admits a parallel null vector field (in which case $\widetilde{g}$ admits two linearly independent parallel null vector fields that are orthogonal to each other).

Proof. Set $\widetilde{\mathfrak{g}} := \mathfrak{hol}(\widetilde{g})$, $\mathfrak{g} := \mathfrak{hol}(g)$ and $V := \mathbb{R}^{1,n-1}$. Let $T = \widetilde{\mathfrak{g}} \cap V$ be the pure translations in $\widetilde{\mathfrak{g}}$. We have to show that $T = V$, in which case we have that $\widetilde{\mathfrak{g}} = \mathfrak{g} \ltimes V$, or that $\mathfrak{g}$ admits an invariant null vector. Hence we assume from now on that $T \neq V$. By Proposition 4.2 we have that $\widetilde{\mathfrak{g}} \subset \mathfrak{g} \ltimes V$ with $\mathfrak{g} = \mathrm{pr}_{\mathfrak{so}(1,n+1)}(\widetilde{\mathfrak{g}})$ and T is $\mathfrak{g}$ invariant.

Since $\mathfrak{g}$ is a holonomy algebra, we can apply the Wu splitting theorem and obtain $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$ and

$$V = \mathbb{R}^{1,n-1} = V_0 \oplus^\perp V_1 \oplus^\perp V_2 \oplus^\perp \ldots \oplus^\perp V_k,$$

with $\mathfrak{g}_i$ acting trivially on $\mathfrak{g}_j$ for $i \neq j$, all the V_i's are non-degenerate, with V_0 a trivial representation and V_i indecomposable for $i = 1, \ldots, k$. Since we assume that $\widetilde{\mathfrak{g}}$ acts indecomposably, $\widetilde{g}$ does not admit non-degenerate parallel vector fields. Therefore, Lemma 4.8 implies that $V_0 = \{0\}$. Hence we can choose the V_i in a way that V_1 is the Minkowski space and indecomposable for $\mathfrak{g}_1$ and the remaining V_i are Euclidean and irreducible for $\mathfrak{g}_i$. Note that for $i = 2, \ldots, k$ we have that $\mathfrak{g}_i \subset \mathfrak{so}(n_i)$, where $n_i = \dim(V_i)$. Moreover, we can write

$$\mathfrak{g} \ltimes V = (\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k) \ltimes (V_1 \oplus \ldots \oplus V_k) = (\mathfrak{g}_1 \ltimes V_1) \oplus \ldots \oplus (\mathfrak{g}_k \ltimes V_k).$$

Not only T but also $T_i = \widetilde{\mathfrak{g}} \cap V_i$ is $\mathfrak{g}$-invariant. Hence we have for $i = 2, \ldots, k$ that $T_i = \{0\}$ or $T_i = V_i$, and that T_1 is degenerate, trivial or equal to V_1. The same holds for $P_i = \mathrm{pr}_{V_i} T$ containing T_i.

Since V_1 is indecomposable but not necessarily irreducible, we have to consider several cases for T:

Case 1: T is indefinite, i.e., of signature $(1, \dim(T) - 1)$. In this case we have that $T \cap V_1 = V_1$ and that $T^\perp$ is positive definite and hence a direct sum of irreducibles that can be arranged such that $T^\perp = V_{\ell+1} \oplus \ldots \oplus V_k$ with $1 \leqslant \ell \leqslant k-1$ (recall that $T \neq \{0\}$ and that we are working under the assumption $T \neq V$). We apply Theorem 5.4 to the following data:

We define $\widetilde{W} := \mathbb{R}e_- \oplus T^\perp \oplus \mathbb{R}e_+$ and a representation $\rho : \tilde{\mathfrak{g}} \to \mathfrak{so}(\widetilde{W})_{e_-}$ by $\rho(X, v) = (X|_{T^\perp}, \mathrm{pr}_{T^\perp}(v))$. Since $T^\perp$ is positive definite, it is $T \cap T^\perp = \{0\}$, so by its very definition $\rho(\tilde{\mathfrak{g}})$ satisfies that $\rho(\tilde{\mathfrak{g}}) \cap T^\perp = \{0\}$. On the other hand, $\rho(\tilde{\mathfrak{g}})$ satisfies the assumptions of Theorem 5.4. Hence, with $\rho(\tilde{\mathfrak{g}}) \cap T^\perp = \{0\}$, the projection of $\rho(\tilde{\mathfrak{g}})$ onto $\mathfrak{so}(T^\perp)$ acts trivially on $T^\perp$. But this contradicts the fact that $T^\perp = V_{\ell+1} \oplus \ldots \oplus V_k$, where the V_i's are irreducible for $\mathrm{pr}_{\mathfrak{so}(1,n-1)}(\tilde{\mathfrak{g}})$ and hence for $\mathrm{pr}_{\mathfrak{so}(T^\perp)}(\rho(\tilde{\mathfrak{g}}))$.

Case 2: T is positive definite (including the case $T = 0$), i.e., $T \cap V_1 = \{0\}$ in virtue of the indecomposability of the $\mathfrak{g}_1$-module V_1. In this case $T^\perp$ is non-degenerate and $V_1 \subset T^\perp$, i.e.,

$$T^\perp = V_1 \oplus \ldots \oplus V_\ell \quad \text{and} \quad T = V_{\ell+1} \oplus \ldots \oplus V_k.$$

Set

$$\mathfrak{g}_- = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_\ell \quad \text{and} \quad \mathfrak{g}_+ = \mathfrak{g}_{\ell+1} \oplus \ldots \oplus \mathfrak{g}_k,$$

where $\mathfrak{g}_+ = \mathfrak{z}_+ + \mathfrak{g}_+'$ is reductive with centre $\mathfrak{z}_+$ and derived algebra $\mathfrak{g}_+'$, and $\mathfrak{g}_1$ is either irreducible or indecomposable but with an invariant null line L.

In the case when $\mathfrak{g}_1$ acts irreducibly on V_1, $\mathfrak{g}$ acts completely reducibly on V and, since T is positive definite, we can apply Corollary 6.1 to get a contradiction to $T \neq V$.

Hence we can assume that $\mathfrak{g}_1$ is contained in the stabiliser of the null line L, i.e., $\mathfrak{g}_1 \subset \mathfrak{so}(V_1)_L$. Since $\mathfrak{g}_+$ acts trivially on $T^\perp$ and the V_i's are irreducible for $i = \ell+1, \ldots, k$, and $\mathfrak{g}_-$ acts trivially on T, we have that

$$\ker(\mathfrak{g}_-) \cap T^\perp = \ker(\mathfrak{g}), \tag{6.1}$$

where ker denotes all vectors in V that are annihilated by all elements from the respective Lie algebra. As in Proposition 5.1, there is a $\varphi \in Z^1(\mathfrak{g}, T^\perp)$, such that $\tilde{\mathfrak{g}} = \mathfrak{h}_\varphi \ltimes T$. Then for $X_\pm \in \mathfrak{g}_\pm$ we have

$$0 = \varphi([X_+, X_-]) = X_- \varphi(X_+).$$

Hence, using equality (6.1), we obtain $\varphi(\mathfrak{g}_+) \subset \ker(\mathfrak{g}_-) \cap T^\perp = \ker(\mathfrak{g})$. If $\varphi|_{\mathfrak{g}_+} \neq 0$, we conclude that $\ker(\mathfrak{g})$ is a non-trivial subspace of $T^\perp$ and thus $\ker(\mathfrak{g}) = L$. Hence, if $\varphi|_{\mathfrak{g}_+} \neq 0$ there is a non-zero vector in L that is annihilated by $\mathfrak{g}$ and therefore the metric g admits a parallel null vector field.

Hence, for Case 2 we can assume that $\varphi|_{\mathfrak{g}_+} = 0$ and are left with

$$\varphi : \mathfrak{g}_- \longrightarrow T^\perp = V_1 \oplus \ldots \oplus V_\ell.$$

Then for $X_i \in \mathfrak{g}_i$ and $X_j \in \mathfrak{g}_j$, with $i, j \in \{1, \ldots, \ell\}$, and $i \neq j$ we have

$$0 = X_i \varphi(X_j) - X_j \varphi(X_i),$$

and hence

$$X_i \varphi(X_j) = 0. \tag{6.2}$$

Since the $V_{j\geqslant 2}$ are irreducible, this relation for $j = 1$ implies that

$$\varphi|_{\mathfrak{g}_1} \in Z^1(\mathfrak{g}_1, V_1).$$

On the other hand, for $j \geqslant 2$ we have that

$$\varphi|_{\mathfrak{g}_j} \in Z^1(\mathfrak{g}_j, L \oplus V_j),$$

where L is the $\mathfrak{g}$-invariant null line. If we write $\varphi = \varphi_1 + \ldots + \varphi_\ell$ with $\varphi_i : \mathfrak{g}_- \to V_i$, then relation (6.2) implies that if there exists $X_j \in \mathfrak{g}_j$ for some $j \geqslant 0$ such that $\varphi_1(X_j) \neq 0$, and thus $\varphi_1(\mathfrak{g}_j) = L$, then $\mathfrak{g}_1$ and hence $\mathfrak{g}$ acts trivially on L. The latter case implies again that the metric g admits a parallel null vector field.

Hence, we have obtained that g admits a parallel null vector field or that $\varphi = \varphi_1 + \ldots + \varphi_\ell$ with $\varphi_i \in Z^1(\mathfrak{g}_i, V_i)$ for $i = 1, \ldots \ell$. Since the V_i for $i \geqslant 2$ are irreducible, we have that $Z^1(\mathfrak{g}_i, V_i) = \mathrm{d}V_i$, by Lemma 5.7. The case $i = 1$ is covered by Theorem 5.8 where we have shown that $H^1(\mathfrak{g}_1, V_1) = 0$ whenever g does not admit a parallel null vector field. Hence, if g does not admit a parallel null vector field we obtain from (1) in Theorem 5.2 that $T^\perp$ is degenerate or zero. But this contradicts $T \neq V$ and that $T^\perp$ in Case 2 is non-degenerate.

Case 3: T is degenerate, i.e., there is a $\mathfrak{g}$-invariant null line $L = T \cap T^\perp$. Our aim is to apply point (2) in Theorem 5.2 and Proposition 5.11. First note that $\mathfrak{g}$ and therefore the indecomposable subalgebra $\mathfrak{g}_1 \subset \mathfrak{so}(V_1)$ both leave T and hence the null line L invariant. If $\mathfrak{g}_1$ acts trivially on L, then $\mathfrak{g}$ acts trivially on L and the metric g admits a parallel null vector field. Therefore we can assume that $\mathfrak{g}_1$ does not act trivially on L. This means that we can apply Proposition 5.11 to $\mathfrak{g}_1$ and $L^\perp \cap V_1$ to get that

$$Z^1(\mathfrak{g}_1, V_1/(L^\perp \cap V_1)) = d(V_1/(L^\perp \cap V_1)).$$

On the other hand, we note that there is a canonical identification

$$V/L^\perp \simeq V_1/(L^\perp \cap V_1),$$

which shows that $\mathfrak{g}_2 \oplus \ldots \oplus \mathfrak{g}_k$ acts trivially on $V/L^\perp$. Hence,

$$Z^1(\mathfrak{g}, V/L^\perp) = Z^1(\mathfrak{g}_1, V_1/(L^\perp \cap V_1)) = d(V/L^\perp).$$

Since we have assumed that $\mathfrak{g}$ does not act trivially on L, (2) in Theorem 5.2 implies that, up to conjugation, $\widetilde{\mathfrak{g}}$ leaves invariant a null line L. This means that $(\widetilde{M}, \widetilde{g})$ admits a recurrent null vector field in the span of ∂_v and L (even a recurrent section in L). But in this situation, Proposition 4.10 ensures the existence of a parallel null vector field on (M, g). This finishes the proof. □

7. Cones with parallel null 2-planes

In this section we consider the base manifolds (M, g) of cones that admit a parallel distribution of totally null 2-planes. Our main result is the description of the most general local form of the metric g. To exclude trivial cases we assume $\dim M > 1$.

7.1. **The induced structure on the base.** If $(\widehat{M}, \widehat{g})$ is a semi-Riemannian manifold and $\widehat{\mathbf{P}}$ a parallel totally null 2-plane bundle, then locally there are two null vector fields χ and ζ that are orthogonal to each other and such that

$$\begin{aligned} \widehat{\nabla}\chi &= \alpha \otimes \chi + \mu \otimes \zeta \\ \widehat{\nabla}\zeta &= \beta \otimes \chi + \nu \otimes \zeta, \end{aligned} \tag{7.1}$$

for 1-forms α, β, μ and ν.

If $(\widehat{M}, \widehat{g})$ is a timelike cone with a parallel null 2-plane bundle $\widehat{\mathbf{P}}$, we can intersect $\widehat{\mathbf{P}}$ with $\xi^{\perp}$, where ξ is the Euler vector field. A subset of $\widehat{M} = \mathbb{R}^{>0} \times M$ will be called *conical* if it is of the form $\widehat{M_0} = \mathbb{R}^{>0} \times M_0$ for some subset $M_0 \subset M$.

Lemma 7.1. *On a conical open dense subset in $\widehat{M}$ the intersection $\widehat{\mathbf{P}} \cap \xi^{\perp}$ is a null-line bundle $\mathbf{L}$ invariant under the flow of ξ. In particular, $\mathbf{L}$ admits local sections, defined on conical open sets, invariant under the flow of ξ and descends to a null line distribution on an open dense subset of M.*

Proof. For this and the following proofs, we note that

$$\left[\xi, \Gamma(\xi^{\perp})\right] \subset \Gamma(\xi^{\perp}) \quad \text{and} \quad \left[\xi, \Gamma(\widehat{\mathbf{P}})\right] \subset \Gamma(\widehat{\mathbf{P}}).$$

This implies that the dimension of the fibres of $\widehat{\mathbf{P}} \cap \xi^{\perp}$ is constant on the integral curves of ξ. At each point $p \in \widehat{M}$, $\xi^{\perp}|_p$ is a hyperplane and $\widehat{\mathbf{P}}|_p$ a 2-plane in $T_p\widehat{M}$. Hence their intersection has dimension one or two. Now let us assume that, over an open set $U \subset \widehat{M}$, $\widehat{\mathbf{P}} \cap \xi^{\perp}$ is of rank 2, i.e. that $\widehat{\mathbf{P}} \subset \xi^{\perp}$. Hence $\widehat{\mathbf{P}} \cap \xi^{\perp}$ a distribution of 2-planes spanned by vector fields V_1 and V_2 on U that are tangential to M. Then formulae (2.2) and (7.1) give us

$$TM \ni \widehat{\nabla}_X V_i = \nabla_X V_i + g(X, V_i)\xi,$$

for all $X \in TM$. Hence, on U it is $g(X, V_i) = 0$ for all $X \in TM$ which is impossible. Consequently, the conical open set over which the fibres of $\widehat{\mathbf{P}} \cap \xi^{\perp}$ are one-dimensional is dense and $\widehat{\mathbf{P}} \cap \xi^{\perp}$ restricts to a line bundle $\mathbf{L}$ over that set. $\square$

Now we project $\widehat{\mathbf{P}}$ to $\xi^{\perp}$.

Lemma 7.2. *Over a conical open dense subset of $\widehat{M}$, the projection $\mathrm{pr}_{\xi^{\perp}}(\widehat{\mathbf{P}}) \subset \xi^{\perp}$ is an involutive 2-plane distribution $\mathbf{P}$. In particular, there is a conical open dense submanifold $\widehat{M_0}$ of $\widehat{M}$ over which the involutive 2-plane distribution $\mathbf{P}$ descends to an involutive 2-plane distribution on M_0.*

Proof. First note that over a conical open dense subset of $\widehat{M}$ the fibres of $\mathrm{pr}_{\xi^{\perp}}(\widehat{\mathbf{P}})$ have dimension 2. Indeed, if $\mathrm{pr}_{\xi^{\perp}}(\widehat{\mathbf{P}})$ had rank one over an open set, then $\xi \in \Gamma(\widehat{\mathbf{P}})$ over that open set. Then $\widehat{\nabla}\xi = \mathrm{Id}$ and $n > 1$ show that over this open set $\widehat{\mathbf{P}}$ cannot be invariant. This shows that over a conical open dense subset of $\widehat{M}$, $\mathrm{pr}_{\xi^{\perp}}(\widehat{\mathbf{P}}) \subset \xi^{\perp}$ is a 2-plane distribution $\mathbf{P}$.

Clearly the projection of a vector field V on $\widehat{M}$ to $\xi^{\perp}$ is given as

$$\mathrm{pr}_{\xi^{\perp}}(V) = V + r^{-2}\widehat{g}(V, \xi)\xi.$$

By a calculation using $\widehat{\nabla}\xi = \mathrm{Id}$ we obtain for all $V_1, V_2 \in \mathfrak{X}(\widehat{M})$:

$$\left[\mathrm{pr}_{\xi^\perp}(V_1), \mathrm{pr}_{\xi^\perp}(V_2)\right] \quad = \quad \mathrm{pr}_{\xi^\perp}\left([V_1, V_2] + r^{-2}\widehat{g}(V_2,\xi)[V_1,\xi] - r^{-2}\widehat{g}(V_1,\xi)[V_2,\xi]\right).$$

Since the distribution $\widehat{\mathbf{P}}$ is invariant under ξ, parallel and hence involutive, the right-hand side is a section of $\mathbf{P}$ for all sections V_1, V_2 of $\widehat{\mathbf{P}}$. This proves the involutivity of $\mathbf{P}$. □

Moreover we obtain:

Lemma 7.3. *There exist local sections V of $\mathbf{L}$ and Z of $\mathbf{P}$, defined on a conical open set, such that V and*

$$\zeta \quad = \quad \xi + Z$$

locally span $\widehat{\mathbf{P}}$ and satisfy

$$[\xi, V] = 0 \quad \textit{and} \quad [\xi, Z] = 0.$$

The vector fields V and Z descend to local vector fields on M.

Proof. We have already seen that there exists a non-vanishing section V of $\mathbf{L}$ over a conical open set such that $[\xi, V] = 0$. In the following we always work locally over conical open sets. Every section of $\widehat{\mathbf{P}}$ that is nowhere a multiple of V is of the form $f\xi + Z$ for Z a (possibly vanishing) local section of $\mathbf{P}$ and f a non-vanishing local function on $\widehat{M}$. Hence, by multiplying with $1/f$ we can assume that we have a section

$$\hat{\zeta} = \xi + \hat{Z}$$

of $\widehat{\mathbf{P}}$. We will now use the freedom to add multiples of V to $\hat{Z}$ without leaving $\widehat{\mathbf{P}}$, in order to find a $Z = \hat{Z} + \varphi V$ for which we have $[\xi, Z] = 0$. Indeed, writing

$$\nabla_\xi \hat{\zeta} = fV + h\hat{\zeta}$$

with functions f and h, we compute

$$[\xi, \hat{Z}] = [\xi, \hat{\zeta}] = fV + (h-1)\hat{\zeta}.$$

Since $[\xi, \hat{Z}]$ belongs to $\xi^\perp$, we must have that $h \equiv 1$ and

$$[\xi, \hat{Z}] = fV.$$

Now if we fix a solution φ of

$$\mathrm{d}\varphi(\xi) + f = 0,$$

and set $Z = \hat{Z} + \varphi V$ we get

$$[\xi, Z] = 0.$$

Clearly, since V is a section of $\widehat{\mathbf{P}}$, the vector field

$$\zeta := \xi + Z = \hat{\zeta} + \varphi V,$$

is also a section in $\widehat{\mathbf{P}}$ that is still linearly independent of V and therefore Z is a section of $\mathbf{P}$ that locally descends to M. □

Theorem 7.4. *Let $(\widehat{M}, \widehat{g})$ be a timelike cone. Then the cone admits locally a parallel, totally null 2-plane field if and only if the base (M, g) admits locally two vector fields V and Z such that*

$$g(V,V) = 0, \quad g(Z,Z) = 1, \quad g(V,Z) = 0, \tag{7.2}$$

and

$$\nabla_X V = \alpha(X)V + g(X,V)Z, \tag{7.3}$$

$$\nabla_X Z = -X + \beta(X)V + g(X,Z)Z, \tag{7.4}$$

for all $X \in TM$, with 1-forms α and β on M.

Proof. First assume that the cone admits a parallel totally null 2-plane $\widehat{\mathbf{P}}$ which is spanned by V and $\zeta = \xi + Z$ as in Lemma 7.3. Equations (7.2) are implied by $\widehat{\mathbf{P}}$ being totally null. Moreover, equations (7.1) with $\chi = V$ and $X \in TM$ become

$$\widehat{\nabla}_X V = \nabla_X V + g(X,V)\xi = \alpha(X)V + \mu(X)(\xi + Z), \tag{7.5}$$

$$\widehat{\nabla}_X \zeta = X + \nabla_X Z + g(X,Z)\xi = \beta(X)V + \nu(X)(\xi + Z), \tag{7.6}$$

and imply

$$\begin{aligned} \mu(X) &= g(X,V), \\ \nu(X) &= g(X,Z), \end{aligned}$$

as well as equations (7.3) and (7.4), but still with r-dependent 1-forms α and β. Hence, it remains to show that α and β, when restricted to $\xi^\perp$, are invariant under the flow of ξ and therefore descend to 1-forms on M, i.e., that

$$\mathcal{L}_\xi \alpha|_{\xi^\perp} = \mathcal{L}_\xi \beta|_{\xi^\perp} = 0.$$

But from

$$\begin{aligned} 0 &= \widehat{R}(\xi, X)V \\ &= (\mathcal{L}_\xi \alpha)(X)V + \alpha(X)V + g(X,V)(\xi + Z) - (\nabla_X V + g(X,V)\xi) \\ &= (\mathcal{L}_\xi \alpha)(X)V, \end{aligned}$$

because of equation (7.5). This proves that $\mathcal{L}_\xi \alpha|_{\xi^\perp} = 0$. Analogously we get

$$0 = \widehat{R}(\xi, X)\zeta = (\mathcal{L}_\xi \beta)(X)V$$

and again $\mathcal{L}_\xi \beta|_{\xi^\perp} = 0$.

Conversely, if we start with a manifold (M, g) and vector fields satisfying conditions (7.2), (7.3) and (7.4), a straightforward computations shows that the cone admits a parallel null plane spanned by V and $\xi + Z$. □

Corollary 7.5. *If the cone (2.1) admits a distribution of parallel totally null 2-planes, then the base (M, g) admits locally a geodesic, shearfree null vector field V.*

Proof. Since V is null, equation (7.3) implies that V is geodesic. Recall that a geodesic null vector field is called *shearfree* if

$$\mathcal{L}_V g = \lambda g + \theta \cdot V^\flat,$$

with a function λ and a 1-form θ and where the dot stands for the symmetric product. From (7.3) and the formula

$$\mathcal{L}_X g = 2(\nabla X^\flat)^{\mathrm{sym}}, \tag{7.7}$$

where 'sym' denotes the projection onto the symmetric part, we compute

$$\mathcal{L}_V g = 2(\alpha + Z^\flat) \cdot V^\flat,$$

i.e., the shear free condition is satisfied with $\lambda = 0$. □

Remark 7.6. We can change the basis of $\mathrm{span}(V, Z)$ to V', Z' such that V' is still null and orthogonal to Z' and such that Z' is a unit vector field,

$$(V, Z) \longmapsto (V' = \mathrm{e}^f V, Z' = Z + hV).$$

Then the 1-forms α and β transform as

$$\begin{array}{rcl} \alpha & \longmapsto & \alpha' = \alpha + \mathrm{d}f - hV^\flat, \\ \beta & \longmapsto & \beta' = e^{-f}(\beta + h\alpha + \mathrm{d}h - hZ^\flat - h^2 V^\flat). \end{array}$$

7.2. **Consequences of the fundamental equations.** Let (M, g) be a semi-Riemannian manifold endowed with two pointwise linearly independent vector fields V, Z which satisfy (7.2), (7.3) and (7.4).

Proposition 7.7. *The fundamental equations (7.2) (7.3) and (7.4) imply*

$$dV^\flat = (\alpha - Z^\flat) \wedge V^\flat, \tag{7.8}$$

$$dZ^\flat = \beta \wedge V^\flat, \tag{7.9}$$

$$[Z, V] = (\alpha(Z) - \beta(V) + 1)V, \tag{7.10}$$

$$\mathcal{L}_V g = 2(\alpha + Z^\flat)V^\flat, \tag{7.11}$$

$$\mathcal{L}_Z g = -2g + 2(Z^\flat)^2 + 2\beta V^\flat, \tag{7.12}$$

where we are using the symmetric product of 1*-forms in the last two formulas.*

Proof. Since ∇ is torsion-free, the differential of any 1-form φ is given by

$$\mathrm{d}\varphi(X, Y) = (\nabla_X \varphi)Y - (\nabla_Y \varphi)X, \quad X, Y \in \mathfrak{X}(M).$$

Now (7.8) and (7.9) follow immediately from (7.3) and (7.4). Using again that ∇ is torsion-free, the fundamental equations easily imply (7.10). Similarly, the last two formulas follow from (7.7). □

Corollary 7.8.

$$\mathcal{L}_V V^\flat = \alpha(V)V^\flat, \tag{7.13}$$

$$\mathcal{L}_Z V^\flat = (\beta(V) - 2)V^\flat \tag{7.14}$$

$$\mathcal{L}_V Z^\flat = (\alpha(Z) + 1)V^\flat, \tag{7.15}$$

$$\mathcal{L}_Z Z^\flat = \beta(Z)V^\flat, \tag{7.16}$$

$$\beta(V) = \alpha(Z) + 1. \tag{7.17}$$

Proof. The first four formulas are obtained from the formulas (7.11) and (7.12). Alternatively one can use Cartan's formula for the Lie derivative and the formulas (7.8) and (7.9). Comparing the results shows (7.17). □

Corollary 7.9. *By multiplying V with a function we can locally assume that*

$$\mathrm{d}V^\flat = 0, \tag{7.18}$$

that is

$$\alpha = Z^\flat + f_\alpha V^\flat$$

for some function f_α. The latter equation implies

$$\alpha(Z) = 1, \quad \alpha(V) = 0, \quad \beta(V) = 2, \quad \mathcal{L}_V V^\flat = 0, \quad \mathcal{L}_V Z^\flat = 2V^\flat, \quad \mathcal{L}_Z V^\flat = 0, \quad [Z, V] = 0.$$

Proof. By equation (7.8) and the Frobenius theorem, the hyperplane distribution $V^\perp$ is integrable, which locally implies that a functional multiple of $V^\flat$ is closed. □

7.3. **The local form of the metric on the base.** In the following we will assume all of the above equations. By (7.18), locally, there exists a function u such that $du = V^\flat$. The function u is constant on each leaf L of the distribution $V^\perp$. Locally, we can decompose M as $M = L \times \mathbb{R}$, such that u corresponds to the coordinate on the $\mathbb{R}$-factor and the leafs of $V^\perp$ are the hypersurfaces $L_u = L \times \{u\}$. Since the vector fields V and Z commute and are tangent to $V^\perp$, we can further decompose each leaf of $V^\perp$ locally as $L_u \cong L = M_0 \times \mathbb{R} \times \mathbb{R}$, such that $V = \partial_t$, $Z = \partial_s$ are the coordinate vector fields tangent to the first and second $\mathbb{R}$-factor, respectively.

Let us denote by $\mathbf{P}$ the integrable distribution spanned by V and Z. Notice that by (7.9) the distribution $\mathbf{P}^\perp = Z^\perp \cap V^\perp$ is also integrable, in virtue of the Frobenius theorem. So we can assume that the level sets of s are tangent to $\mathbf{P}^\perp$. Finally, the decomposition $M = L \times \mathbb{R}$ can be chosen such that the decomposition $L_u = M_0 \times \mathbb{R} \times \mathbb{R}$ is independent of u, that is the vector field ∂_u commutes with V, Z and with the canonical lift of vector fields of M_0.

Theorem 7.10. *Let (M, g) be a semi-Riemannian manifold such that the cone $(\widehat{M}, \widehat{g})$ admits a parallel totally null distribution of 2-planes. In terms of the above local decomposition $M = M_0 \times \mathbb{R}^3$ we have*

$$g = \mathrm{d}s^2 + e^{-2s} g_0(u) + 2\, \mathrm{d}u\, \eta, \tag{7.19}$$

for some 1-form η on M such that $\eta(\partial_t)$ is nowhere vanishing and a family of metrics $g_0(u)$ on M_0 depending on u.

Proof. The restriction of the metric to a leaf $N = M_0 \times \mathbb{R} \times \{(s, u)\}$ of $\mathbf{P}^\perp$ is degenerate with kernel $V = \partial_t \in \mathbf{P}^\perp$ and invariant under the flow of V, see (7.11). Since M_0 is transversal to V, we see that $g|_N = g_0(u, s)$ for some family of metrics on M_0 depending on u and s. The flow of $Z = \partial_s$ is a 1-parameter family of homotheties of weight -2, see (7.12). This shows that $g_0(u, s) = e^{-2s} g_0(u)$ for some 1-parameter family of metrics $g_0(u)$. It follows that on the leafs $L_u = M_0 \times \mathbb{R} \times \mathbb{R} \times \{u\}$ of $V^\perp$ the metric is of the form $\mathrm{d}s^2 + e^{-2s} g_0(u)$. Finally, on M we obtain the general form (7.19) with $\eta(\partial_t) \neq 0$, in view of the non-degeneracy of g. □

It remains to determine the necessary and sufficient conditions for the data $g_0(u)$ and η ensuring that the cone over (M, g) as in (7.19) admits a parallel totally null distribution of 2-planes. Let M_0 be a manifold and let us denote the standard coordinates on $\mathbb{R}^3$ by (t, s, u).

Theorem 7.11. *For any 1-form η on $M := M_0 \times \mathbb{R}^3$ such that $\eta_t := \eta(\partial_t) \neq 0$ and any family of semi-Riemannian metrics $g_0(u)$ on M_0 the tensor field*

$$g = \mathrm{d}s^2 + e^{-2s} g_0(u) + 2\, du\, \eta,$$

cf. (7.19), is a semi-Riemannian metric on M such that the vector fields $V = \partial_t$ and $Z = \partial_s$ satisfy (7.2). The covariant derivatives of V and Z are given by (7.3) and (7.4) for some 1-forms $\alpha = Z^\flat + f_\alpha V^\flat$ and β such that f_α is a function on M and $\beta(V) = 2$, if and only if the coefficients of η solve the following system of first order partial differential equations:

$$\partial_t \eta_t = \partial_s \eta_t = X\eta_t = \partial_t \eta(X) = 0, \quad \partial_t \eta_s = 2\eta_t, \quad \partial_s \eta(X) - X\eta_s = -2\eta(X) \tag{7.20}$$

for all $X \in \mathfrak{X}(M_0)$. Then α and β are determined by

$$f_\alpha = \frac{1}{\eta_t^2} \partial_t \eta_u - \frac{2}{\eta_t} \eta_s, \quad \beta(Z) = \frac{1}{\eta_t} \partial_s \eta_s, \quad \beta(X) = \frac{1}{2\eta_t}(X\eta_s + \partial_s \eta(X) + 2\eta(X)),$$
$$\beta(\partial_u) = \frac{1}{\eta_t}(\partial_s \eta_u - \eta_s^2 + 2\eta_u),$$

Proof. We denote by X the canonical lift of a vector field on M_0. Then X, V, Z and ∂_u commute and using the Koszul formula we obtain

$$\begin{aligned}
&g(\nabla_V V, X) = g(\nabla_V V, V) = g(\nabla_V V, Z) = 0, \quad g(\nabla_V V, \partial_u) = \partial_t \eta_t, \\
&g(\nabla_Z V, X) = g(\nabla_Z V, V) = g(\nabla_Z V, Z) = 0, \quad 2g(\nabla_Z V, \partial_u) = \partial_s \eta_t + \partial_t \eta_s, \\
&g(\nabla_X V, X) = g(\nabla_X V, V) = g(\nabla_X V, Z) = 0, \quad 2g(\nabla_X V, \partial_u) = X\eta_t + \partial_t \eta(X), \\
&2g(\nabla_{\partial_u} V, X) = \partial_t \eta(X) - X\eta_t, \quad g(\nabla_{\partial_u} V, V) = 0, \quad 2g(\nabla_{\partial_u} V, Z) = \partial_t \eta_s - \partial_s \eta_t, \\
&g(\nabla_{\partial_u} V, \partial_u) = \partial_t \eta_u, \\
&g(\nabla_V Z, X) = g(\nabla_V Z, V) = g(\nabla_V Z, Z) = 0, \quad 2g(\nabla_V Z, \partial_u) = \partial_t \eta_s + \partial_s \eta_t, \\
&g(\nabla_Z Z, X) = g(\nabla_Z Z, V) = g(\nabla_Z Z, Z) = 0, \quad g(\nabla_Z Z, \partial_u) = \partial_s \eta_s, \\
&g(\nabla_X Z, X) = -g(X, X), \quad g(\nabla_X Z, V) = g(\nabla_X Z, Z) = 0, \\
&2g(\nabla_X Z, \partial_u) = X\eta_s + \partial_s \eta(X), \\
&2g(\nabla_{\partial_u} Z, X) = \partial_s \eta(X) - X\eta_s, \quad 2g(\nabla_{\partial_u} Z, V) = \partial_s \eta_t - \partial_t \eta_s, \quad g(\nabla_{\partial_u} Z, Z) = 0, \\
&g(\nabla_{\partial_u} Z, \partial_u) = \partial_s \eta_u.
\end{aligned}$$

Comparing with (7.3), (7.4) we obtain the above formulas for α and β and the following system for η:

$$\begin{aligned}
&\partial_t \eta_t = 0, \quad \partial_s \eta_t + \partial_t \eta_s = 2\eta_t, \quad X\eta_t + \partial_t \eta(X) = 0, \quad \partial_t \eta(X) - X\eta_t = 0, \\
&\partial_t \eta_s - \partial_s \eta_t = 2\eta_t, \\
&\partial_s \eta(X) - X\eta_s = -2\eta(X)
\end{aligned}$$

for all $X \in \mathfrak{X}(M_0)$. This system can be brought to the form (7.20). □

For convenience we denote a system of local coordinates on M_0 by $(x^i)_{i=1,\dots,n_0}$ and denote by x the corresponding coordinate vector, where $n_0 = \dim M_0$. The general solution of (7.20) is obtained as follows.

Proposition 7.12. *Let $f_1 = f_1(u)$ be an arbitrary nowhere vanishing smooth function on the real line equipped with the coordinate u and $f_2 = f_2(x, s, u)$ an arbitrary smooth function* *on M which does not depend on t. Let $h_i = h_i(x,s,u)$ be a (t-independent) solution of the ordinary differential equation*

$$\partial_s h_i + 2h_i = \partial_i f_2$$

for all $i = 1,\ldots,n_0$, where $\partial_i = \partial/\partial x^i$. Then

$$\eta_t := f_1(u), \quad \eta_s := 2tf_1(u) + f_2(x,s,u), \quad \eta(\partial_i) := h_i(x,s,u)$$

solves (7.20) and every solution is of this form.

Remark 7.13. Finally we return to the Lorentzian metrics that occurred in Theorem 1.3 and arose from the case where the cone $(\widehat{M},\widehat{g})$ admits a parallel null line: in this case the cone metric $\widehat{g}$ was isometric to the metric $\widetilde{g} = 2\mathrm{d}u\mathrm{d}v + u^2 g_0$ with a Lorentzian metric g_0 and g was isometric to $g = \mathrm{d}s^2 + \mathrm{e}^{2s} g_0$. Then Theorem 1.3 stated that if the holonomy of the cone is not equal to $\mathfrak{hol}(g_0) \ltimes \mathbb{R}^{1,n-1}$, then g_0 admits a parallel null vector field. It is well known (see for example [20, 14]) that locally g_0 is of the form $g_0 = 2\mathrm{d}x\mathrm{d}z + h(z)$, where $h(z)$ is a z-dependent family of Riemannian metrics. Hence, g is of the form

$$g = ds^2 + \mathrm{e}^{2s} h(z) + 2\mathrm{e}^{2s}\mathrm{d}x\mathrm{d}z.$$

This corresponds to the local form in Theorem 7.11, where x corresponds to t and $2\mathrm{e}^{2s}\mathrm{d}x$ to η, z to u and $h(z)$ to $g_0(u)$.

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