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FUNNEL CONTROL FOR BOUNDARY CONTROL SYSTEMS

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Abstract. We study a nonlinear, non-autonomous feedback controller applied to boundary control systems. Our aim is to track a given reference signal with prescribed performance. Existence and uniqueness of solutions to the resulting closed-loop system is proved by using nonlinear operator theory. We apply our results to both hyperbolic and parabolic equations.

1. **Introduction.** In this paper we consider a class of *boundary control systems* (*BCS*) of the form

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), \quad t > 0, \quad x(0) = x_0, \\ u(t) &= \mathfrak{B}x(t), \\ y(t) &= \mathfrak{C}x(t), \end{aligned}$$

where $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ are linear operators. The function u is interpreted as the input, y as the measured output and x is called the state of the system. Typically, $\mathfrak{A}$ is a differential operator on the state space X and $\mathfrak{B}, \mathfrak{C}$ are evaluation operators of the state at the boundary of the spatial domain, that is, the domain of the functions lying in X.

The aim of this paper is to develop an adaptive controller for boundary control systems which, roughly speaking, achieves the following goal:

> *For any prescribed reference signal $y_{\rm ref} \in W^{2,\infty}([0,\infty))$, the output y of the system tracks $y_{\rm ref}$ in the sense that the transient behavior of the error $e(t) := y(t) - y_{\rm ref}(t)$ is controlled.*

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Shortly, we will elaborate on the class of possible reference signals and the meaning of "controlling the transient behavior" in more detail. The goal will be achieved by using a *funnel controller*, which, in the simplest case, has the form

$$u(t) = -\frac{1}{1-\varphi(t)^2\|e(t)\|^2}e(t)$$

for some positive function φ. Under this feedback, the error is supposed to evolve in the *performance funnel*

$$\mathcal{F}_\varphi := \{(t,e) \in [0,\infty) \times \mathbb{C}^m \mid \varphi(t)\|e\| < 1\}$$

and would hence satisfy

$$\|e(t)\| \leq \varphi(t)^{-1}, \qquad \text{for all } t \geq 0.$$

In fact, if φ tends asymptotically to a large value λ, then the error remains bounded by λ^{-1}, see Fig. 1.

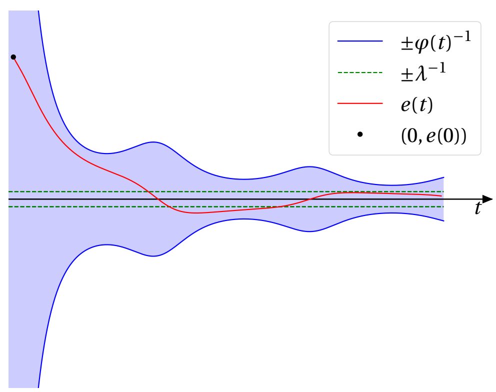

FIGURE 1. Error evolution in a funnel $\mathcal{F}_\varphi$ with boundary $\varphi(t)^{-1}$.

This was first introduced in [20], where feasibility of the funnel controller for a class of functional differential equations has been shown. These encompass infinite dimensional systems with very restrictive assumptions on the operators involved, a special class of nonlinear systems and nonlinear delay systems. In fact, there finite-dimensional linear "prototype" systems with relative degree one are treated. The relative degree is a well-known magnitude for finite-dimensional systems and can roughly be understood as the number of times one needs to differentiate y so that u appears in the equation. This quantity turned out to be relevant when considering the funnel controller and has been used to generalize the results of [20]. For instance, in [14], the funnel controller was proved to be applicable for systems with known but arbitrary relative degree. The problem is that the ansatz used there requires very large powers of the gain factor. This problem has been overcome in [7] by introducing a funnel controller which involves derivatives of the output and reference signal, and feasibility of this controller in the case of nonlinear finite-dimensional systems with *strict relative degree* with *stable internal dynamics* has been proven. The funnel control for infinite-dimensional systems has so far only attracted attention in special configurations [8, 21, 19]. The recent article [8] deals with a linearized model of a moving water tank by showing that this system belongs to the class being treated in [7]. In [21], a class of infinite-dimensional systems has been considered that allows to prove feasibility of the funnel controller in a similar way as for finite-dimensional systems. More precisely, this class consists of systems which possess a so-called *Byrnes-Isidori form* via bounded and boundedly invertible state space transformation. The existence of such a form however requires that the control and observation operators fulfill very strong boundedness conditions, which in particular exclude boundary control and observation. Funnel control of a heat equation with Neumann boundary control and co-located Dirichlet output has been treated in [19]. The proof of feasibility of funnel control uses the spectral properties of the Laplacian, whence this technique is hardly transferable to further classes of boundary control systems.

We consider a class of *boundary control systems* which satisfy a certain energy balance [6, 15]. The feedback law of the funnel controller naturally induces a nonlinear closed-loop system. For the corresponding solution theory, the concept of (nonlinear) m-dissipative operators in a Hilbert space will play an important role. For an appropriate introduction to this classical topic we refer to [16, 17, 22].

The paper is organized as follows. In Section 2 we introduce the system class that is subject of our results. In Section 3 we present the details about the controller and present the main results which refer to the applicability of the funnel controller to the considered system class. In Section 4 we present some examples of partial differential equations for which the funnel controller is applicable. Section 5 contains the proof of the main results together with some preliminary auxiliary results. We provide numerical simulations in Section 6.

The norm in a normed space B will be denoted by $\|\cdot\|_B$ or $\|\cdot\|$, if clear from context. Analogously, the scalar product of an inner product space will be denoted by $\langle\cdot,\cdot\rangle_H$ or $\langle\cdot,\cdot\rangle$. The space $\mathbb{C}^n$ is typically provided with the Euclidean inner product.

The domain of a (possibly nonlinear) operator A is denoted by $\mathcal{D}(A)$, and $\mathcal{R}(A)$ stands for the range of A. Given two Banach spaces X, Y, the set of linear bounded operators from X to Y will be denoted by $\mathcal{L}(X,Y)$ and in the case $X = Y$ simply by $\mathcal{L}(X)$. The identity operator on the space X is I_X, or just I, if clear from context. We further write I_m instead of $I_{\mathbb{C}^m}$. The symbol A^* stands for the adjoint of a linear operator A. In particular, $M^* \in \mathbb{C}^{n\times m}$ is the transposed of the complex conjugate of $M \in \mathbb{C}^{m\times n}$.

Lebesgue and Sobolev spaces from a measurable set $\Omega \subset \mathbb{R}^d$ will be denoted by $L^p(\Omega)$ and $W^{k,p}(\Omega)$. For a domain $\Omega \subset \mathbb{R}^d$ with sufficiently smooth boundary $\partial\Omega := \Gamma$, we denote by $W^{k,p}(\Gamma)$ the Sobolev space at the boundary [1]. The set of infinitely often differentiable functions from Ω with compact support will be denoted by $C_0^\infty(\Omega)$.

We identify spaces of $\mathbb{C}^n$-valued functions with the Cartesian product of spaces of scalar-valued functions, such as, for instance $(W^{k,p}(\Omega))^n \cong W^{k,p}(\Omega;\mathbb{C}^n)$.

For an interval $J \subset \mathbb{R}$ and a Banach space B, we set

$$W^{k,\infty}(J;B) := \{f \in L^\infty(J;B) \mid f^{(j)} \in L^\infty(J;B), j = 0,\dots,k\},$$

which is to be understood in the Bochner sense [10]. The space $W^{k,\infty}_{\mathrm{loc}}(J;B)$ consists of all f whose restriction to any compact interval $K \subset J$ is in $W^{k,\infty}(K;B)$.

The expression $\overline{S}$ indicates the closure of a set S.

2. **System class.** In the following we introduce our system class, define our controller and discuss the solution concept to the resulting nonlinear feedback system.

Definition 2.1 (System class)**.** Let X be a complex Hilbert space and let $m \in \mathbb{N}$ be given. Let $\mathfrak{A} : \mathcal{D}(\mathfrak{A}) \subset X \to X$ be a closed linear operator, $\mathfrak{B}, \mathfrak{C} : \mathcal{D}(\mathfrak{A}) \subset X \to \mathbb{C}^m$ be linear operators to which we associate the system

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), \quad x(0) = x_0, \\ u(t) &= \mathfrak{B}x(t), \\ y(t) &= \mathfrak{C}x(t). \end{aligned} \tag{1}$$

We will refer to (1) by $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ and call it a *boundary control system (BCS)*.

In the sequel we specify the system class.

Assumption 2.2. Let a BCS $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be given.

(i) The system is *(generalized) impedance passive*, i.e., there exists $\alpha \in \mathbb{R}$ such that

$$\operatorname{Re}\langle \mathfrak{A}x, x\rangle_X \le \operatorname{Re}\langle \mathfrak{B}x, \mathfrak{C}x\rangle_{\mathbb{C}^m} + \alpha \|x\|_X^2 \ \text{ for all } x \in \mathcal{D}(\mathfrak{A}). \tag{2}$$

(ii) $\mathfrak{A}|_{\ker \mathfrak{B}}$ (the restriction of $\mathfrak{A}$ to $\ker \mathfrak{B}$) generates a strongly continuous semigroup on X.

(iii) The operator

$$\begin{bmatrix} \mathfrak{B} \\ \mathfrak{C} \end{bmatrix} : \mathcal{D}(\mathfrak{A}) \to \mathbb{C}^{2m} \tag{3}$$

is onto, $\ker \mathfrak{B} \cap \ker \mathfrak{C} \subset X$ is dense and $\mathfrak{C} : \mathcal{D}(\mathfrak{A}|_{\ker \mathfrak{B}}) \to \mathbb{C}^m$ is continuous with respect to the graph norm $\|x\|_{\mathcal{D}(\mathfrak{A})} = (\|x\|_X^2 + \|\mathfrak{A}x\|_X^2)^{1/2}$.

Remark 2.3.

a) By setting $u = 0$, the above assumptions imply that the semigroup $T(\cdot) : [0, \infty) \to \mathcal{L}(X)$ generated by $\mathfrak{A}|_{\ker \mathfrak{B}}$ fulfills $\|T(t)\| \le e^{\alpha t}$. In particular, the semigroup is contractive, if $\alpha \le 0$.

b) The Lumer–Phillips theorem [11, Theorem 3.15] implies that $\mathfrak{A}|_{\ker \mathfrak{B}}$ generates a strongly continuous semigroup $T(\cdot)$ on X with $\|T(t)\| \le e^{\alpha t}$ for all $t > 0$ if, and only if, $\mathcal{R}(\mathfrak{A}|_{\ker \mathfrak{B}} - \lambda I) = X$ for some (and hence any) $\lambda \ge \alpha$, together with $\operatorname{Re}\langle \mathfrak{A}x, x\rangle_X \le \alpha \|x\|_X^2$ for all $x \in D(\mathfrak{A})$. As a consequence, Assumption 2.1(ii) can be replaced by the condition that $\mathcal{R}(\mathfrak{A}|_{\ker \mathfrak{B}} - \lambda I) = X$ for some (and hence any) $\lambda \ge \alpha$.

c) The operator (3) is onto if, and only if, there exist $P, Q : \mathcal{L}(\mathbb{C}^m, \mathcal{D}(\mathfrak{A}))$ with

$$\begin{bmatrix} \mathfrak{B} \\ \mathfrak{C} \end{bmatrix} \begin{bmatrix} P & Q \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_m \end{bmatrix} \tag{4}$$

d) We are dealing with complex spaces in this article for sake of simplicity. A comment on real systems can be found in Remark 3.5(e).

e) An oftentimes considered class in infinite-dimensional linear systems theory is that of *well-posed linear systems*, see e.g. [23]. That is, the controllability map, observability map and input-output map are bounded operators. Note that we do not impose such a well-posedness assumption throughout this article.

Example 2.4. There are several systems which fit in our description. A class of examples of hyperbolic type is given by so-called *port-Hamiltonian systems* such as the *lossy transmission line*

$$\begin{aligned} V_\zeta(\zeta,t) &= -LI_t(\zeta,t) - RI(\zeta,t),\\ I_\zeta(\zeta,t) &= -CV_t(\zeta,t) - GV(\zeta,t),\\ u(t) &= \begin{pmatrix} V(a,t)\\ V(b,t)\end{pmatrix},\\ y(t) &= \begin{pmatrix} I(a,t)\\ -I(b,t)\end{pmatrix}, \end{aligned}$$

where V and I are the voltage and the electric current at a point ζ of a segment (a,b) over the time t. A precise definition of port-Hamiltonian systems will be given in Section 4.1.

In Section 4.3 we will also apply the theoretical results to parabolic systems given through a general second-order elliptic operator on a regular domain Ω. A particular case is the heat equation,

$$\begin{aligned} \partial_t x(t,\zeta) &= \Delta x(t,\zeta),\\ \nu\cdot\nabla x(t,\zeta)|_{\partial\Omega} &= u(t),\\ \int_{\partial\Omega} x(t,\zeta)\mathrm{d}\zeta &= y(t), \end{aligned}$$

where the control variable is the heat flux at the boundary and the observation is the total temperature along the boundary.

3. **Funnel controller.** The following definition presents the cornerstone of our controller, the class of admissible funnel boundaries.

Definition 3.1. Let

$$\Phi := \left\{\varphi \in W^{2,\infty}([0,\infty)) \mid \varphi \text{ is real-valued with } \inf_{t\geq 0}\varphi(t) > 0\right\}.$$

With $\varphi\in\Phi$ we associate the *performance funnel*

$$\mathcal{F}_\varphi := \{(t,e)\in[0,\infty)\times\mathbb{C}^m \mid \varphi(t)\|e\| < 1\}.$$

In this context we refer to $1/\varphi(\cdot)$ as *funnel boundary*, see also Fig. 1.

Now we define our controller, which is a slight modification of the original controller introduced in [20]. For $x_0\in\mathcal{D}(\mathfrak{A})$, we define the funnel controller as

$$u(t) = \left(u_0 + \frac{1}{1-\varphi_0^2\|e_0\|^2}e_0\right)p(t) - \frac{1}{1-\varphi(t)^2\|e(t)\|^2}e(t), \tag{5}$$

where $\varphi_0 = \varphi(0)$, $e_0 := \mathfrak{C}x_0 - y_{\mathrm{ref}}(0)$, $u_0 := \mathfrak{B}x_0$, and p is a function with compact support and $p(0) = 1$. In the following we collect assumptions on the functions involved in the funnel controller and the initial value of the BCS $\mathfrak{S} = (\mathfrak{A},\mathfrak{B},\mathfrak{C})$. Particularly, this includes that the expressions $\mathfrak{C}x_0$, $y_{\mathrm{ref}}(0)$, $\mathfrak{B}x_0$ and $p(0)$ are well-defined.

Assumption 3.2 (Reference signal, performance funnel, initial value)**.** The initial value x_0 of the BCS $\mathfrak{S} = (\mathfrak{A},\mathfrak{B},\mathfrak{C})$ and the functions in the controller (5) fulfill

(i) $y_{\mathrm{ref}}\in W^{2,\infty}([0,\infty);\mathbb{C}^m)$;
(ii) $p\in W^{2,\infty}([0,\infty))$ with compact support and $p(0)=1$;

(iii) $x_0 \in \mathcal{D}(\mathfrak{A})$ and $\varphi \in \Phi$ with $\varphi(0)\|\mathfrak{C}x_0 - y_{\text{ref}}(0)\| < 1$.

Remark 3.3. Apart from smoothness of the reference signal and performance funnel, the assumptions on the controller basically include two points:

a) The initial value is "smooth", i.e., $x_0 \in \mathcal{D}(\mathfrak{A})$. The reason is that - especially for hyperbolic systems - the initialization with $x_0 \in X \setminus \mathcal{D}(\mathfrak{A})$ might result in a discontinuous output. This effect typically occurs when the semigroup generated by $\mathfrak{A}|_{\ker \mathfrak{B}}$ is not analytical, such as, for instance, when a wave equation is considered.

b) The output of the system at $t = 0$ is already in the performance funnel.

The funnel controller (5) differs from the classical one in [20] by the addition of the term

$$\left(u_0 + \tfrac{1}{1-\varphi_0^2\|e_0\|^2} e_0\right) p(t)$$

for some (arbitrary) smooth function with $p(0) = 1$ and compact support. This ensures that the controller is consistent with the initial value, that is, u in (5) satisfies

$$u(0) = \left(u_0 + \tfrac{1}{1-\varphi_0^2\|e_0\|^2} e_0\right) p(0) - \tfrac{1}{1-\varphi(0)^2\|e(0)\|^2} e(0) = u_0 = \mathfrak{B}x_0 = \mathfrak{B}x(0).$$

The funnel controller therefore requires the knowledge of the "initial value of the input" $u_0 = \mathfrak{B}x_0$. This means that, loosely speaking, the "actuator position" has to be known at the initial time, which is —by the opinion of the authors— no restriction from a practical point of view.
We would like to emphasize that the application of the funnel controller does not need any further "internal information" on the system, such as system parameters or the full knowledge of the initial state.

The funnel controller (5) applied to a BCS $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ results in the closed-loop system

$$\begin{aligned}
\dot{x}(t) &= \mathfrak{A}x(t), \qquad\qquad x(0) = x_0,\\
\mathfrak{B}x(t) &= u(t),\\
\mathfrak{C}x(t) &= y(t),\\
e(t) &= y(t) - y_{\text{ref}}(t), \quad e_0 = \mathfrak{C}x_0 - y_{\text{ref}}(0), \quad \varphi_0 = \varphi(0),\\
u(t) &= (\mathfrak{B}x_0 + \psi(\varphi_0, e_0))p(t) - \psi(\varphi(t), e(t)),
\end{aligned} \tag{6a}$$

where

$$\begin{aligned}
\psi(\varphi, e) &:= \frac{1}{1 - \varphi^2\|e\|^2} e,\\
\mathcal{D}(\psi) &:= \{(\varphi, e) \in (0, \infty) \times \mathbb{C}^m \mid \varphi\|e\| < 1\}.
\end{aligned} \tag{6b}$$

We see immediately that the closed-loop system is nonlinear and time-variant. In the sequel we present our main results which state that the funnel controller is functioning in a certain sense. Note that this result includes the specification of the solution concept with which we are working. First we show that the funnel controller applied to any system fulfilling Assumption 2.2 has a solution. Such a solution however might not be bounded on the infinite time horizon. Thereafter, we show that boundedness on $[0, \infty)$ is guaranteed, if the constant α in the energy balance (2) is negative. The proofs of these results can be found in Section 5.

Theorem 3.4 (Feasibility of funnel controller, arbitrary α)**.** *Let a BCS $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be given which satisfies Assumption 2.2 and assume that the initial value x_0 and the functions y_{ref}, p, φ fulfill Assumption 3.2. Then, for all $T > 0$ the closed-loop system (6) has a unique solution $x \in W^{1,\infty}([0,T];X)$ in the following sense:*

(i) $\dot{x}$ is continuous at almost every $t \in [0,T]$, and
(ii) for almost every $t \in [0,T]$ holds $x \in D(\mathfrak{A})$ and (6).

Remark 3.5.

a) The solution concept which is subject of Theorem 3.4 is strong in the sense that the weak derivative of x is evolving in the space X and not in some larger space as used e.g. in [24].
b) The property $x \in W^{1,\infty}([0,T];X)$ of a solution implies $\mathfrak{A}x = \dot{x} \in L^\infty([0,T];X)$, whence $x \in L^\infty([0,T];D(\mathfrak{A}))$. As a consequence, for $u = \mathfrak{B}x$ and $y = \mathfrak{C}x$ holds that $u, y \in L^\infty([0,T];\mathbb{C}^m)$. By the same argumentation, we see that the continuity of $\dot{x}$ at almost every $t \in [0,T]$ implies that u and y are continuous at almost every $t \in [0,T]$.
c) For $T_1 < T_2$ consider solutions x_1 and x_2 of the closed-loop system (6) on $[0,T_1]$ and $[0,T_2]$, respectively. Uniqueness of the solution implies that $x_1 = x_2|_{[0,T_1]}$. As a consequence, there exists a unique $x \in W^{1,\infty}_{\text{loc}}([0,\infty);X)$ with the property that for all $T > 0$ holds that $x|_{[0,T]}$ is a solution of (6). Accordingly, the input satisfies $u \in L^\infty_{\text{loc}}([0,\infty);\mathbb{C}^m)$. Note that, by the fact that the output evolves in the funnel, we have that y is essentially bounded, that is $y \in L^\infty([0,\infty);\mathbb{C}^m)$.
d) The properties $u, y, y_{\text{ref}} \in L^\infty([0,T];\mathbb{C}^m)$ imply that the error $e = y - y_{\text{ref}}$ is uniformly bounded away from the funnel boundary. That is, there exists some $\varepsilon > 0$ such that

$$\varphi(t)\|e(t)\| < 1 - \varepsilon \quad \text{for almost all } t \in [0,T].$$

e) The typical situation is that the system is real in the sense that the input, output and state evolve in the real spaces $\mathbb{R}^m$ and X. By using a complexification $X + iX$, the results presented in this article can be applied to such systems yielding that a (not yet necessarily real) solution $x \in W^{1,\infty}([0,T];X+iX)$ the closed-loop system (6) exists which is moreover unique. A closer look yields that the pointwise complex conjugate $\bar{x}$ is as well a solution of (6), and uniqueness gives $x = \bar{x}$, whence x has to be real in this case.

Though bounded on each bounded interval, the solution x of the closed-loop system (6) might satisfy

$$\limsup_{t\to\infty} \|x(t)\| = \infty, \qquad \limsup_{t\to\infty} \|u(t)\| = \infty$$

In the following we show that this unboundedness does not occur when the constant α in (2) in Assumption 2.2 is negative.

Theorem 3.6. *Let a BCS $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be given which satisfies Assumption 2.2 such that Assumption 2.2(i) holds with $\alpha < 0$. Assume that the initial value x_0 and the functions y_{ref}, p, φ fulfill Assumption 3.2. Then the solution $x : [0,\infty) \to X$ of the closed-loop system (6) (which exists by Theorem 3.4) fulfills*

$$x \in W^{1,\infty}([0,\infty);X) \quad \textit{and} \quad u = \mathfrak{B}x \in L^\infty([0,\infty);\mathbb{C}^m).$$

Remark 3.7. In particular when the input and output of a system have different physical dimensions, it might be essential that the funnel controller is dilated by some constant $k_0 > 0$. More precisely, one might consider the controller

$$u(t) = \left(u_0 + \frac{k_0}{1-\varphi_0^2\|e_0\|^2}e_0\right)p(t) - \frac{k_0}{1-\varphi(t)^2\|e(t)\|^2}e(t). \tag{7}$$

The feasibility of this controller is indeed covered by Theorems 3.4 & 3.6, which can be seen by the following argumentation: Consider the BCS $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ with transformed input $\tilde{u} = k_0^{-1}u$. That is, a system $\mathfrak{S} = (\mathfrak{A}, k_0^{-1}\mathfrak{B}, \mathfrak{C})$. Providing X with the equivalent inner product $\langle\cdot,\cdot\rangle_{\text{new}} := k_0^{-1}\langle\cdot,\cdot\rangle_X$, we obtain

$$\operatorname{Re}\langle\mathfrak{A}x, x\rangle_{\text{new}} \leq \operatorname{Re}\langle k_0^{-1}\mathfrak{B}x, \mathfrak{C}x\rangle_{\mathbb{C}^m} + \alpha\|x\|^2_{\text{new}} \quad \text{for all } x \in \mathcal{D}(\mathfrak{A}).$$

Consequently, by Theorem 3.4, the funnel controller

$$\tilde{u}(t) = \Big(\underbrace{\tilde{u}_0}_{=k_0^{-1}u_0} + \frac{1}{1-\varphi_0^2\|e_0\|^2}e_0\Big)p(t) - \frac{1}{1-\varphi(t)^2\|e(t)\|^2}e(t)$$

results in feasibility of the closed-loop. Now resolving $\tilde{u} = k_0^{-1}u$ in the previous formula, we obtain exactly the controller (7). Further note that, by the same argumentation together with Theorem 3.6, we obtain that all the trajectories are bounded in the case where $\alpha < 0$.

4. **Some PDE examples.** We now present three different system classes for which we can apply the previously presented results. The first two have state variables which are described by hyperbolic PDEs and the third one by a parabolic PDE.

4.1. **Port-Hamiltonian systems in one spatial variable.** The systems considered in this article enclose a class of port-Hamiltonian hyperbolic system in one spatial dimension with boundary control and observation, which has been treated in [3, 4, 5, 6, 15] and is subject of the subsequent definition. Typically they are considered in a bounded interval $(a, b) \subset \mathbb{R}$. We may consider $\mathbb{I} := (a, b) = (0, 1)$ without loss of generality.

Definition 4.1 (Port-Hamiltonian hyperbolic BCS in one spatial variable)**.** Let $N, d \in \mathbb{N}$ and for $k = 0, \ldots, N$ consider $P_k \in \mathbb{C}^{d\times d}$. We assume that $P_k = (-1)^{k+1}P_k^*$ for $k \neq 0$ with P_N invertible and $P_0 + P_0^* \leq 0$. Further let $W_B, W_C \in \mathbb{C}^{Nd\times 2Nd}$ such that the matrix

$$W := \begin{bmatrix} W_B \\ W_C \end{bmatrix} \in \mathbb{C}^{2Nd\times 2Nd}$$

is invertible.

a) Let $\mathcal{H} \in L^\infty\left([0,1];\mathbb{C}^{d\times d}\right)$ with $\mathcal{H}(\zeta) = \mathcal{H}(\zeta)^*$ for almost every $\zeta \in [0,1]$ and assume that there are $m, M > 0$ such that $mI_d \leq \mathcal{H}(\zeta) \leq MI_d$ for almost every $\zeta \in [0,1]$. We consider $X := L^2([0,1];\mathbb{C}^d)$ equipped with the scalar product induced by $\mathcal{H}$,

$$\langle y, x\rangle_X := \langle y, \mathcal{H}x\rangle_{L^2} = \int_0^1 y(\zeta)^*\mathcal{H}(\zeta)x(\zeta)\,\mathrm{d}\zeta\,, \quad x, y \in L^2([0,1];\mathbb{C}^d). \tag{8}$$

The *port-Hamiltonian operator* $\mathfrak{A} : \mathcal{D}(\mathfrak{A}) \subset X \to X$ is given by

$$\mathfrak{A}x = \sum_{k=0}^{N} P_k \frac{\partial^k}{\partial\zeta^k}(\mathcal{H}x), \quad x \in \mathcal{D}(\mathfrak{A}), \tag{9a}$$

with domain

$$\mathcal{D}(\mathfrak{A}) = \left\{x \in X \mid \mathcal{H}x \in W^{N,2}([0,1];\mathbb{C}^d)\right\} \tag{9b}$$

b) Denote the spatial derivative of f by f'. For a port-Hamiltonian operator $\mathfrak{A}$ and $x \in \mathcal{D}(\mathfrak{A})$ we define the *boundary flow* $f_{\partial,\mathcal{H}x} \in \mathbb{C}^{Nd}$ and *boundary effort* $e_{\partial,\mathcal{H}x} \in \mathbb{C}^{Nd}$ by

$$\begin{pmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{pmatrix} := R_0 \begin{pmatrix} (\mathcal{H}x)(1) \\ (\mathcal{H}x)'(1) \\ \vdots \\ (\mathcal{H}x)^{(N-1)}(1) \\ (\mathcal{H}x)(0) \\ (\mathcal{H}x)'(0) \\ \vdots \\ (\mathcal{H}x)^{(N-1)}(0) \end{pmatrix}, \tag{10}$$

where the matrix $R_0 \in \mathbb{C}^{2Nd\times 2Nd}$ is defined by

$$R_0 := \frac{1}{\sqrt{2}} \begin{bmatrix} \Lambda & -\Lambda \\ I_{Nd} & I_{Nd} \end{bmatrix}, \tag{11}$$

with

$$\Lambda := \begin{bmatrix} P_1 & P_2 & \cdots & \cdots & P_N \\ -P_2 & -P_3 & \cdots & -P_N & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^{N-1}P_N & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

c) For a port-Hamiltonian operator $\mathfrak{A}$ we define the *input map* $\mathfrak{B} : \mathcal{D}(\mathfrak{A}) \subset X \to \mathbb{C}^{Nd}$ and the *output map* $\mathfrak{C} : \mathcal{D}(\mathfrak{A}) \subset X \to \mathbb{C}^{Nd}$ as

$$\mathfrak{B}x := W_B \begin{pmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{pmatrix}, \tag{12}$$

$$\mathfrak{C}x := W_C \begin{pmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{pmatrix}. \tag{13}$$

We call $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ a *port-Hamiltonian hyperbolic BCS in one spatial variable* to which we associate the boundary control and observation problem

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), \quad x(0) = x_0, \\ u(t) &= \mathfrak{B}x(t), \\ y(t) &= \mathfrak{C}x(t) \end{aligned} \tag{14}$$

with a state $x(t) := x(t,\cdot) \in X$ and $t \geq 0$.

From the former definition we have the following result.

Lemma 4.2. *With operators $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ as in Definition 4.1, there exist $P, Q \in \mathcal{L}(\mathbb{C}^{2Nd}, \mathcal{D}(\mathfrak{A}))$ with*

$$\begin{aligned} \mathfrak{B}P &= I_{Nd}, & \mathfrak{B}Q &= 0, \\ \mathfrak{C}P &= 0, & \mathfrak{C}Q &= I_{Nd}. \end{aligned}$$

Consequently, $\mathfrak{A}P, \mathfrak{A}Q \in \mathcal{L}(\mathbb{C}^{Nd}; X)$.

Proof. Consider the trace operator $\mathcal{T} : W^{N,2}([0,1];\mathbb{C}^d) \to \mathbb{C}^{2Nd}$ as the linear map

$$\mathcal{T}z = \begin{pmatrix} z(1) \\ z'(1) \\ \vdots \\ z^{(N-1)}(1) \\ z(0) \\ z'(0) \\ \vdots \\ z^{(N-1)}(0) \end{pmatrix},$$

so that

$$\begin{bmatrix} \mathfrak{B}x \\ \mathfrak{C}x \end{bmatrix} = WR_0\mathcal{T}\mathcal{H}x, \qquad \text{where } W = \begin{bmatrix} W_B \\ W_C \end{bmatrix}.$$

Consider the standard orthogonal basis $\{e_j\}_{j=1}^{2Nd}$ in $\mathbb{C}^{2Nd}$ and choose some $f_j \in W^{N,2}([0,1];\mathbb{C}^d)$ with $\mathcal{T}(f_j) = e_j$ for $j = 1, \dots, 2Nd$. Since W, R_0 are invertible, we can define $M_p, M_q \in \mathbb{C}^{2Nd \times Nd}$ by

$$M_p = R_0^{-1}W^{-1}\begin{bmatrix} I_{Nd} \\ 0 \end{bmatrix}, \quad M_q = R_0^{-1}W^{-1}\begin{bmatrix} 0 \\ I_{Nd} \end{bmatrix}.$$

Let M_p, M_q be decomposed as

$$M_p = \begin{bmatrix} M_{p,1} \\ \vdots \\ M_{p,2Nd} \end{bmatrix}, \quad M_q = \begin{bmatrix} M_{q,1} \\ \vdots \\ M_{q,2Nd} \end{bmatrix},$$

with $M_{p,j}, M_{q,j} \in \mathbb{C}^{1\times Nd}$ for $j = 1, \dots, 2Nd$. Now set for almost every $\zeta \in [0,1]$,

$$(Pu)(\zeta) := \mathcal{H}^{-1}(\zeta)\sum_{j=1}^{2Nd} M_{p,j}uf_j(\zeta), \quad \forall u \in \mathbb{C}^{Nd},$$

$$(Qy)(\zeta) := \mathcal{H}^{-1}(\zeta)\sum_{j=1}^{2Nd} M_{q,j}yf_j(\zeta), \quad \forall y \in \mathbb{C}^{Nd}.$$

By construction P, Q have the desired properties. □

Remark 4.3. Note that for a port-Hamiltonian hyperbolic BCS $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ in one spatial variable holds that $C_0^\infty([0,1];\mathbb{C}^d) \subset \ker \mathfrak{B} \cap \ker \mathfrak{C}$ is a dense subspace of X. Integration by parts gives

$$\operatorname{Re}\langle \mathfrak{A}x, x\rangle_X \le \operatorname{Re}\langle \mathfrak{B}x, \mathfrak{C}x\rangle_{\mathbb{C}^{Nd}} + \operatorname{Re}\langle P_0\mathcal{H}x, \mathcal{H}x\rangle_{L^2} \quad \text{for all } x \in \mathcal{D}(\mathfrak{A}). \tag{15}$$

Since $P_0 + P_0^* \le 0$, it follows that the BCS fulfills Assumption 2.2(i) with $\alpha = 0$.

The class of impedance passive port-Hamiltonian systems meets the requirements of Assumption 2.2. We summarize it in the following statement.

Theorem 4.4. *Any port-Hamiltonian hyperbolic BCS $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ with one spatial variable satisfies Assumption 2.2. If, moreover, there exists some $\mu > 0$ such that $P_0 + P_0^* + \mu I$ is pointwise negative definite, then Assumption 2.2(i) holds for some $\alpha < 0$.*

Proof. It is stated in Remark 4.3 that $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ satisfies Assumption 2.2(i) with $\alpha \le 0$. Further, $\mathfrak{A}|_{\ker \mathfrak{B}}$ generates a (contractive) semigroup by [5, Theorem 2.3], whence $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ satisfies Assumption 2.2(ii). We can further infer from Remark 4.3 that $\ker \mathfrak{B} \cap \ker \mathfrak{C}$ is dense in X, and Lemma 4.2 guarantees the existence of P, Q such that (4) holds. This implies that the condition in Assumption 2.2(3) is fulfilled by $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$.
If, moreover, $P_0 + P_0^* + \mu I$ is pointwise negative definite for some $\mu > 0$, then we can conclude from (15) that Assumption 2.2(i) holds with $\alpha := -\mu m/(2M)$, where $m, M > 0$ are given in Definition 4.1. □

Theorem 4.4 allows to directly apply Theorems 3.4 & 3.6. Namely, if the initial value x_0 and the functions y_{ref}, p, φ fulfill Assumption 3.2, the application of the funnel controller (5) results in a unique global solution $x \in W^{1,\infty}_{\mathrm{loc}}([0,\infty); X)$ in the sense of Theorem 3.4. If, moreover, $P_0 + P_0^* + \mu I$ is pointwise negative definite for some $\mu > 0$, then $x, \dot{x}$ and u are moreover essentially bounded by Theorem 3.6.

4.2. **Hyperbolic systems in several spatial variables.** The following setting is presented in [25, Section 8.2]. We give a summary of the main results. For the particular case of the higher dimensional wave equation we refer to [24].

Definition 4.5. Let $d \in \mathbb{N}$ and matrices $P_j \in \mathbb{R}^{n\times n}$ for $j = 0, \dots, d$ such that $P_j^\top = P_j$ for all $j \neq 0$ and $P_0^\top = -P_0$. Let Ω be a bounded open subset of $\mathbb{R}^d$ with smooth boundary Γ and outward unit normal vector field η. We define the first order differential operator

$$\begin{aligned}\mathfrak{A}x &:= P_0 x + \sum_{j=1}^{d} P_j \frac{\partial x}{\partial \zeta_j}, \quad x \in \mathcal{D}(\mathfrak{A}),\\ \mathcal{D}(\mathfrak{A}) &:= \{x \in L^2(\Omega; \mathbb{R}^n) \mid \mathfrak{A}x \in L^2(\Omega;\mathbb{R}^n)\}.\end{aligned} \tag{16}$$

We also define the symmetric operator $Q_\eta := \sum_{j=1}^{d} \eta_j P_j : \Gamma \to \mathbb{R}^{n\times n}$.

Remark 4.6. Note that $\mathcal{D}(\mathfrak{A})$ in (16) is the maximal domain of definition of the operator $\mathfrak{A}$. This is further a Hilbert space, see [18], when endowed with the graph norm.

Assumption 4.7.

(i) Γ is characteristic with constant multiplicity, that is, for all $\zeta \in \Gamma$ we have that
$$\dim \ker Q_\eta(\zeta) = n - 2r \Leftrightarrow \operatorname{rank} Q_\eta(\zeta) = 2r$$
where $n > 2r \in \mathbb{N}$ is constant.

(ii) The spectrum of $Q_\eta(\zeta)$, $\zeta \in \Gamma$, is symmetric with respect to the imaginary axis and the sign of its eigenvalues is independent of $\zeta \in \Gamma$, that is, there are r positive eigenvalues.

Under Assumption 4.7, there exists a unitary operator $U \in \mathcal{L}(L^2(\Gamma;\mathbb{R}^n))$ and a diagonal matrix Λ such that $Q_\eta = U\Lambda U^*$ with $U^*U = I_{L^2(\Gamma;\mathbb{R}^n)}$ and

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 & 0\\ 0 & -\Lambda_1 & 0\\ 0 & 0 & 0\end{bmatrix},$$

see [25]. Here $\Lambda_1 \in \mathcal{L}(L^2(\Gamma;\mathbb{R}^r))$ contains the positive eigenvalues of Q_η. Further we have the following decomposition

$$\Lambda = \begin{bmatrix} R_0^* \Sigma R_0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{17}$$

where

$$R_0 := \frac{1}{\sqrt{2}} \begin{bmatrix} \Lambda_1 & -\Lambda_1 \\ I & I \end{bmatrix} \in \mathcal{L}(L^2(\Gamma;\mathbb{R}^{2r})), \quad \Sigma := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \in \mathcal{L}(L^2(\Gamma;\mathbb{R}^{2r})).$$

According to (17) we partition the unitary operator $U \in \mathcal{L}(L^2(\Gamma)^n)$ as follows

$$U^* = \begin{bmatrix} R^* \\ S^* \end{bmatrix} : L^2(\Gamma;\mathbb{R}^n) \to L^2(\Gamma;\mathbb{R}^{2r}) \times L^2(\Gamma;\mathbb{R}^{n-2r}).$$

Definition 4.8. Let $r \in \mathbb{N}$ be given as in Assumption 4.7 and $\mathcal{T}_0 : W^{1,2}(\Omega;\mathbb{R}^n) \to L^2(\Gamma;\mathbb{R}^n)$ be the trace operator of order zero, i.e., $\mathcal{T}_0 x = x|_\Gamma$ for $x \in W^{1,2}(\Omega;\mathbb{R}^n)$. Then the *boundary port-variables* associated with the differential operator $\mathfrak{A}$ are the operators $e_\partial, f_\partial \in \mathcal{L}(W^{1,2}(\Omega;\mathbb{R}^n), L^2(\Gamma;\mathbb{R}^r))$ defined by

$$\begin{bmatrix} f_\partial x \\ e_\partial x \end{bmatrix} := R_0 R^* \mathcal{T}_0 x, \quad x \in W^{1,2}(\Omega;\mathbb{R}^n).$$

We make the following assumption as in [25], which is a natural extension of the integration by parts formula for this systems. Recall that $W^{1/2,2}(\Gamma;\mathbb{R}^r)$ equals the range of trace operator on $W^{1,2}(\Omega;\mathbb{R}^r)$.

Assumption 4.9. Assume that the mapping

$$\begin{bmatrix} e_\partial \\ f_\partial \end{bmatrix} : W^{1,2}(\Omega;\mathbb{R}^n) \to L^2(\Gamma;\mathbb{R}^r) \times L^2(\Gamma;\mathbb{R}^r)$$

can be continuously extended to a linear mapping

$$\begin{bmatrix} e_\partial \\ f_\partial \end{bmatrix} : \mathcal{D}(\mathfrak{A}) \to W^{-1/2,2}(\Gamma;\mathbb{R}^r) \times W^{1/2,2}(\Gamma;\mathbb{R}^r).$$

Furthermore assume that Green's identity holds for all $x, z \in \mathcal{D}(\mathfrak{A})$, that is

$$\langle \mathfrak{A}x, z \rangle_{L^2} + \langle x, \mathfrak{A}z \rangle_{L^2} = \langle e_\partial x, f_\partial z \rangle_{W^{-1/2,2},W^{1/2,2}} + \langle e_\partial z, f_\partial x \rangle_{W^{-1/2,2},W^{1/2,2}}.$$

Definition 4.10. Let $\mathfrak{A}_0 := \mathfrak{A}$ with $\mathcal{D}(\mathfrak{A}_0) := \{x \in \mathcal{D}(\mathfrak{A}) \mid \exists b \in \mathbb{R}^r : e_\partial x = b\}$. To the operator $\mathfrak{A}_0$ we associate the (BCS) $\mathfrak{S} = (\mathfrak{A}_0, \mathfrak{B}, \mathfrak{C})$ with

$$\mathfrak{B}x = e_\partial x, \quad x \in \mathcal{D}(\mathfrak{A}_0)$$

and

$$\mathfrak{C}x = \int_\Gamma f_\partial x \,\mathrm{d}\sigma, \quad x \in \mathcal{D}(\mathfrak{A}_0).$$

For our purposes, we make the following assumption, which is for instance satisfied by the wave equation.

Remark 4.11. Note that if we restrict $\begin{bmatrix} e_\partial \\ f_\partial \end{bmatrix}$ to $W^{1,2}(\Omega;\mathbb{R}^n)$, we obtain that

$$\begin{bmatrix} e_\partial \\ f_\partial \end{bmatrix} (W^{1,2}(\Omega;\mathbb{R}^n)) = W^{1/2,2}(\Gamma;\mathbb{R}^{2r}),$$

see [25, pp. 212]. Since $\mathbb{R}^{2r} \subset W^{1/2,2}(\Gamma;\mathbb{R}^{2r})$, the former implies that there are $p, q : W^{1,2}(\Omega) \subset \mathcal{D}(\mathfrak{A}_0) \to \mathbb{R}^r$ such that

$$\begin{bmatrix} e_\partial \\ f_\partial \end{bmatrix} \begin{bmatrix} p & q \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & I_r \end{bmatrix}.$$

Thus, by setting $P := p$ and $Q := |\Gamma|^{-1} q$ we have that

$$\begin{bmatrix} \mathfrak{B} \\ \mathfrak{C} \end{bmatrix} \begin{bmatrix} P & Q \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & I_r \end{bmatrix}.$$

Theorem 4.12 ([25, Theorem 8.18])**.** *Under Assumptions 4.7 & 4.9 and the notation of Definitions 4.8 & 4.10, it follows that*

$$\operatorname{Re}\langle \mathfrak{A}_0 x, x\rangle_{L^2} = \operatorname{Re}\langle \mathfrak{B}x, \mathfrak{C}x\rangle_{\mathbb{R}^r} \qquad \forall x \in \mathcal{D}(\mathfrak{A}_0)$$

and that the operator $\mathfrak{A}_0|_{\ker \mathfrak{B}}$ is skew-adjoint and generates a unitary C_0-semigroup.

We show that the class belongs to that which is subject of Section 2, which consequences that the funnel controller is applicable.

Theorem 4.13. *Let $\mathfrak{S} = (\mathfrak{A}_0, \mathfrak{B}, \mathfrak{C})$ be as in Definition 4.10 and let Assumptions 4.7 & 4.9 be satisfied. Then $\mathfrak{S} = (\mathfrak{A}_0, \mathfrak{B}, \mathfrak{C})$ satisfies Assumption 2.2.*

Proof. The result follows immediately from Theorem 4.12 and Remark 4.11 together with the fact that $C_0^\infty(\Omega;\mathbb{R}^n) \subset \ker \begin{bmatrix} e_\partial \\ f_\partial \end{bmatrix}$ is a dense subspace. □

Example 4.14. Consider the 2-dimensional wave equation with boundary control in an open bounded domain Ω with smooth boundary Γ, namely,

$$\begin{aligned} \partial_{tt} w(t,\zeta) &= \Delta w(t,\zeta), \\ u(t) &= \left.\frac{\partial w(t,\zeta)}{\partial \eta}\right|_\Gamma, \\ y(t) &= \int_\Gamma \partial_t w(t,\zeta)|_\Gamma \mathrm{d}\sigma, \end{aligned} \tag{18}$$

and $w(0,\cdot) = a(\cdot) \in W^{2,2}(\Omega)$ with $\partial_\eta a(\cdot)|_\Gamma = 0$, $w_t(0,\cdot) = v(\cdot) \in W^{1,2}(\Omega)$. Then the funnel controller is locally applicable for (18).

Proof. The wave equation can be transformed into a port-Hamiltonian system of the form 16, c.f. [25, Example 8.12] with $P_0 = 0$ and

$$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

and state variable

$$x = \begin{bmatrix} p \\ q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \partial_t w \\ \partial_{\zeta_1} w \\ \partial_{\zeta_2} w \end{bmatrix}.$$

Further

$$\begin{bmatrix} e_\partial x \\ f_\partial x \end{bmatrix} = \begin{bmatrix} \eta \cdot q|_\Gamma \\ p|_\Gamma \end{bmatrix},$$

where η is the normal unit vector. The domain of the operator $\mathfrak{A}_0$ is given by

$$\mathcal{D}(\mathfrak{A}_0) := \left\{ \begin{bmatrix} p \\ q_1 \\ q_2 \end{bmatrix} \in L^2(\Omega;\mathbb{R}^3) \mid p \in W^{1,2}(\Omega), q \in H_{\mathrm{div}}(\Omega), \exists b \in \mathbb{R} : \eta \cdot q|_\Gamma = b \right\},$$

where

$$H_{\rm div}(\Omega) := \{x \in L^2(\Omega) \mid \nabla \cdot x \in L^2(\Omega)\}.$$

It is clear that $x_0 \in \mathcal{D}(\mathfrak{A}_0)$. From [12, Theorem 1.3] the range of f_∂ is precisely $W^{1/2,2}(\Gamma)$ and e_∂ from $H_{\rm div}(\Omega)$ is surjective onto $W^{-1/2,2}(\Gamma)$, see [12, Theorem 2.2] and [12, Corollary 2.4].

In this case P, Q are explicitly given by

$$(Pu)(\zeta) = \begin{bmatrix} 0 \\ \eta_1(\zeta) \\ \eta_2(\zeta) \end{bmatrix} u, \quad (Qy)(\zeta) = \frac{1}{|\Gamma|} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} y, \qquad u, y \in \mathbb{R}.$$

Further $C_0^\infty(\Omega) \subset \ker \mathfrak{B} \cap \ker \mathfrak{C}$ is dense. Hence, Theorem 4.13 gives the result. □

4.3. **A parabolic system.** A particular case of the boundary controlled heat equation was already discussed in [19], with a slightly different funnel controller. Here we present a parabolic problem and refer to [13] for more details on second order elliptic operators.

Definition 4.15. Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary Γ and outward normal unit vector ν. Assume that $a \in C^\infty(\Omega; \mathbb{C}^{n\times n})$ is self-adjoint and satisfies the ellipticity condition

$$\exists \alpha > 0 : \forall v \in \mathbb{C}^n \quad \mathrm{Re} \sum_{i,j=1}^{n} a_{ij}(\zeta) v_i v_j^* \geq \alpha \|v\|^2_{\mathbb{C}^n}.$$

Let $\kappa \geq 0$ and consider the BCS $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ defined by

$$\begin{aligned} \mathfrak{A}x &:= \nabla \cdot (a \nabla x) - \kappa x, \quad x \in \mathcal{D}(\mathfrak{A}), \\ \mathcal{D}(\mathfrak{A}) &:= \left\{ x \in W^{1,2}(\Omega) \mid \nabla \cdot a \nabla x \in L^2(\Omega) \text{ and } \exists b \in \mathbb{C} : \nu \cdot a \nabla x|_\Gamma = b \right\} \\ \mathfrak{B}x &:= \nu \cdot (a \nabla x)|_\Gamma, \\ \mathfrak{C}x &:= \int_\Gamma (\mathcal{T}_0 x) \mathrm{d}\sigma, \end{aligned} \tag{19}$$

where $\mathcal{T}_0 : W^{1,2}(\Omega) \to W^{1/2,2}(\Gamma)$ denotes trace operator, $\mathcal{T}_0 x = x|_\Gamma$.

Remark 4.16. We have the following comments on the former definition.

1. The operator $\mathcal{T}_0 : W^{1,2}(\Omega) \to W^{1/2,2}(\Gamma)$ is onto;
2. it is well-known that the realization of $\mathfrak{A}$ in $\ker \mathfrak{B}$ with $\kappa = 0$ corresponds to the Neumann elliptic problem, e.g. [13, Theorem 2.2.2.5], and $\mathfrak{A}|_{\ker \mathfrak{B}}$ generates a contractive semigroup for $\kappa \geq 0$.
3. for $x \in \mathcal{D}(\mathfrak{A})$

$$\mathrm{Re}\langle \mathfrak{A}x, x \rangle_{L^2} \leq \mathrm{Re}\langle \mathfrak{B}x, \mathfrak{C}x \rangle_{\mathbb{C}} - \kappa \|x\|^2_{L^2}.$$

Lemma 4.17. *There are operators $P, Q : \mathcal{D}(\mathfrak{A}) \to \mathbb{C}$ such that*

$$\begin{bmatrix} \mathfrak{B} \\ \mathfrak{C} \end{bmatrix} \begin{bmatrix} P & Q \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In fact, $Q = |\Gamma|^{-1}$ is constant.

Proof. Let $\mathcal{T}_0 x := x|_\Gamma$ and $\mathcal{T}_\nu x := \nu \cdot a \nabla x|_\Gamma$. From [13, Theorem 1.6.1.3] the combined trace operator

$$\begin{bmatrix} \mathcal{T}_\nu x \\ \mathcal{T}_0 x \end{bmatrix}$$

from $W^{2,2}(\Omega)$ to $W^{1/2,2}(\Gamma) \times W^{3/2,2}(\Gamma)$ is onto. Hence, there are p_ν, q_0 such that

$$\begin{array}{rclcrcl} \nu \cdot a\nabla p_\nu|_\Gamma &=& 1, & \qquad & \nu \cdot a\nabla q_0|_\Gamma &=& 0, \\ p_\nu|_\Gamma &=& 0, & & q_0|_\Gamma &=& 1. \end{array}$$

Note that $q_0 = 1$ is a solution. Considering p_ν, q_0 as operators from $\mathbb{C}$ to $W^{2,2}(\Omega) \subset \mathcal{D}(\mathfrak{A})$ yields that $P := p_\nu$ and $Q := |\Gamma|^{-1}q_0$ have the desired properties. $\square$

Next we show that this class satisfies the preliminaries of Theorem 3.6.

Theorem 4.18. *For any BCS $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ as introduced in Definition 4.15 with, additionally, $\kappa > 0$, satisfies Assumption 2.2 with $\alpha < 0$.*

Proof. It follows immediately from the conditions and previous considerations, together with $C_0^\infty(\Omega) \subset \ker \mathfrak{B} \cap \ker \mathfrak{C}$ being a dense subspace and Theorem 3.6. $\square$

5. **Proof of Theorems 3.4 & 3.6.** We develop some auxiliary results to conclude with the proof of the main results. A part of following lemma has been shown in [9] under the additional assumption of well-posedness, cf. Remark 2.3(e).

Lemma 5.1. *Assume that $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ satisfies Assumption 2.2 with $\alpha \in \mathbb{R}$. For all $\beta > \alpha$, $u \in \mathbb{C}^m$ and $f \in X$ there exist unique $x \in \mathcal{D}(\mathfrak{A})$ and $y \in \mathbb{C}^m$ with*

$$\begin{aligned} (\beta I - \mathfrak{A})x &= f, \\ u &= \mathfrak{B}x, \\ y &= \mathfrak{C}x. \end{aligned} \tag{20}$$

Furthermore, there exist bounded operators $H(\beta) \in \mathcal{L}(X)$, $J(\beta) \in \mathcal{L}(\mathbb{C}^m; X)$, $F(\beta) \in \mathcal{L}(X; \mathbb{C}^m)$ and $G(\beta) \in \mathcal{L}(\mathbb{C}^m) = \mathbb{C}^{m\times m}$ which connect the solution of (20) via

$$\begin{aligned} x &= H(\beta)f + J(\beta)u, \\ y &= F(\beta)f + G(\beta)u. \end{aligned} \tag{21}$$

Thereby, the matrix $G(\beta) + G(\beta)^*$ is positive definite, and $G(\beta)$ is invertible with positive definite $G(\beta)^{-1} + (G(\beta)^*)^{-1}$.

Proof. Step 1: We show uniqueness of the solution of (20). To this end, we have to show that the choice $f = 0$ and $u = 0$ leads to $x = 0$ and $y = 0$. Assuming that $x \in D(\mathfrak{A})$, $y \in \mathbb{C}^m$ fulfills (20) with $f = 0$ and $u = 0$, we obtain from (2) in Assumption 2.2(i) that

$$\beta\|x\|_X^2 = \operatorname{Re}\langle \mathfrak{A}x, x\rangle_X \le \operatorname{Re}\langle \mathfrak{B}x, \mathfrak{C}x\rangle_{\mathbb{C}^m} + \alpha\|x\|_X^2,$$

and thus $(\beta - \alpha)\|x\|^2 \le 0$. Invoking $\beta > \alpha$, we obtain $x = 0$ and, consequently, $y = \mathfrak{C}x = 0$.

Step 2: We show the existence of bounded operators $H(\beta)$, $J(\beta)$, $F(\beta)$ and $G(\beta)$ such that the solutions of (20) fulfill (21): By Remark 2.3b), Assumption 2.2(ii)&(i) imply that $\beta I - \mathfrak{A}|_{\ker \mathfrak{B}}$ is bijective. Further invoking Remark 2.3c), Assumption 2.2(iii) leads to the existence of $P, Q : \mathcal{L}(\mathbb{C}^m, \mathcal{D}(\mathfrak{A}))$, such that (4) holds. Considering

$$\begin{aligned} x &= \underbrace{(\beta I - \mathfrak{A}|_{\ker \mathfrak{B}})^{-1}}_{=:H(\beta)} f + \underbrace{((\beta I - \mathfrak{A}|_{\ker \mathfrak{B}})^{-1}(\mathfrak{A}P - \beta P) + P)}_{=:J(\beta)} u, \\ y &= \underbrace{\mathfrak{C}(\beta I - \mathfrak{A}|_{\ker \mathfrak{B}})^{-1}}_{=:F(\beta)} f + \underbrace{\mathfrak{C}(\beta I - \mathfrak{A}|_{\ker \mathfrak{B}})^{-1}(\mathfrak{A}P - \beta P)}_{=:G(\beta)} u, \end{aligned}$$

a straightforward calculation shows that (20) holds. Further, the operators $H(\beta)$, $J(\beta)$, $F(\lambda)$ and $G(\lambda)$ are bounded as they are compositions of bounded operators. *Step 3:* We show that $G(\beta)+G(\beta)^*$ is positive definite, and $G(\beta)$ is invertible with positive definite $G(\beta)^{-1}+(G(\beta)^*)^{-1}$: Considering (20) with $f=0$ and taking the real part of inner product in X, we obtain

$$\operatorname{Re}\beta\|x\|^2 = \operatorname{Re}\langle\mathfrak{A}x, x\rangle \leq \operatorname{Re}\langle u, y\rangle_{\mathbb{C}^m} + \alpha\|x\|^2 = \langle u, \tfrac{1}{2}(G(\beta)+G(\beta)^*)u\rangle_{\mathbb{C}^m} + \alpha\|x\|^2,$$

whence

$$(\beta-\alpha)\|x\|^2 \leq \langle u, \tfrac{1}{2}(G(\beta)+G(\beta)^*)u\rangle_{\mathbb{C}^m},$$

so that $G(\beta)+G(\beta)^*$ is positive semidefinite. If for $u\in\mathbb{C}^m$ holds $\langle u, \frac{1}{2}(G(\beta)+G(\beta)^*)u\rangle_{\mathbb{C}^m}=0$, then $(\beta-\alpha)\|x\|_X^2\leq 0$ which implies $x=0$ and thus $u=\mathfrak{B}x=0$. This implies the positive definiteness of $G(\beta)+G(\beta)^*$, and we can immediately conclude that $G(\beta)$ is invertible with positive definite $G(\beta)^{-1}+(G(\beta)^*)^{-1}$. □

Next we introduce a special class of nonlinear operators.

Definition 5.2. Let X be a Hilbert space and $A:\mathcal{D}(A)\subset X\to X$ a (possibly nonlinear) operator. We say that A is *dissipative*, if for all $x,y\in\mathcal{D}(A)$ holds $\operatorname{Re}\langle A(x)-A(y), x-y\rangle\leq 0$. If furthermore, for all $\lambda>0$ it holds that $\mathcal{R}(\lambda I-A)=X$, we call A *m-dissipative*.

Remark 5.3. If $A:\mathcal{D}(A)\subset X\to X$ is m-dissipative, then for all $f\in X$ and $\lambda>0$ there exists some $z\in\mathcal{D}(A)$ with $\lambda z-A(z)=f$. The element z is indeed unique, since for any $x\in\mathcal{D}(A)$ with $\lambda x-A(x)=f$, we obtain by taking the difference that

$$\lambda(x-z)-(A(x)-A(z))=0$$

and taking the inner product with $x-z$ gives

$$\lambda\|x-z\|^2=\operatorname{Re}\langle A(x)-A(z), x-z\rangle.$$

Dissipativity of A leads to non-positivity of the latter expression, whence $x=z$.

Proposition 5.4. *Let $\phi:\mathcal{D}(\phi)\subset\mathbb{C}^m\to\mathbb{C}^m$ be defined by*

$$\begin{aligned}\phi(y) &:= \frac{1}{1-\|y\|^2}y,\\ \mathcal{D}(\phi) &:= \{y\in\mathbb{C}^m \mid \|y\|<1\}.\end{aligned}\tag{22}$$

Then $-\phi$ is m-dissipative.

Proof. Step 1: We prove that $-\phi$ is dissipative. We first like to note that the function $g:[0,1)\to\mathbb{R}$ with $r\mapsto\frac{r}{1-r^2}$ is monotonically increasing on $[0,1)$, which follows by nonnegativity of its derivative. As a consequence $(g(a)-g(b))(a-b)\geq 0$ for all $a,b\in[0,1)$. Using this, we obtain that for $w,y\in\mathcal{D}(\phi)$ holds

$$\begin{aligned}\operatorname{Re}\langle\phi(w)-\phi(y), w-y\rangle &= \operatorname{Re}\langle\phi(w),w\rangle+\operatorname{Re}\langle\phi(y),y\rangle-\operatorname{Re}\langle\phi(y),w\rangle-\operatorname{Re}\langle\phi(w),y\rangle\\ &= \left(\tfrac{\|w\|^2}{1-\|w\|^2}+\tfrac{\|y\|^2}{1-\|y\|^2}-\tfrac{\operatorname{Re}\langle w,y\rangle}{1-\|y\|^2}-\tfrac{\operatorname{Re}\langle y,w\rangle}{1-\|w\|^2}\right)\\ &\geq \left(\tfrac{\|w\|^2}{1-\|w\|^2}+\tfrac{\|y\|^2}{1-\|y\|^2}-\tfrac{\|w\|\|y\|}{1-\|y\|^2}-\tfrac{\|y\|\|w\|}{1-\|w\|^2}\right)\\ &= \left(\tfrac{\|w\|}{1-\|w\|^2}-\tfrac{\|y\|}{1-\|y\|^2}\right)(\|w\|-\|y\|)\\ &= (g(\|w\|)-g(\|y\|))\cdot(\|w\|-\|y\|)\geq 0.\end{aligned}$$

Step 2: We show that $\lambda I + \phi(\cdot)$ is surjective for all $\lambda > 0$. Consider $f \in \mathbb{C}^m$ and $\lambda > 0$. Since $\lambda I + \phi(\cdot)$ maps zero to zero, it suffices to prove that any $f \neq 0$ is in the range of $\lambda I + \phi(\cdot)$. To this end, consider the real polynomial p with

$$p(\rho) = \lambda\rho^3 - \|f\|\rho^2 - (\lambda+1)\rho + \|f\|.$$

We observe that $p(0) = \|f\| > 0$ and $p(1) = -1 < 0$, whence there exists some $\tilde{\rho} \in (0,1)$ with $p(\tilde{\rho}) = 0$. Now choosing $y = \frac{\tilde{\rho}}{\|f\|} f$, we obtain by simple arithmetics that

$$\begin{aligned} \lambda y + \phi(y) &= \tfrac{f}{(1-\tilde{\rho}^2)\|f\|^2} \cdot \left(-\lambda\tilde{\rho}^3 + (\lambda+1)\tilde{\rho}\right) \\ &\overset{p(\tilde{\rho})=0}{=} \tfrac{f}{(1-\tilde{\rho}^2)\|f\|^2} \cdot \left(-\|f\| \cdot \tilde{\rho}^2 + \|f\|\right) = f, \end{aligned}$$

which shows that $\lambda I + \phi(\cdot)$ is surjective. □

We now present a result concerning perturbations of m-dissipative operators.

Lemma 5.5. [17, Corollary 6.19 (a)] *Let X be a Hilbert space, $A : \mathcal{D}(A) \subset X \to X$, m-dissipative and $B : \overline{\mathcal{D}(A)} \subset \mathcal{D}(B) \subset X \to X$ continuous. Then if $A + B$ is a dissipative operator, then $A + B$ is m-dissipative.*

The next result is a modification of [4, Theorem 4.3] in which the function ϕ is defined on the whole space $\mathbb{C}^m$ instead of a domain $\mathcal{D}(\phi)$ as in our situation.

Lemma 5.6. *Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be a BCS and let Assumption 2.2 be satisfied with $\alpha \in \mathbb{R}$. Let $\phi : \mathcal{D}(\phi) \subset \mathbb{C}^m \to \mathbb{C}^m$ be given by (22). Then the nonlinear operator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ with*

$$\begin{aligned} \mathcal{A}(z) &:= (\mathfrak{A} - \alpha I)|_{\mathcal{D}(\mathcal{A})} z, \\ \mathcal{D}(\mathcal{A}) &:= \{ z \in \mathcal{D}(\mathfrak{A}) \mid \|\mathfrak{C}z\| < 1, \mathfrak{B}z + \phi(\mathfrak{C}z) = 0 \} \end{aligned} \tag{23}$$

is m-dissipative and $\overline{\mathcal{D}(\mathcal{A})} = X$.

Proof. Step 1: We show that $\mathcal{A}$ is a densely defined: For given $(v, e) \in \mathbb{C}^m \times \mathcal{D}(\phi)$ with $v = -\phi(e)$ we can find $z_0 \in D(\mathcal{A})$ such that

$$\begin{pmatrix} \mathfrak{B}z_0 \\ \mathfrak{C}z_0 \end{pmatrix} = \begin{pmatrix} v \\ e \end{pmatrix},$$

e.g., by setting $z_0 = Pv + Qe$, where P, Q are chosen as in Remark 2.3c). It follows that $z_0 + \ker \mathfrak{B} \cap \ker \mathfrak{C} \subset \mathcal{D}(\mathcal{A})$ is a dense subset of X by Assumption 2.2.

Step 2: For given $\lambda > 0$, we show that $\lambda I - \mathcal{A}$ is surjective:

Let $f \in X$. Our aim is to find some $z \in \mathcal{D}(\mathcal{A})$ with $(\lambda I - \mathcal{A})(z) = f$, that is,

$$\begin{aligned} ((\lambda + \alpha)I - \mathfrak{A})z &= f \\ \mathfrak{B}z &= -\phi(\mathfrak{C}z). \end{aligned} \tag{24}$$

Set $\beta := \lambda + \alpha > \alpha$ and consider the operators $H(\lambda) \in \mathcal{L}(X)$, $J(\lambda) \in \mathcal{L}(\mathbb{C}^m; X)$, $F(\lambda) \in \mathcal{L}(X; \mathbb{C}^m)$ and $G(\lambda) \in \mathcal{L}(\mathbb{C}^m) = \mathbb{C}^{m \times m}$ from Lemma 5.1. Since the matrix $G(\beta)^{-1} + (G(\beta)^*)^{-1}$ is positive definite by Lemma 5.1, there exists some $\delta > 0$ such that $G(\beta)^{-1} + (G(\beta)^*)^{-1} - 2\delta I$ is positive definite. The function $-\phi$ is m-dissipative by Proposition 5.4, whence

$$\Psi(\cdot) := -\phi(\cdot) - G(\beta)^{-1} + \delta I$$

is dissipative. Then Lemma 5.5 gives rise to m-dissipativity of Ψ. In particular, $\Psi(\cdot) - \delta I = -\phi(\cdot) - G(\beta)^{-1} : D(\phi) \to \mathbb{C}^m$ is bijective, whence there exists some $e \in D(\phi)$ with

$$\Psi(e) - \delta e = G(\beta)^{-1} F(\beta) f,$$

which is equivalent to

$$-\phi(e) = G(\beta)^{-1}e - G(\beta)^{-1}F(\beta)f,$$

and thus

$$e = F(\beta)f + G(\beta)(-\phi(e)).$$

Then Lemma 5.1 implies that $z = H(\beta)f + J(\beta)(-\phi(e))$ indeed fulfills (24).
Step 3: We show that $\mathcal{A}$ is dissipative: Let $z_1, z_2 \in \mathcal{D}(\mathcal{A})$, then

$$\begin{aligned}\operatorname{Re}\langle \mathcal{A}(z_1) - \mathcal{A}(z_2), z_1 - z_2\rangle_X &\overset{\text{Assumption}}{\underset{2.2(\text{i})}{=}} \operatorname{Re}\langle \mathfrak{A}_\alpha|_{\mathcal{D}(\mathcal{A})}z_1 - \mathfrak{A}_\alpha|_{\mathcal{D}(\mathcal{A})}z_2, z_1 - z_2\rangle_X \\ &\leq -\operatorname{Re}\langle \phi(\mathfrak{C}z_1) - \phi(\mathfrak{C}z_2), \mathfrak{C}z_1 - \mathfrak{C}z_2\rangle_{\mathbb{C}^n} \\ &\overset{\text{Proposition}}{\underset{5.4}{\leq}} 0.\end{aligned}$$

□

An intrinsic technical problem when investigating solvability of (6) is that the feedback is varying in time, i.e. it depends on t explicitly. To circumvent this problem, we perform a change of variables leading to an evolution equation with a constant operator. This is subject of the subsequent auxiliary result.

Lemma 5.7. *Let a BCS $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be given which satisfies Assumption 2.2 and assume that the initial value x_0 and the functions y_{ref}, p, φ fulfill Assumption 3.2. Then for $\varphi_0 = \varphi(0)$, $e_0 = \mathfrak{C}x_0 - y_{\mathrm{ref}}(0)$, $u_0 = \mathfrak{B}x_0$, operators $P, Q : \mathcal{L}(\mathbb{C}^m, \mathcal{D}(\mathfrak{A}))$ with (4), and the nonlinear m-dissipative operator $\mathcal{A}$ given in (23) and*

$$\begin{aligned}\omega &= \tfrac{\dot\varphi}{\varphi},\\ f &= \varphi\cdot\Big(\mathfrak{A}Qy_{\mathrm{ref}} - Q\dot y_{\mathrm{ref}} + \mathfrak{A}P(u_0 + \psi(\varphi_0, e_0))p - P(u_0 + \psi(\varphi_0, e_0))\dot p(t)\Big)\\ z_0 &= \varphi_0\cdot\Big(x_0 - Qy_{\mathrm{ref}}(0) - P(u_0 + \psi(\varphi_0, e_0))\Big).\end{aligned} \tag{25}$$

holds $\omega \in W^{1,\infty}([0,\infty))$, $f \in W^{1,\infty}([0,\infty); X)$ and $z_0 \in D(\mathcal{A})$. Furthermore, the following holds for $T > 0$:

a) If $x \in W^{1,\infty}([0,T]; X)$ and for almost every $t \in [0,T]$ holds that $\dot x$ is continuous at t, $x \in D(\mathfrak{A})$ and (6), then for

$$z(t) = \varphi(t)\Big(x(t) - Qy_{\mathrm{ref}}(t) - P(\mathfrak{B}x_0 + \psi(\varphi_0, e_0))p(t)\Big), \tag{26}$$

holds $z \in W^{1,\infty}([0,T]; X)$ and for almost every $t \in [0,T]$ holds that $\dot z$ is continuous at t, $z(t) \in D(\mathcal{A})$ and

$$\begin{aligned}\dot z(t) &= \mathcal{A}(z(t)) + (\omega(t) + \alpha)z(t) + f(t),\\ z(0) &= z_0\end{aligned} \tag{27}$$

b) Conversely, if $z \in W^{1,\infty}([0,T]; X)$ and for almost every $t \in [0,T]$ holds that $\dot z$ is continuous at t, $z(t) \in D(\mathcal{A})$ and (27), then for

$$x(t) = \varphi(t)^{-1}z(t) + Qy_{\mathrm{ref}}(t) + P(\mathfrak{B}x_0 + \psi(\varphi_0, e_0))p(t), \tag{28}$$

$x \in W^{1,\infty}([0,T]; X)$ and for almost every $t \in [0,T]$ holds that $\dot x$ is continuous at t, $x \in D(\mathfrak{A})$ and (6).

c) If $z \in W^{1,\infty}([0,\infty); X)$, then x as in (28) fulfills $x \in W^{1,\infty}([0,\infty); X)$.

Proof. The statements $\omega \in W^{1,\infty}([0,\infty))$, $f \in W^{1,\infty}([0,\infty);X)$ follow from the product rule for weak derivatives [2, p. 124]. Since P maps to $D(\mathfrak{A})$, we have $z_0 \in D(\mathfrak{A})$. Further, by using $\mathfrak{B}P = I$, $\mathfrak{B}Q = 0$, $\mathfrak{C}P = 0$ and $\mathfrak{C}Q = I$, we obtain

$$\phi(\mathfrak{C}z_0) = \frac{\varphi_0 \cdot e_0}{1 - \varphi_0^2 \|e_0\|^2} = -\mathfrak{B}z_0,$$

whence $z_0 \in D(\mathcal{A})$.
To prove statement a), assume that $x \in W^{1,\infty}([0,T];X)$ has a derivative which is continuous and in the domain of $\mathfrak{A}$ almost everywhere. First note that the twice weak differentiability of p and φ together with the fact that P and Q map to $D(\mathfrak{A})$ implies that $z \in W^{1,\infty}([0,T];X)$ with $\dot{z}(t)$ being in $D(\mathfrak{A})$ for almost every $t \in [0,T]$. By further using that (6) holds for almost every $t \in [0,T]$, we obtain —analogously to the above computations for z_0— that

$$\phi(\mathfrak{C}z(t)) = \frac{\varphi(t)e(t)}{1 - \varphi(t)^2 \|e(t)\|^2} = -\mathfrak{B}z(t),$$

which implies that $z(t) \in D(\mathcal{A})$ for almost every $t \in [0,T]$. Further, a straightforward calculation shows that (6) implies that $z(t)$ fulfills (27).
Statement b) follows by an argumentation straightforward to that in the proof of a). Statement c) is a simple consequence of $\inf_{t \geq 0} \varphi(t) > 0$, $\varphi, p \in W^{2,\infty}([0,\infty))$, $y_{\mathrm{ref}} \in W^{2,\infty}([0,\infty), \mathbb{C}^m)$ and the product rule for weak derivatives. □

The previous lemma is indeed the key step to prove Theorems 3.4 & 3.6 on the feasibility of the funnel controller. By using the state transformation (26) with inversion (28), the analysis of feasibility of the funnel controller reduces to the proof of existence of a solution to the nonlinear evolution equation (27) in which the time-dependence is now extracted to the inhomogeneity. This is subject of the following result, which is a slight generalization of [22, Thm. IV.4.1], where equations of type (27) with constant ω and m-monotone $\mathcal{A}$ are considered. Thereby we will use Kato's results [16, Thms. 1-3]. Note that these statements deal as well with a slight more special situation, but can be extended to the general case presented below, as directly after the aforementioned results by Kato. Note as well, that in [16] the notion of m-accretive operators A is used, which means that $-A$ is m-dissipative.

Lemma 5.8. *Let $T > 0$, X be a Hilbert space and $A : \mathcal{D}(A) \subset X \to X$ be m-dissipative in X with $0 \in \mathcal{D}(A)$ and $A(0) = 0$. Then for each $z_0 \in \mathcal{D}(A)$, real-valued $\omega \in W^{1,\infty}([0,T])$ and $f \in W^{1,\infty}([0,T];X)$ there exists a unique $z \in W^{1,\infty}([0,T];X)$ with*

(i) $z(t) \in \mathcal{D}(A)$ for almost every $t \in [0,T]$;
(ii) for almost every $t \in [0,T]$ holds

$$\begin{aligned} \dot{z}(t) &= A(z(t)) + \omega(t)z(t) + f(t), \quad t \in [0,T], \\ z(0) &= z_0, \end{aligned} \tag{29}$$

(iii) $\dot{z}$ and $A(z)$ are continuous except at a countable number of values in $[0,T]$.

Proof. Define the operator $\mathcal{A}(t)z := A(z) + \omega(t)z + f(t)$ for $(t,z) \in [0,T] \times \mathcal{D}(A)$ and set $\mathcal{D}(\mathcal{A}(t)) = \mathcal{D}(A)$ for all $t \geq 0$. For $s,t \in [0,T]$ and $z \in \mathcal{D}(A)$ we have

$$\begin{aligned} \|\mathcal{A}(t)z - \mathcal{A}(s)z\| &= \|(\omega(t) - \omega(s))z + f(t) - f(s)\| \\ &\leq (\|\dot{\omega}\|_{L^\infty}\|z\| + \|\dot{f}\|_{L^\infty})|t-s| \\ &\leq \max\{\|\dot{\omega}\|_{L^\infty}, \|\dot{f}\|_{L^\infty}\}|t-s|(1 + \|z\|). \end{aligned}$$

Further, for all $t \in [0,T]$ and $\lambda > \|\omega\|_{L^\infty}$ the operator $-\lambda I + \mathcal{A}(t)$ is m-dissipative. The dissipativity is trivial. For the range condition, let $\mu > 0$ and $u \in X$, then

$$\mu I - (-\lambda z + \mathcal{A}(t)z) = u$$

can be rewritten as

$$(\mu + \lambda - \omega(t))z - A(z) = u + f(t).$$

Since $\lambda - \omega(t) > 0$ uniformly in t and A is m-dissipative, if follows that there is a unique $z \in \mathcal{D}(A)$ such that $\mu z - (-\lambda z + \mathcal{A}(t)z) = u$. Now, the application of [16, Theorems 1-3] and the subsequent remark in [16] deliver the result. □

Proof of Theorem 3.4. Let $T > 0$, and consider the nonlinear operator $\mathcal{A}$ as in (23) and $\omega \in W^{1,\infty}([0,\infty))$, $f \in W^{1,\infty}([0,\infty);X)$ and $z_0 \in D(\mathcal{A})$ as in (25). Then Lemma 5.8 implies that the nonlinear evolution equation (27) has a unique solution $z \in W^{1,\infty}([0,T];X)$ in the sense that for almost all $t \in [0,T]$ holds $z(t) \in D(\mathcal{A})$, $\dot{z}$ is continuous at t, and (27). Then Lemma 5.7b) yields that $x \in W^{1,\infty}([0,T];X)$ as in (28) has the desired properties.
It remains to show uniqueness: Assume that $x_i \in W^{1,\infty}([0,T];X)$ are solutions of the closed-loop system (6) for $i = 1,2$. Then

$$z_i(t) = \varphi(t)\Big(x_i(t) - Qy_{\mathrm{ref}}(t) - P(\mathfrak{B}x_0 + \psi(\varphi_0, e_0))p(t)\Big), \tag{30}$$

fulfills $\dot{z}_i(t) = \mathcal{A}(z_i(t)) + (\omega(t) + \alpha)z(t) + f(t)$ with $z_i(0) = z_0$, and the uniqueness statement in Lemma 5.8 gives $z_1 = z_2$. Now resolving (30) for x_i and invoking $z_1 = z_2$ gives $x_1 = x_2$. □

It remains to prove Theorem 3.6 which states that the global solution and its derivative are bounded in case of negativity of the constant α in Assumption 2.2 (i). To this end a further auxiliary result, which is a generalization of the Grönwall inequality.

Lemma 5.9. [22, Lemma IV.4.1] *Let $a, b \in L^1([0,T])$ be real-valued with $b \geq 0$ almost everywhere and let the absolutely continuous function $v : [0,T] \to (0,\infty)$ satisfy*

$$(1-\rho)\dot{v}(t) \leq a(t)v(t) + b(t)v(t)^\rho, \qquad \text{for almost every } t \in [0,T],$$

where $0 \leq \rho < 1$. Then

$$v(t)^{1-\rho} \leq v(0)^{1-\rho} e^{\int_0^t a(s)\,\mathrm{d}s} + \int_0^t e^{\int_s^t a(r)\,\mathrm{d}r} b(s)\,\mathrm{d}s\,, \quad t \in [0,T].$$

Now we are ready to formulate the proof of Theorem 3.6.

Proof of Theorem 3.6. Let $\alpha < 0$, let $\mathcal{A}$ be the nonlinear operator in (23) and $\omega \in W^{1,\infty}([0,\infty))$, $f \in W^{1,\infty}([0,\infty);X)$ and $z_0 \in D(\mathcal{A})$ as in (25).
Step 1: We show that the solution $z \in W^{1,\infty}_{\mathrm{loc}}([0,\infty);X)$ of (27) (which exists by Lemma 5.8) is bounded:

Then we obtain that for almost all $t \geq 0$ holds

$$
\begin{aligned}
\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|_X &= \mathrm{Re}\langle z(t), \dot z(t)\rangle_X \\
&= \mathrm{Re}\langle z(t), \mathcal{A}z(t) + (\omega(t)+\alpha)z(t) + f(t)\rangle_X \\
&\leq \mathrm{Re}\langle z(t), \mathcal{A}z(t)\rangle_X + (\omega(t)+\alpha)\|z(t)\|_X^2 + \|z(t)\|_X\|f(t)\|_X \\
&\overset{(23)}{=} \mathrm{Re}\langle z(t), \mathfrak{A}z(t)\rangle_X + \omega(t)\|z(t)\|_X^2 + \|z(t)\|_X\|f(t)\|_X \\
&\overset{(2)}{\leq} \mathrm{Re}\langle \mathfrak{B}z(t), \mathfrak{C}z(t)\rangle_{\mathbb{C}^n} + \alpha\|z(t)\|_X^2 + \omega(t)\|z(t)\|_X^2\|z(t)\|_X^2 + \|z(t)\|_X\|f(t)\|_X \\
&\overset{(23)}{=} -\frac{\|\mathfrak{C}z(t)\|^2}{1-\|\mathfrak{C}z(t)\|^2} + \alpha\|z(t)\|_X^2 + \omega(t)\|z(t)\|_X^2 + \|z(t)\|_X\|f(t)\|_X \\
&\leq \alpha\|z(t)\|_X^2 + \omega(t)\|z(t)\|_X^2 + \|z(t)\|_X\|f(t)\|_X .
\end{aligned}
$$

Now applying Lemma 5.9 with $\rho = 1/2$, using that the definition of ω in (25) leads to $\omega = \frac{\mathrm{d}}{\mathrm{d}t}\log(\varphi)$ and setting $\varepsilon := \inf_{t\geq 0}\varphi(t) > 0$, we obtain that for almost all $t \geq 0$ holds

$$
\|z(t)\|_X \leq \varepsilon^{-1}\|z_0\|_X\varphi(t)e^{\alpha t} + \varphi(t)e^{\alpha t}\int_0^t \varphi(s)^{-1}e^{-\alpha s}\|f(s)\|_X\,\mathrm{d}s\,.
$$

The definition of f in (25) leads to the existence of $c_0, c_1 > 0$ such that for almost all $t \geq 0$ holds $\|f(t)\|_X \leq \varphi(t)(c_0 + c_1\|y_{\mathrm{ref}}\|_{W^{1,\infty}})$. Thus,

$$
\|z(t)\|_X \leq \varepsilon^{-1}\|z_0\|_X\varphi(t)e^{\alpha t} - \alpha^{-1}\varphi(t)(c_0 + c_1\|y_{\mathrm{ref}}\|_{W^{1,\infty}})(1-e^{\alpha t}),
$$

whence $z \in L^\infty([0,\infty);X)$.
Step 2: We show that $\dot z \in L^\infty([0,\infty);X)$:
To this end, let $h > 0$ and, by using the dissipativity of $\mathcal{A}$, consider

$$
\begin{aligned}
\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|z(t+h)-z(t)\|_X^2 \leq \alpha\|z(t+h)-z(t)\|_X^2 &+ \omega_0(t+h)\|z(t+h)-z(t)\|_X^2 \\
&+ |\omega_0(t+h)-\omega_0(t)|\|z(t)\|_X\|z(t+h)-z(t)\|_X \\
&+ \|f(t+h)-f(t)\|_X\|z(t+h)-z(t)\|_X,
\end{aligned}
$$

Again applying the Grönwall type inequality from Lemma 5.9 with $\rho = 1/2$, dividing by h and letting $h \to 0$ yields

$$
\begin{aligned}
&\|\dot z(t)\|_X \\
&\leq \varepsilon^{-1}\|\dot z(0)\|_X\varphi(t)e^{\alpha t} + \varphi(t)e^{\alpha t}\int_0^t e^{-\alpha s}\varphi(s)^{-1}(\|z\|_{L^\infty}|\dot\omega_0(s)| + \|\dot f(s)\|_X)\,\mathrm{d}s \\
&\leq \varepsilon^{-1}\|\mathcal{A}(z_0) + \omega_0(0)z_0 + f(0)\|_X\|\varphi\|_{L^\infty} \\
&\quad + \|\varphi\|_{L^\infty}\left\|\varphi^{-1}\right\|_{L^\infty}(\|z\|_{L^\infty}\|\dot\omega_0\|_{L^\infty} + d_0 + d_1\|y_{\mathrm{ref}}\|_{W^{2,\infty}})
\end{aligned}
$$

for some $d_0, d_1 > 0$. Hence, $\dot z(t) \in L^\infty([0,\infty);X)$.
Step 3: We conclude that the solution x in (6) (which exists by Theorem 3.4) fufills $x \in W^{1,\infty}([0,\infty);X)$:
We know from the first two steps that $z \in W^{1,\infty}([0,\infty);X)$. Then Lemma 5.7c) leads to $x \in W^{1,\infty}([0,\infty);X)$.
Step 4: We finally show that $u = \mathfrak{B}x$ fufills $u \in L^\infty([0,\infty);\mathbb{C}^m)$:
We know from the third step, we know that $x \in W^{1,\infty}([0,\infty);X)$. Since we have $x(t) \in D(\mathfrak{A})$ with $\dot x(t) = \mathfrak{A}x(t)$ for almost all $t \geq 0$, we can conclude that $\mathfrak{A}x \in L^\infty([0,\infty);X)$, and thus $x \in L^\infty([0,\infty);D(\mathfrak{A}))$. Then $\mathfrak{B} \in \mathcal{L}(D(\mathfrak{A}),\mathbb{C}^m)$ gives $u = \mathfrak{B}x \in L^\infty([0,\infty);\mathbb{C}^m)$. □

6. **Simulations.** Here we show some examples which correspond to the classes mentioned in Section 3. The implementation of all simulations has been done with Python.

6.1. **Lossy transmission line.** We consider the dissipative version of the Telegrapher's Equation with constant coefficients given by

$$\begin{aligned} V_\zeta(\zeta,t) &= -LI_t(\zeta,t) - RI(\zeta,t), \\ I_\zeta(\zeta,t) &= -CV_t(\zeta,t) - GV(\zeta,t), \\ u(t) &= \begin{pmatrix} V(a,t) \\ V(b,t) \end{pmatrix}, \\ y(t) &= \begin{pmatrix} I(a,t) \\ -I(b,t) \end{pmatrix}. \end{aligned}$$

R is the resistance, C the capacitance, L the inductance and G the conductance —all of them per unit length.

The system can be written in port-Hamiltonian form as

$$\begin{aligned} \partial_t x(\zeta,t) &= P_1 \partial_\zeta(\mathcal{H}(\zeta)x(\zeta,t)) + P_0\mathcal{H}(\zeta)x(\zeta,t), \\ u(t) &= W_B R_0 \begin{pmatrix} (\mathcal{H}x)(b,t) \\ (\mathcal{H}x)(a,t) \end{pmatrix}, \\ y(t) &= W_C R_0 \begin{pmatrix} (\mathcal{H}x)(b,t) \\ (\mathcal{H}x)(a,t) \end{pmatrix}, \end{aligned}$$

where

$$P_1 := \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, P_0 := \begin{bmatrix} -R & 0 \\ 0 & -G \end{bmatrix}, \mathcal{H}(\zeta) := \begin{bmatrix} L^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix},$$

$$W_B := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}, W_C := \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix},$$

$$x(\zeta,t) := \begin{pmatrix} LI(\zeta,t) \\ CV(\zeta,t) \end{pmatrix}.$$

We have chosen the reference signals and funnel boundary of the following form

$$\begin{aligned} y_{\text{ref}}(t) &= \begin{pmatrix} A_1 \sin(\omega_1 t)\sin(\omega_2 t) \\ A_2 \sin(\omega_3 t) \end{pmatrix}, \\ \varphi(t) &= \varphi_0 \varepsilon^{-2} \tanh(\omega t + \varepsilon). \end{aligned}$$

In this case the system is impedance passive and $P_0 + P_0^* \leq -2\min\{R,G\}I_2$ and Theorem 4.4 implies that $u, y \in L^\infty([0,\infty);\mathbb{R}^2)$. The simulated system is shown in Fig. 2.

The parameter values are $\zeta \in (a,b)$ with $a = 0$ m, $b = 1$ m,

$$\begin{aligned} R &= 463.59\Omega\text{m}^{-1}, & L &= 0.5062 \text{ mH m}^{-1}, \\ G &= 29.111\ \mu\text{S m}^{-1}, & C &= 51.57 \text{ nF m}^{-1}. \end{aligned}$$

Further, set $c_0 = (LC)^{-1/2}$, $f = 1$ MHz, $\omega = 2\pi f$, $\varphi_0 = 1$ A^{-1}, $\varepsilon = 0.1$ and amplitudes $A_1 = -0.3$ A, $A_2 = 0.4$ A. The other angular frequencies are $\omega_1 = \omega$, $\omega_2 = 16\omega$ and $\omega_3 = \omega/2$. For the time interval we have defined $T_0 = f^{-1}$ and

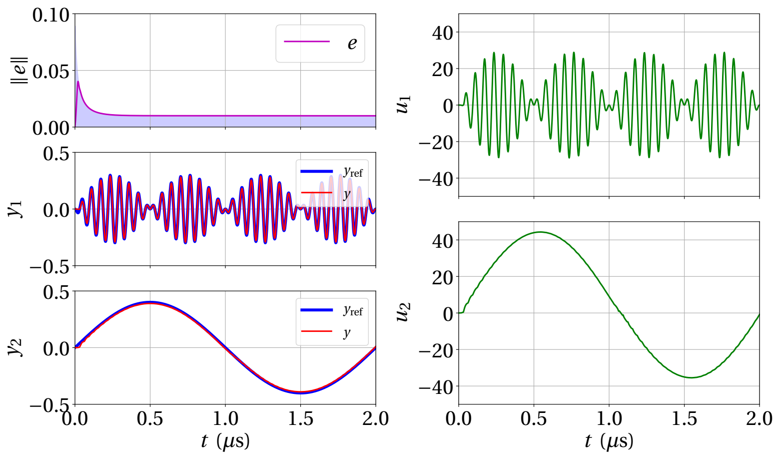

FIGURE 2. Left: Norm of the error within the funnel boundary followed by the two reference signals and the respective outputs. Right: Inputs obtained from the feedback law.

$t \in [0, T]$, where $T = 2T_0$. We have used semi-explicit finite differences with a tolerance of 10^{-3}. The mesh in ζ has $M = 1000$ points and the mesh in t has

$$N = \left\lfloor \frac{b-a}{2c_0 T} M \right\rfloor$$

points. We further assume that the initial state is zero, i.e., $x_0 = 0$ and we apply the controller (7) from Remark 3.7 with $k_0 = 1\ \Omega$.

6.2. **Wave equation in two spatial dimensions.** Here we consider the situation described in Example 4.14, given by the system in polar coordinates on the unit disc

$$\begin{aligned}
\partial_{tt} w(t,r,\theta) &= \partial_{rr} w(t,r,\theta) + r^{-1}\partial_r w(t,r,\theta) + r^{-2}\partial_{\theta\theta} w(t,r,\theta), \\
u(t) &= (\partial_r w(t,r,\theta))|_{r=1}, \\
y(t) &= \int_0^{2\pi} \partial_t w(t,1,\theta)\mathrm{d}\theta,
\end{aligned}$$

and use again a funnel boundary of the form $\varphi(t) = \varphi_0 \varepsilon^{-2} \tanh(\omega t + \varepsilon)$ and a reference signal of the form $y_{\rm ref}(t) = A \tanh(\omega t) + B \sin(\omega t)$. The results are given in Fig. 3. Note that by setting the speed of propagation to 1, the units of t coincide with the ones of r.

The parameter values are $r \in (a, b)$ with $a = 0$ m, $b = 1$ m, $\theta \in (0, 2\pi)$, $f = 1\ \mathrm{m}^{-1}$, $\omega = 2\pi f$, $\varepsilon = 10^{-2}$, $\varphi_0 = 1$. The amplitudes are $A = 1$ and $B = 0.1$. We define $T_0 = f^{-1}$ and $T = 4T_0$. The initial state of the system is

$$w(0, r, \theta) = 0\ \mathrm{m}, \quad w_t(0, r, \theta) = 0,$$

which leads to a problem with radial symmetry, so the partial derivatives with respect to θ vanish and we use explicit finite differences in r with $M = 2000$ points

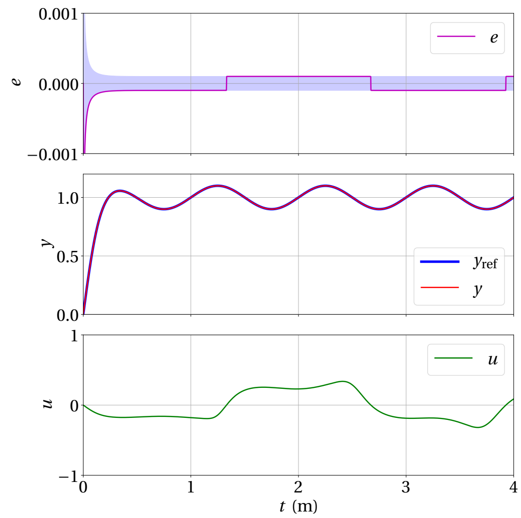

FIGURE 3. Performance funnel with the error, reference signal with the output of the closed-loop system and input of the closed-loop.

and in $t \in [0, T]$ with N points, where

$$N = \left\lfloor \frac{b-a}{2T} \right\rfloor M.$$

6.3. **Heat equation.** Here we consider the following boundary controlled 2D heat equation on the unit disc given by

$$\begin{aligned}\partial_t x(t,r,\theta) &= \alpha(\partial_{rr} x(t,r,\theta) + r^{-1}\partial_r x(t,r,\theta) + r^{-2}\partial_{\theta\theta} x(t,r,\theta)),\\ u(t) &= \alpha(\partial_r x(t,r,\theta))|_{r=1},\\ y(t) &= \int_0^{2\pi} x(t,1,\theta)\mathrm{d}\theta,\end{aligned}$$

where $\alpha > 0$ is the thermal diffusivity.

In this case, making use of Theorem 4.18, we choose a funnel boundary of the form $\varphi(t) = \varphi_0 \varepsilon^{-2} \tanh(\omega t + \varepsilon)$. The reference signal is given by $y_{\mathrm{ref}}(t) = A \sin(\omega t)$ and the simulated system is shown in Fig. 4. In Fig. 5 we show the evolution of the plate at four different times.

The parameter values are $\alpha = 1\ \mathrm{m}^2\mathrm{s}^{-1}$, $r \in (r_0, r_1)$, with $r_0 = 0$ m and $r_1 = 1$ m, and $\theta \in (0, 2\pi)$. The amplitude values are $A = 1$ J, $\varphi_0 = 0.1$ J and $\varepsilon = 10^{-1}$. We have set $T_0 = 1$ s, $\omega = 2\pi T_0^{-1}$ and $T = 5T_0$. We have used explicit finite differences with a partition in r and θ of $N = 25$ points for each variable and in $t \in [0, T]$ of

$$M = \left\lfloor 10T \left(\frac{N^2}{(r_1 - r_0)^2} + \frac{N}{r_1 - r_0} + \frac{N^2}{4\pi^2} \right) \right\rfloor$$

points. The initial state of the system is

$$x(0, r, \theta) = x_0 (r_1 - r)^2 \sin(\theta),$$

where $x_0 = 0.5\ \mathrm{Jm}^{-2}$. We apply the controller (7) from Remark 3.7 with $k_0 = 1\ \mathrm{s}^{-2}$.

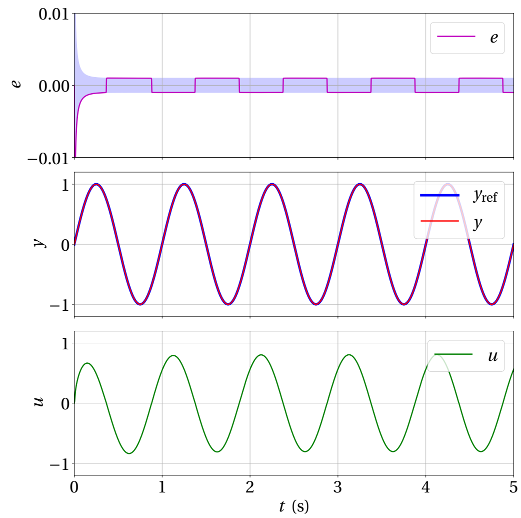

FIGURE 4. Performance funnel with the error, reference signal with the output of the closed-loop system and input of the closed-loop.

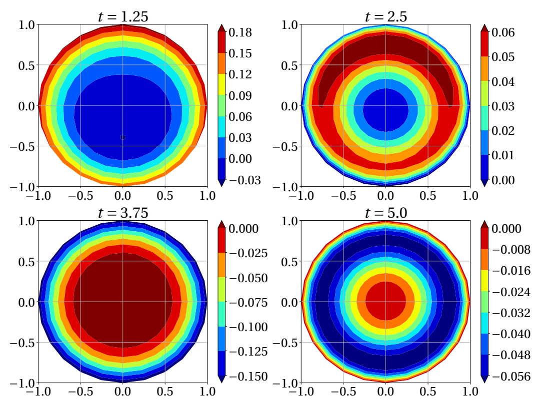

FIGURE 5. From left to right, top to bottom, the temperature of the plate for different increasing times.

REFERENCES

[1] R.A. Adams. *Sobolev Spaces*. Number 65 in Pure and Applied Mathematics. Academic press, New York, London, 1975.

[2] H.W. Alt. *Linear Functional Analysis*. Universitext. Springer-Verlag, London, 2016. An application-oriented introduction, Translated from the German edition by Robert Nürnberg.

[3] B. Augner. Well-posedness and stability of linear port-hamiltonian systems with nonlinear boundary feedback. Submitted for publication. Available at arXiv: https://arxiv.org/abs/1506.02921.

[4] B. Augner. *Stabilisation of Infinite-Dimensional Port-Hamiltonian Systems via Dissipative Boundary Feedback*. PhD thesis, Bergische Universität Wuppertal, 2016.

[5] B. Augner and B. Jacob. Stability and stabilization of infinite-dimensional linear port-hamiltonian systems. *Evol. Equ. Control The.*, 3(2):207—229, 2014.

[6] G. Bastin and J.-M. Coron. *Stability and Boundary Stabilization of 1-D Hyperbolic Systems*, volume 88 of *Progress in Nonlinear Differential Equations and Their Applications: Subseries in Control*. Birkhäuser, 2016.

[7] T. Berger, L. H. Hoang, and T. Reis. Funnel control for nonlinear systems with known strict relative degree. *Automatica*, 87:345–357, 2018.

[8] T. Berger, M. Puche, and F.L. Schwenninger. Funnel control for a moving water tank. Submitted for publication. Available at arXive: https://arxiv.org/abs/1902.00586.

[9] A. Cheng and K. Morris. Well-posedness of boundary control systems. *SIAM J. Control Optim.*, 2003.

[10] J. Diestel and J.J. Uhl. *Vector Measures*, volume 15 of *Mathematical surveys and monographs*. American Mathematical Society, Providence, RI, 1977.

[11] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer, New York, 2000.

[12] V. Girault and P.-A. Raviart. *Finite element approximation of the Navier-Stokes equations*, volume 749 of *Lecture Notes in Mathematics*. Springer, Berlin Heidelberg, 1979.

[13] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman Advanced Publishing Program, Boston, Londonm Melbourne, 1985.

[14] A. Ilchmann, E. Ryan, and P. Townsend. Tracking with prescribed transient behaviour for non-linear systems of known relative degree. *SIAM J. Control Optim.*, 46(1):210–230, 2007.

[15] B. Jacob and H. Zwart. *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces*, volume 223 of *Operator Theory: Advances and Applications*. Birkhäuser, Basel, 2012.

[16] T. Kato. Nonlinear semigroups and evolution equations. *J. Math. Soc. Japan*, 19(4), 1967.

[17] I. Miyadera. *Nonlinear Semigroups*, volume 109 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1992.

[18] J. Rauch. Symmetric positive systems with boundary characteristics of constant multiplicity. *Trans. Amer. Math. Soc*, 291(1):167—187, 1985.

[19] T. Reis and T. Selig. Funnel control for the boundary controlled heat equation. *SIAM J. Control Optim.*, 53(1):547—574, 2015.

[20] E. P. Ryan, A. Ilchmann, and C. J. Sangwin. Tracking with prescribed transient behaviour. *ESAIM: Control, Optimisation and Calculus of Variations*, 7:471–493, 2002.

[21] T. Selig, A. Ilchmann, and C. Trunk. The Byrnes–Isidori form for infinite-dimensional systems. *SIAM J. Control Optim.*, 54:1504–1534, 2016.

[22] R. E. Showalter. *Monotone operators in Banach space and nonlinear partial differential equations*. American Mathematical Society, Providence, RI, (1996).

[23] O. J. Staffans. *Well-Posed Linear Systems*. 103. Cambridge University Press, 2005.

[24] M. Tucsnak and G. Weiss. *Observation and Control for Operator Semigroups*. Birkhäuser Advanced Texts Basler Lehrbücher. Birkhäuser, Basel, 2009.

[25] J. A. Villegas. *A port-Hamiltonian approach to distributed parameter systems*. PhD thesis, University of Twente, 2007.

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