

PARTITIONING EDGE-COLOURED COMPLETE SYMMETRIC DIGRAPHS INTO MONOCHROMATIC COMPLETE SUBGRAPHS

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ABSTRACT. Let $\vec{K}_{\mathbb{N}}$ be the complete symmetric digraph on the positive integers. Answering a question of DeBiasio and McKenney [3], we construct a 2-colouring of the edges of $\vec{K}_{\mathbb{N}}$ in which every monochromatic path has density 0.

However, if we restrict the length of monochromatic paths in one colour, then no example as above can exist: We show that every $(r+1)$ -edge-coloured complete symmetric digraph (of arbitrary infinite cardinality) containing no directed paths of edge-length ℓ_i for any colour $i \leq r$ can be covered by $\prod_{i \leq r} \ell_i$ pairwise disjoint monochromatic complete symmetric digraphs in colour $r+1$.

Furthermore, we present a stability version for the countable case of the latter result: We prove that the edge-colouring is uniquely determined on a large subgraph, as soon as the upper density of monochromatic paths in colour $r+1$ is bounded by $\prod_{i \in [r]} \frac{1}{\ell_i}$.

1. INTRODUCTION

Let $K_{\mathbb{N}}$ be the complete graph on the positive integers and $\vec{K}_{\mathbb{N}}$ be the complete symmetric digraph on the positive integers. The *upper density* of a set $A \subseteq \mathbb{N}$ is

$$\bar{d}(A) = \limsup_{n \rightarrow \infty} \frac{|A \cap \{1, \dots, n\}|}{n}$$

For a graph or digraph G with vertex set $V(G) \subseteq \mathbb{N}$, we define the *upper density* of G to be that of $V(G)$. Throughout this paper, by a k -colouring, we mean a k -edge-colouring. In a 2-colouring, we will assume that the colours are red and blue. Given a directed graph D and sets $A, B \subseteq V(D)$, we write $[A, B]$ -edges to mean all edges $(x, y) \in E(D)$ with $x \in A$ and $y \in B$.

For finite graphs, Gerencsér and Gyárfás [7] proved that in every 2-colouring of K_n there is a monochromatic path on at least $(2n+1)/3$ vertices; furthermore, this is best possible. Erdős and Galvin [5] proved an infinite analogue, showing that in every 2-colouring of $K_{\mathbb{N}}$ there exists a monochromatic infinite path with upper density at least $\frac{2}{3}$; they also give a 2-colouring of $K_{\mathbb{N}}$ in which every monochromatic path has upper density at most $\frac{8}{9}$. Recently, DeBiasio and McKenney [3]

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improved the lower bound, showing that, in every 2-colouring of $K_{\mathbb{N}}$, there exists a monochromatic infinite path with upper density at least $\frac{3}{4}$. For more colours, Rado [10] showed that, in every r -colouring of $K_{\mathbb{N}}$, there is a partition of the vertices into at most r disjoint paths of distinct colours¹, which implies that there is a monochromatic path of upper density at least $1/r$. Elekes, Soukup, Soukup and Szentmiklössy [4] have recently extended Rado's result for two colours to the complete graph on \aleph_1 where \aleph_1 is the smallest uncountable cardinal. Shortly after, Soukup [13] extended this further to any finite number of colours and complete graphs of any infinite cardinality.

For directed graphs, the picture is a little different. In the finite case, Raynaud [11] showed that, in any 2-colouring of \vec{K}_n , there is a directed Hamiltonian cycle C with the following property: there are two vertices a and b such that the directed path from a to b along C is red and the directed path from b to a along C is blue. As a corollary, in any 2-colouring of \vec{K}_n , there is a monochromatic directed path on at least $n/2 + 1$ vertices; this is easily seen to be best possible by partitioning the vertices of \vec{K}_n into two sets A, B with $||A| - |B|| \leq 1$ and colouring the edge (x, y) red if $x \in A$ and blue if $x \in B$. In this paper, we will be interested in the infinite directed case. In particular, we will be considering edge-colourings of $\vec{K}_{\mathbb{N}}$ and prove a variety of results relating to the upper density of paths in $\vec{K}_{\mathbb{N}}$.

Let $P = v_1 v_2 \dots$ be a path in $\vec{K}_{\mathbb{N}}$. We say that P is a *directed path* if every edge in P is oriented in the same direction. By the *length* of a path, we mean the number of edges in the path. DeBiasio and McKenney [3] recently proved the following result.

Theorem 1.1. *For every $\varepsilon > 0$, there exists a 2-colouring of $\vec{K}_{\mathbb{N}}$ such that every monochromatic directed path has upper density less than ε .*

DeBiasio and McKenney also asked the following natural question: does there exist a 2-colouring of $\vec{K}_{\mathbb{N}}$ in which every monochromatic directed path has upper density 0? In Section 2, we will give a positive answer to this question (taken from the manuscript [8] by the third author). We note that the same example was independently obtained by Jan Corsten [2].

Theorem 1.2. *There exists a 2-colouring of $\vec{K}_{\mathbb{N}}$ such that every monochromatic directed path has upper density 0.*

In light of this result, it is natural to ask under what conditions we can guarantee the existence of a monochromatic path of positive density. It is easy to see (from Ramsey's Theorem) that every r -colouring of $\vec{K}_{\mathbb{N}}$ contains a monochromatic directed path of infinite length. The third author observed in [8] that if one restricts the maximal length of directed paths in the first colour, then there must be monochromatic paths in the second colour with non-vanishing upper density. More generally, the authors proved the following sequence of results:

¹In fact, Rado proved that in any r -colouring of the edges of $\vec{K}_{\mathbb{N}}$ there is a partition of the vertices into at most r disjoint *anti-directed* paths of distinct colours. This implies the undirected version stated above.

- In the manuscript [8], Guggiari shows that for any $(r + 1)$ -edge-colouring of $\vec{K}_{\mathbb{N}}$ for which there are no directed paths of length ℓ_i in colour i for any $i \in [r]$, there is a monochromatic directed path in colour $r + 1$ with upper density at least $\prod_{i \leq r} \frac{1}{\ell_i}$.
- Confirming and extending a conjecture by Guggiari, Bürger and Pitz showed in the manuscript [1] that under the same assumptions as above, the vertex set \mathbb{N} can be covered by $\prod_{i \leq r} \ell_i$ pairwise disjoint monochromatic directed paths in colour $r + 1$.

In Section 3 of this paper, we will present a proof of the following statement, which nicely generalizes the results above. Write \vec{K} for the complete symmetric digraph on a finite or (not-necessarily-countable) infinite number of vertices.

Theorem 1.3. *Let $c: E(\vec{K}) \rightarrow [r + 1]$ be an edge-colouring of \vec{K} for which there is no directed path of length ℓ_i in colour i for any $i \in [r]$. Then there is a partition of $V(\vec{K})$ into $\prod_{i \in [r]} \ell_i$ complete symmetric digraphs monochromatic in colour $r + 1$.*

In particular, for the countable directed complete digraph $\vec{K}_{\mathbb{N}}$, we reobtain the result, now with a much shorter proof, that for any $(r + 1)$ -edge-colouring of $\vec{K}_{\mathbb{N}}$ for which there are no monochromatic directed paths of length ℓ_i in colour i for any $i \in [r]$, the vertex set \mathbb{N} can be covered by $\prod_{i \leq r} \ell_i$ pairwise disjoint monochromatic directed paths in colour $r + 1$.

It is not hard to see that this result is best possible: There are colourings of $\vec{K}_{\mathbb{N}}$, which we will call *cube colourings* for their geometric structure, that witness optimality of the above result, see Theorem 3.6 below. For example, it is not hard to see that any blue monochromatic path in the colouring of Figure 1 has upper density at most $\frac{1}{6}$.

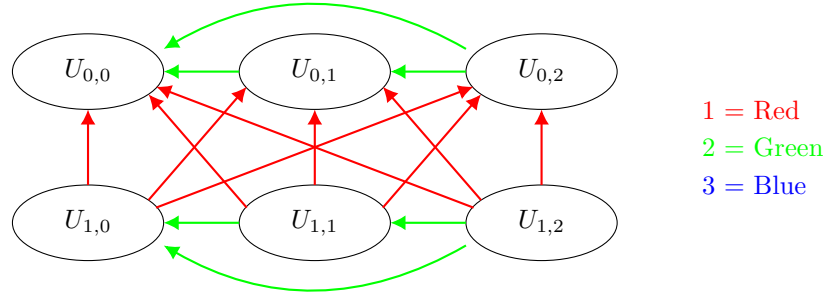


FIGURE 1. Cube colouring for $\ell_1 = 2$ and $\ell_2 = 3$. All edges not shown are blue. All partition classes $U_{x,y}$ have density $\frac{1}{6}$.

Extending the work of Guggiari in [8, Theorem 1.4], we will prove a stability version of Theorem 1.3 in Section 4 of this paper. This stability result, Theorem 4.2, says that any colouring $c: E(\vec{K}_{\mathbb{N}}) \rightarrow [r + 1]$ for which there is no directed path of length ℓ_i in colour i for any $i \in [r]$ and every directed path in colour $r + 1$ has upper density at most $\prod_{i \in [r]} \frac{1}{\ell_i}$, must essentially be of the same cubic structure as the *cube colouring*, where the meaning of ‘essentially’ will be explained in Section 4.

2. AN EXAMPLE WHERE ALL MONOCHROMATIC PATHS HAVE DENSITY 0

Proof of Theorem 1.2. We colour the edges of $\vec{K}_{\mathbb{N}}$ in the following way. Let $m, n \in \mathbb{N}$ be distinct positive integers. Set $t = \min\{s \in \mathbb{N} : m \not\equiv n \pmod{2^s}\}$. Exchanging m and n if necessary, we may assume that $m \equiv x \pmod{2^t}$ where $x \in \{0, \dots, 2^{t-1} - 1\}$ and $n \equiv 2^{t-1} + x \pmod{2^t}$. We colour (m, n) red and (n, m) blue.

Let P be any monochromatic directed path. If P is a finite path, then $\bar{d}(P) = 0$. Therefore, we may assume that P is an infinite path. Without loss of generality, P is red.

Inductive Hypothesis. For any $k \in \mathbb{N}$, there exists $i \in \{0, \dots, 2^k - 1\}$ such that P is eventually contained within the set $\{n \in \mathbb{N} : n \equiv i \pmod{2^k}\}$. Hence, $\bar{d}(P) \leq 2^{-k}$.

Base Case. For $i \in \{0, 1\}$, let $A_i = \{n \in \mathbb{N} : n \equiv i \pmod{2}\}$. The sets A_0 and A_1 partition the vertices of $\vec{K}_{\mathbb{N}}$. Suppose P contains a vertex $u \in A_1$. Then all of the vertices occurring after u in P must also be in A_1 because P is a red directed path and hence P is eventually contained within A_1 . If P does not contain a vertex from A_1 , then P must be completely contained within A_0 . Hence, for some $i \in \{0, 1\}$, we have $\bar{d}(P) \leq \bar{d}(A_i) = \frac{1}{2}$.

Inductive Step. Fix $k \geq 2$ and partition the vertices of $\vec{K}_{\mathbb{N}}$ into 2^k sets A_0, \dots, A_{2^k-1} based on their residue modulo 2^k (see Figure 2). By the inductive hypothesis, there exists $i \in \{0, \dots, 2^{k-1} - 1\}$ such that P is eventually contained within the set $A_i \cup A_{2^{k-1}+i}$. By using the same argument as in the base case, we find that, if P contains a vertex in $A_{2^{k-1}+i}$, then it is eventually contained within this set; otherwise, it is eventually contained within A_i . Hence, there exists $j \in \{0, \dots, 2^k - 1\}$ such that P is eventually contained within A_j and so $\bar{d}(P) \leq \bar{d}(A_j) = 2^{-k}$.

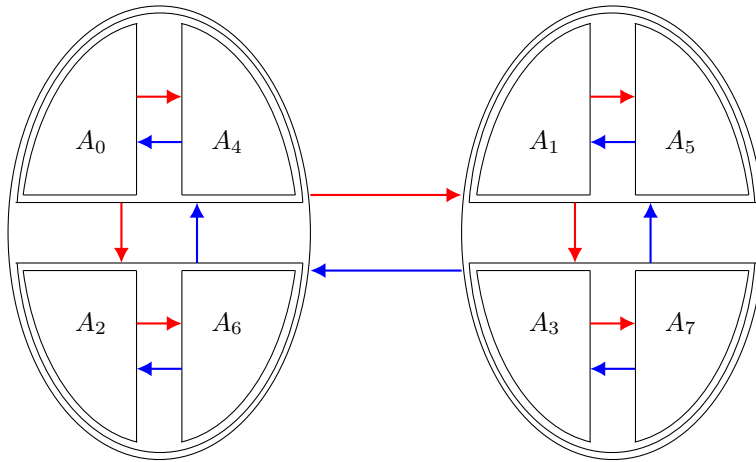


FIGURE 2. Diagram showing the edges between sets for $k = 3$.

The inductive hypothesis holds for every $k \in \mathbb{N}$. Therefore, if P is any monochromatic directed path, we have that the upper density of P is at most 2^{-k} for every $k \in \mathbb{N}$. Hence, P has upper density 0. \square

3. PARTITIONING COMPLETE SYMMETRIC DIGRAPHS

We will make use of the following well-known result. We give a proof both for completeness and since many references only handle the finite case.

Theorem 3.1 (Gallai [6], Hasse [9], Roy [11], Vitaver [14]). *Let D be a (not-necessarily-countable) directed graph and let G be the underlying graph of D . If the longest path in D has length $0 \leq k < \infty$, then $\chi(G) \leq k + 1$.*

Proof. Let D' be a maximal acyclic subgraph of D ; that is, take a well ordering of the edges of D and add them one at a time subject to the condition that a (finite) cycle is not created (or take the poset of acyclic subgraphs of D ordered by inclusion and choose a maximal element by Zorn's lemma). For each $0 \leq i \leq k$, let

$$U_i = \{v \in V(D) : \text{the length of the longest path in } D' \text{ starting at } v \text{ is } i\}.$$

By the hypothesis, $\{U_0, U_1, \dots, U_k\}$ partitions $V(D)$.

We first claim that if $(y, x) \in E(D')$ with $x \in U_i$ and $y \in U_j$, then $i < j$. Let P be a path in D' of length i which starts at x . Since D' is acyclic, $y \notin V(P)$ and thus yP is a path of length $i + 1$ in D' which, since $y \in U_j$, implies $j \geq i + 1$. Next we claim that if $(x, y) \in E(D) \setminus E(D')$ with $x \in U_i$ and $y \in U_j$, then $i < j$. Since $(x, y) \in E(D) \setminus E(D')$, the addition of (x, y) to D' must create a cycle, which implies that there is a $y - x$ -path in D' . By the previous claim, this implies that $i < j$. Together, these two claims imply that there is no edge from D with both endpoints in U_i for any $0 \leq i \leq k$ and thus $\{U_0, U_1, \dots, U_k\}$ is a proper colouring of G . \square

Proof of Theorem 1.3. For $r = 0$, the result is trivial (note $\prod_{i=1}^r \ell_i = 1$ in this case). Let $r \geq 1$ and suppose the result holds for r -colourings satisfying the required path length condition. Now consider an $(r + 1)$ -colouring in which every path of colour $1 \leq i \leq r$ has length at most $\ell_i - 1$. Apply Theorem 3.1 to the digraph induced by the edges of colour r to get a partition $\{U_0, \dots, U_{\ell_r-1}\}$ of V such that each U_j contains no edges of colour r . So each U_j is an r -coloured complete symmetric digraph such that, for all $i \in [r - 1]$, every path of colour i has length at most $\ell_i - 1$. Thus by induction, there is a partition of each U_j into $\prod_{i=1}^{r-1} \ell_i$ complete symmetric digraphs of colour $r + 1$, giving a partition of \vec{K} into a total of $\ell_r \prod_{i=1}^{r-1} \ell_i = \prod_{i=1}^r \ell_i$ complete symmetric digraphs of colour $r + 1$. \square

Corollary 3.2 (cf. [8]). *Let $c: E(\vec{K}_{\mathbb{N}}) \rightarrow [r + 1]$ be an edge-colouring of $\vec{K}_{\mathbb{N}}$ for which there is no directed path of length ℓ_i in colour i for any $i \in [r]$. Then there exists a directed path of colour $r + 1$ with upper density at least $\prod_{i \in [r]} \frac{1}{\ell_i}$.*

Corollary 3.3 (cf. [1]). *Let $c: E(\vec{K}_{\mathbb{N}}) \rightarrow [r + 1]$ be an edge-colouring of $\vec{K}_{\mathbb{N}}$ for which there is no directed path of length ℓ_i in colour i for any $i \in [r]$. Then the vertex set \mathbb{N} can be partitioned into at most $\prod_{i \in [r]} \ell_i$ many monochromatic directed paths of colour $r + 1$.*

This is best possible as shown by the *cube colouring* on any *cube partition* of \mathbb{N} with equally upper-dense partition classes:

Definition 3.4 (Cube partition). For positive integers ℓ_1, \dots, ℓ_r a partition

$$\mathcal{U} = (U_i : i \in \prod_{i \in [r]} \{0, \dots, \ell_i - 1\})$$

of \mathbb{N} indexed by the *cube* $\prod_{i \in [r]} \{0, \dots, \ell_i - 1\}$ is called a *cube partition* (of \mathbb{N} of order (ℓ_1, \dots, ℓ_r)).

Definition 3.5 (Cube colouring). For a cube partition \mathcal{U} of \mathbb{N} , define the cube colouring $c_{\mathcal{U}}$ on $\vec{K}_{\mathbb{N}}$ as follows: Consider an edge $(m, n) \in E(\vec{K}_{\mathbb{N}})$. If $m, n \in U_{i_1, \dots, i_r}$, then colour both (m, n) and (n, m) with colour $r + 1$. If not, suppose that $m \in U_{i_1, \dots, i_r}$ and $n \in U_{j_1, \dots, j_r}$. Let $k = \min\{k' \in [r] : i_{k'} \neq j_{k'}\}$ and $i_k < j_k$ (say). Colour (m, n) with colour $r + 1$ and (n, m) with colour k .

See Figure 1 in the introduction for the case with three colours and $\ell_1 = 2$ and $\ell_2 = 3$.

Theorem 3.6. *Let $c_{\mathcal{U}}$ be the cube colouring on a cube partition \mathcal{U} with equally (upper-) dense partition classes. Then there is no directed path of length ℓ_i in colour i for any $i \in [r]$ and every directed monochromatic path of colour $r + 1$ has upper density at most $\prod_{i \in [r]} \frac{1}{\ell_i}$.*

Proof. Let f map a vertex $v \in \vec{K}_{\mathbb{N}}$ to the index of the partition class of \mathcal{U} that contains v . For $i \in [r]$, let f_i map a vertex v to the i th entry of $f(v)$.

First, consider a monochromatic forward directed path $P = v_0 \dots v_m$ in some colour $i \in [r]$. Then $(f_i(v_0), \dots, f_i(v_m))$ is a strictly decreasing sequence in $\{0, \dots, \ell_i - 1\}$ and thus P has length smaller than ℓ_i . The proof for backward directed case is analogous.

Second, consider a monochromatic forward directed path P with colour $r + 1$. If P is finite, then it has upper density 0 so we may assume that P is infinite, $P = v_0 v_1 v_2 \dots$ (say). The sequence $f(v_0) f(v_1) f(v_2) \dots$ is an increasing sequence of indices with respect to the lexicographic order on the cube $\prod_{i \in [r]} \{0, \dots, \ell_i - 1\}$. Hence the vertices of P are eventually contained in some partition class of \mathcal{U} . On the other hand, each partition class in \mathcal{U} has upper density exactly $\prod_{i \in [r]} \frac{1}{\ell_i}$ completing the proof. The proof for the backward directed monochromatic paths of colour $r + 1$ is analogue. \square

4. CUBE-LIKE STRUCTURES AND STABILITY THEOREM

The main result of this section, our stability result, says that any optimal colouring $c : E(\vec{K}_{\mathbb{N}}) \rightarrow [r + 1]$ for which there is no directed path of length ℓ_i in colour i for any $i \in [r]$ and every directed path in colour $r + 1$ has upper density at most $\prod_{i \in [r]} \frac{1}{\ell_i}$ must agree with our *cube colouring* from above on the following spanning subgraph, which we call the *slide digraph*.

Definition 4.1 (Slide digraph). For a cube partition \mathcal{U} , the digraph $D_{\mathcal{U}}$ on \mathbb{N} is defined as follows: A pair (m, n) of distinct integers $m \in U_{i_1, \dots, i_r}$ and $n \in U_{j_1, \dots, j_r}$ is an edge of $D_{\mathcal{U}}$ if

- there is an index k such that $i_k > j_k$ and $i_{k'} = j_{k'}$ for all $k' \neq k$ (cf. Figure 3), or
- $i_k \leq j_k$ for every $k \in [r]$ (cf. Figure 4).

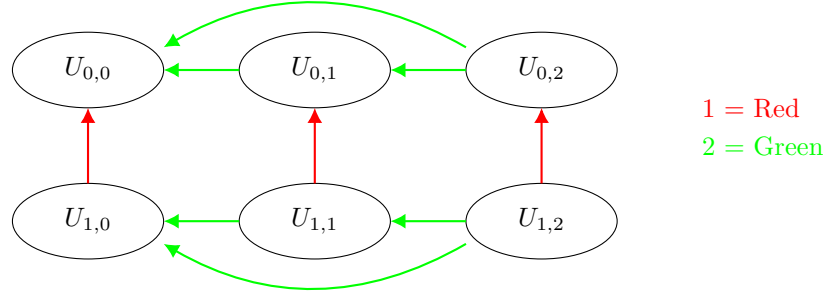


FIGURE 3. First type of edges of the slide digraph for $\ell_1 = 2$ and $\ell_2 = 3$.

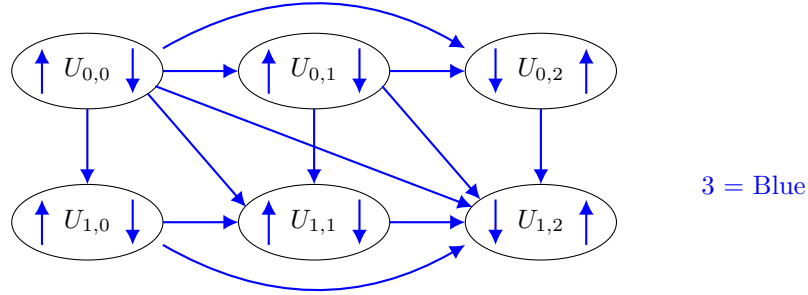


FIGURE 4. Second type of edges of the slide digraph for $\ell_1 = 2$ and $\ell_2 = 3$.

Theorem 4.2. *Let $c: E(\vec{K}_{\mathbb{N}}) \rightarrow [r+1]$ be an edge-colouring of $\vec{K}_{\mathbb{N}}$ for which there is no directed path of length ℓ_i in colour i for any $i \in [r]$. Moreover, assume that every directed path in colour $r+1$ has upper density at most $\prod_{i \in [r]} \frac{1}{\ell_i}$. Then there exists a cube partition \mathcal{U} of order (ℓ_1, \dots, ℓ_r) of equally (upper-) dense partition classes and a finite set $F \subseteq \mathbb{N}$ such that $c_{\mathcal{U}} = c$ on $D_{\mathcal{U}} - F$.*

For the proof, we first need the following lemma.

Lemma 4.3. *For any $n \in \mathbb{N}$, a spanning subgraph $G \subseteq \vec{K}_n$ contains a spanning transitive tournament if and only if $\vec{K}_n - E(G)$ is acyclic.*

Proof. If the spanning subgraph G has a spanning transitive tournament T , then $\vec{K}_n - E(G)$ is a subgraph of the transitive tournament $\vec{K}_n - E(T)$ and hence acyclic.

The converse is proved via induction on n . The base case is clear. Assume that the statement is true for $n-1$ and consider a spanning subgraph $G \subseteq \vec{K}_n$ such

that $\vec{K}_n - E(G)$ is acyclic. Recall that every finite directed acyclic graph has a *source*, i.e. a vertex of in-degree 0. Fix a source v in $\vec{K}_n - G$ and claim that v has in-degree $n - 1$ in G : Indeed, since v has in-degree $n - 1$ in \vec{K}_n and is a source of $\vec{K}_n - E(G)$, it follows that all these incoming edges must be contained in the subgraph G . By induction, the graph $G - v$ contains a transitive tournament T spanning $G - v$. Then $E(T) \cup \{(w, v) : w \in T\}$ is the desired transitive spanning tournament contained in G . \square

Proof of Theorem 4.2. Apply Theorem 1.3 to obtain a partition \mathcal{U} of \mathbb{N} into $\prod_{i \in [r]} \ell_i$ complete symmetric digraphs in colour $r + 1$. Up to deleting finitely many vertices, these partition classes are the partition classes of the slide digraph that we are going to construct. Note that every such partition class $U \in \mathcal{U}$ has density precisely $\prod_{i \in [r]} \frac{1}{\ell_i}$ in \mathbb{N} .

For $U, U' \in \mathcal{U}$, either there is an infinite matching of $[U, U']$ -edges in colour $r + 1$ or there is a finite set of vertices whose deletion leaves no $[U, U']$ -edges of colour $r + 1$ (by König's theorem). In the latter case, delete such a finite set. Let K be the complete symmetric digraph with vertex set \mathcal{U} and let G be the digraph with vertex set \mathcal{U} and edge set

$$E(G) := \{(U, U') : \text{every } [U, U']\text{-edge has colour in } [r]\}.$$

If there is a cycle in $K - E(G)$, this implies that we can construct a path of colour $r + 1$ of density larger than $\prod_{i \in [r]} \frac{1}{\ell_i}$ in \mathbb{N} ; so suppose not. By Lemma 4.3, it follows that G has a spanning transitive tournament T .

Let \vec{T} be the lift of T to $\vec{K}_{\mathbb{N}}$, i.e. \vec{T} is the spanning subgraph of $\vec{K}_{\mathbb{N}}$ which contains all edges (m, n) where $m \in U$, $n \in U'$ and (U, U') is an auxiliary edge in $E(T)$. Then \vec{T} is also acyclic and spanning. Following the proof of Theorem 1.3 (but applied to \vec{T} instead of all of $\vec{K}_{\mathbb{N}}$), we define sets W_{i_1, \dots, i_j} for all $j \in [r]$ and $0 \leq i_j < \ell_j$ by recursively defining W_{i_1, \dots, i_m} to consist of all vertices $w \in W_{i_1, \dots, i_{m-1}}$ such that the longest path in $\vec{T}[W_{i_1, \dots, i_{m-1}}]$ in colour m starting at w has length i_m .

Claim 1. The family $\{W_{i_1, \dots, i_j, 0}, \dots, W_{i_1, \dots, i_j, \ell_{j+1}-1}\}$ is a partition of W_{i_1, \dots, i_j} .

Proof of Claim 1. Clear, because there is no path of length ℓ_{m+1} in colour $m + 1$. \diamond

Claim 2. If $w, w' \in W_{i_1, \dots, i_j}$ and $(w, w') \in E(\vec{T})$, then $c(w, w') \notin [j]$.

Proof of Claim 2. Let $i_j < \ell_j$ and suppose for a contradiction that $w, w' \in W_{i_1, \dots, i_j}$, $(w, w') \in E(\vec{T})$ and $c(w, w') \in [j]$. Let us write $k := c(w, w')$. By Claim 1, we have $w, w' \in W_{i_1, \dots, i_k}$. Take a longest path P in $\vec{T}[W_{i_1, \dots, i_k}]$ in colour k starting in w' . Since \vec{T} is acyclic, we know that wP is a path in $\vec{T}[W_{i_1, \dots, i_k}]$ in colour k starting in w contradicting that w is contained in W_{i_1, \dots, i_k} . \diamond

So far, we have defined the partition $\mathcal{W} := \{W_{i_1, \dots, i_r} : 0 \leq i_j < \ell_j \text{ for all } j \in [r]\}$ in terms of the acyclic spanning graph \vec{T} . It turns out, however, that we still arrive at the same partition as earlier, when we did this construction with respect to all of $\vec{K}_{\mathbb{N}}$.

Claim 3. We have $\mathcal{U} = \mathcal{W}$.

Proof of Claim 3. Each $W \in \mathcal{W}$ meets at most one partition class of \mathcal{U} . Indeed, suppose for a contradiction that $u \in U \cap W$ and $u' \in U' \cap W$ for different partition classes $U, U' \in \mathcal{U}$ and some $W \in \mathcal{W}$. Since $u \in U$ and $u' \in U'$ we know that either (u, u') or (u', u) is contained in $E(\vec{T})$, say (u, u') . Then (u, u') has a colour in $[r]$, in violation of Claim 2. But then it follows from the fact that \mathcal{W} is a partition with $|\mathcal{W}| \leq |\mathcal{U}|$ that $\mathcal{U} = \mathcal{W}$. \diamond

Consider the slide graph with regard to the partition \mathcal{W} . The following claim shows that the first type of edges in the definition of the slide graph have the right colour:

Claim 4. Let $w \in W_{i_1, \dots, i_r}$ and $w' \in W_{j_1, \dots, j_r}$ where for some k we have $i_k < j_k$ and $i_{k'} = j_{k'}$ for $k' \neq k$. Then (w', w) is an edge of \vec{T} and has colour k .

Proof of Claim 4. The claim is proved via induction on $r - k$. For the base case let $r - k = 0$, i.e., $k = r$. By Claim 3 and the construction of T , either (w, w') or (w', w) is contained in $E(\vec{T})$. Suppose, for a contradiction, that $(w, w') \in E(\vec{T})$.

By definition of \mathcal{W} , we find a path P in $\vec{T}[W_{i_1, \dots, i_{r-1}}]$ of length j_r in colour r starting at w' . However, since \vec{T} is acyclic, we see that wP is a path in $\vec{T}[W_{i_1, \dots, i_{r-1}}]$ of length $j_r + 1$ in colour r starting at w , implying that $i_r \geq j_r + 1$, contradicting our assumption.

Now, assume inductively that that $r - k \geq 1$ and Claim 4 holds for integers less than $r - k$. Let $w \in W_{i_1, \dots, i_r}$ and $w' \in W_{j_1, \dots, j_r}$. Furthermore, assume that $i_k < j_k$ and $i_{k'} = j_{k'}$ if $k' \neq k$. Again, either (w, w') or (w', w) is contained in $E(\vec{T})$. Let \vec{e} be the unique edge in $E(\vec{T}) \cap \{(w, w'), (w', w)\}$. We first show that \vec{e} has colour k .

By Claim 2, we know the colour of \vec{e} is at least k . Suppose for a contradiction that \vec{e} has a colour k' at least $k + 1$.

- If $\vec{e} = (w, w')$, fix, for all $i > i_{k'}$, vertices $w_i \in W_{i_1, \dots, i_{k'-1}, i, i_{k'+1}, \dots, i_r}$ and, for all $j < j_{k'}$, vertices $w_j \in W_{j_1, \dots, j_{k'-1}, j, j_{k'+1}, \dots, j_r}$. Then define Q as the path

$$Q := w_{\ell_{k'}-1} \dots w_{i_{k'}+1} \vec{e} w_{i_{k'}-1} \dots w_2 w_1 w_0.$$

(cf. Figure 5.)

- If $\vec{e} = (w', w)$, fix, for all $j > j_{k'}$, vertices $w_j \in W_{j_1, \dots, j_{k'-1}, j, j_{k'+1}, \dots, j_r}$ and, for all $i < i_{k'}$, vertices $w_i \in W_{i_1, \dots, i_{k'-1}, i, i_{k'+1}, \dots, i_r}$. Then define Q as the path

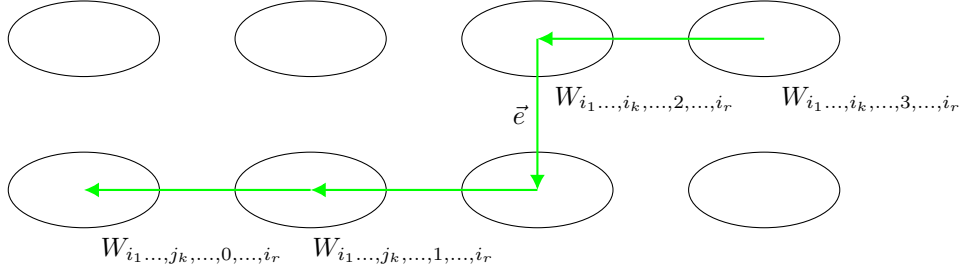
$$Q := w_{\ell_{k'}-1} \dots w_{i_{k'}+1} \vec{e} w_{i_{k'}-1} \dots w_2 w_1 w_0.$$

In either case, it follows from our inductive hypothesis that Q is a monochromatic path in colour k' . However, Q now has length $\ell_{k'}$, a contradiction.

Finally, to show that $\vec{e} = (w', w) \in E(\vec{T})$, we may now argue as in the base case: By construction of \mathcal{W} , find a longest path P starting at w' in colour k of length j_k in $\vec{T}[W_{i_1, \dots, i_{k-1}}]$. If $\vec{e} = (w, w')$, then $\vec{e}P$ is a path starting at w in colour k in $\vec{T}[W_{i_1, \dots, i_{k-1}}]$ of length $j_k + 1$, forcing that $i_k \geq j_k + 1$, a contradiction. \diamond

The following claim completes the proof:

Claim 5. Let $w \in W_{i_1, \dots, i_r}$ and $w' \in W_{j_1, \dots, j_r}$ with $i_k \leq j_k$ for every $k \in [r]$. Then (w, w') has colour $r + 1$.

FIGURE 5. The path Q (green) for $\ell_{k'} = 4$

Proof of Claim 5. Suppose for a contradiction, that (w, w') has a colour $k' \in [r]$. Fix, for all $i > i_{k'}$, vertices $w_i \in W_{i_1, \dots, i_{k'-1}, i, i_{k'+1}, \dots, i_r} \setminus \{w, w'\}$ and, for all $j < j_{k'}$, vertices $w_j \in W_{j_1, \dots, j_{k'-1}, j, j_{k'+1}, \dots, j_r} \setminus \{w, w'\}$ (all distinct). By Claim 4, the path

$$w_{\ell_{k'}-1} \dots w_{i_{k'}+1} \vec{e} w_{i_{k'}-1} \dots w_2 w_1 w_0$$

has colour k' and since $i_k \leq j_k$ it has length at least $\ell_{k'}$ (contradiction). \diamond

With this final claim established, the proof is complete. \square

Since the slide graph is just the complete symmetric digraph on \mathbb{N} for $r = 1$, we obtain the stability result of Guggiari in [8, Theorem 1.4] as a corollary:

Corollary 4.4 (cf. [8, Theorem 1.4]). *Take any 2-colouring of $\vec{K}_{\mathbb{N}}$ in which there are no monochromatic directed paths of length ℓ in colour 1 and every monochromatic directed path in colour 2 has upper density at most $\frac{1}{\ell}$. Then there exists a finite set of vertices U such that the 2-colouring induced on $\mathbb{N} \setminus U$ is isomorphic to the cube colouring on $\vec{K}_{\mathbb{N}}$.*

Unless $r = 1$, there are still edges whose colours we did not specify—those which are not part of the slide graph. In fact, the colours of these edges can vary depending on the choice of the colouring:

Example 4.5. Let $r \geq 2$ and ℓ_1, \dots, ℓ_r positive integers. Consider the cube colouring $c_{\mathcal{U}}$ and the slide graph $D_{\mathcal{U}}$ on a cube partition \mathcal{U} of order (ℓ_1, \dots, ℓ_r) with equally (upper-) dense partition classes. Fix U_{i_1, \dots, i_r} and U_{j_1, \dots, j_r} such that the $[U_{i_1, \dots, i_r}, U_{j_1, \dots, j_r}]$ edges are not part of $D_{\mathcal{U}}$. Colour the $[U_{i_1, \dots, i_r}, U_{j_1, \dots, j_r}]$ -edges with arbitrary colours from $\{k \in [r] : i_k > j_k\}$ and all the other edges as in the cube colouring. Almost the same proof as in Theorem 3.6 shows that there is no directed path of length ℓ_i in colour i for any $i \in [r]$ and that every directed monochromatic path of colour $r + 1$ has upper density at most $\prod_{i \in [r]} \frac{1}{\ell_i}$.

However, we observe (using the terminology of the proof of Theorem 4.2):

- If $i_k \leq j_k$, then no $[W_{i_1, \dots, i_r}, W_{j_1, \dots, j_r}]$ -edge has colour k (cf. the proof of Claim 5).
- If $i_k > j_k$ for every $k \in [r]$, then there is a finite set of vertices F such that all $[(W_{j_1, \dots, j_r} \setminus F), (W_{i_1, \dots, i_r} \setminus F)]$ -edges have a colour in $[r]$.

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REFERENCES

- [1] C. Bürger, M. Pitz, Decomposing edge-coloured complete symmetric digraphs into monochromatic paths, manuscript, <http://arxiv.org/abs/1711.08711> (2017).
- [2] J. Corsten, *Personal communication* (2017).
- [3] L. DeBiasio and P. McKenney, Density of monochromatic infinite subgraphs, <http://arxiv.org/abs.1611.05423> (2016).
- [4] M. Elekes, D. Soukup, L. Soukup and Z. Szentmiklőssy, Decompositions of edge-coloured infinite complete graphs into monochromatic paths, *Discrete Mathematics* **340** (2017), 2053–2069.
- [5] P. Erdős and F. Galvin, Monochromatic infinite paths, *Discrete Mathematics* **113** (1993), 59–70.
- [6] T. Gallai, On directed paths and circuits, *Theory of graphs* (1968), 115–118.
- [7] L. Gerencsér and A. Gyárfás, On Ramsey-type problems, *Ann. Sci. Budapest. Eötvös Sect. Math.* **10** (1967), 167–170.
- [8] H. Guggiari, Monochromatic paths in the complete symmetric infinite digraph, manuscript, <http://arxiv.org/abs.1710.10900> (2017).
- [9] M. Hasse, Zur algebraischen Begründung der Graphentheorie. I, *Mathematische Nachrichten* **28** (1965), no. 5-6, 275–290.
- [10] R. Rado, Monochromatic paths in graphs, *Ann Discrete Math.* **3** (1978), 191–194.
- [11] H. Raynaud, Sur le circuit hamiltonien bi-colouré dans les graphes orientés, *Periodica Mathematica Hungarica* **3** (1973), 289–297.
- [12] B. Roy, Nombre chromatique et plus longs chemins d’un graphe, *Revue française d’informatique et de recherche opérationnelle* **1** (1967), no. 5, 129–132.
- [13] D. Soukup, Decompositions of edge-colored infinite complete graphs into monochromatic paths II *Israel Journal of Mathematics* **221** (2017), 235–273.
- [14] L. M. Vitaver, Determination of minimal colouring of vertices of a graph by means of boolean powers of the incidence matrix, *Dokl. Akad. Nauk SSSR*, **147** (1962), p. 728.

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