Sufficient conditions for unique global solutions in optimal control of semilinear equations with C^1-nonlinearity

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Dedicated to Günter Leugering on the occasion of his 65th birthday.

Abstract

We consider a semilinear elliptic optimal control problem possibly subject to control and/or state constraints. Generalizing previous work in [2] we provide a condition which guarantees that a solution of the necessary first order conditions is a global minimum. A similiar result also holds at the discrete level where the corresponding condition can be evaluated explicitly. Our investigations are motivated by Günter Leugering, who raised the question whether the problem class considered in [2] can be extended to the nonlinearity $\phi(s) = s|s|$. We develop a corresponding analysis and present several numerical test examples demonstrating its usefulness in practice.

1 Introduction and problem setting

Let $\Omega \subset \mathbb{R}^d\,(d = 2, 3)$ be a bounded, convex polygonal/polyhedral domain, in which we consider the semilinear elliptic PDE

$$-\Delta y + \phi(\cdot, y) = u \quad \text{in } \Omega, \tag{1.1}$$

$$y = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

We assume that $\phi : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with $\phi(x, 0) = 0$ a.e. in Ω and that

$$y \mapsto \phi(x, y) \text{ is of class } C^1 \text{ with } \phi_y(x, y) \geq 0 \text{ for almost all } x \in \Omega; \tag{1.3}$$

$$\forall L \geq 0\ \exists c_L \geq 0 \quad \phi_y(x, y) \leq c_L \quad \text{for almost all } x \in \Omega \text{ and all } |y| \leq L. \tag{1.4}$$

Under the above conditions it can be shown that for every $u \in L^2(\Omega)$ the boundary value problem (1.1), (1.2) has a unique solution $y =: \mathcal{G}(u) \in H^2(\Omega) \cap H^1_0(\Omega)$. Next, let us introduce

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$U_{ad} := \{v \in L^2(\Omega) : u_a \le v(x) \le u_b$ a.e. in $\Omega\}$, where $u_a, u_b \in \mathbb{R}$ with $-\infty \le u_a \le u_b \le \infty$. For given $y_0 \in L^2(\Omega)$, $\alpha > 0$ we then consider the optimal control problem

$$(\mathbb{P}) \qquad \begin{aligned} &\min_{u\in U_{ad}} J(u) := \tfrac{1}{2}\|y - y_0\|^2_{L^2(\Omega)} + \tfrac{\alpha}{2}\|u\|^2_{L^2(\Omega)} \\ &\text{subject to } y = \mathcal{G}(u) \text{ and } y_a(x) \le y(x) \le y_b(x) \text{ for all } x \in K. \end{aligned}$$

Here, $y_a, y_b \in C^0(\bar\Omega)$ satisfy $y_a(x) < y_b(x)$ for all $x \in K$, where $K \subset \bar\Omega$ is compact and either $K \subset \Omega$ or $K = \bar\Omega$. In the latter case we suppose in addition that $y_a(x) < 0 < y_b(x), x \in \partial\Omega$. It is well–known that $(\mathbb{P})$ has a solution provided that a feasible point exists (compare [5]). Under some constraint qualification, such as the linearized Slater condition, a local solution $\bar u \in U_{ad}$ of $(\mathbb{P})$ then satisfies the following necessary first order conditions, see [5, Theorem 5.2]: There exist $\bar p \in L^2(\Omega)$ and a regular Borel measure $\bar\mu \in \mathcal{M}(K)$ such that

$$\int_\Omega \nabla\bar y \cdot \nabla v + \phi(\cdot, \bar y)v\,dx = \int_\Omega \bar u v\,dx \quad \forall\, v \in H^1_0(\Omega), \qquad y_a \le \bar y \le y_b \text{ in } K, \tag{1.5}$$

$$\int_\Omega \bar p(-\Delta v) + \phi_y(\cdot, \bar y)\,\bar p\,v\,dx = \int_\Omega (\bar y - y_0)v\,dx + \int_K v\,d\bar\mu \quad \forall\, v \in H^1_0(\Omega) \cap H^2(\Omega), \tag{1.6}$$

$$\int_\Omega (\bar p + \alpha\bar u)(u - \bar u)\,dx \ge 0 \qquad \forall\, u \in U_{ad}, \tag{1.7}$$

$$\int_K (z - \bar y)\,d\bar\mu \le 0 \qquad \forall\, z \in C^0(K), y_a \le z \le y_b \text{ in } K. \tag{1.8}$$

In view of the nonlinearity of the state equation problem $(\mathbb{P})$ is in general nonconvex and hence there may be several solutions of the conditions (1.5)–(1.8). The problem we are interested in is whether it is possible to establish sufficient conditions which guarantee that a solution of (1.5)–(1.8) is actually a global minimum of $(\mathbb{P})$. A first result in this direction was obtained by the authors in [2] and holds for a class of nonlinearities which satisfy a certain growth condition:

Theorem 1.1. *([2, Theorem 3.2]) Let $d = 2$; suppose that $y \mapsto \phi(x, y)$ belongs to C^2 for almost all $x \in \Omega$ and that there exist $r > 1$ and $M \ge 0$ such that*

$$|\phi_{yy}(x, y)| \le M(\phi_y(x, y))^{\frac{1}{r}} \qquad \textit{for almost all } x \in \Omega \textit{ and all } y \in \mathbb{R}. \tag{1.9}$$

Assume that $(\bar u, \bar y, \bar p, \bar\mu)$ solves (1.5)–(1.8) and that

$$\|\bar p\|_{L^q} \le \Big(\frac{r-1}{2r-1}\Big)^{\frac{1-r}{r}} M^{-1} C_q^{\frac{2-2r}{r}} \alpha^{\frac{\rho}{2}} q^{1/q} r^{1/r} \rho^{\rho/2} (2-\rho)^{\frac{\rho}{2}-1}, \tag{1.10}$$

where $q := \frac{3r-2}{r-1}$, $\rho := \frac{r+q}{rq}$ and C_q denotes the constant in (2.6) below. Then $\bar u$ is a global minimum for Problem $(\mathbb{P})$. If the above inequality is strict, then $\bar u$ is the unique global minimum.

Assumption (1.9) is satisfied for $\phi_q(y) := |y|^{q-2}y$ provided that $q > 3$ if we choose $r = \frac{q-2}{q-3}$. Günter Leugering recently raised the question whether our theory can be extended to include the case $q = 3$. The corresponding nonlinearity $\phi_3(y) = |y|y$ appears for example in the mathematical modeling of gas flow through pipes with PDEs [16, (5.1)], so that an extension of Theorem 1.1 to this case could be helpful in understanding the optimal control of pipe networks. As ϕ_3 is no longer C^2 it does not fit directly into the theory above. However it turns out that instead the analysis can be built on the fact that $\phi_{3,y}$ satisfies a global Lipschitz condition.

The purpose of this paper is to generalize Theorem 1.1 in several directions. To begin, we shall replace (1.9) by a condition that can be formulated for C^1–nonlinearities ϕ and is satisfied by the functions ϕ_q for every $q \geq 3$ thus including the case suggested by Günter Leugering, see (2.4). A second generalization concerns the choice of the norm $\|\bar{p}\|_{L^q}$ in condition (1.10). Even though the integration index $q = \frac{3r-2}{r-1}$ is quite natural (solve $r = \frac{q-2}{q-3}$ for q), it is nevertheless possible to formulate a corresponding result not just for one index but for q belonging to a suitable interval, see (2.9), thus giving additional flexibility in its application. Our arguments are natural extensions of the analysis presented in [2] and will also cover the case $d = 3$ left out in Theorem 1.1.

There is a lot of literature available considering the problem ($\mathbb{P}$). For a broad overview, we refer the reader to the references of the respective citations. In [5] this problem is studied for boundary controls. The regularity of optimal controls of ($\mathbb{P}$) and their associated multipliers is investigated in [12] and [11]. Sufficient second order conditions are discussed in e.g. [9, 7, 8] when the set K contains finitely/infinitely many points. For the role of those conditions in PDE constrained optimization see e.g. [13].

The finite element discretization of problem ($\mathbb{P}$) in rather general settings is studied in [4, 10, 19]. Convergence rates for sets K containing only finitely many points are established in [23] for finite dimensional controls, and in [6] for control functions. Only in [27, 3] an error analysis is provided for general pointwise state constraints in K. Error analysis for linear-quadratic control problems can be found in e.g. [11], [14, 15] and [24]. Improved error estimates for the state in the case of weakly active state constraints are provided in [28]. A detailed discussion of discretization concepts and error analysis in PDE-constrained control problems can be found in [20, 21] and [17, Chapter 3].

The organization of the paper is as follows: in § 2 we shall develop the optimality conditions outlined above. In addition to the criteria based on an L^q–norm of $\bar{p}$ we shall also include a result that uses a sign of $\bar{p}$. The variational discretization of ($\mathbb{P}$) is considered in § 3 and is based on a finite element approximation of (1.1), (1.2) that uses numerical integration for the nonlinear term. We obtain corresponding optimality criteria for discrete stationary points and apply these conditions in a series of numerical tests in § 4 including the nonlinearity $\phi(y) = y|y|$.

2 Optimality conditions for ($\mathbb{P}$)

In what follows we assume that $(\bar{u}, \bar{y}, \bar{p}, \bar{\mu})$ is a solution of (1.5)–(1.8). Let $u \in U_{ad}$ be a feasible control, $y = \mathcal{G}(u)$ the associated state such that $y_a \leq y \leq y_b$ in K. A straightforward calculation shows that

$$J(u) - J(\bar{u}) = \frac{1}{2}\|y - \bar{y}\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u - \bar{u}\|^2_{L^2(\Omega)} + \alpha \int_\Omega \bar{u}(u - \bar{u})\, dx + \int_\Omega (\bar{y} - y_0)(y - \bar{y})\, dx. \quad (2.1)$$

Combining (1.6) for $v := y - \bar{y}$ with (1.8) and (1.1) we deduce that

$$\int_\Omega (\bar{y} - y_0)(y - \bar{y})\,dx = -\int_\Omega \bar{p}\,\Delta(y - \bar{y})\,dx + \int_\Omega \phi_y(\cdot,\bar{y})\,\bar{p}\,(y - \bar{y})\,dx - \int_K (y - \bar{y})d\bar{\mu}$$
$$\geq \int_\Omega (u - \bar{u})\bar{p}\,dx - \int_\Omega \big(\phi(\cdot,y) - \phi(\cdot,\bar{y}) - \phi_y(\cdot,\bar{y})(y - \bar{y})\big)\bar{p}\,dx.$$

Inserting this relation into (2.1) and recalling (1.7) we finally obtain

$$J(u) - J(\bar{u}) \geq \frac{1}{2}\|y - \bar{y}\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u - \bar{u}\|^2_{L^2(\Omega)} - R(u), \tag{2.2}$$

where

$$R(u) = \int_\Omega \big(\phi(\cdot,y) - \phi(\cdot,\bar{y}) - \phi_y(\cdot,\bar{y})(y - \bar{y})\big)\bar{p}\,dx. \tag{2.3}$$

2.1 Conditions involving a sign of $\bar{p}$

A natural first idea to deduce global optimality from (2.2) consists in identifying situations in which $R(u) \leq 0$ for all $u \in U_{ad}$. We have the following result:

Theorem 2.1. *Suppose that there exists an interval $I \subset \mathbb{R}$ such that $y \mapsto \phi(x,y)$ is convex (concave) on I for almost all $x \in \Omega$. Furthermore, assume that for every $u \in U_{ad}$ the solution $y = \mathcal{G}(u)$ with $y_a \leq y \leq y_b$ in K satisfies $y(x) \in I$ for all $x \in \Omega$. If $\bar{p} \leq 0$ ($\bar{p} \geq 0$) a.e. on Ω, then $\bar{u}$ is the unique global minimum of ($\mathbb{P}$).*

Proof. Suppose that $y \mapsto \phi(x,y)$ is convex. Then our assumptions imply that

$$\phi(x,y(x)) - \phi(x,\bar{y}(x)) - \phi_y(x,\bar{y}(x))(y(x) - \bar{y}(x)) \geq 0 \quad \text{for almost all } x \in \Omega$$

which yields that $R(u) \leq 0$ since $\bar{p} \leq 0$ a.e. in Ω. Hence $J(u) > J(\bar{u})$ for $u \neq \bar{u}$ by (2.2). ∎

In general we cannot expect the adjoint variable $\bar{p}$ to have a sign without additional conditions on the data of the problem. The following result is similar in spirit to a sufficient condition involving a suitable bound on y_0 obtained in [25, Theorem 5.4] and [22, Section 5.2] for the optimal control of the obstacle problem.

Lemma 2.2. *Suppose that $K = \emptyset$ and that $u_a = 0, u_b < \infty$. Let $y_b \in H^2(\Omega)$ satisfy*

$$-\Delta y_b + \phi(\cdot,y_b) \geq u_b \ \text{ in } \Omega, \quad y_b \geq 0 \ \text{ on } \partial\Omega.$$

Then $0 \leq \mathcal{G}(u) \leq y_b$ in $\bar{\Omega}$ for every $u \in U_{ad}$. Also, if $y_0 \geq y_b$ a.e. in Ω, then $\bar{p} \leq 0$ in Ω.

Proof. Let $u \in U_{ad}$ and set $y = \mathcal{G}(u)$. If we test (1.5) with $v = y^-$ we have

$$\int_\Omega |\nabla y^-|^2\,dx = -\int_\Omega \phi(\cdot,y^-)\,y^-\,dx + \int_\Omega u\,y^-\,dx \leq 0$$

using (1.3), the fact that $\phi(\cdot,0) = 0$ as well as $u \geq 0$. We infer that $y^- \equiv 0$ and hence $y \geq 0$ in $\bar{\Omega}$. Next, $y - y_b$ satisfies

$$-\Delta(y - y_b) + [\phi(\cdot,y) - \phi(\cdot,y_b)] \leq u - u_b \leq 0 \quad \text{a.e. in } \Omega.$$

Testing with $(y - y_b)^+$ then gives $y \leq y_b$ in $\bar{\Omega}$. Finally, since $K = \emptyset$, the adjoint state satisfies

$$-\Delta\bar{p} + \phi_y(\cdot,\bar{y})\bar{p} = \bar{y} - y_0 \leq y_b - y_0 \leq 0 \quad \text{a.e. in } \Omega$$

since $\bar{y} \leq y_b$ by what we have already shown. We infer that $\bar{p} \leq 0$ in a similar way as above. ∎

Example 2.3. Let $a \in L^\infty(\Omega)$ with $a \geq 0$ a.e. in Ω. Then the functions $\phi(x,y) = e^{a(x)y} - 1$ and $\phi(x,y) = a(x)|y|^{q-2}y$ $(q \geq 3)$ are convex on $\mathbb{R}$ and $[0,\infty)$ respectively. Hence if $K = \emptyset$ and u_a, u_b and y_0 are chosen as in Lemma 2.2, then Theorem 2.1 and Lemma 2.2 imply that a solution of the necessary first order conditions will be the unique global minimum of $(\mathbb{P})$.

2.2 Conditions involving a bound on $\|\bar{p}\|_{L^q}$

As mentioned above it will in general not be possible to establish a sign on the adjoint variable $\bar{p}$, so that one is left with trying to bound $|R(u)|$ in terms of $\frac{1}{2}\|y-\bar{y}\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u-\bar{u}\|^2_{L^2(\Omega)}$. In what follows we shall assume that there exists $\gamma \in [0,1)$ and $M \geq 0$ such that

$$\Big|\frac{\phi_y(x,y_2)-\phi_y(x,y_1)}{y_2-y_1}\Big| \leq M\Big(\frac{\phi(x,y_2)-\phi(x,y_1)}{y_2-y_1}\Big)^\gamma \tag{2.4}$$

for almost all $x \in \Omega$ and for all $y_1, y_2 \in \mathbb{R}, y_1 \neq y_2$. Note that (2.4) holds with $\gamma = 0$ if $y \mapsto \phi_y(x,y)$ is globally Lipschitz uniformly in $x \in \Omega$. Furthermore, it is not difficult to verify that (2.4) is satisfied with $\gamma = \frac{1}{r}$ provided that (1.9) holds.

Example 2.4. Let $\phi(x,y) = a(x)|y|^{q-2}y$, where $q \geq 3$ and $a \in L^\infty(\Omega)$ with $a(x) \geq 0$ a.e. in Ω. Then, ϕ satisfies (2.4) with $\gamma = \frac{q-3}{q-2}$ and $M = (q-2)(q-1)^{\frac{1}{q-2}}\|a\|^{\frac{1}{q-2}}_{L^\infty(\Omega)}$.

In what follows we shall make use of the elementary inequality (see e.g. [2, Lemma 7.1])

$$a^\lambda b^\mu \leq \frac{\lambda^\lambda \mu^\mu}{(\lambda+\mu)^{\lambda+\mu}}(a+b)^{\lambda+\mu}, \quad a,b \geq 0, \lambda,\mu > 0, \tag{2.5}$$

as well as of the Gagliardo–Nirenberg interpolation inequality

$$\|f\|_{L^q} \leq C_q \|f\|^{1-\theta}_{L^2}\|\nabla f\|^\theta_{L^2} \tag{2.6}$$

where $\theta = d(\frac{1}{2} - \frac{1}{q})$ and $2 \leq q < \infty$ if $d = 2$ and $2 \leq q \leq 6$ if $d = 3$. Explicit values for the constant C_q in (2.6) can e.g. be found in [26] and [29], see also [2, Theorem 7.3].
Before we state our main result we mention that it is well–known that $\bar{p} \in W^{1,s}_0(\Omega)$ for all $s \in [1, \frac{d}{d-1})$. In particular we infer with the help of a standard embedding result that

$$\bar{p} \in L^q(\Omega) \begin{cases} \text{for every } 1 \leq q < \infty & \text{if } d = 2; \\ \text{for every } 1 \leq q < 3 & \text{if } d = 3. \end{cases} \tag{2.7}$$

Furthermore, we have that

$$\bar{p} \in L^\infty(\Omega) \text{ if } K = \emptyset \text{ or } K = \bar{\Omega} \text{ with } y_a, y_b \in W^{2,\infty}(\Omega). \tag{2.8}$$

In order to see (2.8) we note that $\bar{p} \in H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ by elliptic regularity theory if $K = \emptyset$. On the other hand, if $K = \bar{\Omega}$ with $y_a, y_b \in W^{2,\infty}(\Omega)$ we may apply Theorem 3.1 and Section 4.2 in [11] to obtain that $\bar{p} \in L^\infty(\Omega)$.

Theorem 2.5. *Assume that ϕ satisfies (2.4) and let $(\bar{u},\bar{y},\bar{p},\bar{\mu}) \in U_{ad} \times (H^2(\Omega)\cap H_0^1(\Omega)) \times L^2(\Omega) \times \mathcal{M}(K)$ be a solution of (1.5)–(1.8). Furthermore, choose $q > 1$ such that*

$$\frac{1}{1-\gamma} < q < \infty \ \text{ if } d=2; \quad \frac{3}{2(1-\gamma)} \le q < 3 \ \text{ if } d=3 \tag{2.9}$$

and define for $t := \frac{2q(1-\gamma)}{q(1-\gamma)-1}$ and $\rho := \frac{d}{2q}+\gamma$ the quantity

$$\eta(\alpha,q,d) := \Big(\frac{1-\gamma}{2-\gamma}\Big)^{\gamma-1} M^{-1} C_t^{2(\gamma-1)} \alpha^{\frac{\rho}{2}} (\frac{d}{2q})^{-\frac{d}{2q}} \gamma^{-\gamma} (2-\rho)^{\frac{\rho}{2}-1} \rho^{\frac{\rho}{2}}, \tag{2.10}$$

where C_t is the constant in (2.6). If the inequality

$$\|\bar{p}\|_{L^q} \le \eta(\alpha,q,d) \tag{2.11}$$

is satisfied, then $\bar{u}$ is a global minimum for Problem $(\mathbb{P})$. If the inequality (2.11) is strict, then $\bar{u}$ is the unique global minimum. The assertions hold for $\frac{3}{2(1-\gamma)} \le q < \infty$ and $d = 3$ provided that $K = \emptyset$ or $K = \bar{\Omega}$ with $y_a, y_b \in W^{2,\infty}(\Omega)$.

Proof. To begin, note that (2.7) and (2.8) imply that $\bar{p} \in L^q(\Omega)$ for the cases that we consider. Our starting point is again (2.2) in which we write the remainder term as

$$R(u) = \int_\Omega \bar{p}(y-\bar{y}) \int_0^1 [\phi_y(\cdot,\bar{y}+t(y-\bar{y})) - \phi_y(\cdot,\bar{y})]dt\,dx. \tag{2.12}$$

We claim that for all $y_1, y_2 \in \mathbb{R}, y_1 \neq y_2$ we have

$$\begin{aligned} \Big| \int_0^1 [\phi_y(\cdot,y_1+t(y_2-y_1)) - \phi_y(\cdot,y_1)]dt \Big| & \\ \le\ L_\gamma |y_2-y_1|^{1-2\gamma} \big((\phi(\cdot,y_2)-\phi(\cdot,y_1))(y_2-y_1)\big)^\gamma, & \end{aligned} \tag{2.13}$$

where $L_\gamma = M\big(\frac{1-\gamma}{2-\gamma}\big)^{1-\gamma}$. To see this, let us suppress temporarily the dependence on x and introduce

$$\phi_\epsilon(y) := \int_\mathbb{R} \zeta_\epsilon(z)\,\phi(y-z)\,dz, \quad y \in \mathbb{R},$$

where $(\zeta_\epsilon)_{0<\epsilon<1} \subset C_0^\infty(\mathbb{R})$ is a sequence of mollifiers satisfying

$$\zeta_\epsilon \ge 0,\ \text{supp}\zeta_\epsilon \subset [-\epsilon,\epsilon],\ \text{and} \ \int_\mathbb{R} \zeta_\epsilon(z)\,dz = 1.$$

Since $\phi'_\epsilon(y) = \int_\mathbb{R} \zeta_\epsilon(z)\phi'(y-z)\,dz$ we have that

$$\phi''_\epsilon(y) = \lim_{h\to 0} \int_\mathbb{R} \zeta_\epsilon(z) \frac{\phi'(y+h-z)-\phi'(y-z)}{h}\,dz$$

so that we obtain with the help of (2.4) and Hölder's inequality

$$\begin{aligned} |\phi''_\epsilon(y)| &\le M \int_\mathbb{R} \zeta_\epsilon(z)(\phi'(y-z))^\gamma\,dz = M\int_\mathbb{R} (\zeta_\epsilon(z))^{1-\gamma}\big(\zeta_\epsilon(z)\,\phi'(y-z)\big)^\gamma\,dz \\ &\le M\Big(\int_\mathbb{R} \zeta_\epsilon(z)\phi'(y-z)\,dz\Big)^\gamma = M(\phi'_\epsilon(y))^\gamma. \end{aligned}$$

We may therefore apply Lemma 7.2 in [2] for $\gamma \in (0,1)$ to deduce that

$$\Big|\int_0^1 [\phi_\epsilon'(y_1+t(y_2-y_1))-\phi_\epsilon'(y_1)]dt\Big| \le L_\gamma |y_2-y_1|\Big(\int_0^1 \phi_\epsilon'(y_1+t(y_2-y_1))\,dt\Big)^\gamma,$$

but the above estimate easily extends to the case $\gamma = 0$. The bound (2.13) now follows by sending $\epsilon \to 0$. If we insert (2.13) into (2.12) we find that

$$\begin{aligned} |R(u)| &\le L_\gamma \int_\Omega |\bar p|\,|y-\bar y|^{2-2\gamma}\big((\phi(\cdot,y)-\phi(\cdot,\bar y))(y-\bar y)\big)^\gamma\,dx \\ &\le L_\gamma\,\|\bar p\|_{L^q}\|y-\bar y\|^{2(1-\gamma)}_{L^{2s(1-\gamma)}}\Big(\int_\Omega(\phi(\cdot,y)-\phi(\cdot,\bar y))(y-\bar y)dx\Big)^\gamma, \end{aligned} \tag{2.14}$$

where we have used Hölder's inequality with exponents $q, r = \frac{1}{\gamma}$ and $s = \frac{q}{q(1-\gamma)-1}$. Note that

$$2s(1-\gamma) = \frac{2q(1-\gamma)}{q(1-\gamma)-1} = t \in \begin{cases} (2,\infty), & \text{if } d=2; \\ (2,6], & \text{if } d=3 \end{cases}$$

in view of our assumptions on q. We may therefore use (2.6) in order to estimate $\|y-\bar y\|_{L^t}$ and obtain with

$$\theta = d\big(\frac{1}{2}-\frac{1}{t}\big) = \frac{d}{2q(1-\gamma)} \quad \text{and hence} \quad 2(1-\gamma)\theta = \frac{d}{q}$$

that

$$\begin{aligned} &|R(u)| \\ &\le L_\gamma\, C_t^{2(1-\gamma)}\|\bar p\|_{L^q}\|y-\bar y\|_{L^2}^{2(1-\gamma)-\frac{d}{q}}\|\nabla(y-\bar y)\|_{L^2}^{\frac{d}{q}}\Big(\int_\Omega(\phi(\cdot,y)-\phi(\cdot,\bar y))(y-\bar y)dx\Big)^\gamma. \end{aligned}$$

Applying (2.5) with $\lambda = \frac{d}{2q}$ and $\mu = \gamma$ and recalling that $\rho = \frac{d}{2q}+\gamma$ we may continue

$$\begin{aligned} |R(u)| &\le L_\gamma\, C_t^{2(1-\gamma)}\|\bar p\|_{L^q}\|y-\bar y\|_{L^2}^{2(1-\gamma)-\frac{d}{q}} \\ &\qquad \times\frac{\big(\frac{d}{2q}\big)^{\frac{d}{2q}}\gamma^\gamma}{\rho^\rho}\Big(\|\nabla(y-\bar y)\|_{L^2}^2+\int_\Omega(\phi(\cdot,y)-\phi(\cdot,\bar y))(y-\bar y)dx\Big)^\rho. \end{aligned}$$

If we take the difference of the PDEs satisfied by $\bar y$ and y and test it with $y-\bar y$ we easily deduce that

$$\|\nabla(y-\bar y)\|_{L^2}^2+\int_\Omega\big(\phi(\cdot,y)-\phi(\cdot,\bar y)\big)(y-\bar y)\,dx \le \|y-\bar y\|_{L^2}\|u-\bar u\|_{L^2},$$

which yields

$$\begin{aligned} |R(u)| &\le L_\gamma\, C_t^{2(1-\gamma)}\frac{\big(\frac{d}{2q}\big)^{\frac{d}{2q}}\gamma^\gamma}{\rho^\rho}\|\bar p\|_{L^q}\|y-\bar y\|_{L^2}^{2(1-\gamma)-\frac{d}{q}+\rho}\|u-\bar u\|_{L^2}^\rho \\ &= 2L_\gamma\, C_t^{2(1-\gamma)}\alpha^{-\frac{\rho}{2}}\frac{\big(\frac{d}{2q}\big)^{\frac{d}{2q}}\gamma^\gamma}{\rho^\rho}\|\bar p\|_{L^q}\big(\frac{1}{2}\|y-\bar y\|_{L^2}^2\big)^{1-\frac{\rho}{2}}\big(\frac{\alpha}{2}\|u-\bar u\|_{L^2}^2\big)^{\frac{\rho}{2}}. \end{aligned}$$

Using once more (2.5), this time with $\lambda = 1 - \frac{\rho}{2}, \mu = \frac{\rho}{2}$ we finally deduce that

$$\begin{aligned} |R(u)| &\leq 2L_\gamma\, C_t^{2(1-\gamma)} \alpha^{-\frac{\rho}{2}} \frac{\left(\frac{d}{2q}\right)^{\frac{d}{2q}} \gamma^\gamma}{\rho^\rho} \left(1 - \frac{\rho}{2}\right)^{1-\frac{\rho}{2}} \left(\frac{\rho}{2}\right)^{\frac{\rho}{2}} \|\bar{p}\|_{L^q} \left(\frac{1}{2}\|y - \bar{y}\|_{L^2}^2 + \frac{\alpha}{2}\|u - \bar{u}\|_{L^2}^2\right) \\ &= L_\gamma\, C_t^{2(1-\gamma)} \alpha^{-\frac{\rho}{2}} \left(\frac{d}{2q}\right)^{\frac{d}{2q}} \gamma^\gamma (2-\rho)^{1-\frac{\rho}{2}} \rho^{-\frac{\rho}{2}} \|\bar{p}\|_{L^q} \left(\frac{1}{2}\|y - \bar{y}\|_{L^2}^2 + \frac{\alpha}{2}\|u - \bar{u}\|_{L^2}^2\right). \end{aligned}$$

If we use this estimate in (2.2) and recall (2.10) as well as $L_\gamma = M\left(\frac{1-\gamma}{2-\gamma}\right)^{1-\gamma}$ we infer that $J(u) - J(\bar{u}) \geq 0$ provided that (2.11) holds, so that $\bar{u}$ is a global solution of problem $(\mathbb{P})$. If the inequality in (2.11) is strict, then $\bar{u}$ is the unique global minimum of problem $(\mathbb{P})$. ∎

Remark 2.6. Suppose that $d = 2$ and that ϕ satisfies (1.9) for some $r > 1, M \geq 0$, so that (2.4) holds with $\gamma = \frac{1}{r}$. If we set $q := \frac{3r-2}{r-1}$, then q satisfies (2.9) while $t = q$ and $\rho = \frac{1}{q} + \frac{1}{r} = \frac{r+q}{rq}$, so that Theorem 1.1 is a special case of Theorem 2.5.

3 Variational discretization

In this section we consider the case $d = 2$ and let $\mathcal{T}_h$ be an admissible triangulation of $\Omega \subset \mathbb{R}^2$. We introduce the following spaces of linear finite elements:

$$\begin{aligned} X_h &:= \{v_h \in C^0(\bar{\Omega}) : v_{h|T} \text{ is a linear polynomial on each } T \in \mathcal{T}_h\}, \\ X_{h0} &:= \{v_h \in X_h : v_{h|\partial\Omega} = 0\}. \end{aligned}$$

The Lagrange interpolation operator I_h is defined by

$$I_h : C^0(\bar{\Omega}) \to X_h, \qquad I_h y := \sum_{i=1}^n y(x_i)\phi_i,$$

where $x_1, \ldots, x_n$ denote the nodes in the triangulation $\mathcal{T}_h$ and $\{\phi_1, \ldots, \phi_n\}$ is the set of basis functions of the space X_h which satisfy $\phi_i(x_j) = \delta_{ij}$. We discretize (1.1), (1.2) using numerical integration for the nonlinear part: for a given $u \in L^2(\Omega)$, find $y_h \in X_{h0}$ such that

$$\int_\Omega \nabla y_h \cdot \nabla v_h + I_h[\phi(\cdot, y_h)v_h]\, dx = \int_\Omega u v_h\, dx \quad \forall\, v_h \in X_{h0}. \tag{3.1}$$

Using the monotonicity of $y \mapsto \phi(\cdot, y)$ and the Brouwer fixed-point theorem one can show that (3.1) admits a unique solution $y_h =: \mathcal{G}_h(u) \in X_{h0}$. The variational discretization (see [18]) of Problem $(\mathbb{P})$ then reads:

$$(\mathbb{P}_h) \qquad \begin{aligned} &\min\nolimits_{u \in U_{ad}} J_h(u) := \tfrac{1}{2}\|y_h - y_0\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|u\|_{L^2(\Omega)}^2 \\ &\text{subject to } y_h = \mathcal{G}_h(u),\ y_a(x_j) \leq y_h(x_j) \leq y_b(x_j),\ x_j \in \mathcal{N}_h, \end{aligned}$$

where $\mathcal{N}_h := \{x_j \,|\, x_j \text{ is a node of } T \in \mathcal{T}_h,\ \text{such that } T \cap K \neq \emptyset\}$. It can be shown that $(\mathbb{P}_h)$ has a solution, provided that a feasible point exists. In practice, candidates for solutions are calculated by solving the system of necessary first order conditions which reads: find

$\bar{u}_h \in U_{ad}, \bar{y}_h \in X_{h0}, \bar{p}_h \in X_{h0}, \bar{\mu}_j \in \mathbb{R}, x_j \in \mathcal{N}_h$ such that $y_a(x_j) \le y_h(x_j) \le y_b(x_j), x_j \in \mathcal{N}_h$ and

$$\int_\Omega \nabla \bar{y}_h \cdot \nabla v_h + I_h[\phi(\cdot, \bar{y}_h) v_h] \, dx = \int_\Omega \bar{u}_h v_h \, dx \quad \forall \, v_h \in X_{h0}, \tag{3.2}$$

$$\int_\Omega \nabla \bar{p}_h \cdot \nabla v_h + I_h[\phi_y(\cdot, \bar{y}_h) \bar{p}_h v_h] \, dx = \int_\Omega (\bar{y}_h - y_0) v_h \, dx + \sum_{x_j \in \mathcal{N}_h} \bar{\mu}_j v_h(x_j) \; \forall \, v_h \in X_{h0}, \tag{3.3}$$

$$\int_\Omega (\bar{p}_h + \alpha \bar{u}_h)(u - \bar{u}_h) \, dx \ge 0 \qquad \forall \, u \in U_{ad}, \tag{3.4}$$

$$\sum_{x_j \in \mathcal{N}_h} \bar{\mu}_j (y_j - \bar{y}_h(x_j)) \le 0 \qquad \forall \, (y_j)_{x_j \in \mathcal{N}_h}, y_a(x_j) \le y_j \le y_b(x_j), x_j \in \mathcal{N}_h. \tag{3.5}$$

In order to formulate the analogue of Theorem 2.5 we introduce the following h–dependent norm on X_h:

$$\| v_h \|_{h,q} := \Big(\int_\Omega I_h[|v_h|^q] dx \Big)^{\frac{1}{q}}, \quad v_h \in X_h, \; 1 \le q < \infty.$$

Theorem 3.1. *Suppose that ϕ and $q > 1$ satisfy the conditions (2.4) and (2.9) respectively and let $\bar{u}_h \in U_{ad}$, $\bar{y}_h \in X_{h0}$, $\bar{p}_h \in X_{h0}$, $(\bar{\mu}_j)_{x_j \in \mathcal{N}_h}$ be a solution of (3.2)–(3.5). If*

$$\| \bar{p}_h \|_{h,q} \le \Big(\frac{1}{4}\Big)^{1-\gamma-\frac{1}{q}} \eta(\alpha, q, 2), \tag{3.6}$$

then $\bar{u}_h$ is a global minimum for Problem $(\mathbb{P}_h)$. If the inequality (3.6) is strict, then $\bar{u}_h$ is the unique global minimum.

Proof. Just as in the continuous case we obtain for $u \in U_{ad}$ with $y_h = \mathcal{G}_h(u)$

$$J_h(u) - J_h(\bar{u}_h) \ge \frac{1}{2} \| y_h - \bar{y}_h \|^2_{L^2(\Omega)} + \frac{\alpha}{2} \| u - \bar{u}_h \|^2_{L^2(\Omega)} - R_h(u), \tag{3.7}$$

where

$$\begin{aligned} R_h(u) &= \int_\Omega I_h \left[\big(\phi(\cdot, y_h) - \phi(\cdot, \bar{y}_h) - \phi_y(\cdot, \bar{y}_h)(y_h - \bar{y}_h) \big) \bar{p}_h \right] \, dx \\ &= \int_\Omega I_h \left[\bar{p}_h (y_h - \bar{y}_h) \int_0^1 (\phi_y(\cdot, \bar{y}_h + t(y_h - \bar{y}_h)) - \phi_y(\cdot, \bar{y}_h)) dt \right] dx. \end{aligned} \tag{3.8}$$

If we use (2.13) then we obtain as above with the help of Hölder's inequality

$$\begin{aligned} |R_h(u)| &\le M 2^{\gamma - 1} \int_\Omega I_h \left[|\bar{p}_h| \, |y_h - \bar{y}_h|^{2-2\gamma} \big((\phi(\cdot, y_h) - \phi(\cdot, \bar{y}_h))(y_h - \bar{y}_h) \big)^\gamma \right] \, dx \\ &\le M 2^{\gamma-1} \| \bar{p}_h \|_{h,q} \| y_h - \bar{y}_h \|^{2(1-\gamma)}_{h, 2s(1-\gamma)} \Big(\int_\Omega I_h \left[(\phi(\cdot, y_h) - \phi(\cdot, \bar{y}_h))(y_h - \bar{y}_h) \right] dx \Big)^\gamma, \end{aligned}$$

where $s = \frac{q}{q(1-\gamma)-1}$. Applying Lemma 5.1 in the Appendix we derive

$$|R_h(u)| \le M 2^{\gamma-1} 4^{\frac{1}{s}} \| \bar{p}_h \|_{h,q} \| y_h - \bar{y}_h \|^{2(1-\gamma)}_{L^{2s(1-\gamma)}} \Big(\int_\Omega I_h \left[(\phi(\cdot, y_h) - \phi(\cdot, \bar{y}_h))(y_h - \bar{y}_h) \right] dx \Big)^\gamma,$$

which is the analogue of (2.14). The rest of the proof now follows in the same way as in Theorem 2.5, where we use (3.1) instead of the PDEs. ∎

We shall investigate condition (3.6) for different choices of ϕ and q in the numerics section. From the numerical analysis point of view it is also possible to examine the convergence of a sequence of solutions $(\bar{u}_h, \bar{y}_h, \bar{p}_h, (\bar{\mu}_j)_{x_j \in \mathcal{N}_h})_{0<h<h_0}$ of (3.2)–(3.5) that satisfy (3.6) uniformly in h. Based on Theorem 1.1, convergence in $L^2(\Omega)$ of $(\bar{u}_h)_{0<h<h_0}$ to a solution $\bar{u}$ of $(\mathbb{P})$ has been obtained in [2, Theorem 4.2], while an error estimate is proved in [1, 3]. We expect that these results carry over to the generalized framework considered in this paper. In this context we also refer to [27] as a further contribution to the error analysis for optimal control of semilinear equations with pointwise bounds on the state. Contrary to our approach this work is based on second order sufficient optimality conditions for a local solution of the control problem and requires in particular a C^2–nonlinearity ϕ.

4 Numerical experiments

In this section we conduct several numerical experiments related to Theorem 3.1. We consider $(\mathbb{P})$ with different choices for the nonlinearity ϕ. For each choice we fix $\Omega := (0,1) \times (0,1)$, while for the desired state y_0 we consider the following two scenarios:

A1: (Reachable desired state) $y_0(x) := 2\sin(2\pi x_1)\sin(2\pi x_2)$.
A2: (Not reachable desired state) $y_0(x) := 60 + 160(x_1(x_1 - 1) + x_2(x_2 - 1))$.

For the control and state bounds we consider these three cases:

Case 1: (Unconstrained problem) $u_b = -u_a = \infty$, $K = \emptyset$.
Case 2: (Control constrained problem) $u_b = -u_a = 5$, $K = \emptyset$.
Case 3: (State constrained problem) $u_b = -u_a = \infty$, $K = \bar{\Omega}, y_b \equiv -y_a \equiv 1$.

For α we report numerical results for the values $\alpha = 10^i$, $i = -6, -5, \ldots, 3$. The domain Ω is partitioned using a uniform triangulation with mesh size $h = 2^{-5}\sqrt{2}$, and the discrete counterpart of the problem is as in Section 3. The resulting discrete optimality system (3.2)–(3.5) is solved using the semismooth Newton method.

Example 4.1. We consider $\phi(y) := y|y|$. Then, $\gamma = 0$ with $M = 2$. Taking $q = 2$, the condition reads

$$\|\bar{p}_h\|_{h,2} \leq \frac{1}{2}\,\eta(\alpha, 2, 2)$$

with

$$C_4^{-2} \approx 2.381297723376159.$$

The results are reported in Figure 1. We see that in the light of Theorem 3.1, the unique global solution of the considered control problem has been computed for all given values of α, except for case 2 when $\alpha \leq 10^{-3}$. There, no conclusion can be derived. However, with the coefficient $a(x) := \frac{1}{8}$ we obtain a global unique solution for the whole considered parameter range, see Fig. 2.

Example 4.2. We consider $\phi(y) := y^3$. Then, $\gamma = 0.5$ with $M = 2\sqrt{3}$. Taking $q = 3$, the condition reads

$$\|\bar{p}_h\|_{L^3(\Omega)} \leq \eta(\alpha, 3, 2)$$

with

$$C_6^{-1} \approx 1.616080082127768.$$

The choice of $q = 3$ is motivated by fact that among the possible choices of the Gagliardo-Nirenberg constant the value of C_6 is among the smallest possible ones, see [2, Figure 4]. The integrals involving ϕ, and the norm $\|\bar{p}_h\|_{L^3(\Omega)}$ are computed exactly. The results are reported in Figure 3. We for comparison also include the results for $q = 4$ which correspond to the findings of [2, Example 2]. As one can see this choice in some situations delivers larger uniqueness intervalls for α. Overall, uniqueness of the global solution can be deduced for certain ranges of the parameter α, where it is more likely in the case of a reachable desired state y_0.

Example 4.3. We consider $\phi(y) := y^5$. Then, $\gamma = 3/4$ with $M = 4 \times (5)^{1/4}$. Taking $q = 6$, the condition reads

$$\|\bar{p}_h\|_{L^6(\Omega)} \leq \eta(\alpha, 6, 2)$$

with

$$C_6^{-1/2} \approx 1.271251384316953.$$

The choice of $q = 6$ is motivated as in the previous example. This then is the situation of [2, Example 3]. For comparison we also include the results obtained with quadrature based on the estimate (3.6). As one can see the differences in both approaches (exact integration versus quadrature) is negligible. The results are reported in Figure 4.

5 Appendix

Lemma 5.1. *Let $d = 2$ and $2 \leq q < \infty$. Then*

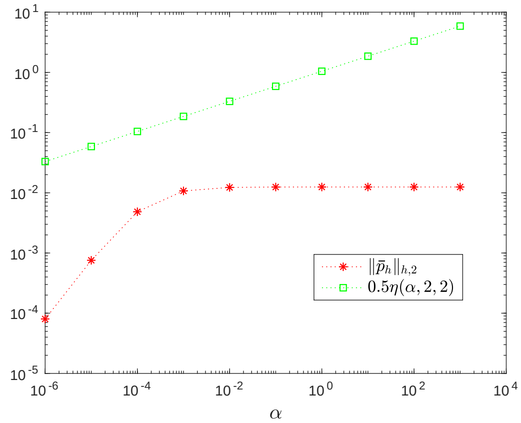
$$\|v_h\|_{L^q} \leq \|v_h\|_{h,q} \leq 4^{\frac{1}{q}} \|v_h\|_{L^q} \quad \textit{for all } v_h \in X_h.$$

Proof. Let us denote by $\hat{T} \subset \mathbb{R}^2$ the unit simplex with vertices $\hat{a}_0 = (0,0), \hat{a}_1 = (1,0)$ and $\hat{a}_2 = (0,1)$. Using a scaling argument it is sufficient to show that

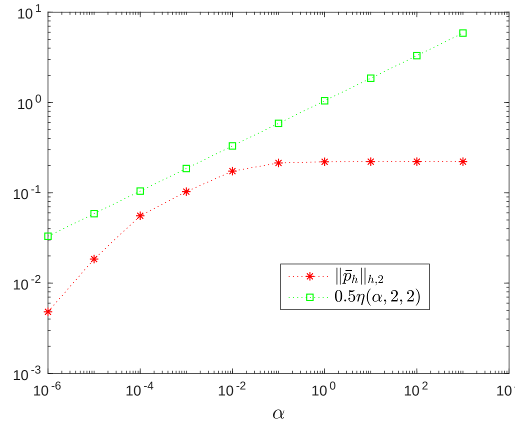
$$\int_{\hat{T}} |p|^q \, d\hat{x} \leq \int_{\hat{T}} \hat{I}_h[|p|^q] d\hat{x} \leq 4 \int_{\hat{T}} |p|^q \, d\hat{x} \quad \text{for all } p \in P_1(\hat{T}), \tag{5.1}$$

where $\hat{I}_h f = \sum_{j=0}^{2} f(\hat{a}_j)\hat{\phi}_j$ and $\hat{\phi}_j(\hat{a}_i) = \delta_{ij}$. In order to see the first inequality in (5.1) we observe that

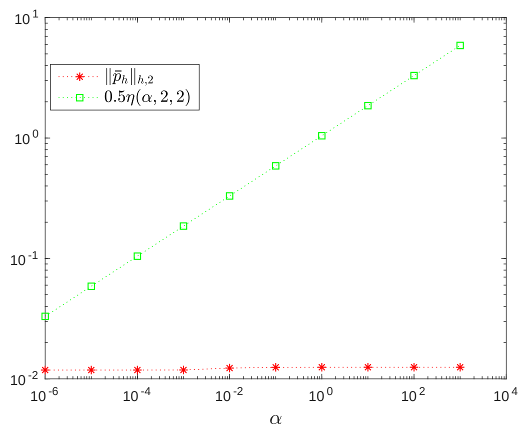
$$\int_{\hat{T}} |p|^q \, d\hat{x} = \int_{\hat{T}} |\sum_{j=0}^{2} p(\hat{a}_j)\hat{\phi}_j|^q d\hat{x} \leq \int_{\hat{T}} \sum_{j=0}^{2} |p(\hat{a}_j)|^q \hat{\phi}_j \, d\hat{x} = \int_{\hat{T}} \hat{I}_h[|p|^q] d\hat{x}$$



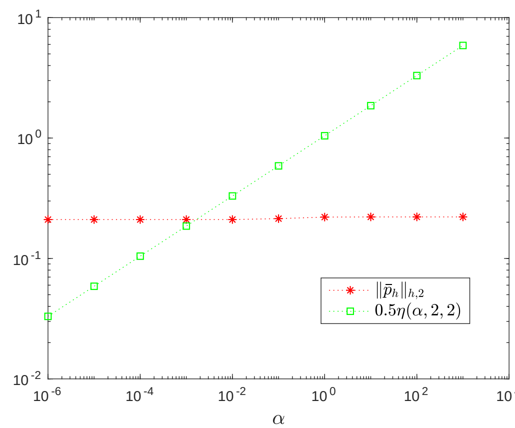
(a) Case 1 with A1



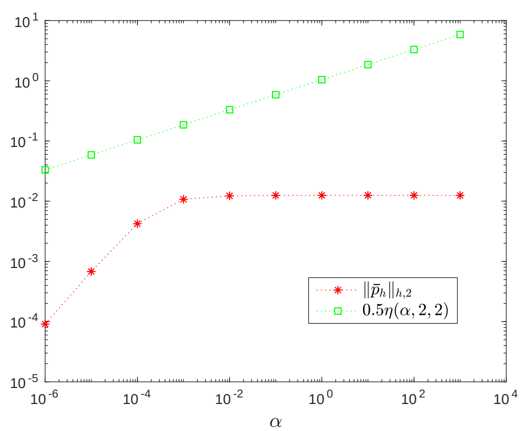
(b) Case 1 with A2



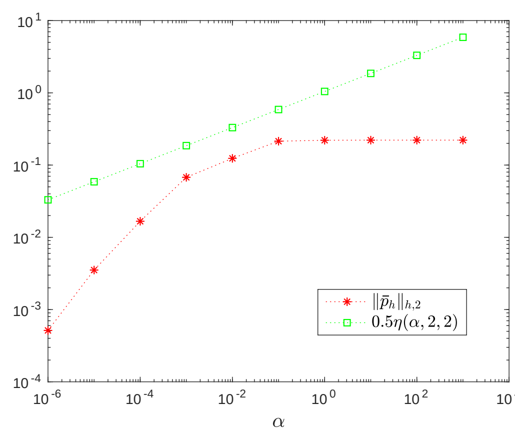
(c) Case 2 with A1



(d) Case 2 with A2



(e) Case 3 with A1



(f) Case 3 with A2

Figure 1: Results for $\phi(s) = s|s|$

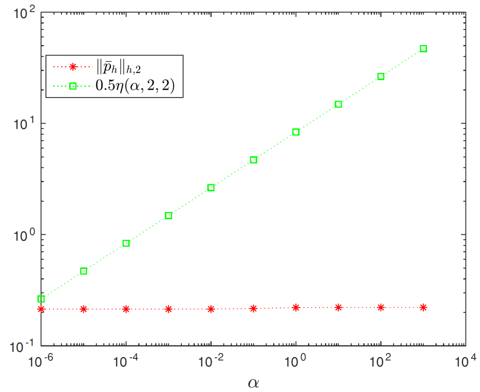


Figure 2: Case 2 with A2 for $\phi(s) = \frac{1}{8}s|s|$

in view of the convexity of $t \mapsto |t|^q$ and the properties of $\hat{\phi}_j, j = 0, 1, 2$. Let us next consider the remaining estimate and first focus on the case $q = 2$. A straightforward calculation shows that

$$\int_{\hat{T}} \hat{I}_h[|p|^2] d\hat{x} = \frac{1}{6} \sum_{j=0}^{2} |p(\hat{a}_j)|^2, \quad \int_{\hat{T}} |p|^2 \, d\hat{x} = \frac{1}{24} \sum_{j=0}^{2} |p(\hat{a}_j)|^2 + \frac{1}{24} |p(\frac{\hat{a}_0 + \hat{a}_1 + \hat{a}_2}{3})|^2,$$

which implies that

$$\int_{\hat{T}} \hat{I}_h[|p|^2] d\hat{x} \leq 4 \int_{\hat{T}} |p|^2 \, d\hat{x}. \tag{5.2}$$

Let us introduce the measure $\mu := \sum_{j=0}^{2} m_j \delta_{\hat{a}_j}$ with $m_j = \int_{\hat{T}} \hat{\phi}_j d\hat{x} = \frac{1}{6}, j = 0, 1, 2$. Clearly,

$$\|p\|^q_{L^q(\mu)} := \int_{\hat{T}} |p|^q d\mu = \sum_{j=0}^{2} |p(\hat{a}_j)|^q m_j = \int_{\hat{T}} \hat{I}_h[|p|^q] d\hat{x}.$$

Now, (5.2) yields that $\|p\|_{L^2(\mu)} \leq 2\|p\|_{L^2(d\hat{x})}$, while $\|p\|_{L^\infty(\mu)} \leq \|p\|_{L^\infty(d\hat{x})}$, so that the Riesz–Thorin convexity theorem implies that

$$\|p\|_{L^q(\mu)} \leq 2^{\frac{2}{q}} \|p\|_{L^q(d\hat{x})} \qquad \text{for all } p \in P_1(\hat{T}),$$

which is (5.1). ∎

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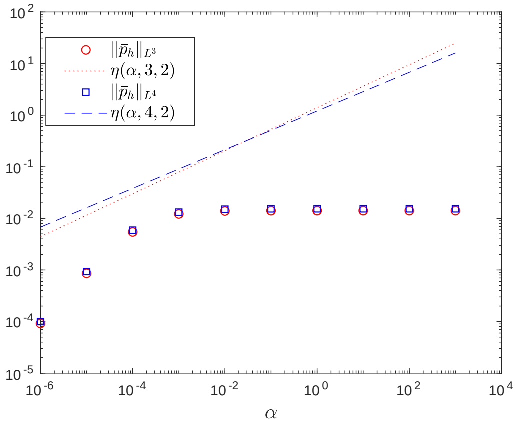
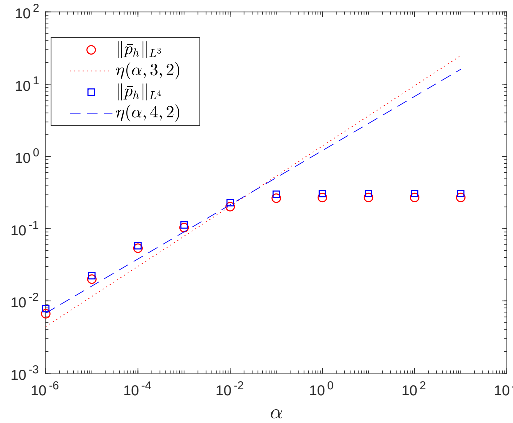
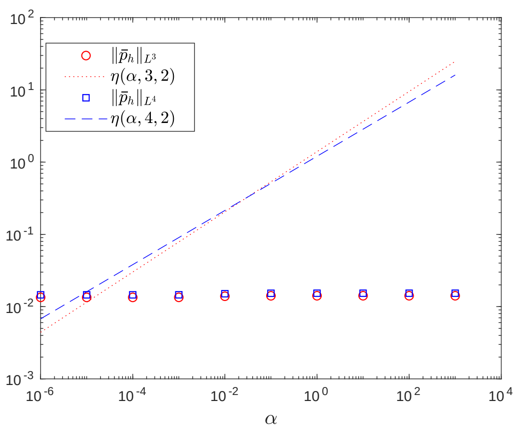


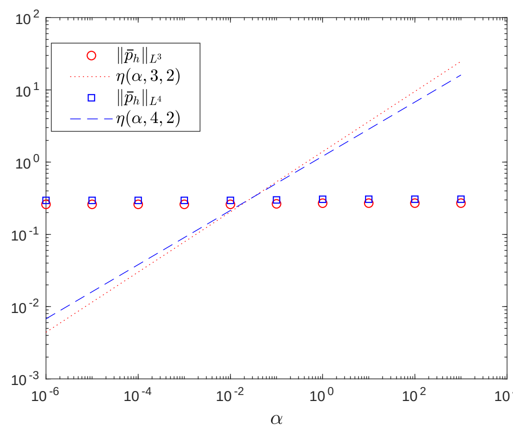
(a) Case 1 with A1



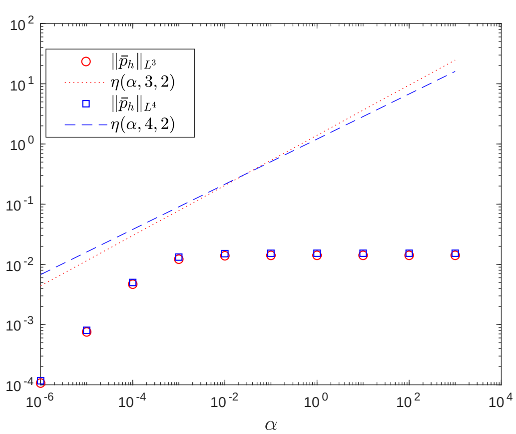
(b) Case 1 with A2



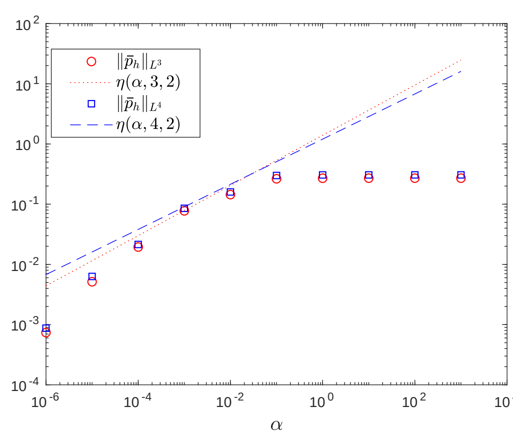
(c) Case 2 with A1



(d) Case 2 with A2

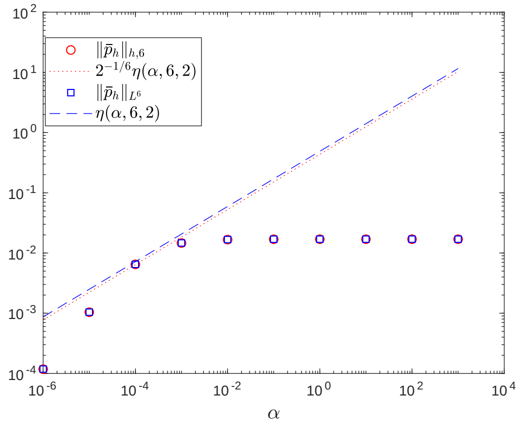


(e) Case 3 with A1

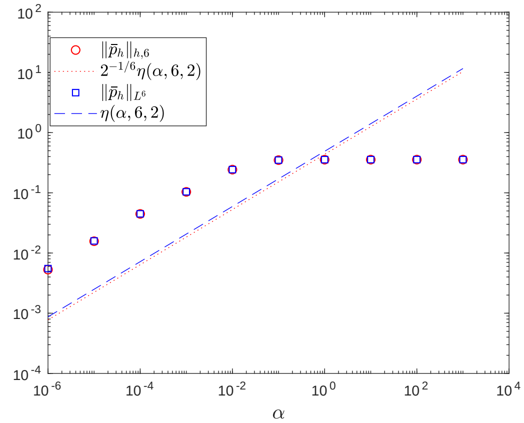


(f) Case 3 with A2

Figure 3: Results for $\phi(s) = s^3$



(a) Case 1 with A1



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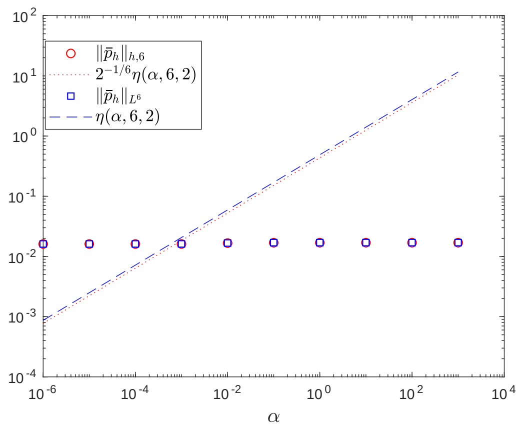
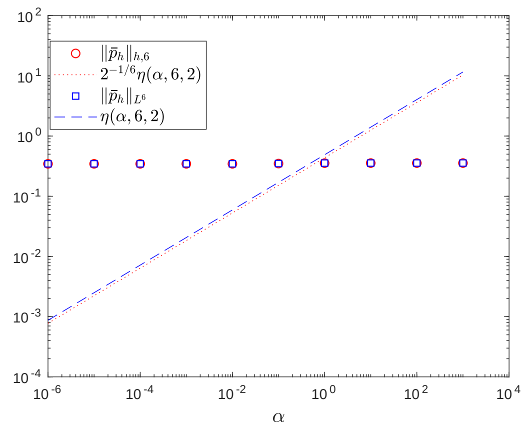
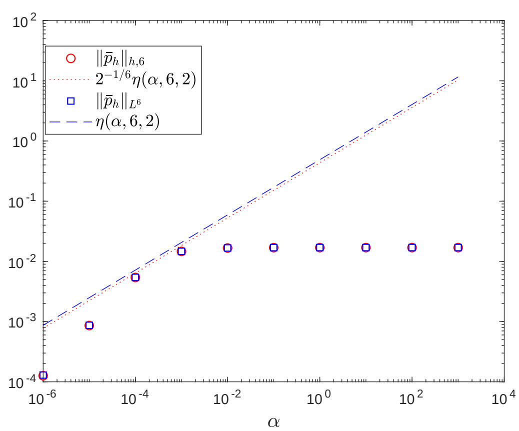


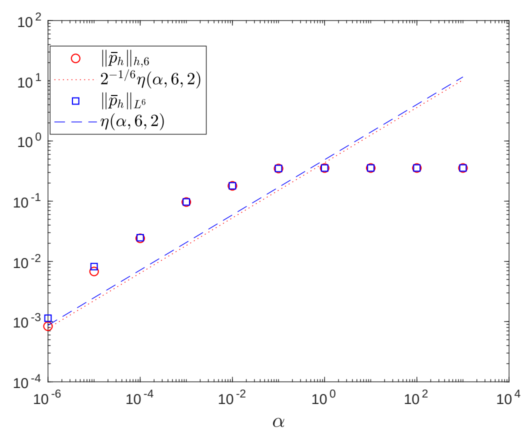
(c) Case 2 with A1



(d) Case 2 with A2



(e) Case 3 with A1



(f) Case 3 with A2

Figure 4: Results for $\phi(s) = s^5$

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