Splitting groups with cubic Cayley graphs of connectivity two

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Abstract

A group G splits over a subgroup C if G is either a free product with amalgamation $A \underset{C}{*} B$ or an HNN-extension $G = A \underset{C}{*} (t)$. We invoke Bass-Serre theory and classify all infinite groups which admit cubic Cayley graphs of connectivity two in terms of splittings over a subgroup.

1 Introduction

A finitely generated group G is called *planar* if it admits a generating set S such that the Cayley graph $Cay(G, S)$ is planar. In that case, S is called a *planar generating set*. For the first time, in 1896, Maschke [14] characterized all finite groups admitting planar Cayley graphs. Infinite planar groups attracted more attention, as some of them are related to surface and Fuchsian groups [18, section 4.10] which play a substantial role in complex analysis, see survey [18]. Hamann [11] uses a combinatorial method in order to show that planar groups are finitely presented. His method is based on tree-decompositions, a crucial tool of graph minor theory which we also utilize extensively in this paper.

A related topic to infinite planar Cayley graphs is the connectivity of Cayley graphs, see [7, 9, 10]. Studying connectivity of infinite graphs goes back to 1971 by Jung, see [13]. In [7], Droms et. al. characterized planar groups with low connectivity in terms of the fundamental group of the graph of groups. Indeed, they showed that

Theorem. [7, Theorem 4.4] *If a group G has planar connectivity*[1] *2, then either G is a finite cyclic or dihedral group, or it is the fundamental group of a graph of groups whose edge groups all have order two or less and whose vertex groups all have planar connectivity at least three. In the latter case, the vertex groups have planar generating sets which include the nontrivial elements of the incident edge groups.*

Later, Georgakopoulos [9] determines the presentations of all groups whose Cayley graphs are cubic with connectivity 2. His method does not assert anything regarding (and is, in a sense, independent of) splitting the group over

[1] The planar connectivity $\kappa(G)$ of a planar group G is the minimum connectivity of all its planar Cayley graphs.

subgroups to obtain its structure. By combining tree-decompositions and Bass-Serre theory, we give a short proof for the full characterization of groups with cubic Cayley graphs of connectivity 2 via the following theorem:

Theorem 1.1. *Let $G = \langle S \rangle$ be a group such that $\Gamma = Cay(G, S)$ is a cubic graph of connectivity two. Then G is isomorphic to one of the following groups:*

(i) $\mathbb{Z}_n * \mathbb{Z}_2$,

(ii) $D_{2n} \underset{\mathbb{Z}_2}{*} (t)$,

(iii) $D_{2n} \underset{\mathbb{Z}_2}{*} D_{2m}$,

(iv) $\mathbb{Z}_{2n} \underset{\mathbb{Z}_2}{*} D_{2m}$,

(v) $D_\infty \underset{\mathbb{Z}_2}{*} D_{2m}$.

Theorem 1.1 is a direct consequence of Theorems 4.3, 4.5, 5.4 and 5.8, where we also discuss in detail the planarity of the corresponding Cayley graphs in each case, as well as their presentations. This allows us to obtain as a corollary the results of [9].

2 Preliminaries

Our terminology of groups and graphs is standard. We refer the reader to [16] for Bass-Serre theory and [6] for graph theory for any notation missing.

2.1 Graphs

Throughout this paper, Γ always denotes a connected locally finite graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. A *ray* is a one-way infinite path and a *tail* of a ray is an infinite subpath of the ray. Two rays R_1 and R_2 are equivalent if there is no finite set S of vertices such that R_1 and R_2 have tails in different components of $G \setminus S$. The equivalence classes of rays are called *ends*. We refer the reader to surveys [4, 5] for a detailed study of the end structure of graphs.

A *separation* of Γ is an ordered pair (A, B) such that $\Gamma[A] \cup \Gamma[B] = \Gamma$. Clearly $A \cap B$ separates A from B. The *order* of (A, B) is the size of $A \cap B$ and we denote it by $|(A, B)|$. If $|(A, B)| = k$, we say that (A, B) is a k-separation. The set of separations of Γ can be equipped with the following partial order: $(A, B) \leq (C, D)$ if $A \subseteq C$ and $B \supseteq D$. We say that (A, B) is *nested* with (C, D) if (A, B) is comparable to either (C, D) or (D, C).

Let S be a set of vertices of Γ. The set of neighbours of S is denoted by $N(S)$ and also $N[S]$ denotes $S \cup N(S)$. A component C of $G \setminus S$ is called *tight* if $N(C) = S$. A separation (A, B) is called *tight* if both $A \setminus B$ and $B \setminus A$ have tight components. A separation (A, B) *distinguishes* two ends ω_1 and ω_2 if ω_1 has a tail in $A \setminus B$ and ω_2 has a tail in $B \setminus A$ or vise versa. Moreover, it distinguishes ω_1 and ω_2 *efficiently* if there is no separation (C, D) distinguishing ω_1 and ω_2 such that $|(C, D)| < |(A, B)|$. We note that if (A, B) distinguishes two ends efficiently, then (A, B) is a tight separation. Two ends ω_1 and ω_2 are *k-distinguishable* if there is a separation of order k distinguishing ω_1 and ω_2 efficiently. Hamann and et al. [3] proved the following theorem:

Theorem 2.1. *Let Γ be a locally finite graph with more than one end. For each $k \in \mathbb{N}$, there is a nested set $\mathcal{N}$ of tight separations of Γ distinguishing all k-distinguishable ends efficiently.* $\square$

Let Γ be an arbitrary connected graph. A *tree-decomposition* of Γ is a pair $(T, \mathcal{V})$ of a tree T and a family $\mathcal{V} = (V_t)_{t \in T}$ of vertex sets $V_t \subseteq V(\Gamma)$, which are called *parts*, one for every node of T such that:

(T1) $V(\Gamma) = \bigcup_{t \in T} V_t$,

(T2) for every edge $e \in E(\Gamma)$, there exists a $t \in T$ such that both ends of e lie in V_t,

(T3) $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever t_2 lies on the (t_1, t_3)-path in T.

An *adhesion set* of $(T, \mathcal{V})$ is a set of the form $V_t \cap V_{t'}$, where $tt' \in E(T)$. It is not hard to see that each adhesion set leads to a separation of Γ. More precisely, assume that T_t and T'_t are the components of $T - tt'$ containing t and t' respectively. Then the adhesion set $V_t \cap V_{t'}$ induces the separation $(W_{t \setminus t'}, W_{t' \setminus t})$ of Γ, where $W_{t \setminus t'} = \bigcup_{s \in T_t} V_s$ and $W_{t' \setminus t} = \bigcup_{s \in T_{t'}} V_s$. When every such separation is tight, we call the tree-decomposition *tight* as well.

It is known that every nested set $\mathcal{N}$ of separations gives rise to a tree-decomposition whose adhesion sets are exactly the elements of $\mathcal{N}$, see [2]. As an application of Theorem 2.1, consider any orbit of a separation under the action of a group G that acts on Γ in a nested set $\mathcal{N}$ satisfying the conclusion of the theorem, as well as the corresponding tree decomposition it gives rise to. One can show that that G also acts not only on the adhesion sets, but also on the parts. We have the following Lemma.

Lemma 2.2. [12, Corollary 4.3] *Let Γ be a locally finite graph with more than one end such that a group G acts on Γ. Then there exists a tree-decomposition $(T, \mathcal{V})$ with the following properties:*

(i) *$(T, \mathcal{V})$ distinguishes at least two ends.*

(ii) *All adhesion sets of $(T, \mathcal{V})$ are finite.*

(iii) *The action of G on Γ induces an action on $\Gamma[\mathcal{V}]$ and a transitive action on the set of separations corresponding to the adhesion sets.*

Notice that the transitive action on the set of separations in Lemma 2.2 (iii) implies at most two orbits for $\Gamma(\mathcal{V})$ under the action of G. Moreover, we can translate the action of item (iii) to an action of G on T in the natural way (and G will clearly act transitively on $E(T)$):

$$gt = t' \Leftrightarrow gV_t = V_{t'}.$$

Let G be a locally finite graph with a tree-decomposition $(T, \mathcal{V})$. We call the *torso* of a part V_t the supergraph of $G[V_t]$ obtained by adding to it all possible edges in the adhesion sets incident to V_t. The following general lemma for tree-decompositions is folklore.

Lemma 2.3. *Let $(T, \mathcal{V})$ be a tree-decomposition of a connected graph Γ and $t \in V(T)$ such that every adhesion set of t induces a connected subgraph. Then $\Gamma[V_t]$ is connected. In particular, the torso of every part of $(T, \mathcal{V})$ is connected.*

In this paper, we are studying groups admitting cubic Cayley graphs of connectivity two. The next Lemma implies that such a graph has at least two ends.

Lemma 2.4. [1, Lemma 2.4] *Let Γ be a connected vertex-transitive d-regular graph. Assume Γ has one end. Then the connectivity of Γ is $\geq 3(d+1)/4$.*

2.2 Groups

Let G be a group acting on a set X. Then the *setwise stabilizer* of a subset Y of X is the set of all elements $g \in G$ stabilizing Y setwise, i.e

$$\mathsf{St}_G(Y) := \{g \in G \mid gy \in Y, \forall y \in Y\}.$$

Let G be a group acting on a graph Γ. Then this action induces an action on $E(\Gamma)$. We say that G acts without *inversion* on Γ if $g(uv) \neq vu$ for all $uv \in E(\Gamma)$ and $g \in G$. In the case that $g(uv) = vu$, we say that g *inverts* u, v. Notice that when G acts transitively with inversion on the set $E(T)$ of edges of a tree T without leaves, it must also act transitively on the set $V(T)$ of its vertices.

Let $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$ be two groups. Suppose that a subgroup H_1 of G_1 is isomorphic to a subgroup H_2 of G_2, say an isomorphic map $\phi \colon H_1 \to H_2$. The *free-product with amalgamation* of G_1 and G_2 over H_1 is

$$G_1 \underset{H_1}{*} G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup h\phi(h)^{-1}, \forall h \in H_1 \rangle.$$

If H_1 and $\phi(H_1)$ are isomorphic subgroups of G_1, then the HNN-*extension* of G_1 over H_1 with respect to ϕ is

$$G_1 \underset{H_1}{*} (t) = \langle S_1, t \mid R_1 \cup tht^{-1}\phi(h)^{-1}, \forall h \in H_1 \rangle$$

The crux of Bass-Serre theory is captured in the next Lemma which determines the structure of groups acting on trees.

Lemma 2.5. [16] *Let G act without inversion on a tree that has no vertices of degree one and let G act transitively on the set of (undirected) edges. If G acts transitively on the tree, then G is an* HNN-*extension of the stabilizer of a vertex over the pointwise stabilizer of an edge. If there are two orbits on the vertices of the tree, then G is the free product of the stabilizers of two adjacent vertices with amalgamation over the pointwise stabilizer of an edge.*

There is a standard way to deal with the case where we cannot apply Lemma 2.5 directly when G acts with inversion on the tree.

Lemma 2.6. *Let G act transitively with inversion on a tree T without leaves. Then G is the free product of the stabilizers of a vertex and an edge with amalgamation over their intersection.*

Proof. Subdivide every edge tt' of T to obtain tree T' and let $v_{tt'}$ the corresponding new node. Notice that G now acts transitively on $E(T')$ without inversion and with two orbits on $V(T')$. Each old node t of T has the same pointwise stabilizer in T'. Observe that for each new node $v_{tt'}$ we have $\mathsf{St}_G(v_{tt'}) = \mathsf{St}_G(e)$, where $tt' = e \in E(T)$. The result follows from Lemma 2.5. $\square$

In our applications of Lemmas 2.5 and 2.6, the setwise stabilizers of the parts and the adhesion sets of $(T, \mathcal{V})$ will play the role of the stabilizer of a vertex and pointwise stabilizer of an edge of T, respectively.

Finally, $\mathbb{Z}_n$ denotes the *cyclic group of order* n. A finite *dihedral* group is defined by the presentation $\langle a, b \mid b^2 = a^n = (ba)^2 \rangle$ and denoted by D_{2n}. Moreover, the infinite dihedral group D_∞ is defined by $\langle a, b \mid b^2 = (ba)^2 \rangle$.

3 General structure of the tree-decomposition

For the rest of the paper, we assume that $G = \langle S \rangle$ be an infinite finitely generated group such that $\Gamma = Cay(G, S)$ is cubic with connectivity two. Let $\mathcal{N}$ be a nested set of separations of order two in such a way that $\mathcal{N}$ gives a tree-decomposition as in Lemma 2.2. Then we notice that every 2-separation of Γ such that $A \cap B$ is a proper subset of A and B distinguishes at least two ends, see [8, Lemma 3.4]. For an arbitrary element $(A, B) \in \mathcal{N}$, there are three cases:

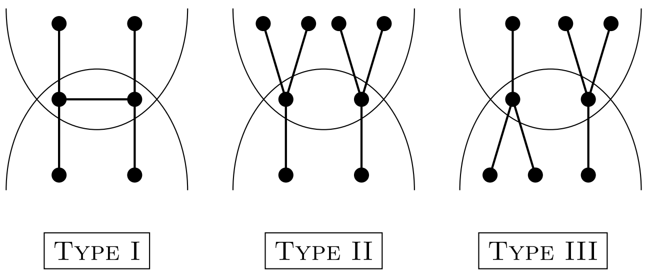

Figure 1: The three types of splitting 2-separations in cubic Cayley graphs of connectivity 2.

First, we dismiss the case of TYPE III separations by easily showing that we can always choose TYPE II instead for the nested set of separations and the respective tree-decomposition obtained by Lemma 2.1 and Lemma 2.2.

Lemma 3.1. *Assume that* Γ *contains a* TYPE III *separation distinguishing efficiently at least two ends. Then it also contains a* TYPE II *separation distinguishing efficiently the same ends.*

Proof. Let (A, B) be a TYPE III separation on $A \cap B = \{x, y\}$ distinguishing efficiently at least two ends. We can assume that $|N(x) \cap A| = 1$ and $|N(x) \cap B| = 2$. Let x' be the unique neighbor of x in A. Then $(A \setminus \{x\}, B \cup \{x\})$ is a tight TYPE II separation on $\{x', y\}$, clearly distinguishing efficiently the same ends as (A, B). $\square$

In what follows, $(T, \mathcal{V})$ will always be as in Lemma 2.2, either of TYPE I or TYPE II if not specified. For a node $t \in V(T)$, we define

$$n(t) := \Gamma \left[\bigcup_{t \in N_T[t]} V_t \right].$$

Recall that every adhesion set $V_t \cap V_{t'}$ of $(T, \mathcal{V})$ induces the separation $(W_{t \setminus t'}, W_{t' \setminus t})$ of Γ. Assume that $(T, \mathcal{V})$ and the separations $(W_{t \setminus t'}, W_{t' \setminus t})$ it induces is of Type II. We call such a separation $(W_{t\setminus t'}, W_{t'\setminus t})$ *small* if the vertices of the separator $V_t \cap V_{t'}$ have degree 1 in $W_{t'\setminus t}$ and *big* if they have degree 2 in $W_{t'\setminus t}$.

One of our main goals towards the general structure of the tree-decomposition of Γ is to eventually prove in Lemma 3.4 that all adhesion sets of $(T, \mathcal{V})$ are disjoint. As a preparatory step for that, we need the following Lemma.

Lemma 3.2. *Every vertex u belongs in at least one and at most two different adhesion sets of $(T, \mathcal{V})$ (as subsets of $V(\Gamma)$ and not as intersections of different pairs of parts).*

Proof. The lower bound follows directly from the transitivity of the actions of G on Γ and $E(T)$. For the upper bound, let $\{x, u\}$ and $\{y, u\}$ be two adhesion sets of the tree-decomposition meeting on u. Since G acts transitively on $E(T)$, there is a $1 \neq g \in G$ such that $g\{x, u\} = \{y, u\}$. Observe that since $g \neq 1$, we must have $gx = u$ and $gu = y$, from which we obtain $ux^{-1}u = y$. Since $\{x, u\}$ and $\{y, u\}$ were arbitrary adhesion sets containing u, the upper bound follows. □

Let H be an arbitrary graph with a set $U \subseteq V(H)$ and a subgraph H' of H. The set U is called *connected in H'* if for every pair of vertices $u, u' \in U$ there is a (u, u')-path in H'.

Lemma 3.3. *Let t be an arbitrary vertex of T. Then for every $t' \in N_T(t)$, the following holds:*

(i) *The adhesion set $V_t \cap V_{t'}$ is connected in at least one of $V_t, V_{t'}$.*

(ii) *V_t is connected in $n(t)$.*

Proof. (i) Let $V_t \cap V_{t'} = \{u, u'\}$ and P be a path between u and u'. Since P is finite, we eventually find a part V_s of $(T, \mathcal{V})$ such that $P' = V(P) \cap V_s$ is a subpath of P whose end vertices constitute exactly one of the adhesion sets S of V_s. Recall that G acts transitively on the set of adhesion sets of $(T, \mathcal{V})$. Hence, we can map S to $V_t \cap V_{t'}$, say $gS = V_t \cap V_{t'}$. Then $gs \in \{t, t'\}$. Thus, gP' is a (u, u')-path that either lies in V_t or V_t'.

(ii) Since Γ is connected, the torso of V_t is a connected graph. The result follows by replacing the virtual edges of a path within the torso of V_t by paths obtained by (i). □

The next crucial lemma implies that all adhesion sets in $\mathcal{N}$ are disjoint.

Lemma 3.4. *Let t be a node of T. Then for every $t_1, t_2 \in N_T(t)$, we have $V_{t_1} \cap V_{t_2} = \emptyset$.*

Proof. Suppose that there are $t_1, t_2 \in N_T(t)$ such that $V_{t_1} \cap V_{t_2} \neq \emptyset$. Clearly, $|V_{t_1} \cap V_{t_2}| \leq 2$.

First, let $|V_{t_1} \cap V_{t_2}| = 2$. It follows from the definition of a tree-decomposition that $V_{t_1} \cap V_{t_2} \subseteq V_t$ and so $V_{t_1} \cap V_{t_2}$ is a subset of both $V_t \cap V_{t_1}$ and $V_t \cap V_{t_2}$. Therefore, we have $V_{t_1} \cap V_t = V_{t_2} \cap V_t = V_{t_1} \cap V_{t_2} := S$. Let T_S be the subtree of T whose corresponding parts contain S. Then $|V(T_S)| \geq 3$.

Assume $|V(T_S)| \geq 4$. Since all separations of $\mathcal{N}$ are tight, observe that $\Gamma \setminus S$ has at least four tight components. Hence, $|T_S| = 3$ and so $V(T_S) = \{t_1, t_2, t\}$.

Consequently, since Γ is cubic, we easily see that $C_1 = W_{t_1\setminus t}$, $C_2 = W_{t_2\setminus t}$ and $C_3 = (W_{t\setminus t_1}) \setminus (W_{t_2\setminus t}) = (W_{t\setminus t_2}) \setminus (W_{t_1\setminus t})$ must be the components of $G \setminus S$, all of them tight.

This means that both vertices of S must have degree one in each of V_{t_1}, V_{t_2}, V_t and that S induces an independent set. Since G acts transitively on Γ and $E(T)$, it follows that every vertex has degree one in every part it belongs in. We conclude that every part of $\mathcal{V}$ induces a matching where every pair of vertices in the same adhesion set is unmatched. This yields a contradiction to Part (i) of Lemma 3.3.

Next, let $|V_{t_1} \cap V_{t_2}| = 1$. Let $V_{t_1} = \{x, y\}$, $V_{t_2} = \{x, z\}$. Again since $\mathcal{N}$ is a set containing tight separations and Γ is cubic, we deduce that $\Gamma \setminus V_{t_1} \cap V_{t_2}$ has at most three components and so every vertex of Γ lies in exactly three parts of $\mathcal{V}$. We can assume that $(T, \mathcal{V})$ is of Type III: indeed, assume that $(T, \mathcal{V})$ is of Type I. By the tightness of all separations in $\mathcal{N}$, we have that x has at least one neighbor in each of $V_{t_1} \setminus V_t$ and $V_{t_2} \setminus V_t$ in addition to y and z, a contradiction to Γ being cubic. Hence, $(T, \mathcal{V})$ is of Type II.

Now, assume that the separations $(W_{t\setminus t_1}, W_{t_1\setminus t})$ and $(W_{t\setminus t_2}, W_{t_2\setminus t})$ are not in the same orbit under the action of G on $E(T)$. Then, there is $g \in G$, such that

$$(W_{t\setminus t_1}, W_{t_1\setminus t}) = (gW_{t_2\setminus t}, gW_{t\setminus t_2}) \tag{1}$$

and we can assume w.l.o.g. that they are small separations. We observe that it must be $\deg_{V_{t_1}}(x) = 1$, $\deg_{V_t}(x) = 0$ and $\deg_{V_{t_2}}(x) = 2$.

By the transitive action of G on Γ and $E(T)$, we have for an arbitrary vertex u that $\deg_{V_s}(u) = 0$, where s is the middle node of the path of length two in T containing u. Since $\deg_{V_{t_1}}(y) \neq 0$, the node t_1 cannot be the middle node of the path of length two in T, whose nodes contain y. It follows that $\deg_{V_t}(y) = 0$. By the fact that $(W_{t\setminus t_1}, W_{t_1\setminus t})$ is small, we conclude that $\deg_{V_{t_1}}(y) = 1$ and that there exists $t_3 \in N_T(t)$ with $\deg_{V_{t_3}}(y) = 2$. Similarly, we have $\deg_{V_t}(z) = 0$, $\deg_{V_{t_2}}(z) = 2$ and there exists $t_4 \in N_T(t)$ with $\deg_{V_{t_4}}(z) = 1$.

Therefore, every $v \in V_t$ has degree 0 in V_t. By Lemma 3.3, there is an (x, y)-path P lying completely within V_t, but by (1) we have that gP lies within V_t, which yields a contradiction.

Otherwise, $(W_{t\setminus t_1}, W_{t_1\setminus t})$ and $(W_{t\setminus t_2}, W_{t_2\setminus t})$ are in the same orbit of the action of G on $E(T)$. Subsequently, there is $g \in G$ such that

$$(W_{t\setminus t_1}, W_{t_1\setminus t}) = (gW_{t\setminus t_2}, gW_{t_2\setminus t}).$$

Since Γ is cubic, we observe that it must be $\deg_{V_t}(x) = \deg_{V_{t_1}}(x) = \deg_{V_{t_2}}(x) = 1$. As before, by the transitive action of G on Γ and $E(T)$ we have that every $u \in \Gamma$ has degree one in all three parts of $\mathcal{V}$ it is contained. Hence, every part induces a matching. Consequently, there is no (x, y)-path in V_t or V_{t_1}, which violates Lemma 3.3. $\square$

Lemma 3.4 has some important consequences. Combined with Lemma 3.2, we immediately obtain the following.

Corollary 3.5. *Every vertex u of Γ is contained in exactly two parts $t, t' \in V(T)$. In addition, $N_\Gamma(u) \subseteq V_t \cup V_{t'}$ and every part is the disjoint union of its adhesion sets.* $\square$

Moreover, let $\{x, y\}$ be an adhesion set. Observe that $xy^{-1}\{x, y\}$ is again an adhesion set containing x, so $xy^{-1}\{x, y\} = \{x, y\}$ with $xy^{-1}x = y$. We obtain:

Lemma 3.6. *For every adhesion set $\{x, y\}$, we have $(xy^{-1})^2 = 1$.* □

Lemma 3.6 implies the following Corollary for the edge stabilizers of T.

Corollary 3.7. *Let $tt' \in E(T)$. Then $\mathsf{St}_G(V_t \cap V_{t'}) \cong \mathbb{Z}_2$.* □

Lastly, we will invoke the following folklore Lemma from the well-known theory of tree decompositions into 3-connected components (see [15, 17] as an example) when we argue about the planarity of Γ and G in each case that arises.

Lemma 3.8. *Let $(T, \mathcal{V})$ be a tight tree-decomposition of a (locally finite) connected graph H with finite parts and adhesion at most 2. Then Γ is planar if and only if the torso of every part of $(T, \mathcal{V})$ is planar.*

Proof. The forward implication follows from the fact that the torso of a part in $(T, \mathcal{V})$ is a topological minor of H: for every virtual edge of the part realized by an adhesion set of size exactly two, there is always a path outside of the part that connects the two vertices of the adhesion set.

For the backward implication, embed T on the plane. It is straightforward to combine the planar embeddings of every torso along the adhesion sets according to T following its embedding. □

Our goal in the following sections is to determine the structure of the parts of the tree-decomposition of Γ obtained by Corollary 2.2 in order to compute their stabilizers and apply Lemma 2.5 or 2.6.

4 Tree-decomposition of Type I

In this section, we assume that $(T, \mathcal{V})$ is of Type I. Suppose that b is the label of the edge induced by the adhesion sets of $(T, \mathcal{V})$, which by Lemma 3.6 is an involution. It will be enough to study two neighboring parts $V_t, V_{t'}$ to obtain the general structure of $(T, \mathcal{V})$. In order to simplify this, we can assume w.l.o.g that $V_t \cap V_{t'} = \{1, b\}$, so $\mathsf{St}_G(V_t \cap V_{t'}) = \langle b \rangle$.

Notice that if G acts on $(T, \mathcal{V})$ with inversion, there is an element in $g \in \mathsf{St}_G(V_t \cap V_{t'}) = \langle b \rangle$ that inverts $V_t, V_{t'}$. Let us express this easy fact with the following lemma.

Lemma 4.1. *G acts with inversion on $(T, \mathcal{V})$ if and only if b inverts V_t and $V_{t'}$.* □

Moreover, the following Lemma holds regardless of the number of generators in S.

Lemma 4.2. *Every part of $\mathcal{V}$ induces a finite cycle.*

Proof. Let $t \in V(T)$. Since every adhesion set induces a connected subgraph, we conclude by Lemma 2.3 that $\Gamma[V_t]$ is connected. Moreover, Corollary 3.5 implies that $\Gamma[V_t]$ is 2-regular. It follows that $\Gamma[V_t]$ is either a finite cycle or a double ray. Recall that by Lemma 3.4 all adhesion sets are disjoint. The conclusion follows by observing that every vertex of V_t is a cut vertex when V_t induces a double ray and hence, the graph Γ is not 2-connected. □

It will be clear by Lemma 3.8 that we will obtain in all subcases planar Cayley graphs.

4.1 Two Generators

Assume that $G = \langle a, b\rangle$, where b is an involution. We distinguish the following cases depending on the colors of the edges incident to the adhesion sets, depicted as in the following Figure.

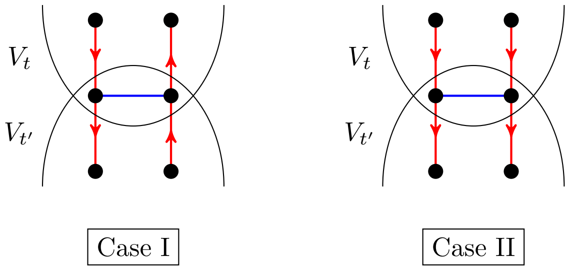

Figure 2: Cases of TYPE I with two generators

4.1.1 Case I

Suppose that the edges incident to each adhesion set in $\mathcal{N}$ are as in Case I of Figure 2. Observe that $\{a^{-1}, ba\} \subseteq V_t$ and $\{a, ba^{-1}\} \subseteq V_{t'}$ are the neighbors of 1 and b in V_t and $V_{t'}$, respectively. Since $b\{a^{-1}, ba\} = \{a, ba^{-1}\}$, it must be that $bV_t = V_{t'}$ and $bV_{t'} = V_t$. Lemma 4.1 implies that G acts on $E(T)$ with inversion (and hence transitively on V(T)).

By Lemma 4.2, there is an $n \in \mathbb{N}$ such that that $(ba)^n = 1$ and

$$V_t = \{1, b, ba, \dots, (ba)^{n-1}b = a^{-1}\}.$$

This gives a partition $\langle ba\rangle \sqcup \langle ba\rangle b$ of V_t. We next conclude that $\mathsf{St}_G(V_t) \subseteq V_t$ by noting that $1 \in V_t$. Clearly, we have $\langle ba\rangle \subseteq \mathsf{St}_G(V_t)$. Moreover, for the element $ba \in V_t$, we observe that

$$(ba)^i b(ba) = (ba)^i a \notin V_t.$$

Since $V_t = \langle ba\rangle \sqcup \langle ba\rangle b$, we conclude that $\mathsf{St}_G(V_t) = \langle ba\rangle \cong \mathbb{Z}_n$. Moreover, $\mathsf{St}_G(V_t) \cap \mathsf{St}_G(V_t \cap V_{t'}) = \langle ba\rangle \cap \langle b\rangle = 1$.

We apply Lemma 2.6 and obtain that

$$G \cong \mathbb{Z}_n * \mathbb{Z}_2.$$

4.1.2 Case II

By the structure of the neighbourhood of $\{1, b\}$ and Lemma 4.1 we see that b cannot invert V_t and $V_{t'}$, hence G acts on $(T, \mathcal{V})$ without inversion.

Now, consider the adhesion set $a^{-1}\{1, b\} = (a^{-1}V_t)\cap(a^{-1}V_{t'})$. From $a\{1, b\} \subseteq V_t$ we deduce that $\{V_t, V_{t'}\} = \{a^{-1}V_t, a^{-1}V_{t'}\}$. Since the adhesion set $\{1, b\}$ has ingoing a-edges but $a\{1, b\}$ has outgoing a-edges in V_t, we cannot have that

$a^{-1}V_t = V_t$. Consequently, it must be that $a^{-1}V_{t'} = V_t$. The fact that two adjacent parts lie in the same orbit under the action of G implies that G acts transitively on $\mathcal{V}$ (and $V(T)$).

By Lemma 4.2, there is in this case an $n \in \mathbb{N}$ such that $(ba^{-1}ba)^n = 1$ and

$$V_t = \{1, b, ba^{-1}, ba^{-1}b, \ldots, (ba^{-1}ba)^{n-1}ba^{-1}b = a^{-1}\}.$$

In other words, $\langle ba^{-1}ba\rangle \sqcup \langle ba^{-1}ba\rangle b \sqcup \langle ba^{-1}ba\rangle ba^{-1} \sqcup \langle ba^{-1}ba\rangle ba^{-1}b$ forms a partition of V_t. Notice that $\langle ba^{-1}ba\rangle$ is the trivial group when $ba^{-1}ba = 1$. As before, since $1 \in V_t$ we infer that $\mathsf{St}_G(V_t) \subseteq V_t$. Clearly, we have $\langle ba^{-1}ba\rangle \subseteq \mathsf{St}_G(V_t)$. Moreover, we see that $\langle ba^{-1}ba\rangle ba^{-1} \not\subseteq \mathsf{St}_G(V_t)$ because we have $(ba^{-1}ba)^i ba^{-1}(ba^{-1}ba) \notin V_t$ and that $\langle ba^{-1}ba\rangle ba^{-1}a \not\subseteq \mathsf{St}_G(V_t)$ because $(ba^{-1}ba)^i ba^{-1}b(a^{-1}ba) \notin V_t$.

Lastly, observe that since b is an involution and all adhesion sets induce a b-edge, we have that the action of b on Γ fixes every adhesion set. Hence, we have that $b \in \mathsf{St}_G(V_t)$. It follows that $\langle ba^{-1}ba, b\rangle \subseteq \mathsf{St}_G(V_t)$. Therefore, we conclude that

$$\mathsf{St}_G(V_t) = \langle ba^{-1}ba, b \mid b^2, (ba^{-1}ba)^n, (a^{-1}ba)^2\rangle \cong D_{2n}.$$

By Lemma 2.5, we have that

$$G \cong D_{2n} \underset{\mathbb{Z}_2}{*} (t).$$

We collect both cases in the following theorem.

Theorem 4.3. *If $(T, \mathcal{V})$ is of* Type I *with two generators, then G satisfies one of the following cases:*

(i) $G \cong \mathbb{Z}_n * \mathbb{Z}_2$.

(ii) $G \cong D_{2n} \underset{\mathbb{Z}_2}{*} (t)$. □

The definitions of a free product with amalgamation, an HNN-extention and the proof of Theorem 4.3 immediately imply:

Corollary 4.4. [9, Theorem 1.1] *If $(T, \mathcal{V})$ is of* Type I *with two generators, then G has one of the following presentations:*

(i) $\langle a, b \mid b^2, (ba)^n\rangle$.

(ii) $\langle a, b \mid b^2, (ba^{-1}ba)^n\rangle$. □

4.2 Three Generators

Let $G = \langle a, b, c\rangle$, where a, b and c are involutions. Suppose that the edges induced by the adhesion sets in $\mathcal{N}$ are colored with b. Up to rearranging a, b, c, there are two cases for the local structure of the separators in $\mathcal{N}$, as in the following figure:

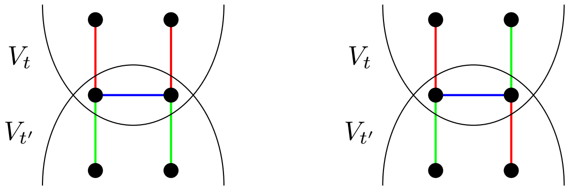

Figure 3: Cases of TYPE I with three generators

4.2.1 Case I

First, we observe by Lemma 4.1 that G acts on T without inversion, since by the structure of the neighbourhood of $\{1, b\}$ we see that b must stabilize both V_t and $V_{t'}$. Consequently, G must act with two orbits O_1, O_2 on $\Gamma[\mathcal{V}]$, where the parts in O_1 contain the a-edges and the parts in O_2 contain the c-edges. By Lemma 4.2 we deduce that $(ba)^n = 1$ and $(bc)^m = 1$ and so $V_t = \langle ba \rangle \sqcup \langle ba \rangle b$ and $V_{t'} = \langle bc \rangle \sqcup \langle bc \rangle b$

To compute the stabilizers of the parts, observe that we can escape a part in O_1 only with c-edges. Hence, we have $\mathsf{St}_G(V_t) = V_t = \langle ba, b \mid b^2 = (ba)^n = a^2 \rangle \cong D_{2n}$ and similarly $\mathsf{St}_G(V_{t'}) = V_{t'} = \langle bc, b \mid b^2 = (bc)^m = c^2 \rangle \cong D_{2m}$. Therefore, by Lemma 2.5 we obtain

$$G \cong D_{2n} \underset{\mathbb{Z}_2}{*} D_{2m}.$$

4.2.2 Case II

In this case, we see that b inverts V_t and $V_{t'}$, so G acts on T with inversion by Lemma 4.1. Hence, G also acts transitively on $V(T)$.

Let $x := bcba$. By Lemma 4.2 we see that $(bcba)^n = 1$ and that $\langle x \rangle \sqcup \langle x \rangle b \sqcup \langle x \rangle bc \sqcup \langle x \rangle bcb$ is a partition of V_t. Clearly, we have that $\langle bcba \rangle \subseteq \mathsf{St}_G(V_t)$. We show that we actually have equality:

- $x^i b \cdot bc = x^i c \notin V_t$, hence $\langle x \rangle b \notin \mathsf{St}_G(V_t)$,
- $x^i bc \cdot a \notin V_t$, hence $\langle x \rangle bc \notin \mathsf{St}_G(V_t)$,
- $x^i bcb \cdot c \notin V_t$, hence $\langle x \rangle bcb \notin \mathsf{St}_G(V_t)$.

We conclude that $\mathsf{St}_G(t) = \langle bcba \rangle \cong \mathbb{Z}_n$ and consequently we also have that $\mathsf{St}_G(V_t) \cap \mathsf{St}_G(V_t \cap V_{t'}) = \langle bcba \rangle \cap \langle b \rangle = 1$. It follows from Lemma 2.6 that

$$G \cong \mathbb{Z}_n * \mathbb{Z}_2.$$

In conclusion, we have proved:

Theorem 4.5. *If $(T, \mathcal{V})$ is of* TYPE I *with three generators, then G satisfies one of the following cases:*

(i) $G \cong D_{2n} \underset{\mathbb{Z}_2}{*} D_{2m}$.

(ii) $G \cong \mathbb{Z}_n * \mathbb{Z}_2$. □

Corollary 4.6. [9, Theorem 1.1] *If* $(T, \mathcal{V})$ *is of* Type I *with three generators, then G has one of the following presentations:*

(i) $\langle a, b, c \mid a^2, b^2, c^2, (ba)^n, (bc)^m \rangle$.

(ii) $\langle a, b, c \mid a^2, b^2, c^2, (bcba)^n \rangle$. □

5 Tree-decomposition of Type II

Even though at first glance there can be several cases for Type II separations, we will in fact be able to quickly exclude most of them using appropriately the following lemma.

Lemma 5.1. *Let* $G = \langle a, b, c \rangle$ *(with possibly* $c = a^{-1}$*), where b is an involution and let* $\{x, y\}$ *be a* Type II *separation in* Γ *as in Lemma 2.1. Let* v_1, v_2, v_3 *be any consecutive vertices in a shortest* (x, y)*-path P with at least two edges and suppose there is* $g \in G$ *such that* $gv_2 \in \{x, y\}$*. Then* gv_1 *and* gv_3 *lie in the same component of* $\Gamma \setminus \{x, y\}$*.*

Proof. Suppose not. We observe that gx, gy must then lie in different components of $\Gamma \setminus \{x, y\}$ as well: if not, then gx, gy lie in the same component and since $gv_2 \in gP \cap \{x, y\}$, we have that both $x, y \in V(gP)$. Since $gv_2 \in \{x, y\}$ is an inner vertex of gP, the subpath of gP from x to y contradicts the choice of P.

Hence, $g\{x, y\}$ is a separator where gx, gy lie in different components of $\Gamma \setminus \{x, y\}$. It easily follows that $\{x, y\}$ and $\{gx, gy\}$ are not nested, a contradiction to Lemma 2.1. □

Now, let $V_{2n}, n \geq 2$ denote the cubic graph obtained by the $2n$-cycle along with the "diagonal" edges (Fig. 4).

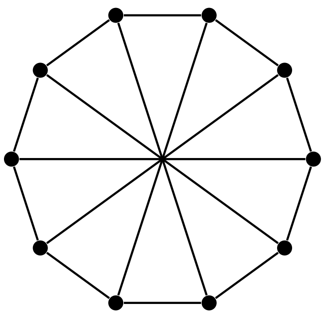

Figure 4: The graph V_{10}.

Moreover, let R_{2m+1} be the cubic graph obtained by a double ray with vertex set $\mathbb{Z}$ (defined in the natural way) and by adding the edges of the form $\{2i, 2i + 2m + 1\}$ (Fig. 5).

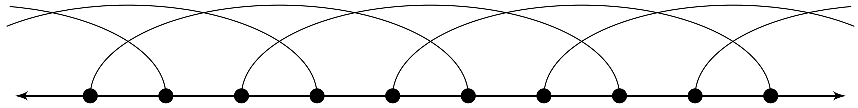

Figure 5: The graph R_5.

We note that we will see in the next subsections that the tree-decomposition of Γ obtained by Corollary 2.2 will have two orbits of parts and that the torsos of the parts of one of the two orbits will always be isomorphic to either V_{2n} or R_{2n+1}, depending on whether the part is finite or infinite. The fact that V_{4n} and R_{2m+1} are planar if and only if $n = 2$ and $m = 1$, respectively, will allow us by Lemma 3.8 to determine exactly when Γ will be planar.

5.1 Two generators

Let $G = \langle a, b\rangle$, where b is an involution. Let $\mathcal{N}$ be as in Lemma 2.1 and $(T, \mathcal{V})$ the corresponding tree-decomposition obtained by Lemma 2.2. Then we have the following cases for the neighbourhood of a separation of $\mathcal{N}$ on $\{x, y\}$:

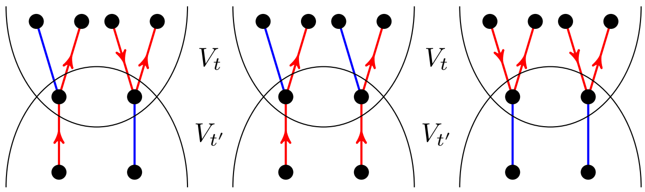

Figure 6: Cases of TYPE II with two generators.

Lemma 5.2. *The adhesion sets of $(T, \mathcal{V})$ satisfy Case III.*

Proof. Let $\{x, y\}$ be an adhesion set. First, observe that no path in Γ contains two consecutive b-edges, hence every path of length two contains at least one a-edge. Let P be a shortest (x, y)-path[2], necessarily of length at least two.

Assume that either Case I or Case II happen. Notice that –in both cases– for every possible edge-coloring of a path of length two there exists a path Q of length two whose middle vertex belongs in $\{x, y\}$ and its two endpoints lie in different components of $\Gamma \setminus \{x, y\}$ that realizes the same edge-coloring. Consider an arbitrary subpath $P' = v_1v_2v_3$ of P of length two and an appropriate Q as above that realizes the edge-coloring of P'. Let w be the middle vertex of Q and $g = wv_2^{-1}$. Then $gP = Q$ and gv_1, gv_3 lie in different components of $\Gamma \setminus \{x, y\}$, contradicting Lemma 5.1. $\square$

Consequently, we can assume for the rest of this subsection that only Case III happens. It follows that no part of $(T, \mathcal{V})$ contains edges of all colors: otherwise, by Corollary 3.5 we see for such a part V_t that the a-edges and the b-edges induce different connected components in the torso of V_t, a contradiction to the

[2]By Lemma 3.3(i) we can see that P lies completely within V_t or $V_{t'}$, but this is irrelevant to the proof of the Lemma.

connectivity of Γ. Hence, $(T, \mathcal{V})$ has two orbits of parts O_1, O_2, where parts in O_1 contain only edges colored with a and parts in O_2 contain edges colored with b. Moreover, G acts on $(T, \mathcal{V})$ without inversion. The structure of the parts in O_2 is clear: their edges induce a perfect b-matching in the bag. We are ready to obtain the full structure of the parts in O_1 as well.

Lemma 5.3. *There is an $n \geq 2$, such that for every adhesion set $\{x, y\}$ we have $x = ya^n$ or $x = ya^{-n}$. Moreover, every part in O_1 induces an a-cycle of length $2n$.*

Proof. Let $V_t \in O_1$ and $\{x, y\} = V_t \cap V_{t'}$ be an adhesion set of t. For every $s \in N_T(t)$, we have that $V_s \in O_2$ and consequently that V_s induces a b-matching. By Lemma 3.3(ii), it follows that $G[V_t]$ is connected.

Consider an (x, y)-path P within V_t and let $n \geq 2$ be its length. Hence, $x = ya^n$ or $x = ya^{-n}$. By Lemma 3.6, we have $(xy^{-1})^2 = 1$, from which we obtain $a^{2n} = 1$ after substituting x.

We have inferred that the 2-regular graph $\Gamma[V_t]$ is connected. Notice that $\Gamma[V_t]$ can be a double a-ray only if $xy^{-1}P = P$. But since P is an a-path, it can only intersect $xy^{-1}P$ on x, y. Recall that a has order $2n$. This directly implies the Lemma. $\square$

Observe that the torso of a part $V_s \in O_2$ induces a connected, 2-regular graph. It cannot be a double ray: in that case every vertex is a cut vertex (as is easily seen), which violates the 2-connectivity of Γ. Hence, the torso of V_s induces a finite cycle, whose edges we can label by Lemma 5.3 with a^n (corresponding to the virtual edges of the torso) and b in an alternating fashion. Therefore, there is a $m \geq 2$ such that $(ba^n)^m = 1$.

It remains to compute the vertex stabilizers of T.

Let $V_{t_1} \in O_1$ such that $1 \in V_{t_1}$. By Lemma 5.3, we clearly have $\langle a \rangle = V_{t_1}$ and therefore $\mathsf{St}_G(V_{t_1}) = \langle a \rangle \cong \mathbb{Z}_{2n}$. Next, let $V_{t_2} \in O_2$ such that $1 \in V_{t_2}$. Recall that $(ba^n)^m = 1$ and notice that $(b(ba^n))^2 = a^{2n} = 1$. By the structure of the torso of V_{t_2}, we observe that the elements of V_{t_2} form a group generated by b and ba^n with presentation $\langle ba^n, b \mid ((ba)^n)^m, b^2, (b(ba^n))^2 \rangle$. Since V_{t_2} forms a subgroup of G, we deduce that

$$\mathsf{St}_G(V_{t_2}) = V_{t_2} = \langle ba^n, b \mid ((ba)^n)^m, b^2, (b(ba^n))^2 \rangle \cong D_{2m}.$$

Finally, by Lemma 2.5 we obtain $G \cong \mathbb{Z}_{2n} \underset{\mathbb{Z}_2}{*} D_{2m}$.

We observe that the torso of V_{t_1} is isomorphic to V_{2n}. Since V_{2n} is planar if and only if $n = 2$, we conclude by Lemma 3.8 that Γ is planar if and only if $n = 2$. We have obtained the following theorem, along with its corollary by the definition of a free product with amalgamation:

Theorem 5.4. *If $(T, \mathcal{V})$ is of* Type II *with two generators, then*

$$G \cong \mathbb{Z}_{2n} \underset{\mathbb{Z}_2}{*} D_{2m}.$$

In particular, G is planar if and only if $n = 2$.

Corollary 5.5. [9, Theorem 1.1] *If $(T, \mathcal{V})$ is of* Type I *with two generators, then*

$$G = \langle a, b \mid b^2, a^{2n}, (ba^n)^m \rangle.$$

In particular, G is planar if and only if $n = 2$.

5.2 Three generators

Let $G = \langle a, b, c \rangle$, where a, b and c are involutions. Then –up to rearranging a, b, c– we have the following cases for the separations in $\mathcal{N}$:

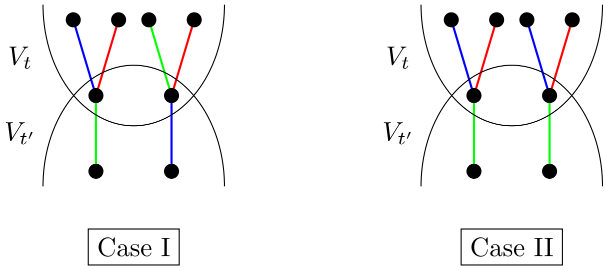

Figure 7: TYPE II cases with three generators

As in Subsection 5.1, by properly applying Lemma 5.1 we obtain the analogue of Lemma 5.2 for three generators with exactly the same proof.

Lemma 5.6. *The adhesion sets of* $(T, \mathcal{V})$ *satisfy Case II.* □

Since the torso of every part of $(T, \mathcal{V})$ is a connected graph, we deduce that the tree-decomposition has two orbits of parts: parts in O_1 contain only b- and c-edges and parts in O_2 induce perfect a-matchings. Clearly, G then acts on $(T, \mathcal{V})$ without inversion. Let us quickly obtain the analogue of Lemma 5.3.

Lemma 5.7. *Every part in* O_1 *induces an alternating* (b, c)*-cycle of length a multiple of* 4 *or an alternating double* (b, c)*-ray.*

Proof. Let $V_t \in O_1$ and $\{x, y\} = V_t \cap V_{t'}$ be an adhesion set of t. Since all neighbours of t induce an a-matching, it follows by Lemma 3.3(ii) that $\Gamma[V_t]$ is connected.

Hence, there exists an (x, y)-path P of length i within V_t, necessarily alternating with b- and c-edges. Then, either $x = y(bc)^n$ or $x = y(bc)^n b$, up to swapping b and c. To obtain the structure of the 2-regular, connected graph V_t we distinguish two cases.

- If $x = y(bc)^n$, then the (x, y)-path $xy^{-1}P$ intersects P only in x, y and by Lemma 3.6, we obtain $(bc)^{2n} = 1$. In this case, V_t induces an alternating (b, c)-cycle of length $4n$.
- If $x = y(bc)^n b$, then $xy^{-1}P = P$ and, consequently, V_t induces an alternating double (b, c)-ray. □

By the 2-connectivity of Γ, the connected, 2-regular torso of a part $V_s \in O_2$ must be a finite cycle. Depending on which of the cases of Lemma 5.7 we have, we can label its edges with $(bc)^n$ or $(bc)^n b$ (corresponding to the virtual edges of the torso) and a in an alternating fashion. Therefore, there is an $m \geq 2$ such that $(a(bc)^n)^m = 1$ or $(a(bc)^n b)^m = 1$. It remains to infer the structure of G in each case.

(i) Suppose that every part in O_1 is an alternating (b, c)-cycle of length $4n$ and $(a(bc)^n)^m = 1$.

In order to compute the vertex stabilizers of T, let $V_{t_1} \in O_1$ with $1 \in V_{t_1}$. Since $(b(bc))^2 = c^2 = 1$, we have that

$$V_{t_1} = \langle bc \rangle \cup \langle bc \rangle b = \langle bc, b \mid (bc)^{2n}, b^2, (b(bc))^2 \rangle \cong D_{4n}.$$

Then $\mathsf{St}_G(V_{t_1}) = V_{t_1} \cong D_{4n}$, as V_{t_1} forms a group. Next, let $V_{t_2} \in O_2$ with $1 \in V_{t_2}$. Notice that $(a(bc)^n)^m = a^2 = 1$ and $(a(a(bc)^n))^2 = (bc)^{2n} = 1$. We can deduce that V_{t_2} is a group (and hence $\mathsf{St}_G(V_{t_2}) = V_{t_2}$), along with its presentation:

$$\mathsf{St}_G(V_{t_2}) = V_{t_2} = \langle a(bc)^n, a \mid (a(bc)^n)^m, a^2, (a(a(bc)^n))^2 \rangle \cong D_{2m}.$$

By Lemma 2.5, we have

$$G \cong D_{4n} \underset{\mathbb{Z}_2}{*} D_{2m}.$$

In this case, the torso of V_{t_1} is isomorphic to V_{4n}, which is planar if and only if $n = 1$.

(ii) Assume that every part in O_1 is an alternating double (b,c)-ray and $(a(bc)^n b)^m = 1$.

Let $V_{t_1} \in O_1$ and $V_{t_2} \in O_2$, both containing 1 in the respective parts. Similarly, we see that

$$\mathsf{St}_G(V_{t_1}) = V_{t_1} = \langle bc, b \mid b^2, (b(bc))^2 \rangle \cong D_\infty,$$

$$\mathsf{St}_G(V_{t_2}) = V_{t_2} = \langle a(bc)^n b, a \mid (a(bc)^n b)^m, a^2, (a(a(bc)^n b))^2 \rangle \cong D_{2m}.$$

By Lemma 2.5,

$$G \cong D_\infty \underset{\mathbb{Z}_2}{*} D_{2m}.$$

Notice that the torso of V_{t_1} is isomorphic to R_{2n+1}, which is planar if and only if $n = 1$.

By Lemma 3.8 and the above discussion, we have deduced:

Theorem 5.8. *If $(T, \mathcal{V})$ is of* Type II *with three generators, then G satisfies one of the following cases:*

(i) $G \cong D_{4n} \underset{\mathbb{Z}_2}{*} D_{2m}$.

(ii) $G \cong D_\infty \underset{\mathbb{Z}_2}{*} D_{2m}$. □

Corollary 5.9. [9, Theorem 1.1] *If $(T, \mathcal{V})$ is of* Type I *with three generators, then G has one of the following presentations:*

(i) $G = \langle a, b, c \mid a^2, b^2, c^2, (bc)^{2n}, (a(bc)^n)^m \rangle$ *and Γ is planar if and only if $n = 1$.*

(ii) $G = \langle a, b, c \mid a^2, b^2, c^2, (a(bc)^n b)^m \rangle$ *and Γ is planar if and only if $n = 1$.* □

6 Open Questions

Having obtained the full characterization of groups admitting cubic Cayley graphs of connectivity two, some further open questions can naturally be raised. In light of Lemma 2.4, we can ask the following.

Problem 1. *Characterize all groups admitting* 4*-regular Cayley graphs of connectivity at most three in terms of splitting over subgroups.*

A graph is called quasi-transitive if it has a finite number of orbits under the action of its automorphism group. Looking back at Theorem 1.1, we see that cubic Cayley graphs of connectivity two can be expressed as a tree decomposition whose torsos induce two cycles or the double ray and a cycle. The main tools from our proof seem to go through to support that this is in general the case for every cubic transitive graph of connectivity two. We can go a step futher and ask the following question:

Problem 2. *Characterize all cubic quasi-transitive graphs of connectivity two in terms of "canonical" tree decompositions with the property that the automorphism group of the graph acts transitively on the set of the adhesion sets.*

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