# Model-independent comparison between factorization algebras and algebraic quantum field theory on Lorentzian manifolds

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## Abstract

This paper investigates the relationship between algebraic quantum field theories and factorization algebras on globally hyperbolic Lorentzian manifolds. Functorial constructions that map between these two types of theories in both directions are developed under certain natural hypotheses, including suitable variants of the local constancy and descent axioms. The main result is an equivalence theorem between (Cauchy constant and additive) algebraic quantum field theories and (Cauchy constant, additive and time-orderable) prefactorization algebras.

## Contents

# 1 Introduction and summary

Factorization algebras and algebraic quantum field theory are two mathematical frameworks to axiomatize the algebraic structure of observables in a quantum field theory. While from a superficial point of view these two approaches look similar, there are subtle differences. A prefactorization algebra $\mathfrak{F}$ assigns to each spacetime $M$ a *vector space* $\mathfrak{F}(M)$ of observables and to each tuple $\underline{f} = (f_1 : M_1 \to N, \ldots, f_n : M_n \to N)$ of pairwise disjoint spacetime embeddings a factorization product $\mathfrak{F}(\underline{f}) : \bigotimes_{i=1}^{n} \mathfrak{F}(M_i) \to \mathfrak{F}(N)$ satisfying suitable properties, cf. [CG17] and Section 2.2. On the other hand, an algebraic quantum field theory $\mathfrak{A}$ assigns to each spacetime $M$ an associative and unital *algebra* $\mathfrak{A}(M)$ of observables and to each spacetime embedding $f : M \to N$ an algebra morphism $\mathfrak{A}(f) : \mathfrak{A}(M) \to \mathfrak{A}(N)$ such that suitable axioms hold true, cf. [BFV03, FV12, BDFY15, BSW17] and Section 2.3. The main difference is that, in contrast to an algebraic quantum field theory $\mathfrak{A}$, a prefactorization algebra $\mathfrak{F}$ does *not* in general come endowed with a multiplication of observables in $\mathfrak{F}(M)$, i.e. on the same spacetime $M$, because $(\mathrm{id}_M : M \to M, \mathrm{id}_M : M \to M)$ is not a pair of disjoint spacetime embeddings.

In this paper we shall develop functorial constructions (cf. Theorems 3.11 and 4.7) that allow us to relate prefactorization algebras and algebraic quantum field theories, provided that we assume certain natural hypotheses on both sides. We shall focus on the case where spacetimes are described by oriented and time-oriented globally hyperbolic Lorentzian manifolds, i.e. on the case of relativistic quantum field theory. Our main result is an equivalence theorem between (Cauchy constant and additive) algebraic quantum field theories and (Cauchy constant, additive and time-orderable) prefactorization algebras, cf. Theorem 5.1. Our equivalence theorem is considerably stronger and more general than the earlier comparison result by Gwilliam and Rejzner [GR17]: (1) We work in a model-independent setup, supplemented by natural additional hypotheses such as Cauchy constancy, additivity and time-orderability, while [GR17] only studies the example of a free Klein-Gordon field. (2) We investigate in detail uniqueness, associativity, naturality and Einstein causality of the multiplications $\mu_M : \mathfrak{F}(M) \otimes \mathfrak{F}(M) \to \mathfrak{F}(M)$ determined by a Cauchy constant additive prefactorization algebra $\mathfrak{F}$, which requires rather sophisticated arguments from Lorentzian geometry. These questions were not addressed in [GR17]. (3) Our equivalence theorem admits an interpretation in terms of operad theory (cf. Remark 5.2), which provides a suitable starting point for generalizations to higher categorical quantum field theories [CG17, BSS15, BS17, BSW19b, BS19] such as gauge theories. (The present paper does not study this generalization and will focus on the case of 1-categorical quantum field theories.)

Let us now explain in more detail our constructions and results while outlining the content of the present paper: In Section 2 we recall the necessary preliminaries from Lorentzian geometry, factorization algebras and algebraic quantum field theory. All prefactorization algebras and algebraic quantum field theories will be defined on the usual category **Loc** of oriented and time-oriented globally hyperbolic Lorentzian manifolds. We introduce an additivity axiom for both prefactorization algebras and algebraic quantum field theories, which roughly speaking demands that the observables in a spacetime $M$ are generated by the observables in the *relatively compact* and causally convex open subsets $U \subseteq M$. It is shown that factorization algebras, i.e. prefactorization algebras satisfying Weiss descent, are in particular additive prefactorization algebras. We also introduce a Cauchy constancy (or time-slice) axiom for both kinds of theories, which formalizes a concept of time evolution in a globally hyperbolic Lorentzian manifold. In Section 3 we construct a functor $\mathfrak{A}_{(-)} : \mathbf{PFA}^{\mathrm{add},c} \to \mathbf{AQFT}^{\mathrm{add},c}$ that assigns a Cauchy constant additive algebraic quantum field theory $\mathfrak{A}_{\mathfrak{F}}$ to each Cauchy constant additive prefactorization algebra $\mathfrak{F}$, see Theorem 3.11 for the main result. The crucial step is to define canonical multiplications $\mu_M : \mathfrak{F}(M) \otimes \mathfrak{F}(M) \to \mathfrak{F}(M)$ for such $\mathfrak{F}$ (cf. (3.1)), which is done by using Cauchy constancy. Proving naturality and Einstein causality of these multiplications requires the additivity axiom, cf. Propositions 3.7 and 3.10. In Section 4 we construct a functor $\mathfrak{F}_{(-)} : \mathbf{AQFT} \to \mathbf{tPFA}$ that assigns a *time-orderable* prefactorization algebra $\mathfrak{F}_{\mathfrak{A}}$ to each algebraic quantum field the-

ory $\mathfrak{A}$, see Theorem 4.7 for the main result. The difference between time-orderable and ordinary prefactorization algebras on **Loc** is that the former just encode factorization products $\mathfrak{F}(\underline{f}) : \bigotimes_{i=1}^{n} \mathfrak{F}(M_i) \to N$ for tuples of pairwise disjoint morphisms $\underline{f}$ that are in a suitable sense time-orderable, see Definition 4.1. There is a natural forgetful functor **PFA** $\to$ **tPFA** from ordinary to time-orderable prefactorization algebras, which is however not full, see Remarks 4.2 and 4.5. Our results suggest that the concept of time-orderable prefactorization algebras from Section 4 is better suited to the category of Lorentzian spacetimes **Loc** than the more naive concept from Section 2.2 that allows also for factorization products for non-time-orderable tuples of pairwise disjoint morphisms. In Section 5 we explain that the construction $\mathfrak{A}_{(-)} : \mathbf{PFA}^{\mathrm{add},c} \to \mathbf{AQFT}^{\mathrm{add},c}$ from Section 3 factors through the forgetful functor $\mathbf{PFA}^{\mathrm{add},c} \to \mathbf{tPFA}^{\mathrm{add},c}$, thereby defining a functor $\mathfrak{A}_{(-)} : \mathbf{tPFA}^{\mathrm{add},c} \to \mathbf{AQFT}^{\mathrm{add},c}$ that assigns a Cauchy constant additive algebraic quantum field theory to each Cauchy constant additive *time-orderable* prefactorization algebra. Our main Equivalence Theorem 5.1 proves that this functor admits an inverse that is given by the restriction $\mathfrak{F}_{(-)} : \mathbf{AQFT}^{\mathrm{add},c} \to \mathbf{tPFA}^{\mathrm{add},c}$ of the functor from Section 4 to Cauchy constant and additive theories. Hence, Cauchy constant additive algebraic quantum field theories are naturally identified with Cauchy constant additive time-orderable prefactorization algebras. In Section 5.2, we apply our general results to the simple example given by the free Klein-Gordon field $\mathfrak{A}_{\mathrm{KG}} \in \mathbf{AQFT}^{\mathrm{add},c}$. We observe as in [GR17] that the corresponding time-orderable prefactorization algebra $\mathfrak{F}_{\mathrm{KG}} \in \mathbf{tPFA}^{\mathrm{add},c}$ describes the time-ordered products from perturbative algebraic quantum field theory, cf. [FR13, Rej16].

## 2 Preliminaries

### 2.1 Lorentzian geometry

In order to fix our notations, we shall briefly recall some basic definitions and properties of Lorentzian manifolds. We refer to [BGP07] for a concise introduction.

A *Lorentzian manifold* is a manifold $M$ together with a metric $g$ of signature $(-+\cdots+)$. A non-zero tangent vector $0 \neq v \in T_x M$ at a point $x \in M$ is called *time-like* if $g(v,v) < 0$, *light-like* if $g(v,v) = 0$ and *space-like* if $g(v,v) > 0$. It is called *causal* if it is either time-like of light-like, i.e. $g(v,v) \leq 0$. A curve $\gamma : I \to M$, where $I \subseteq \mathbb{R}$ is an open interval, is called *time-like/light-like/space-like/causal* if all its tangent vectors $\dot{\gamma}$ are time-like/light-like/space-like/causal. A Lorentzian manifold is called *time-orientable* if there exists a vector field $\mathfrak{t} \in \Gamma^\infty(TM)$ that is everywhere time-like. A choice of such $\mathfrak{t}$ is called a *time-orientation*.

In what follows we always consider time-oriented Lorentzian manifolds, denoted collectively by symbols like $M$, suppressing the metric $g$ and time-orientation $\mathfrak{t}$ from our notation. A time-like or causal curve $\gamma : I \to M$ is called *future directed* if $g(\mathfrak{t}, \dot{\gamma}) < 0$ and *past directed* if $g(\mathfrak{t}, \dot{\gamma}) > 0$. The *chronological future/past* of a point $x \in M$ is the subset $I_M^\pm(x) \subseteq M$ of all points that can be reached from $x$ by future/past directed time-like curves. The *causal future/past* of a point $x \in M$ is the subset $J_M^\pm(x) \subseteq M$ of all points that can be reached from $x$ by future/past directed causal curves and $x$ itself. Given any subset $S \subseteq M$, we define $I_M^\pm(S) := \bigcup_{x \in S} I_M^\pm(x)$ and $J_M^\pm(S) := \bigcup_{x \in S} J_M^\pm(x)$.

**Definition 2.1.** Let $M$ be a time-oriented Lorentzian manifold. A subset $S \subseteq M$ is called *causally convex* if $J_M^+(S) \cap J_M^-(S) \subseteq S$. Two subsets $S, S' \subseteq M$ are called *causally disjoint* if $\left(J_M^+(S) \cup J_M^-(S)\right) \cap S' = \emptyset$.

**Remark 2.2.** In words, a subset $S \subseteq M$ is causally convex if every causal curve that starts and ends in $S$ is contained entirely in $S$. Two subsets $S, S' \subseteq M$ are causally disjoint if there exists no causal curve in $M$ connecting $S$ and $S'$. $\triangle$

**Definition 2.3.** A time-oriented Lorentzian manifold $M$ is called *globally hyperbolic* if the following two conditions hold true: (1) Every open neighborhood of every point $x \in M$ contains a

causally convex open neighborhood of $x$, and (2) $J_M^+(x) \cap J_M^-(y) \subseteq M$ is compact, for all $x, y \in M$.

**Remark 2.4.** Global hyperbolicity of $M$ is equivalent to the existence of a *Cauchy surface*, i.e. a subset $\Sigma \subset M$ that is met exactly once by each inextensible time-like curve in $M$. $\triangle$

The following category of Lorentzian manifolds plays a fundamental role in algebraic quantum field theory, see e.g. [BFV03, FV12, BDFY15, BSW17].

**Definition 2.5.** We denote by **Loc** the category whose objects are all oriented and time-oriented globally hyperbolic Lorentzian manifolds $M$ and morphisms are all orientation and time-orientation preserving isometric embeddings $f : M \to N$ with causally convex and open image $f(M) \subseteq N$.

We introduce the following terminology to specify important (tuples of) **Loc**-morphisms that enter the definitions of algebraic quantum field theories and factorization algebras.

**Definition 2.6.**   (a) A **Loc**-morphism $f : M \to N$ is called a *Cauchy morphism* if its image $f(M) \subseteq N$ contains a Cauchy surface of $N$. We shall write $f : M \xrightarrow{c} N$ for Cauchy morphisms.

(b) A pair of **Loc**-morphisms $(f_1 : M_1 \to N, f_2 : M_2 \to N)$ to a common target is called *causally disjoint* if the images $f_1(M_1) \subseteq N$ and $f_2(M_2) \subseteq N$ are causally disjoint subsets of $N$. We shall write $f_1 \perp f_2$ for causally disjoint morphisms.

(c) A tuple of **Loc**-morphisms $(f_1 : M_1 \to N, \ldots, f_n : M_n \to N)$ to a common target is called *pairwise disjoint* if the images $f_i(M_i) \subseteq N$ are pairwise disjoint subsets of $N$, i.e. $f_i(M_i) \cap f_j(M_j) = \emptyset$, for all $i \neq j$. We shall write $\underline{f} : \underline{M} \to N$ for tuples $\underline{f} = (f_1, \ldots, f_n)$ of pairwise disjoint morphisms.

**Remark 2.7.** By convention, a 1-tuple $\underline{f} = (f) : \underline{M} \to N$ of pairwise disjoint morphisms is just a **Loc**-morphism $f : M \to N$ and there exists a unique empty tuple $\emptyset \to N$ for each $N \in \mathbf{Loc}$. $\triangle$

## 2.2   Factorization algebras

Factorization algebras are typically considered in the context of topological, complex or Riemannian manifolds, see [CG17] for a detailed study. In order to obtain a meaningful comparison to algebraic quantum field theory, which is typically considered in the context of globally hyperbolic Lorentzian manifolds, we shall introduce below a variant of factorization algebras on the category **Loc** from Definition 2.5. A similar concept of factorization algebras on **Loc** appeared before in [GR17]. For what follows let us fix any cocomplete closed symmetric monoidal category $(\mathbf{C}, \otimes, I, \tau)$, e.g. the category of vector spaces $\mathbf{Vec}_{\mathbb{K}}$ over a field $\mathbb{K}$.

A *prefactorization algebra* $\mathfrak{F}$ on **Loc** with values in $\mathbf{C}$ is given by the following data:

(i) for each $M \in \mathbf{Loc}$, an object $\mathfrak{F}(M) \in \mathbf{C}$;

(ii) for each tuple $\underline{f} = (f_1, \ldots, f_n) : \underline{M} \to N$ of pairwise disjoint morphisms, a $\mathbf{C}$-morphism $\mathfrak{F}(\underline{f}) : \bigotimes_{i=1}^{n} \mathfrak{F}(\overline{M}_i) \to \mathfrak{F}(N)$ (called *factorization product*), with the convention that to the empty tuple $\emptyset \to N$ is assigned a morphism $I \to \mathfrak{F}(N)$ from the monoidal unit.

These data are required to satisfy the following conditions:

1. for every $\underline{f} = (f_1, \ldots, f_n) : \underline{M} \to N$ and $\underline{g}_i = (g_{i1}, \ldots, g_{ik_i}) : \underline{L}_i \to M_i$, for $i = 1, \ldots, n$, the diagram

$$\bigotimes_{i=1}^{n} \bigotimes_{j=1}^{k_i} \mathfrak{F}(L_{ij}) \xrightarrow{\otimes_i \mathfrak{F}(\underline{g}_i)} \bigotimes_{i=1}^{n} \mathfrak{F}(M_i)$$

with $\mathfrak{F}(\underline{f}(\underline{g}_1, \ldots, \underline{g}_n))$ to $\mathfrak{F}(N)$ and $\mathfrak{F}(\underline{f})$ to $\mathfrak{F}(N)$. $\qquad (2.1)$

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in **C** commutes, where $\underline{f}(\underline{g}_1, \ldots, \underline{g}_n) := (f_1 \, g_{11}, \ldots, f_n \, g_{nk_n}) : (\underline{L}_1, \ldots, \underline{L}_n) \to N$ is given by composition in **Loc**;

2. for every $M \in \mathbf{Loc}$, $\mathfrak{F}(\mathrm{id}_M) = \mathrm{id}_{\mathfrak{F}(M)} : \mathfrak{F}(M) \to \mathfrak{F}(M)$;

3. for every $\underline{f} = (f_1, \ldots, f_n) : \underline{M} \to N$ and every permutation $\sigma \in \Sigma_n$, the diagram

$$
\begin{array}{ccc}
\displaystyle\bigotimes_{i=1}^{n} \mathfrak{F}(M_i) & \xrightarrow{\ \mathfrak{F}(\underline{f})\ } & \mathfrak{F}(N) \\[2.5ex]
{\scriptstyle\text{permute}}\Big\downarrow & \nearrow {\scriptstyle \mathfrak{F}(\underline{f}\sigma)} & \\[2.5ex]
\displaystyle\bigotimes_{i=1}^{n} \mathfrak{F}(M_{\sigma(i)}) & &
\end{array}
\tag{2.2}
$$

in **C** commutes, where $\underline{f}\sigma := (f_{\sigma(1)}, \ldots, f_{\sigma(n)}) : \underline{M}\sigma \to N$ is given by right permutation.

A morphism $\zeta : \mathfrak{F} \to \mathfrak{G}$ of prefactorization algebras is a family $\zeta_M : \mathfrak{F}(M) \to \mathfrak{G}(M)$ of **C**-morphisms, for all $M \in \mathbf{Loc}$, that is compatible with the factorization products, i.e. for all $\underline{f} : \underline{M} \to N$ the diagram

$$
\begin{array}{ccc}
\displaystyle\bigotimes_{i=1}^{n} \mathfrak{F}(M_i) & \xrightarrow{\ \mathfrak{F}(\underline{f})\ } & \mathfrak{F}(N) \\[2.5ex]
{\scriptstyle \otimes_i \zeta_{M_i}}\Big\downarrow & & \Big\downarrow {\scriptstyle \zeta_N} \\[2.5ex]
\displaystyle\bigotimes_{i=1}^{n} \mathfrak{G}(M_i) & \xrightarrow[\ \mathfrak{G}(\underline{f})\ ]{} & \mathfrak{G}(N)
\end{array}
\tag{2.3}
$$

in **C** commutes.

**Definition 2.8.** We denote by **PFA** the category of prefactorization algebras on **Loc**.

Factorization algebras are prefactorization algebras that satisfy a suitable descent condition with respect to Weiss covers [CG17]. For proving our results in this paper, it is sufficient to assume a weaker descent condition that we shall call *additivity* in reference to a similar property in algebraic quantum field theory [Few13]. As explained below, this includes in particular all factorization algebras on **Loc**. Before we can formalize the additivity property, we have to introduce some further terminology and notations.

**Definition 2.9.** For $M \in \mathbf{Loc}$, we denote by $\mathbf{RC}_M$ the *category of all relatively compact and causally convex open subsets* $U \subseteq M$ with morphisms given by subset inclusions.

**Remark 2.10.** Note that the assignment $M \mapsto \mathbf{RC}_M$ may be promoted to a functor $\mathbf{RC}_{(-)} : \mathbf{Loc} \to \mathbf{Cat}$ with values in the category of (small) categories. Concretely, given any **Loc**-morphism $f : M \to N$, then the functor $\mathbf{RC}_f : \mathbf{RC}_M \to \mathbf{RC}_N$ sends each relatively compact and causally convex open subset $U \subseteq M$ to its image $f(U) \subseteq N$. Since $f$ is continuous, it follows that this is a relatively compact and causally convex open subset of $N$. △

**Lemma 2.11.** *For every $M \in \mathbf{Loc}$, the category $\mathbf{RC}_M$ is a directed set.*

*Proof.* Let $U_1, U_2 \in \mathbf{RC}_M$. We shall construct $U \in \mathbf{RC}_M$ such that $U_i \subseteq U$, for $i = 1, 2$. Since $K := \overline{U_1} \cup \overline{U_2}$ is compact, there exists a Cauchy surface $\Sigma$ of $M$ such that $K \subseteq I_M^-(\Sigma)$. We set $S := J_M^+(K) \cap J_M^-(\Sigma)$ and observe that this is a compact subset of $M$ by [BGP07, Corollary A.5.4]. Using also [BGP07, Lemma A.5.12], it follows that $U := I_M^+(K) \cap I_M^-(S)$ belongs to $\mathbf{RC}_M$. By construction, $U$ contains both $U_1$ and $U_2$. $\qquad\square$

We may restrict the orientation, time-orientation and metric on $M$ to the causally convex open subsets $U \in \mathbf{RC}_M$ and thereby define objects $U \in \mathbf{Loc}$. Every inclusion $U \subseteq V$ in $\mathbf{RC}_M$ then defines a $\mathbf{Loc}$-morphism $\iota_U^V : U \to V$. Hence, we can regard $\mathbf{RC}_M \subseteq \mathbf{Loc}$ as a subcategory, for every $M \in \mathbf{Loc}$, and restrict any prefactorization algebra $\mathfrak{F} \in \mathbf{PFA}$ to a functor $\mathfrak{F}|_M : \mathbf{RC}_M \to \mathbf{C}$.

**Definition 2.12.** A prefactorization algebra $\mathfrak{F} \in \mathbf{PFA}$ is called *additive* if, for every $M \in \mathbf{Loc}$, the canonical morphism

$$\mathrm{colim}\Big(\mathfrak{F}|_M : \mathbf{RC}_M \to \mathbf{C}\Big) \xrightarrow{\;\cong\;} \mathfrak{F}(M) \tag{2.4}$$

is an isomorphism in $\mathbf{C}$. We denote by $\mathbf{PFA}^{\mathrm{add}} \subseteq \mathbf{PFA}$ the full subcategory of additive prefactorization algebras.

**Remark 2.13.** The additivity condition formalizes the idea that $\mathfrak{F}(M)$ is "generated" by the images of the maps $\mathfrak{F}(U) \to \mathfrak{F}(M)$, for all relatively compact and causally convex open subsets $U \subseteq M$. Interpreting $\mathfrak{F}(M)$ as a collection of observables for a quantum field theory, this means that all observables described by $\mathfrak{F}(M)$ arise from relatively compact regions $U \subseteq M$. $\triangle$

**Proposition 2.14.** *Every factorization algebra $\mathfrak{F}$ on $\mathbf{Loc}$ is an additive prefactorization algebra.*

*Proof.* Suppose that $\mathfrak{F}$ is a factorization algebra [CG17], i.e. it satisfies a cosheaf condition with respect to all Weiss covers of every $M \in \mathbf{Loc}$. For every $M \in \mathbf{Loc}$, the cover defined by $\mathbf{RC}_M$ is a Weiss cover. Indeed, given finitely many points $x_1, \ldots, x_n \in M$, there exist $U_i \in \mathbf{RC}_M$ with $x_i \in U_i$ and hence $U \in \mathbf{RC}_M$ with $x_1, \ldots, x_n \in U$ because $\mathbf{RC}_M$ is directed by Lemma 2.11. The property of being a factorization algebra then implies that the canonical diagram

$$\coprod_{\substack{U,V \in \mathbf{RC}_M \\ U \cap V \neq \emptyset}} \mathfrak{F}(U \cap V) \rightrightarrows \coprod_{U \in \mathbf{RC}_M} \mathfrak{F}(U) \longrightarrow \mathfrak{F}(M) \tag{2.5}$$

is a coequalizer in $\mathbf{C}$. Our claim then follows by observing that the cocones of (2.4) are canonically identified with the cocones of (2.5). Indeed, any cocone $\{\alpha_U : \mathfrak{F}(U) \to Z\}$ of (2.4) defines a cocone of (2.5) because $U \cap V \in \mathbf{RC}_M$ (whenever nonempty) and hence the diagram

$$\tag{2.6a}$$



in $\mathbf{C}$ commutes. Vice versa, any cocone $\{\alpha_U : \mathfrak{F}(U) \to Z\}$ of (2.5) defines a cocone of (2.4) because $U \cap V = U$, for all $U \subseteq V$, and hence the diagram

$$\tag{2.6b}$$



in $\mathbf{C}$ commutes. $\qquad\square$

As a last definition, we would like to introduce a suitable local constancy property that is adapted to the category $\mathbf{Loc}$. This property will play a crucial role in establishing our comparison results. Recall from Definition 2.6 the concept of Cauchy morphisms.

**Definition 2.15.** A prefactorization algebra $\mathfrak{F} \in \mathbf{PFA}$ is called *Cauchy constant* if $\mathfrak{F}(f) : \mathfrak{F}(M) \to \mathfrak{F}(N)$ is an isomorphism in $\mathbf{C}$, for every Cauchy morphism $f : M \xrightarrow{c} N$. We denote by $\mathbf{PFA}^c \subseteq \mathbf{PFA}$ the full subcategory of Cauchy constant prefactorization algebras. The full subcategory $\mathbf{PFA}^{\mathrm{add},c} \subseteq \mathbf{PFA}^{\mathrm{add}}$ of Cauchy constant additive prefactorization algebras is defined analogously.

## 2.3 Algebraic quantum field theories

Let $\mathbf{C}$ be a cocomplete closed symmetric monoidal category as in the previous subsection. We briefly review the basic definitions for $\mathbf{C}$-valued algebraic quantum field theories on $\mathbf{Loc}$ following [BSW17]. See also [BFV03, FV12, BDFY15] for a broader introduction to algebraic quantum field theories and their applications to physics.

Let us denote by $\mathbf{Alg} := \mathbf{Alg}_{\mathsf{As}}(\mathbf{C})$ the category of associative and unital algebras in $\mathbf{C}$. An *algebraic quantum field theory* $\mathfrak{A}$ on $\mathbf{Loc}$ with values in $\mathbf{C}$ is a functor $\mathfrak{A} : \mathbf{Loc} \to \mathbf{Alg}$ that satisfies the Einstein causality axiom: for every pair of causally disjoint morphisms $(f_1 : M_1 \to N) \perp (f_2 : M_2 \to N)$, the diagram

$$\begin{array}{ccc}
\mathfrak{A}(M_1) \otimes \mathfrak{A}(M_2) & \xrightarrow{\mathfrak{A}(f_1) \otimes \mathfrak{A}(f_2)} & \mathfrak{A}(N) \otimes \mathfrak{A}(N) \\
{\scriptstyle \mathfrak{A}(f_1) \otimes \mathfrak{A}(f_2)} \downarrow & & \downarrow {\scriptstyle \mu_N^{\mathrm{op}}} \\
\mathfrak{A}(N) \otimes \mathfrak{A}(N) & \xrightarrow{\mu_N} & \mathfrak{A}(N)
\end{array} \qquad (2.7)$$

in $\mathbf{C}$ commutes, where $\mu_N^{(\mathrm{op})}$ denotes the (opposite) multiplication on $\mathfrak{A}(N)$. A morphism $\kappa : \mathfrak{A} \to \mathfrak{B}$ of algebraic quantum field theories is a natural transformation between the underlying functors.

**Definition 2.16.** We denote by $\mathbf{AQFT}$ the category of algebraic quantum field theories on $\mathbf{Loc}$.

For proving some of the results of this paper, we require a relatively mild variant of an additivity property in the sense of [Few13]. Recall from Definition 2.9 the category $\mathbf{RC}_M$ of relatively compact and causally convex open subsets of $M \in \mathbf{Loc}$.

**Definition 2.17.** An algebraic quantum field theory $\mathfrak{A} \in \mathbf{AQFT}$ is called *additive* if, for every $M \in \mathbf{Loc}$, the canonical morphism

$$\mathrm{colim}\Big( \mathfrak{A}|_M : \mathbf{RC}_M \to \mathbf{Alg} \Big) \xrightarrow{\;\cong\;} \mathfrak{A}(M) \qquad (2.8)$$

is an isomorphism in $\mathbf{Alg}$. We denote by $\mathbf{AQFT}^{\mathrm{add}} \subseteq \mathbf{AQFT}$ the full subcategory of additive algebraic quantum field theories.

**Remark 2.18.** Because $\mathbf{RC}_M$ is a directed set by Lemma 2.11, the colimit in Definition 2.17 can be computed in the underlying category $\mathbf{C}$, see e.g. [Fre17, Proposition 1.3.6]. Hence, additivity of $\mathfrak{A}$ can be tested by studying its underlying functor $\mathfrak{A} : \mathbf{Loc} \to \mathbf{C}$ to the category $\mathbf{C}$. $\triangle$

Furthermore, we introduce a suitable local constancy property that is also known in the literature as the time-slice axiom.

**Definition 2.19.** An algebraic quantum field theory $\mathfrak{A} \in \mathbf{AQFT}$ is called *Cauchy constant* if $\mathfrak{A}(f) : \mathfrak{A}(M) \to \mathfrak{A}(N)$ is an isomorphism in $\mathbf{Alg}$, for every Cauchy morphism $f : M \xrightarrow{c} N$. We denote by $\mathbf{AQFT}^c \subseteq \mathbf{AQFT}$ the full subcategory of Cauchy constant algebraic quantum field theories. The full subcategory $\mathbf{AQFT}^{\mathrm{add},c} \subseteq \mathbf{AQFT}^{\mathrm{add}}$ of Cauchy constant additive algebraic quantum field theories is defined analogously.

## 3 From PFA to AQFT

In this section we show that every Cauchy constant additive prefactorization algebra $\mathfrak{F} \in \mathbf{PFA}^{\mathrm{add},c}$ (cf. Definitions 2.12 and 2.15) defines a Cauchy constant additive algebraic quantum field theory (cf. Definitions 2.17 and 2.19). This construction will define a functor $\mathfrak{A}_{(-)} : \mathbf{PFA}^{\mathrm{add},c} \to \mathbf{AQFT}^{\mathrm{add},c}$.

Our construction consists of three steps, which will be carried out in detail in individual subsections below. Step (1) consists of proving that, for each $M \in \mathbf{Loc}$, the object $\mathfrak{F}(M) \in \mathbf{C}$ carries canonically the structure of an associative and unital algebra in $\mathbf{C}$. This step relies on Cauchy constancy, while it does not require that the additivity property holds true. Step (2) consists of proving that these algebra structures are compatible with the maps $\mathfrak{F}(f) : \mathfrak{F}(M) \to \mathfrak{F}(N)$ induced by $\mathbf{Loc}$-morphisms $f : M \to N$. Here our additivity property turns out to be crucial. Finally, in step (3) we show that the resulting functor $\mathbf{Loc} \to \mathbf{Alg}$ satisfies the properties of a Cauchy constant additive algebraic quantum field theory, cf. Section 2.3.

## 3.1 Object-wise algebra structure

All results of this subsection do not use the additivity property from Definition 2.12. Hence, we let $\mathfrak{F} \in \mathbf{PFA}^c$ be any Cauchy constant prefactorization algebra.

Let us fix any $M \in \mathbf{Loc}$. The basic idea to define a multiplication map $\mu_M : \mathfrak{F}(M) \otimes \mathfrak{F}(M) \to \mathfrak{F}(M)$ is as follows: Consider two causally convex open subsets $U_+, U_- \subseteq M$ satisfying (i) there exists a Cauchy surface $\Sigma$ of $M$ such that $U_\pm \subseteq I_M^\pm(\Sigma)$, and (ii) $\iota_{U_\pm}^M : U_\pm \xrightarrow{c} M$ are Cauchy morphisms. In particular, $U_+ \cap U_- = \emptyset$ are disjoint and hence provide a pair of disjoint morphisms $\iota_{\underline{U}}^M = (\iota_{U_+}^M, \iota_{U_-}^M) : \underline{U} \to M$. We define $\mu_M$ by the commutative diagram

$$
\begin{array}{ccc}
\mathfrak{F}(M) \otimes \mathfrak{F}(M) & \xrightarrow{\quad\quad\mu_M\quad\quad} & \mathfrak{F}(M) \\
& \underset{\mathfrak{F}(\iota_{U_+}^M) \otimes \mathfrak{F}(\iota_{U_-}^M)}{\overset{\cong}{\nwarrow}} \quad \underset{\mathfrak{F}(U_+) \otimes \mathfrak{F}(U_-)}{} \quad \overset{\mathfrak{F}(\iota_{\underline{U}}^M)}{\nearrow} &
\end{array}
\tag{3.1}
$$

where the upward-left pointing arrow is an isomorphism because $\mathfrak{F}$ is by hypothesis Cauchy constant. A priori, it is not clear whether different choices of such $\iota_{\underline{U}}^M : \underline{U} \to M$ lead to the same multiplication map in (3.1). The possible choices are recorded in the following category.

**Definition 3.1.** Let $M \in \mathbf{Loc}$. We denote by $\mathbf{P}_M$ the category whose objects are all pairs of disjoint morphisms $\iota_{\underline{U}}^M = (\iota_{U_+}^M, \iota_{U_-}^M) : \underline{U} \to M$ corresponding to causally convex open subsets $U_+, U_- \subseteq M$ that satisfy

(i) there exists a Cauchy surface $\Sigma$ of $M$ such that $U_\pm \subseteq I_M^\pm(\Sigma)$, and

(ii) $\iota_{U_\pm}^M : U_\pm \xrightarrow{c} M$ are Cauchy morphisms.

There exists a unique morphism $(\iota_{\underline{U}}^M : \underline{U} \to M) \to (\iota_{\underline{V}}^M : \underline{V} \to M)$ if and only if $U_\pm \subseteq V_\pm$.

**Lemma 3.2.** *For every $M \in \mathbf{Loc}$, the category $\mathbf{P}_M$ is non-empty and connected.*

*Proof. Non-empty:* Choose any Cauchy surface $\Sigma$ of $M$ and define $\Sigma_\pm := I_M^\pm(\Sigma)$. Then $\iota_{\underline{\Sigma}}^M = (\iota_{\Sigma_+}^M, \iota_{\Sigma_-}^M) : \underline{\Sigma} \to M$ defines an object in $\mathbf{P}_M$.

*Connected:* We have to prove that there exists a zig-zag of morphisms in $\mathbf{P}_M$ between every pair of objects $\iota_{\underline{U}}^M : \underline{U} \to M$ and $\iota_{\underline{V}}^M : \underline{V} \to M$. For every object $\iota_{\underline{U}}^M : \underline{U} \to M$ in $\mathbf{P}_M$, there exists by hypothesis a Cauchy surface $\Sigma$ of $M$ such that $U_\pm \subseteq \Sigma_\pm := I_M^\pm(\Sigma)$. Hence, there exists a morphism $(\iota_{\underline{U}}^M : \underline{U} \to M) \to (\iota_{\underline{\Sigma}}^M : \underline{\Sigma} \to M)$. As a consequence, our original problem reduces to finding a zig-zag of morphisms in $\mathbf{P}_M$ between $\iota_{\underline{\Sigma}}^M : \underline{\Sigma} \to M$ and $\iota_{\underline{\Sigma'}}^M : \underline{\Sigma'} \to M$, for any two Cauchy surfaces $\Sigma, \Sigma'$ of $M$. To exhibit such a zig-zag, let us introduce $\widetilde{U}_+ := \Sigma_+ \cap \Sigma'_+$ and $\widetilde{U}_- := \Sigma_- \cap \Sigma'_-$. If we could prove that $\iota_{\widetilde{U}_\pm}^M : \widetilde{U}_\pm \xrightarrow{c} M$ are Cauchy morphisms, then

$$
\left( \iota_{\underline{\Sigma}}^M : \underline{\Sigma} \to M \right) \longleftarrow \left( \iota_{\underline{\widetilde{U}}}^M : \underline{\widetilde{U}} \to M \right) \longrightarrow \left( \iota_{\underline{\Sigma'}}^M : \underline{\Sigma'} \to M \right)
\tag{3.2}
$$

would provide a zig-zag that proves connectedness of $\mathbf{P}_M$.

It remains to show that $\widetilde{U}_+ = \Sigma_+ \cap \Sigma'_+ = I_M^+(\Sigma) \cap I_M^+(\Sigma') \subseteq M$ contains a Cauchy surface of $M$. (A similar argument shows that $\widetilde{U}_- \subseteq M$ also contains a Cauchy surface of $M$.) Because $\Sigma, \Sigma'$ are by hypothesis Cauchy surfaces of $M$, there exists a Cauchy surface $\Sigma_1 \subset I_M^+(\Sigma)$ of $M$ in the future of $\Sigma$ and a Cauchy surface $\Sigma'_1 \subset I_M^+(\Sigma')$ of $M$ in the future of $\Sigma'$. We define the subset

$$\widetilde{\Sigma} := \left(\Sigma_1 \cap J_M^+(\Sigma'_1)\right) \cup \left(J_M^+(\Sigma_1) \cap \Sigma'_1\right) \subset \widetilde{U}_+ \subseteq M \tag{3.3}$$

and claim that $\widetilde{\Sigma}$ is a Cauchy surface of $M$. This can be easily confirmed by checking that every inextensible time-like curve $\gamma : I \to M$ meets $\widetilde{\Sigma}$ exactly once. $\qquad \square$

**Corollary 3.3.** *For every $M \in \mathbf{Loc}$, the multiplication map $\mu_M$ in (3.1) does not depend on the choice of object $\iota_{\underline{U}}^M : \underline{U} \to M$ in $\mathbf{P}_M$.*

*Proof.* By Lemma 3.2, it is sufficient to prove that $\iota_{\underline{U}}^M : \underline{U} \to M$ and $\iota_{\underline{V}}^M : \underline{V} \to M$ define the same multiplication if $U_+ \subseteq V_+$ and $U_- \subseteq V_-$. This is a consequence of the commutative diagram

$$\tag{3.4}$$



where one also uses the composition properties (2.1) of prefactorization algebras. $\qquad \square$

To obtain a unit for $\mathfrak{F}(M)$, we recall that there exists a unique empty tuple of disjoint morphisms $\emptyset \to M$ to which the prefactorization algebra assigns a $\mathbf{C}$-morphism that we shall denote by $\eta_M : I \to \mathfrak{F}(M)$. The main result of this subsection is as follows.

**Proposition 3.4.** *Let $\mathfrak{F} \in \mathbf{PFA}^c$ be any Cauchy constant prefactorization algebra. For every $M \in \mathbf{Loc}$, the object $\mathfrak{F}(M) \in \mathbf{C}$ carries the structure of an associative and unital algebra in $\mathbf{C}$ with multiplication $\mu_M : \mathfrak{F}(M) \otimes \mathfrak{F}(M) \to \mathfrak{F}(M)$ given by (3.1) and unit $\eta_M : I \to \mathfrak{F}(M)$ given by evaluating $\mathfrak{F}$ on the empty tuple $\emptyset \to M$.*

*Proof.* To prove that the multiplication $\mu_M$ is associative, we consider two Cauchy surfaces $\Sigma_0, \Sigma_1$ of $M$ such that $\Sigma_1 \subset I_M^+(\Sigma_0)$, i.e. $\Sigma_1$ is in the future of $\Sigma_0$. Using the independence result from Corollary 3.3 and the composition properties of prefactorization algebras from Section 2.2, one easily confirms that $\mu_M (\mathrm{id} \otimes \mu_M)$ is the upper path and $\mu_M (\mu_M \otimes \mathrm{id})$ the lower path from $\mathfrak{F}(M)^{\otimes 3}$ to $\mathfrak{F}(M)$ in the commutative diagram

$$\tag{3.5}$$



where as before we denote by $\Sigma_\pm := I_M^\pm(\Sigma) \subseteq M$ the chronological future/past of a Cauchy surface $\Sigma$ of $M$. Unitality of the product follows immediately from the fact that there exists a unique empty tuple $\emptyset \to N$ for each $N \in \mathbf{Loc}$ and the composition properties (2.1) of prefactorization algebras. $\qquad \square$

## 3.2 Naturality of algebra structures

The aim of this subsection is to investigate compatibility between the algebra structures from Proposition 3.4 and the maps $\mathfrak{F}(f) : \mathfrak{F}(M) \to \mathfrak{F}(N)$ induced by **Loc**-morphisms. For our main statement to be true it will be crucial to assume that $\mathfrak{F} \in \mathbf{PFA}^{\mathrm{add},c}$ is a Cauchy constant additive prefactorization algebra in the sense of Definitions 2.12 and 2.15. As a first partial result, we prove the following general statement.

**Lemma 3.5.** *Let $\mathfrak{F} \in \mathbf{PFA}^c$ be any Cauchy constant prefactorization algebra (not necessarily additive). Let further $f : M \to N$ be a **Loc**-morphism such that the image $f(M) \subseteq N$ is relatively compact. Then $\mathfrak{F}(f) : \mathfrak{F}(M) \to \mathfrak{F}(N)$ preserves the multiplications and units from Proposition 3.4, i.e. $\mu_N \left( \mathfrak{F}(f) \otimes \mathfrak{F}(f) \right) = \mathfrak{F}(f) \mu_M$ and $\eta_N = \mathfrak{F}(f) \eta_M$.*

*Proof.* The units are clearly preserved for every **Loc**-morphism $f : M \to N$ because composing the unique empty tuple $\emptyset \to M$ with $f : M \to N$ yields the unique empty tuple $\emptyset \to N$.

Let us focus now on the multiplications. Because $f(M) \subseteq N$ is by hypothesis relatively compact, its closure $\overline{f(M)} \subseteq N$ is compact. Let us take any Cauchy surface $\Sigma$ of $M$ and note that $\overline{f(\Sigma)} \subseteq N$ is a compact subset. Using further that $f(M) \subseteq N$ is causally convex and that the causality relation induced by time-like curves is open (cf. [ONe83, Lemma 14.3]), it follows that $\overline{f(\Sigma)} \subseteq N$ is achronal, i.e. every time-like curve in $N$ meets this subset at most once. By [BS06, Theorem 3.8], there exists a Cauchy surface $\widetilde{\Sigma}$ of $N$ such that $f(\Sigma) \subseteq \widetilde{\Sigma}$.

Using the Cauchy surfaces constructed above, we can define the multiplication $\mu_M$ in terms of $\Sigma_\pm := I_M^\pm(\Sigma)$ and the multiplication $\mu_N$ in terms of $\widetilde{\Sigma}_\pm := I_N^\pm(\widetilde{\Sigma})$, cf. (3.1). By construction, $f : M \to N$ restricts to **Loc**-morphisms $f : \Sigma_\pm \to \widetilde{\Sigma}_\pm$. Our claim that $\mathfrak{F}(f) : \mathfrak{F}(M) \to \mathfrak{F}(N)$ preserves the multiplications then follows by observing that the diagram

$$
\begin{array}{ccccc}
\mathfrak{F}(M) \otimes \mathfrak{F}(M) & \xleftarrow[\cong]{\mathfrak{F}(\iota_{\Sigma_+}^M) \otimes \mathfrak{F}(\iota_{\Sigma_-}^M)} & \mathfrak{F}(\Sigma_+) \otimes \mathfrak{F}(\Sigma_-) & \xrightarrow{\mathfrak{F}(\iota_{\underline{\Sigma}}^M)} & \mathfrak{F}(M) \\
\mathfrak{F}(f) \otimes \mathfrak{F}(f) \downarrow & & \downarrow \mathfrak{F}(f) \otimes \mathfrak{F}(f) & & \downarrow \mathfrak{F}(f) \\
\mathfrak{F}(N) \otimes \mathfrak{F}(N) & \xleftarrow[\mathfrak{F}(\iota_{\widetilde{\Sigma}_+}^N) \otimes \mathfrak{F}(\iota_{\widetilde{\Sigma}_-}^N)]{\cong} & \mathfrak{F}(\widetilde{\Sigma}_+) \otimes \mathfrak{F}(\widetilde{\Sigma}_-) & \xrightarrow{\mathfrak{F}(\iota_{\underline{\widetilde{\Sigma}}}^N)} & \mathfrak{F}(N)
\end{array}
\tag{3.6}
$$

commutes. $\qquad\square$

**Remark 3.6.** We would like to emphasize that our assumption that the image $f(M) \subseteq N$ is relatively compact was crucial for the proof of Lemma 3.5. In fact, if one *does not* assume that the image of the **Loc**-morphism $f : M \to N$ is relatively compact, then it is *not* true that the image $f(\Sigma) \subset N$ of a Cauchy surface $\Sigma$ of $M$ can be extended to a Cauchy surface $\widetilde{\Sigma}$ of $N$. A simple example that demonstrates this feature is given by the subset inclusion $\iota_U^V : U \to V$ of the following two diamond regions in 2-dimensional Minkowski spacetime (note that $U$ is not relatively compact as a subset of $V$):



$$\tag{3.7}$$
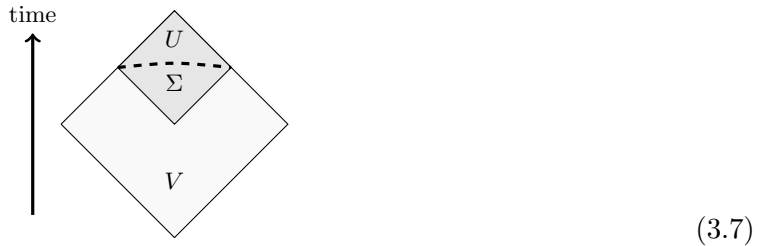
It is evident that no Cauchy surface $\Sigma$ of $U$ admits an extension to a Cauchy surface of $V$. Hence, $\mathfrak{F}(\iota_U^V) : \mathfrak{F}(U) \to \mathfrak{F}(V)$ may fail to preserve the multiplications. We shall show below that the issues explained in this remark are solved by considering *additive* prefactorization algebras as in Definition 2.12. $\qquad\triangle$

The main result of this subsection is as follows.

**Proposition 3.7.** *Let $\mathfrak{F} \in \mathbf{PFA}^{\mathrm{add},c}$ be any Cauchy constant additive prefactorization algebra. For every $\mathbf{Loc}$-morphism $f : M \to N$, the $\mathbf{C}$-morphism $\mathfrak{F}(f) : \mathfrak{F}(M) \to \mathfrak{F}(N)$ preserves the multiplications and units from Proposition 3.4.*

*Proof.* We already observed in the proof of Lemma 3.5 that $\mathfrak{F}(f)$ preserves the units.

For the multiplications we have to prove that $\mu_N \left( \mathfrak{F}(f) \otimes \mathfrak{F}(f) \right) = \mathfrak{F}(f) \, \mu_M$ as $\mathbf{C}$-morphisms from $\mathfrak{F}(M) \otimes \mathfrak{F}(M)$ to $\mathfrak{F}(N)$. Because $\mathfrak{F}$ is by hypothesis additive (cf. Definition 2.12) and the monoidal product $\otimes$ in a cocomplete closed symmetric monoidal category preserves colimits in both entries, it follows that

$$\mathfrak{F}(M) \otimes \mathfrak{F}(M) \cong \mathrm{colim}_{U,V \in \mathbf{RC}_M} \left( \mathfrak{F}(U) \otimes \mathfrak{F}(V) \right) \cong \mathrm{colim}_{U \in \mathbf{RC}_M} \left( \mathfrak{F}(U) \otimes \mathfrak{F}(U) \right) \quad , \qquad (3.8)$$

where in the last step we also used that $\mathbf{RC}_M$ is directed by Lemma 2.11. For every $U \in \mathbf{RC}_M$, consider the diagram

$$(3.9)$$



The top and bottom squares of this diagram commute because of Lemma 3.5 and the fact that both $U \subseteq M$ and $f(U) \subseteq N$ are relatively compact subsets. The two triangles commute by direct inspection. By universality of the colimit in (3.8), this implies that the front square in (3.9) commutes, proving our claim. $\qquad \square$

**Corollary 3.8.** *Every Cauchy constant additive prefactorization algebra $\mathfrak{F} \in \mathbf{PFA}^{\mathrm{add},c}$ defines a functor $\mathfrak{A}_{\mathfrak{F}} : \mathbf{Loc} \to \mathbf{Alg}$ to the category of associative and unital algebras. Explicitly, this functor acts on objects $M \in \mathbf{Loc}$ as $\mathfrak{A}_{\mathfrak{F}}(M) := (\mathfrak{F}(M), \mu_M, \eta_M)$ and on $\mathbf{Loc}$-morphisms $f : M \to N$ as $\mathfrak{A}_{\mathfrak{F}}(f) := \mathfrak{F}(f)$. The assignment $\mathfrak{F} \mapsto \mathfrak{A}_{\mathfrak{F}}$ canonically extends to a functor $\mathfrak{A}_{(-)} : \mathbf{PFA}^{\mathrm{add},c} \to \mathbf{Alg}^{\mathbf{Loc}}$, where $\mathbf{Alg}^{\mathbf{Loc}}$ denotes the category of functors from $\mathbf{Loc}$ to $\mathbf{Alg}$.*

*Proof.* It remains to prove that every morphism $\zeta : \mathfrak{F} \to \mathfrak{G}$ in $\mathbf{PFA}^{\mathrm{add},c}$ defines a natural transformation $\zeta : \mathfrak{A}_{\mathfrak{F}} \to \mathfrak{A}_{\mathfrak{G}}$ between $\mathbf{Alg}$-valued functors on $\mathbf{Loc}$, i.e. that all components $\zeta_M : \mathfrak{F}(M) \to \mathfrak{G}(M)$ preserve the multiplications and units. For the units this is immediate, while for the multiplications it follows from the fact that the diagram

$$(3.10)$$



commutes by the compatibility properties (2.3) of prefactorization algebra morphisms. $\qquad \square$

## 3.3 Algebraic quantum field theory axioms

The goal of this subsection is to show that the construction above assigns to each Cauchy constant additive prefactorization algebra a Cauchy constant additive algebraic quantum field theory. More precisely, we shall prove that the functor $\mathfrak{A}_{(-)} : \mathbf{PFA}^{\mathrm{add},c} \to \mathbf{Alg}^{\mathbf{Loc}}$ established in Corollary 3.8 factors through the full subcategory $\mathbf{AQFT}^{\mathrm{add},c} \subseteq \mathbf{Alg}^{\mathbf{Loc}}$ of Cauchy constant additive algebraic quantum field theories.

**Lemma 3.9.** *Let $\mathfrak{F} \in \mathbf{PFA}^c$ be any Cauchy constant prefactorization algebra (not necessarily additive). Let further $(f_1 : M_1 \to N) \perp (f_2 : M_2 \to N)$ be any causally disjoint pair of $\mathbf{Loc}$-morphisms such that the images $f_1(M_1), f_2(M_2) \subseteq N$ are relatively compact. Then $\mu_N^{\mathrm{op}} (\mathfrak{F}(f_1) \otimes \mathfrak{F}(f_2)) = \mu_N (\mathfrak{F}(f_1) \otimes \mathfrak{F}(f_2))$, where $\mu_N^{(\mathrm{op})}$ denotes the (opposite) multiplication on $\mathfrak{F}(N)$ from Proposition 3.4.*

*Proof.* In order to compare the two morphisms $\mu_N (\mathfrak{F}(f_1) \otimes \mathfrak{F}(f_2))$ and $\mu_N^{\mathrm{op}} (\mathfrak{F}(f_1) \otimes \mathfrak{F}(f_2))$ from $\mathfrak{F}(M_1) \otimes \mathfrak{F}(M_2)$ to $\mathfrak{F}(N)$, we introduce convenient ways to compute these composites. Let us choose arbitrary Cauchy surfaces $\Sigma_1$ of $M_1$ and $\Sigma_2$ of $M_2$. As in the proof of Lemma 3.5, we deduce that $\overline{f_1(\Sigma_1)}, \overline{f_2(\Sigma_2)} \subseteq N$ are achronal compact subsets. Causal disjointness of the pair $f_1 \perp f_2$ entails achronality of the union $\overline{f_1(\Sigma_1)} \cup \overline{f_2(\Sigma_2)} \subseteq N$. By [BS06, Theorem 3.8], there exists a Cauchy surface $\widetilde{\Sigma}$ of $N$ that contains the union $\overline{f_1(\Sigma_1)} \cup \overline{f_2(\Sigma_2)} \subseteq \widetilde{\Sigma}$. Similarly, choosing any Cauchy surface $\Sigma_1' \subset I_{M_1}^+(\Sigma_1)$ of $M_1$ that lies in the future of $\Sigma_1$ and any Cauchy surface $\Sigma_2' \subset I_{M_2}^-(\Sigma_2)$ of $M_2$ that lies in the past of $\Sigma_2$, there exists a Cauchy surface $\widetilde{\Sigma}'$ of $N$ that contains the union $\overline{f_1(\Sigma_1')} \cup \overline{f_2(\Sigma_2')} \subseteq \widetilde{\Sigma}'$. Let us introduce

$$U_1 := I_{M_1}^+(\Sigma_1) \cap I_{M_1}^-(\Sigma_1') \subseteq M_1 \quad , \qquad U_2 := I_{M_2}^+(\Sigma_2') \cap I_{M_2}^-(\Sigma_2) \subseteq M_2 \quad , \qquad (3.11)$$

and also consider $\widetilde{\Sigma}_\pm := I_N^\pm(\widetilde{\Sigma}) \subseteq N$ and $\widetilde{\Sigma}'_\pm := I_N^\pm(\widetilde{\Sigma}') \subseteq N$. By construction, $\iota_{U_i}^{M_i} : U_i \xrightarrow{c} M_i$, for $i = 1, 2$, and $\iota_{\widetilde{\Sigma}_\pm^{(\prime)}}^N : \widetilde{\Sigma}_\pm^{(\prime)} \xrightarrow{c} N$ are Cauchy morphisms. The following picture illustrates in dark gray the chosen subsets $U_1 \subseteq M_1$ and $U_2 \subseteq M_2$:
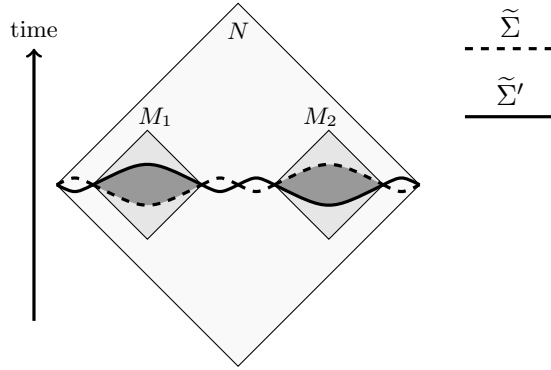


$$(3.12)$$

With these preparations, we can compute $\mu_N (\mathfrak{F}(f_1) \otimes \mathfrak{F}(f_2))$ by

$$(3.13)$$

Similarly, $\mu_N^{\mathrm{op}} (\mathfrak{F}(f_1) \otimes \mathfrak{F}(f_2))$ can be computed by

$$(3.14)$$

The claim follows from the equivariance property (2.2) of prefactorization algebras. $\qquad\square$

The main result of this subsection is as follows.

**Proposition 3.10.** *Let $\mathfrak{F} \in \mathbf{PFA}^{\mathrm{add},c}$ be any Cauchy constant additive prefactorization algebra. Let further $(f_1 : M_1 \to N) \perp (f_2 : M_2 \to N)$ be any causally disjoint pair of $\mathbf{Loc}$-morphisms. Then $\mu_N^{\mathrm{op}} (\mathfrak{F}(f_1) \otimes \mathfrak{F}(f_2)) = \mu_N (\mathfrak{F}(f_1) \otimes \mathfrak{F}(f_2))$, where $\mu_N^{(\mathrm{op})}$ denotes the (opposite) multiplication on $\mathfrak{F}(N)$ from Proposition 3.4.*

*Proof.* Because $\mathfrak{F}$ is by hypothesis additive (cf. Definition 2.12) and the monoidal product $\otimes$ in a cocomplete closed symmetric monoidal category preserves colimits in both entries, it follows that

$$\mathfrak{F}(M_1) \otimes \mathfrak{F}(M_2) \cong \mathrm{colim}_{(U_1,U_2) \in \mathbf{RC}_{M_1} \times \mathbf{RC}_{M_2}} \big( \mathfrak{F}(U_1) \otimes \mathfrak{F}(U_2) \big) \quad . \tag{3.15}$$

For every $(U_1, U_2) \in \mathbf{RC}_{M_1} \times \mathbf{RC}_{M_2}$, consider the diagram



$$\tag{3.16}$$

The two triangles coincide and commute by direct inspection. Furthermore, for every $(U_1, U_2) \in \mathbf{RC}_{M_1} \times \mathbf{RC}_{M_2}$, the outer square commutes as a consequence of Lemma 3.9 applied to the causally disjoint pair $(f_1 : U_1 \to N) \perp (f_2 : U_2 \to N)$, whose images $f_1(U_1), f_2(U_2) \subseteq N$ are relatively compact subsets. Hence, by universality of the colimit in (3.15), the inner square commutes as well, which is our claim. $\qquad\square$

Proposition 3.10 leads to the following refinement of Corollary 3.8.

**Theorem 3.11.** *Every Cauchy constant additive prefactorization algebra $\mathfrak{F} \in \mathbf{PFA}^{\mathrm{add},c}$ defines a Cauchy constant additive algebraic quantum field theory $\mathfrak{A}_{\mathfrak{F}} \in \mathbf{AQFT}^{\mathrm{add},c}$. Hence, the functor $\mathfrak{A}_{(-)} : \mathbf{PFA}^{\mathrm{add},c} \to \mathbf{Alg}^{\mathbf{Loc}}$ from Corollary 3.8 factors through the full subcategory $\mathbf{AQFT}^{\mathrm{add},c} \subseteq \mathbf{Alg}^{\mathbf{Loc}}$.*

*Proof.* Proposition 3.10 implies that the functor $\mathfrak{A}_{\mathfrak{F}} : \mathbf{Loc} \to \mathbf{Alg}$ defined in Corollary 3.8 is an algebraic quantum field theory, i.e. it satisfies the Einstein causality axiom (2.7). Because $\mathfrak{F}$ is by hypothesis Cauchy constant, it follows that $\mathfrak{A}_{\mathfrak{F}}$ is Cauchy constant too. Recalling Remark 2.18, additivity of $\mathfrak{F} \in \mathbf{PFA}^{\mathrm{add},c}$ immediately implies additivity of $\mathfrak{A}_{\mathfrak{F}}$, i.e. $\mathfrak{A}_{\mathfrak{F}} \in \mathbf{AQFT}^{\mathrm{add},c}$. $\qquad\square$

# 4 From AQFT to PFA

In this section we show that every algebraic quantum field theory $\mathfrak{A} \in \mathbf{AQFT}$ defines a variant of a prefactorization algebra on $\mathbf{Loc}$ where the factorization products are defined only for those tuples of pairwise disjoint morphisms $\underline{f} : \underline{M} \to N$ that are in a suitable sense time-orderable. We shall call this type of prefactorization algebras *time-orderable* and denote the corresponding category by $\mathbf{tPFA}$. Our construction defines a functor $\mathfrak{F}_{(-)} : \mathbf{AQFT} \to \mathbf{tPFA}$ to the category of time-orderable prefactorization algebras. Cauchy constancy and additivity do not play a role in this section, however we shall prove that these properties are preserved by our functor.

Let $\mathfrak{A} \in \mathbf{AQFT}$ be an algebraic quantum field theory. Our aim is to construct from this data factorization products $\mathfrak{F}_{\mathfrak{A}}(\underline{f}) : \bigotimes_{i=1}^{n} \mathfrak{A}(M_i) \to \mathfrak{A}(N)$, for suitable tuples of pairwise disjoint morphisms $\underline{f} = (f_1, \ldots, f_n) : \underline{M} \to N$. For $n = 0$, i.e. the empty tuples $\emptyset \to N$, we may take the unit $\eta_N : I \to \mathfrak{A}(N)$ of the associative and unital algebra $\mathfrak{A}(N)$ that is assigned by $\mathfrak{A}$ to $N \in \mathbf{Loc}$. For $n = 1$, the tuples of pairwise disjoint morphisms are just $\mathbf{Loc}$-morphisms $f : M \to N$, hence we may take the $\mathbf{C}$-morphism $\mathfrak{F}_{\mathfrak{A}}(f) := \mathfrak{A}(f) : \mathfrak{A}(M) \to \mathfrak{A}(N)$ that is obtained from the $\mathbf{Alg}$-morphism assigned by $\mathfrak{A}$ to $f : M \to N$ via the forgetful functor $\mathbf{Alg} \to \mathbf{C}$. For $n \geq 2$, the envisaged construction becomes far less obvious. Let us consider for the moment $n = 2$ and a pair of disjoint morphisms $\underline{f} = (f_1, f_2) : \underline{M} \to N$. Inspired by our previous construction (3.1) of multiplications from factorization products, we propose to define $\mathfrak{F}_{\mathfrak{A}}(\underline{f}) : \mathfrak{A}(M_1) \otimes \mathfrak{A}(M_2) \to \mathfrak{A}(N)$ by the commutative diagram

$$
\begin{array}{ccc}
\mathfrak{A}(M_1) \otimes \mathfrak{A}(M_2) & \xrightarrow{\;\;\mathfrak{F}_{\mathfrak{A}}(\underline{f})\;\;} & \mathfrak{A}(N) \\
& \searrow{\scriptstyle \mathfrak{A}(f_1) \otimes \mathfrak{A}(f_2)} \qquad \nearrow{\scriptstyle \mu_N} & \\
& \mathfrak{A}(N) \otimes \mathfrak{A}(N) &
\end{array}
\tag{4.1}
$$

in $\mathbf{C}$. This is however problematic in view of the equivariance property (2.2) of prefactorization algebras. In fact, if we would use (4.1) for *all* pairs of disjoint morphisms $\underline{f} = (f_1, f_2) : \underline{M} \to N$, then (2.2) would be satisfied if and only if the diagram in (2.7) commutes, which is in general not the case unless $f_1 \perp f_2$ are causally disjoint. By closer inspection of (3.1), one observes that (4.1) is *not* supposed to be the correct definition for all pairs of disjoint morphisms, but only for those pairs $\underline{f} = (f_1, f_2) : \underline{M} \to N$ where $f_1(M_1) \subseteq N$ is "later" (in a suitable sense) than $f_2(M_2) \subseteq N$. This would solve the problem concerning the equivariance property discussed above. The following definition formalizes a concept of time-ordering that allows us to prove our desired statements.

**Definition 4.1.** (a) Let $M \in \mathbf{Loc}$. A tuple $(U_1, \ldots, U_n)$ of causally convex open subsets $U_i \subseteq M$ is called *time-ordered* if $J_M^+(U_i) \cap U_j = \emptyset$, for all $i < j$.

(b) A tuple of pairwise disjoint morphisms $\underline{f} = (f_1, \ldots, f_n) : \underline{M} \to N$ is called *time-ordered* if the tuple $(f_1(M_1), \ldots, f_n(M_n))$ of causally convex open subsets $f_i(M_i) \subseteq N$ is time-ordered.

(c) A tuple of pairwise disjoint morphisms $\underline{f} = (f_1, \ldots, f_n) : \underline{M} \to N$ is called *time-orderable* if there exists a permutation $\rho \in \Sigma_n$ such that the tuple $\underline{f}\rho = (f_{\rho(1)}, \ldots, f_{\rho(n)}) : \underline{M}\rho \to N$ is time-ordered. We call $\rho$ a *time-ordering permutation* for $\underline{f}$.

**Remark 4.2.** By convention, all empty tuples $\emptyset \to N$ and all 1-tuples $f : M \to N$ are time-ordered. However, we would like to stress that for $n \geq 2$ not every tuple of pairwise disjoint morphisms $\underline{f} : \underline{M} \to N$ is time-orderable. For example, consider $n = 2$ and $\underline{f} = (f_1, f_2) : \underline{M} \to N$ the inclusion of the following causally convex open subsets into the Lorentzian cylinder $N$:



$$\tag{4.2}$$
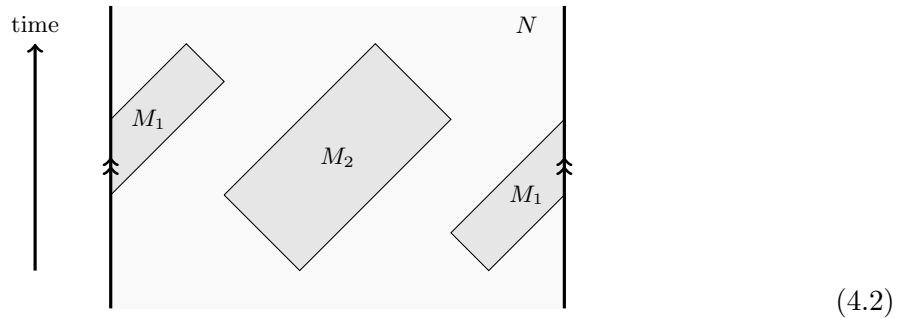
In this picture the left and right boundaries are identified as indicated, thereby producing the Lorentzian cylinder $N = (\mathbb{R} \times \mathbb{S}^1, g = -\mathrm{d}t^2 + \mathrm{d}\phi^2, \mathfrak{t} = \frac{\partial}{\partial t})$. $\triangle$

The following technical lemma is the crucial ingredient for our proofs below.

**Lemma 4.3.** *(i) Let $\rho \in \Sigma_n$ be a time-ordering permutation for the tuple of pairwise disjoint morphisms $\underline{f} = (f_1, \dots, f_n) : \underline{M} \to N$ and $\sigma \in \Sigma_n$ a permutation. Then $\sigma^{-1}\rho \in \Sigma_n$ is a time-ordering permutation for $\underline{f}\sigma = (f_{\sigma(1)}, \dots, f_{\sigma(n)}) : \underline{M}\sigma \to N$.*

*(ii) Let $\rho \in \Sigma_n$ be a time-ordering permutation for $\underline{f} = (f_1, \dots, f_n) : \underline{M} \to N$ and $\rho_i \in \Sigma_{k_i}$ a time-ordering permutation for $\underline{g}_i = (g_{i1}, \dots, g_{ik_i}) : \underline{L}_i \to M_i$, for $i = 1, \dots, n$. Then the permutation*

$$\rho\langle k_1, \dots, k_n\rangle \, (\rho_{\rho(1)} \oplus \dots \oplus \rho_{\rho(n)}) \in \Sigma_{k_1 + \dots + k_n} \quad , \tag{4.3}$$

*where $\rho\langle k_1, \dots, k_n\rangle$ denotes the block permutation corresponding to $\rho$ and $\rho_{\rho(1)} \oplus \dots \oplus \rho_{\rho(n)}$ the sum permutation of the $\rho_{\rho(i)}$, is a time-ordering permutation for*

$$\underline{f}(\underline{g}_1, \dots, \underline{g}_n) := (f_1 \, g_{11}, \dots, f_n \, g_{nk_n}) : (\underline{L}_1, \dots, \underline{L}_n) \longrightarrow N \quad . \tag{4.4}$$

*(iii) Let $\underline{f} : \underline{M} \to N$ be a time-orderable tuple of pairwise disjoint morphisms and $\rho, \rho' \in \Sigma_n$ time-ordering permutations for $\underline{f}$. Then the right permutation $\rho^{-1}\rho' : \underline{f}\rho \to \underline{f}\rho'$ is generated by transpositions of adjacent causally disjoint pairs of morphisms.*

*Proof.* (i): Trivial.

(ii): Since

$$\underline{f}(\underline{g}_1, \dots, \underline{g}_n) \, \rho\langle k_1, \dots, k_n\rangle \, (\rho_{\rho(1)} \oplus \dots \oplus \rho_{\rho(n)}) = (\underline{f}\rho)(\underline{g}_{\rho(1)}\rho_{\rho(1)}, \dots, \underline{g}_{\rho(n)}\rho_{\rho(n)}) \quad , \tag{4.5}$$

it is sufficient to prove that the composition of time-ordered tuples of pairwise disjoint morphisms is time-ordered. Therefore, assuming that $\underline{f}$ and $\underline{g}_i$, for $i = 1, \dots, n$, are time-ordered, we have to show that $(f_1 \, g_{11}, \dots, f_n \, g_{nk_n})$ is time-ordered, i.e. $J_N^+(f_i g_{ii'}(L_{ii'})) \cap f_j g_{jj'}(L_{jj'}) = \emptyset$ for the following two cases: Case 1 is $i < j$ and arbitrary $i' = 1, \dots, k_i$ and $j' = 1, \dots, k_j$. Case 2 is $i = j$ and $j < j'$. Case 1 follows immediately from the hypothesis that $\underline{f}$ is time-ordered, i.e. $J_N^+(f_i(M_i)) \cap f_j(M_j) = \emptyset$ for all $i < j$. For case 2 we use that $\underline{g}_i$ is time-ordered, i.e. $J_{M_i}^+(g_{ii'}(L_{ii'})) \cap g_{ij'}(L_{ij'}) = \emptyset$ for all $j < j'$, and hence by the properties of **Loc**-morphisms

$$J_N^+(f_i g_{ii'}(L_{ii'})) \cap f_i g_{ij'}(L_{ij'}) = f_i\Big(J_{M_i}^+(g_{ii'}(L_{ii'})) \cap g_{ij'}(L_{ij'})\Big) = \emptyset \quad . \tag{4.6}$$

This proves that $(f_1 \, g_{11}, \dots, f_n \, g_{nk_n})$ is time-ordered.

(iii): Suppose that $\rho^{-1}\rho' : \underline{f}\rho \to \underline{f}\rho'$ reverses the time-ordering between $f_k$ and $f_\ell$, i.e. $\rho(i) = k = \rho'(i')$ and $\rho(j) = \ell = \rho'(j')$ with $i < j$ and $j' < i'$ or vice versa with $j < i$ and $i' < j'$. Let us consider the case $i < j$ and $j' < i'$, the other one being similar. It follows that $J_N^+(f_{\rho(i)}(M_{\rho(i)})) \cap f_{\rho(j)}(M_{\rho(j)}) = \emptyset$ and $J_N^+(f_{\rho'(j')}(M_{\rho'(j')})) \cap f_{\rho'(i')}(M_{\rho'(i')}) = \emptyset$, which implies that $f_k \perp f_\ell$ are causally disjoint. This observation implies the existence of a factorization $\rho^{-1}\rho' = \tau_1 \cdots \tau_N : \underline{f}\rho \to \underline{f}\rho'$, where $\tau_l : \underline{f}\rho\tau_1 \cdots \tau_{l-1} \to \underline{f}\rho\tau_1 \cdots \tau_l$ is a transposition of adjacent causally disjoint pairs of morphisms, for all $l = 1, \dots, N$. $\square$

Lemma 4.3 plays a crucial role in the following definition of time-orderable prefactorization algebras because it ensures that time-orderable tuples of pairwise disjoint morphisms are composable and carry permutation actions. A *time-orderable prefactorization algebra* $\mathfrak{F}$ on **Loc** with values in **C** is given by the following data:

(i) for each $M \in$ **Loc**, an object $\mathfrak{F}(M) \in$ **C**;

(ii) for each time-orderable tuple $\underline{f} = (f_1, \dots, f_n) : \underline{M} \to N$ of pairwise disjoint morphisms, a **C**-morphism $\mathfrak{F}(\underline{f}) : \bigotimes_{i=1}^n \mathfrak{F}(M_i) \to \mathfrak{F}(N)$ (called *time-ordered product*), with the convention that to the empty tuple $\emptyset \to N$ is assigned a morphism $I \to \mathfrak{F}(N)$ from the monoidal unit.

These data are required to satisfy the analogs of the prefactorization algebra axioms from Section 2.2 for time-orderable tuples. A morphism $\zeta : \mathfrak{F} \to \mathfrak{G}$ of time-orderable prefactorization algebras is a family $\zeta_M : \mathfrak{F}(M) \to \mathfrak{G}(M)$ of $\mathbf{C}$-morphisms, for all $M \in \mathbf{Loc}$, that is compatible with the time-ordered products as in (2.3).

**Definition 4.4.** We denote by $\mathbf{tPFA}$ the category of time-orderable prefactorization algebras on $\mathbf{Loc}$. In analogy to Definitions 2.12 and 2.15, we introduce the full subcategories $\mathbf{tPFA}^{\mathrm{add}}, \mathbf{tPFA}^c, \mathbf{tPFA}^{\mathrm{add},c} \subseteq \mathbf{tPFA}$ of additive, Cauchy constant and Cauchy constant additive time-orderable prefactorization algebras.

**Remark 4.5.** Each ordinary prefactorization algebra on $\mathbf{Loc}$ defines a time-orderable one by restriction to time-orderable tuples of pairwise disjoint morphisms. This defines a functor $\mathbf{PFA} \to \mathbf{tPFA}$, which is faithful, but not necessarily full due to the fact that not all pairwise disjoint tuples $\underline{f} : \underline{M} \to N$ are time-orderable, cf. Remark 4.2. This functor clearly preserves both additivity and Cauchy constancy. $\triangle$

With these preparations we can now carry out our envisaged construction of a time-orderable prefactorization algebra $\mathfrak{F}_{\mathfrak{A}} \in \mathbf{tPFA}$ from a given algebraic quantum field theory $\mathfrak{A} \in \mathbf{AQFT}$. In particular, we can now complete our attempt from the beginning of this section to define the time-ordered factorization products. Let $\underline{f} = (f_1, \dots, f_n) : \underline{M} \to N$ be a *time-orderable* tuple of pairwise disjoint morphisms with time-ordering permutation $\rho \in \Sigma_n$. We define the corresponding time-ordered product $\mathfrak{F}_{\mathfrak{A}}(\underline{f}) : \bigotimes_{i=1}^n \mathfrak{A}(M_i) \to \mathfrak{A}(N)$ by the commutative diagram

$$
\begin{array}{ccc}
\displaystyle\bigotimes_{i=1}^n \mathfrak{A}(M_i) & \xrightarrow{\quad \mathfrak{F}_{\mathfrak{A}}(\underline{f}) \quad} & \mathfrak{A}(N) \\
{\scriptstyle \text{permute}} \downarrow & & \uparrow {\scriptstyle \mu_N^{(n)}} \\
\displaystyle\bigotimes_{i=1}^n \mathfrak{A}(M_{\rho(i)}) & \xrightarrow{\quad \otimes_i \mathfrak{A}(f_{\rho(i)}) \quad} & \mathfrak{A}(N)^{\otimes n}
\end{array}
\tag{4.7}
$$

in $\mathbf{C}$, where $\mu_N^{(n)}$ denotes the $n$-ary multiplication in the associative and unital algebra $\mathfrak{A}(N)$ in the given order, i.e. $\mu_N^{(n)}(a_1 \otimes \cdots \otimes a_n) = a_1 \cdots a_n$ with juxtaposition denoting multiplication in $\mathfrak{A}(N)$. As before, for $n = 0$ we assign to the empty tuple $\emptyset \to N$ the $\mathbf{C}$-morphism $\eta_N : I \to \mathfrak{A}(N)$ corresponding to the unit of $\mathfrak{A}(N)$.

**Lemma 4.6.** *The $\mathbf{C}$-morphism $\mathfrak{F}_{\mathfrak{A}}(\underline{f}) : \bigotimes_{i=1}^n \mathfrak{A}(M_i) \to \mathfrak{A}(N)$ defined in (4.7) does not depend on the choice of time-ordering permutation for $\underline{f} : \underline{M} \to N$.*

*Proof.* Consider time-ordering permutations $\rho, \rho' \in \Sigma_n$ for $\underline{f}$. Recalling Lemma 4.3 (iii), the right permutation $\rho^{-1}\rho' : \underline{f}\rho \to \underline{f}\rho'$ is generated by transpositions of adjacent causally disjoint pairs of morphisms. Hence, the claim follows from the Einstein causality axiom (2.7) of the algebraic quantum field theory $\mathfrak{A} \in \mathbf{AQFT}$. $\square$

**Theorem 4.7.** *Let $\mathfrak{A} \in \mathbf{AQFT}$ be an algebraic quantum field theory. Then the following data defines a time-orderable prefactorization algebra $\mathfrak{F}_{\mathfrak{A}} \in \mathbf{tPFA}$:*

*(i) for each $M \in \mathbf{Loc}$, define $\mathfrak{F}_{\mathfrak{A}}(M) := \mathfrak{A}(M) \in \mathbf{C}$ via the forgetful functor $\mathbf{Alg} \to \mathbf{C}$;*

*(ii) for each time-orderable tuple of pairwise disjoint morphisms $\underline{f} = (f_1, \dots, f_n) : \underline{M} \to N$, define the time-ordered product $\mathfrak{F}_{\mathfrak{A}}(\underline{f}) : \bigotimes_{i=1}^n \mathfrak{F}_{\mathfrak{A}}(M_i) \to \mathfrak{F}_{\mathfrak{A}}(N)$ according to (4.7) and Lemma 4.6 and, for each empty tuple $\emptyset \to N$, assign the unit $\eta_N : I \to \mathfrak{F}_{\mathfrak{A}}(N)$ of $\mathfrak{A}(N)$.*

*The assignment $\mathfrak{A} \mapsto \mathfrak{F}_{\mathfrak{A}}$ canonically extends to a functor $\mathfrak{F}_{(-)} : \mathbf{AQFT} \to \mathbf{tPFA}$.*

*Proof.* Lemma 4.3 immediately implies that $\mathfrak{F}_{\mathfrak{A}}$ satisfies the axioms of time-orderable prefactorization algebras. More explicitly, Lemma 4.3 (i) implies the equivariance axiom (2.2) for all time-orderable tuples and Lemma 4.3 (ii) implies the composition axiom (2.1) for all time-orderable tuples. By definition, we also have that $\mathfrak{F}_{\mathfrak{A}}(\mathrm{id}_M) = \mathrm{id}_{\mathfrak{F}_{\mathfrak{A}}(M)}$, for all $M \in \mathbf{Loc}$.

Concerning functoriality of the assignment $\mathfrak{A} \mapsto \mathfrak{F}_{\mathfrak{A}}$, we have to show that every **AQFT**-morphism $\kappa : \mathfrak{A} \to \mathfrak{B}$ canonically defines a **tPFA**-morphism $\mathfrak{F}_{\mathfrak{A}} \to \mathfrak{F}_{\mathfrak{B}}$. Observe that, for every time-orderable tuple $\underline{f} : \underline{M} \to N$ with time-ordering permutation $\rho \in \Sigma_n$, the diagram

$$
\begin{array}{ccccccc}
\bigotimes\limits_{i=1}^{n} \mathfrak{A}(M_i) & \xrightarrow{\text{permute}} & \bigotimes\limits_{i=1}^{n} \mathfrak{A}(M_{\rho(i)}) & \xrightarrow{\otimes_i \mathfrak{A}(f_{\rho(i)})} & \mathfrak{A}(N)^{\otimes n} & \xrightarrow{\mu_N^{(n)}} & \mathfrak{A}(N) \\
{\scriptstyle\otimes_i \kappa_{M_i}}\downarrow & & {\scriptstyle\otimes_i \kappa_{M_{\rho(i)}}}\downarrow & & {\scriptstyle\kappa_N^{\otimes n}}\downarrow & & \downarrow{\scriptstyle\kappa_N} \\
\bigotimes\limits_{i=1}^{n} \mathfrak{B}(M_i) & \xrightarrow[\text{permute}]{} & \bigotimes\limits_{i=1}^{n} \mathfrak{B}(M_{\rho(i)}) & \xrightarrow[\otimes_i \mathfrak{B}(f_{\rho(i)})]{} & \mathfrak{B}(N)^{\otimes n} & \xrightarrow[\mu_N^{(n)}]{} & \mathfrak{B}(N)
\end{array}
\tag{4.8}
$$

in **C** commutes. Hence, the family $\kappa_M : \mathfrak{A}(M) \to \mathfrak{B}(M)$ of **C**-morphisms defines a **tPFA**-morphism $\kappa : \mathfrak{F}_{\mathfrak{A}} \to \mathfrak{F}_{\mathfrak{B}}$. $\qquad\square$

**Proposition 4.8.** *$\mathfrak{A} \in \mathbf{AQFT}$ is additive (respectively Cauchy constant) if and only if $\mathfrak{F}_{\mathfrak{A}} \in$ **tPFA** is additive (respectively Cauchy constant). In particular, the functor $\mathfrak{F}_{(-)} : \mathbf{AQFT} \to \mathbf{tPFA}$ from Theorem 4.7 restricts to full subcategories as $\mathfrak{F}_{(-)} : \mathbf{AQFT}^{\mathrm{add}} \to \mathbf{tPFA}^{\mathrm{add}}$, $\mathfrak{F}_{(-)} : \mathbf{AQFT}^{c} \to \mathbf{tPFA}^{c}$ and $\mathfrak{F}_{(-)} : \mathbf{AQFT}^{\mathrm{add},c} \to \mathbf{tPFA}^{\mathrm{add},c}$.*

*Proof.* For additivity, this is a consequence of Remark 2.18 and the fact that $\mathfrak{F}_{\mathfrak{A}}(f) = \mathfrak{A}(f) : \mathfrak{A}(M) \to \mathfrak{A}(N)$ as **C**-morphisms, for all $f : M \to N$ in **Loc**. For Cauchy constancy, this is a consequence of the latter and the fact that the forgetful functor $\mathbf{Alg} \to \mathbf{C}$ preserves and detects isomorphisms. $\qquad\square$

# 5 Equivalence theorem

## 5.1 Main result

The aim of this section is to prove that our two constructions from Sections 3 and 4 are inverse to each other when restricted to their common domain of validity. Recall that in Section 3 we considered Cauchy constant additive prefactorization algebras and constructed a functor $\mathfrak{A}_{(-)} : \mathbf{PFA}^{\mathrm{add},c} \to \mathbf{AQFT}^{\mathrm{add},c}$ to the category of Cauchy constant additive algebraic quantum field theories, cf. Theorem 3.11. Because the construction presented in Section 3 only involves *time-orderable* tuples of disjoint morphisms, this functor factors through the forgetful functor $\mathbf{PFA}^{\mathrm{add},c} \to \mathbf{tPFA}^{\mathrm{add},c}$ (cf. Remark 4.5) to the category of Cauchy constant additive *time-orderable* prefactorization algebras, cf. Definition 4.4. We shall denote the resulting functor by the same symbol $\mathfrak{A}_{(-)} : \mathbf{tPFA}^{\mathrm{add},c} \to \mathbf{AQFT}^{\mathrm{add},c}$. Let us further recall the functor $\mathfrak{F}_{(-)} : \mathbf{AQFT}^{\mathrm{add},c} \to \mathbf{tPFA}^{\mathrm{add},c}$ from Theorem 4.7 and Proposition 4.8. Our main result is the following equivalence theorem.

**Theorem 5.1.** *The two functors $\mathfrak{A}_{(-)} : \mathbf{tPFA}^{\mathrm{add},c} \to \mathbf{AQFT}^{\mathrm{add},c}$ and $\mathfrak{F}_{(-)} : \mathbf{AQFT}^{\mathrm{add},c} \to \mathbf{tPFA}^{\mathrm{add},c}$ are inverses of each other. As a consequence, the category $\mathbf{AQFT}^{\mathrm{add},c}$ of Cauchy constant additive algebraic quantum field theories is isomorphic to the category $\mathbf{tPFA}^{\mathrm{add},c}$ of Cauchy constant additive time-orderable prefactorization algebras.*

*Proof.* The only non-trivial check to confirm that $\mathfrak{A}_{(-)} \circ \mathfrak{F}_{(-)} = \mathrm{id}_{\mathbf{AQFT}^{\mathrm{add},c}}$ amounts to show that, for every $\mathfrak{A} \in \mathbf{AQFT}^{\mathrm{add},c}$, the multiplications on $\mathfrak{A}_{\mathfrak{F}_{\mathfrak{A}}}(M)$ and on $\mathfrak{A}(M)$ coincide, for all

$M \in \textbf{Loc}$. By (3.1) and (4.7), the multiplication on $\mathfrak{A}_{\mathfrak{F}_\mathfrak{A}}(M)$ is given by

$$\mathfrak{A}(M) \otimes \mathfrak{A}(M) \xleftarrow[\cong]{\mathfrak{A}(\iota_{U_+}^M) \otimes \mathfrak{A}(\iota_{U_-}^M)} \mathfrak{A}(U_+) \otimes \mathfrak{A}(U_-) \xrightarrow{\mathfrak{A}(\iota_{U_+}^M) \otimes \mathfrak{A}(\iota_{U_-}^M)} \mathfrak{A}(M)^{\otimes 2} \xrightarrow{\mu_M} \mathfrak{A}(M) \quad , \quad (5.1)$$

where $\iota_{\underline{U}}^M = (\iota_{U_+}^M, \iota_{U_-}^M) : \underline{U} \to M$ is any object of $\textbf{P}_M$. This clearly coincides with the original multiplication $\mu_M$ on $\mathfrak{A}(M)$.

Conversely, to show that $\mathfrak{F}_{(-)} \circ \mathfrak{A}_{(-)} = \text{id}_{\textbf{tPFA}^{\text{add},c}}$, we have to confirm that the time-ordered products of $\mathfrak{F}_{\mathfrak{A}_\mathfrak{F}} \in \textbf{tPFA}^{\text{add},c}$ coincide with the original time-ordered products of $\mathfrak{F} \in \textbf{tPFA}^{\text{add},c}$. In arity $n = 0$ and $n = 1$ this is obvious. For $n \geq 2$, this is more complicated and requires some preparations. Using equivariance under permutation actions, it is sufficient to compare the time-ordered products for *time-ordered* (in contrast to time-orderable) tuples $\underline{f} = (f_1, \ldots, f_n) : \underline{M} \to N$. Because of additivity, we can further restrict to the case where $\underline{f} : \underline{M} \to N$ has relatively compact images, i.e. $f_i(M_i) \subseteq N$ is relatively compact, for all $i = 1, \ldots, n$. We shall now show that, due to Cauchy constancy, we can further restrict our attention to time-ordered tuples $\underline{h} = (h_1, \ldots, h_n) : \underline{L} \to N$ with relatively compact images for which there exists a Cauchy surface $\Sigma$ of $N$ such that

$$h_1(L_1), \ldots, h_{n-1}(L_{n-1}) \subseteq \Sigma_+ := I_N^+(\Sigma) \subseteq N \qquad \text{and} \qquad h_n(L_n) \subseteq \Sigma_- := I_N^-(\Sigma) \subseteq N \quad . \quad (5.2)$$

Indeed, given any time-ordered tuple $\underline{f} : \underline{M} \to N$ with relatively compact images, we shall prove below that there exists a family of Cauchy morphisms $g_i : L_i \xrightarrow{c} M_i$, for $i = 1, \ldots, n$, such that $\underline{h} := \underline{f}(g_1, \ldots, g_n) = (f_1 \, g_1, \ldots, f_n \, g_n) : \underline{L} \to N$ admits a Cauchy surface $\Sigma$ that satisfies (5.2). Cauchy constancy and the fact that the time-ordered products of $\mathfrak{F}_{\mathfrak{A}_\mathfrak{F}}$ and $\mathfrak{F}$ agree in arity $n = 1$ then implies that $\mathfrak{F}_{\mathfrak{A}_\mathfrak{F}}(\underline{f}) = \mathfrak{F}(\underline{f})$ if and only if $\mathfrak{F}_{\mathfrak{A}_\mathfrak{F}}(\underline{h}) = \mathfrak{F}(\underline{h})$. To exhibit such a family of Cauchy morphisms for $\underline{f} : \underline{M} \to N$, let us choose Cauchy surfaces $\Sigma_i$ of $M_i$, for $i = 1, \ldots, n$, and define $L_i := I_{M_i}^+(\Sigma_i)$, for $i = 1, \ldots, n-1$, and $L_n := I_{M_n}^-(\Sigma_n)$. Let us further define $g_i := \iota_{L_i}^{M_i} : L_i \xrightarrow{c} M_i$ by subset inclusion, for $i = 1, \ldots, n$. A Cauchy surface $\Sigma$ of $N$ is constructed by extending via [BS06, Theorem 3.8] the compact and achronal subset

$$\widetilde{\Sigma} := \bigcup_{i=1}^n \left( \overline{f_i(\Sigma_i)} \Big\backslash I_N^+ \Big( \bigcup_{j=i+1}^n \overline{f_j(\Sigma_j)} \Big) \right) \subseteq N \quad . \quad (5.3)$$

By direct inspection one observes that $\Sigma$ fulfills (5.2).

Using (5.2), we obtain a factorization

$$\underline{h} = \iota_{\underline{\Sigma}}^N \big( (h_1, \ldots, h_{n-1}), h_n \big) \quad , \quad (5.4)$$

where on the right-hand side we regard $h_i : L_i \to \Sigma_+$ as morphisms to $\Sigma_+$, for $i = 1, \ldots, n-1$, and $h_n : L_n \to \Sigma_-$ as a morphism to $\Sigma_-$. Iterating this construction, we observe that it is sufficient to prove that $\mathfrak{F}_{\mathfrak{A}_\mathfrak{F}}(\iota_{\underline{\Sigma}}^N) = \mathfrak{F}(\iota_{\underline{\Sigma}}^N)$, for all $\iota_{\underline{\Sigma}}^N = (\iota_{\Sigma_+}^N, \iota_{\Sigma_-}^N) : \underline{\Sigma} \to N$, where $N \in \textbf{Loc}$ and the Cauchy surface $\Sigma$ of $N$ is arbitrary. Using (4.7) and (3.1), we obtain that $\mathfrak{F}_{\mathfrak{A}_\mathfrak{F}}(\iota_{\underline{\Sigma}}^N) : \mathfrak{F}(\Sigma_+) \otimes \mathfrak{F}(\Sigma_-) \to \mathfrak{F}(N)$ is given by

$$\mathfrak{F}(\Sigma_+) \otimes \mathfrak{F}(\Sigma_-) \xrightarrow{\mathfrak{F}(\iota_{\Sigma_+}^N) \otimes \mathfrak{F}(\iota_{\Sigma_-}^N)} \mathfrak{F}(N) \otimes \mathfrak{F}(N) \xleftarrow[\cong]{\mathfrak{F}(\iota_{\Sigma_+}^N) \otimes \mathfrak{F}(\iota_{\Sigma_-}^N)} \mathfrak{F}(\Sigma_+) \otimes \mathfrak{F}(\Sigma_-) \xrightarrow{\mathfrak{F}(\iota_{\underline{\Sigma}}^N)} \mathfrak{F}(N) \quad , \quad (5.5)$$

which clearly coincides with the original time-ordered product $\mathfrak{F}(\iota_{\underline{\Sigma}}^N) : \mathfrak{F}(\Sigma_+) \otimes \mathfrak{F}(\Sigma_-) \to \mathfrak{F}(N)$. This concludes our proof. $\qquad \square$

**Remark 5.2.** We would like to mention very briefly a more abstract operadic perspective on the Equivalence Theorem 5.1. Recall from [BSW17] that there exists a **Set**-valued colored operad

$\mathcal{O}_{(\mathbf{Loc},\perp)}$ whose category of $\mathbf{C}$-valued algebras is the category of algebraic quantum field theories, i.e. $\mathbf{AQFT} = \mathbf{Alg}_{\mathcal{O}_{(\mathbf{Loc},\perp)}}(\mathbf{C})$. We can also define a $\mathbf{Set}$-valued colored operad $\mathcal{P}_{\mathbf{Loc}}$ such that $\mathbf{tPFA} = \mathbf{Alg}_{\mathcal{P}_{\mathbf{Loc}}}(\mathbf{C})$. Concretely, the colors of $\mathcal{P}_{\mathbf{Loc}}$ are the objects of $\mathbf{Loc}$ and the sets of operations are $\mathcal{P}_{\mathbf{Loc}}\binom{N}{\underline{M}} := \big\{\text{all time-orderable tuples } \underline{f} : \underline{M} \to N\big\}$. Operadic composition is given by (4.4), the operadic units are $\mathrm{id}_M \in \mathcal{P}_{\mathbf{Loc}}\binom{M}{M}$ and the permutation actions are $\mathcal{P}_{\mathbf{Loc}}(\sigma) : \mathcal{P}_{\mathbf{Loc}}\binom{N}{\underline{M}} \to \mathcal{P}_{\mathbf{Loc}}\binom{N}{\underline{M}\sigma}$, $\underline{f} \mapsto \underline{f}\sigma$, for $\sigma \in \Sigma_n$. Using Lemma 4.3 and the definition of the colored operad $\mathcal{O}_{(\mathbf{Loc},\perp)}$ given in [BSW17], one immediately observes that the component maps

$$\Phi : \mathcal{P}_{\mathbf{Loc}}\binom{N}{\underline{M}} \longrightarrow \mathcal{O}_{(\mathbf{Loc},\perp)}\binom{N}{\underline{M}} \ , \quad \underline{f} \longmapsto [\rho^{-1}, \underline{f}] \tag{5.6}$$

define a colored operad morphism $\Phi : \mathcal{P}_{\mathbf{Loc}} \to \mathcal{O}_{(\mathbf{Loc},\perp)}$, where $\rho \in \Sigma_n$ is a time-ordering permutation for $\underline{f}$. The associated pullback functor $\Phi^* : \mathbf{Alg}_{\mathcal{O}_{(\mathbf{Loc},\perp)}}(\mathbf{C}) \to \mathbf{Alg}_{\mathcal{P}_{\mathbf{Loc}}}(\mathbf{C})$ is then precisely our functor $\mathfrak{F}_{(-)} : \mathbf{AQFT} \to \mathbf{tPFA}$ from Theorem 4.7. By operadic left Kan extension, there exists an adjunction

$$\Phi_! : \mathbf{tPFA} \; \underset{\longleftarrow}{\overset{\longrightarrow}{\rightleftarrows}} \; \mathbf{AQFT} : \Phi^* = \mathfrak{F}_{(-)} \quad . \tag{5.7}$$

Theorem 5.1 then states that restricting both sides of this adjunction to Cauchy constant and additive theories induces an adjoint equivalence $\mathfrak{A}_{(-)} : \mathbf{tPFA}^{\mathrm{add},c} \overset{\sim}{\rightleftarrows} \mathbf{AQFT}^{\mathrm{add},c} : \mathfrak{F}_{(-)}$.

We expect that this operadic perspective will become important when considering the case where the target category $\mathbf{C}$ is a higher category or model category. This generalization is crucial for the description of quantum gauge theories in terms of factorization algebras [CG17] or algebraic quantum field theories [BSS15, BS17, BSW19b, BS19]. The adjunction (5.7) then becomes a Quillen adjunction between model categories, and a reasonable equivalence theorem would state that suitable restrictions to homotopy-invariant analogs of Cauchy constant and additive theories induce a Quillen equivalence. Proving such an equivalence theorem in a higher categorical context is technically complicated and will not be considered in the present paper. $\triangle$

**Remark 5.3.** If the underlying category $\mathbf{C}$ carries an involutive structure, the category $\mathbf{AQFT}$ of algebraic quantum field theories carries an induced involutive structure, which is used to formalize $*$-objects, cf. [BSW19a]. Working in the standard involutive category $\mathbf{Vec}_{\mathbb{C}}$ of complex vector spaces, these $*$-object are the usual algebraic quantum field theories that assign $*$-*algebras* to spacetimes. Theorem 5.1 allows us to transfer the involutive structure on $\mathbf{AQFT}^{\mathrm{add},c}$ to the category $\mathbf{tPFA}^{\mathrm{add},c}$ of Cauchy constant additive time-ordered prefactorization algebras. The resulting involutive structure, however, involves Cauchy constancy explicitly. As a consequence, it seems to lack a geometric interpretation, which is why we refrain from describing it explicitly. $\triangle$

## 5.2 Example: The free Klein-Gordon field

We apply our general Equivalence Theorem 5.1 to the simple example given by the free Klein-Gordon field and thereby recover the results from [GR17]. Let us briefly recall the algebraic quantum field theory description of the free Klein-Gordon field. For every $M \in \mathbf{Loc}$, consider the Klein-Gordon operator $P_M := -\Box_M + m^2 : C^\infty(M) \to C^\infty(M)$, where $\Box_M$ is the d'Alembert operator and $m^2 \geq 0$ is a mass parameter. $P_M$ admits a unique retarded/advanced Green's operator $G_M^\pm : C_c^\infty(M) \to C^\infty(M)$, where the subscript $c$ denotes compactly supported functions. The $\mathbb{R}$-vector space $\mathcal{V}(M)$ of linear observables on $M$ is defined as the cokernel

$$C_c^\infty(M) \xrightarrow{P_M} C_c^\infty(M) \longrightarrow \mathcal{V}(M) := C_c^\infty(M)/P_M(C_c^\infty(M)) \quad . \tag{5.8}$$

Because $C_c^\infty : \mathbf{Loc} \to \mathbf{Vec}_{\mathbb{R}}$ is a cosheaf for (causally convex) open covers and $P : C_c^\infty \to C_c^\infty$ is a natural transformation, it follows that $\mathcal{V} : \mathbf{Loc} \to \mathbf{Vec}_{\mathbb{R}}$ is a cosheaf too. Consider the complexified symmetric algebra $\mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M)) \in \mathbf{CAlg}$, which is a commutative algebra in the

closed symmetric monoidal category $(\mathbf{Vec}_{\mathbb{C}}, \otimes, \mathbb{C}, \tau)$ of complex vector spaces. This algebra is deformed to a noncommutative algebra by introducing a $\star$-product. For this we first define a (de Rham type) differential $\mathrm{d} : \mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M)) \to \mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M)) \otimes \mathcal{V}(M)$ by setting on monomials

$$\mathrm{d}\big(\varphi_1 \cdots \varphi_n\big) := \sum_{i=1}^{n} \varphi_1 \cdots \overset{i}{\vee} \cdots \varphi_n \otimes \varphi_i \quad , \tag{5.9}$$

where $\overset{i}{\vee}$ means omission of $\varphi_i$. Using the causal propagator $G_M := G_M^+ - G_M^- : \mathcal{V}(M) \to \ker P_M$ and the integration map $\int_M : \mathcal{V}(M) \otimes \ker P_M \to \mathbb{R}, \ \varphi \otimes \Phi \mapsto \int_M \varphi \, \Phi \, \mathrm{vol}_M$, we define the bi-differential operator

$$\begin{array}{ccc}
\mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M))^{\otimes 2} & \xrightarrow{\ \langle G_M, \mathrm{d} \otimes \mathrm{d} \rangle\ } & \mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M))^{\otimes 2} \\
{\scriptstyle (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\mathrm{d} \otimes \mathrm{d})} \Big\downarrow & & \Big\uparrow {\scriptstyle \mathrm{id} \otimes \int_M} \\
\mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M))^{\otimes 2} \otimes \mathcal{V}(M) \otimes \mathcal{V}(M) & \xrightarrow{\ \mathrm{id} \otimes \mathrm{id} \otimes G_M\ } & \mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M))^{\otimes 2} \otimes \mathcal{V}(M) \otimes \ker P_M
\end{array} \tag{5.10}$$

The $\star$-product $\star_M : \mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M))^{\otimes 2} \to \mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M))$ is then defined by composing

$$\mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M))^{\otimes 2} \xrightarrow{\ \exp\left(\frac{i}{2} \langle G_M, \mathrm{d} \otimes \mathrm{d} \rangle\right)\ } \mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M))^{\otimes 2} \xrightarrow{\ \cdot_M\ } \mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M)) \quad , \tag{5.11}$$

where $\cdot_M$ denotes the commutative product on $\mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M))$. (The exponential series converges because it terminates for polynomials.) Setting $\mathfrak{A}_{\mathrm{KG}}(M) := \big(\mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M)), \star_M, \eta_M\big) \in \mathbf{Alg}$ with $\eta_M$ the unit of $\mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M))$, for all $M \in \mathbf{Loc}$, defines a Cauchy constant additive algebraic quantum field theory $\mathfrak{A}_{\mathrm{KG}} \in \mathbf{AQFT}^{\mathrm{add},c}$. Note that additivity is a consequence of $\mathcal{V} : \mathbf{Loc} \to \mathbf{Vec}_{\mathbb{R}}$ being a cosheaf.

Theorem 5.1 provides a corresponding Cauchy constant additive time-orderable prefactorization algebra $\mathfrak{F}_{\mathrm{KG}} := \mathfrak{F}_{\mathfrak{A}_{\mathrm{KG}}} \in \mathbf{tPFA}^{\mathrm{add},c}$. To get some intuition on what this prefactorization algebra does, let us analyze the explicit form of the binary time-ordered products $\mathfrak{F}_{\mathrm{KG}}(\underline{f}) : \mathfrak{F}_{\mathrm{KG}}(M_1) \otimes \mathfrak{F}_{\mathrm{KG}}(M_2) \to \mathfrak{F}_{\mathrm{KG}}(N)$. In the case where $\underline{f} = (f_1, f_2) : \underline{M} \to N$ is time-ordered, i.e. $J_N^+(f_1(M_1)) \cap f_2(M_2) = \emptyset$, we obtain from (4.7), (5.11) and the support properties of $G_N^\pm$ that

$$\mathfrak{F}_{\mathrm{KG}}(\underline{f}) = \cdot_N \circ \exp\big(\tfrac{i}{2} \langle G_N^+, \mathrm{d} \otimes \mathrm{d} \rangle\big) \circ \big(\mathfrak{A}_{\mathrm{KG}}(f_1) \otimes \mathfrak{A}_{\mathrm{KG}}(f_2)\big) \quad (\underline{f} \text{ time-ordered}) \quad . \tag{5.12a}$$

In the case where $\underline{f} = (f_1, f_2) : \underline{M} \to N$ is anti-time-ordered, i.e. $J_N^+(f_2(M_2)) \cap f_1(M_1) = \emptyset$, we obtain

$$\mathfrak{F}_{\mathrm{KG}}(\underline{f}) = \cdot_N \circ \exp\big(\tfrac{i}{2} \langle G_N^-, \mathrm{d} \otimes \mathrm{d} \rangle\big) \circ \big(\mathfrak{A}_{\mathrm{KG}}(f_1) \otimes \mathfrak{A}_{\mathrm{KG}}(f_2)\big) \quad (\underline{f} \text{ anti-time-ordered}) \quad . \tag{5.12b}$$

Using again the support properties of $G_N^\pm$, we observe that the the two cases in (5.12) can be combined into a single formula

$$\mathfrak{F}_{\mathrm{KG}}(\underline{f}) = \cdot_N \circ \exp\big(i \langle G_N^{\mathrm{D}}, \mathrm{d} \otimes \mathrm{d} \rangle\big) \circ \big(\mathfrak{A}_{\mathrm{KG}}(f_1) \otimes \mathfrak{A}_{\mathrm{KG}}(f_2)\big) \quad (\underline{f} \text{ time-orderable}) \quad , \tag{5.13}$$

where $G_N^{\mathrm{D}} := \tfrac{1}{2}(G_N^+ + G_N^-)$ is the so-called Dirac propagator, that is valid for every time-orderable tuple $(f_1, f_2)$. In perturbative algebraic quantum field theory (see e.g. [FR13, Rej16]), the products $\cdot_{\mathcal{T}_N} := \cdot_N \circ \exp\big(i \langle G_N^{\mathrm{D}}, \mathrm{d} \otimes \mathrm{d} \rangle\big)$ are called time-ordered products.

Our observations in this subsection can thus be summarized as follows: The prefactorization algebra $\mathfrak{F}_{\mathrm{KG}} \in \mathbf{tPFA}^{\mathrm{add},c}$ corresponding to the free Klein-Gordon theory $\mathfrak{A}_{\mathrm{KG}} \in \mathbf{AQFT}^{\mathrm{add},c}$ encodes the usual time-ordered products obtained by the Dirac propagator. This agrees with the observations in [GR17].

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