Iterated total variation regularization with finite element methods for reconstruction the source term in elliptic systems

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Abstract: In this paper we investigate the problem of recovering the source f in the elliptic system

$$\begin{aligned} -\nabla \cdot \big(\alpha \nabla u\big) + \beta u &= f \text{ in } \Omega, \\ \alpha \nabla u \cdot \vec{n} + \sigma u &= j \text{ on } \partial\Omega \end{aligned}$$

from an observation z of the state u on a part Γ of the boundary $\partial\Omega$, where the functions α, β, σ and j are given. For the particular interest in reconstructing probably discontinuous sources, we use the standard least squares method with the total variation regularization, i.e. we consider a minimizer of the minimization problem

$$\min_{f \in F_{ad}} J(f), \quad J(f) := \frac{1}{2}\|u(f) - z\|^2_\Gamma + \rho TV(f) \tag{$\mathcal{P}$}$$

as reconstruction. Here $u(f)$ denotes the unique weak solution of the above elliptic system which depends on the source term f, $TV(f)$ is the total variation of f, and $\rho > 0$ is the regularization parameter. Let u^h be the approximation of u in the finite dimensional space of piecewise linear, continuous finite elements. We then consider the discrete regularized problem corresponding to $(\mathcal{P})$, i.e. the following minimization problem

$$\min_{f \in F^h_{ad}} J^h(f), \quad J^h(f) := \frac{1}{2}\|u^h(f) - z\|^2_\Gamma + \rho TV(f). \tag{$\mathcal{P}^h$}$$

We show the stability and the convergence of solutions to $\big(\mathcal{P}^h\big)$. Furthermore, based on the Bartels' work [SIAM J. Numer. Anal. (2012)], we have proposed an algorithm to stably solve the minimization problem $\big(\mathcal{P}^h\big)$. We prove the iteration sequence $\big(f^h_n\big)_n$ generated by the derived algorithm conversing to a minimizer of $\big(\mathcal{P}^h\big)$, and a convergence measurement of the kind

$$\|f^h_{n+1} - f^h_n\| = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$$

is also established. Finally, a numerical experiment is presented to illustrate our theoretical findings.

Key words and phrases: Inverse source problem, total variation regularization, ill-posedness, finite element method, stability and convergence, elliptic boundary value problem.

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1 Introduction

Let Ω be an open bounded connected domain of $\mathbb{R}^d$, $d \ge 2$ with the polygonal boundary $\partial\Omega$. In this paper we investigate the inverse problem of identifying the *source term* $f \in L^2(\Omega)$ in the elliptic system

$$\begin{aligned} -\nabla \cdot \big(\alpha \nabla u\big) + \beta u &= f \quad \text{in} \quad \Omega, \\ \alpha \nabla u \cdot \vec{n} + \sigma u &= j \quad \text{on} \quad \partial\Omega \end{aligned} \tag{1.1}$$

from a boundary measurement $z \in L^2(\Gamma)$ of the exact data $g^\dagger := u_{|\Gamma}$, where $\vec{n}$ is the unit outward normal on $\partial\Omega$ and $\Gamma \subset \partial\Omega$ is a relative open subset of the boundary.

In the system (1.1) the Robin boundary condition $j \in H^{-1/2}(\partial\Omega) := \big(H^{1/2}(\partial\Omega)\big)^*$ and special functions α, β, σ are assumed to be given, where $\sigma \in L^\infty(\partial\Omega)$ with $\sigma(x) \ge 0$ a.e. on $\partial\Omega$, $\beta \in L^\infty(\Omega)$ with $\beta(x) \ge 0$

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a.e. in Ω and $\alpha := (\alpha_{rs})_{1\le r,s\le d} \in L^\infty(\Omega)^{d\times d}$ is a symmetric diffusion matrix satisfying the uniformly elliptic condition

$$\alpha(x)\xi\cdot\xi = \sum_{1\le r,s\le d} \alpha_{rs}(x)\xi_r\xi_s \ge \underline{\alpha}|\xi|^2 \text{ a.e. in } \Omega$$

for all $\xi = (\xi_r)_{1\le r\le d} \in \mathbb{R}^d$ with some constant $\underline{\alpha} > 0$.

We start with some notations. The expressions $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_{\partial\Omega}$ stand respectively for the scalar products of the Lebesgue spaces $L^2(\Omega)$ and $L^2(\partial\Omega)$, while $[\cdot,\cdot]_{\partial\Omega}$ denotes the dual pair $(\cdot,\cdot)_{\left(H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega)\right)}$. Let

$$a(u,v) := (\alpha\nabla u,\nabla v)_\Omega + (\beta u,v)_\Omega + (\sigma u,v)_{\partial\Omega} \quad\text{and}\quad l^f_j(v) := (f,v)_\Omega + [j,v]_{\partial\Omega}, \tag{1.2}$$

where $u,v\in H^1(\Omega)$.

We first assume that $\inf_\Omega \beta(x) + \inf_{\partial\Omega}\sigma(x) > 0$. Then, there exist positive constants c_1, c_2 such that

$$c_1\|u\|^2_{1,\Omega} \le a(u,u) \le c_2\|u\|^2_{1,\Omega} \tag{1.3}$$

for all $u\in H^1(\Omega)$, where $\|u\|_{1,\Omega} := \left((\nabla u,\nabla u)_\Omega + (u,u)_\Omega\right)^{1/2}$ denotes the usual norm of the Sobolev space $H^1(\Omega)$. Thus, the expression $a(u,v)$ generates a scalar product on the space $H^1(\Omega)$ equivalent to the usual one. Consequently, for each $f\in L^2(\Omega)$ the Robin boundary value problem (1.1) defines a unique weak solution u in the sense that $u := u_j(f) := u(f) \in H^1(\Omega)$ such that the variational equation

$$a(u,v) = l^f_j(v) \tag{1.4}$$

is satisfied for all $v\in H^1(\Omega)$. Furthermore, the estimate

$$\|u\|_{1,\Omega} \le \frac{1}{c_1}\max\left(1, \|\gamma\|_{\mathcal{L}\left(H^1(\Omega),H^{1/2}(\partial\Omega)\right)}\right)\left(\|j\|_{H^{-1/2}(\partial\Omega)} + \|f\|_\Omega\right) \tag{1.5}$$

holds true, where $\|f\|_\Omega := \|f\|_{L^2(\Omega)}$ and $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ denotes the continuous Dirichlet trace operator. Recall that $\gamma : H^1(\Omega)\to L^2(\partial\Omega)$ is compact (cf., e.g., [24, 27]) and there is a positive constant c_γ such that

$$\|\gamma u\|_{\partial\Omega} \le c_\gamma\|u\|_{1,\Omega} \tag{1.6}$$

for all $u\in H^1(\Omega)$.

In case $\beta = 0$ and $\sigma = 0$, by the Poincaré-Friedrichs inequality (cf., e.g., [28])

$$C_\Omega(v,v)_\Omega \le (\nabla v,\nabla v)_\Omega \quad\text{for all}\quad v\in H^1_\diamond(\Omega) := \left\{u\in H^1(\Omega) \mid (u,1)_\Omega = 0\right\}$$

for some constant $C_\Omega > 0$ depending only on Ω, the expression $(\alpha\nabla u,\nabla v)_\Omega$ is a scalar product on the space $H^1_\diamond(\Omega)$ which is equivalent to the usual one. Therefore, in view of Riesz's representation theorem, for each $f\in L^2(\Omega)$ the problem (1.1) also has a unique weak solution $u := u_j(f) := u(f) \in H^1_\diamond(\Omega)$ which is defined via the variational equation $(\alpha\nabla u,\nabla v)_\Omega = l^f_j(v)$ for all $v\in H^1_\diamond(\Omega)$ and further satisfied the estimate $\|u\|_{1,\Omega} \le C\left(\|j\|_{H^{-1/2}(\partial\Omega)} + \|f\|_\Omega\right)$. In the subsequent we thus consider the case $\inf_\Omega\beta(x) + \inf_{\partial\Omega}\sigma(x) > 0$ only. All results in present paper are still valid for the case $\beta = \sigma = 0$.

The inverse problem is stated as follows:

Given a boundary measurement $z\in L^2(\Gamma)$ of the exact $g^\dagger := u_{|\Gamma}$ of (1.1), find $f\in L^2(\Omega)$.

For this purpose and with the particular interest in reconstructing probably discontinuous sources, we use the standard least squares method with the total variation regularization, i.e. we consider a minimizer of the minimization problem

$$\min_{f\in F_{ad}} J(f), \quad J(f) := \frac{1}{2}\|u(f) - z\|^2_\Gamma + \rho TV(f) \tag{$\mathcal{P}$}$$

as reconstruction. Here $\rho \in (0,1)$ is the regularization parameter and

$$F_{ad} := \left\{ f \in BV(\Omega) \mid -\infty < \underline{f} \le f(x) \le \overline{f} < \infty \quad \text{for a.e. in} \quad \Omega \right\}$$

with $\underline{f}$, $\overline{f}$ being given constants. We mention that the admissible set F_{ad} is acted under the constraint constants $\underline{f}$, $\overline{f}$ ensures the existence of a minimizer to $(\mathcal{P})$ (cf. [1, 13, 35]).

Let u^h be the approximation of u in the finite dimensional space $\mathcal{V}_1^h := \left\{ v^h \in C(\overline{\Omega}) \mid {v^h}_{|T} \in \mathcal{P}_1(T),\ \forall T \in \mathcal{T}^h \right\}$ of piecewise linear, continuous finite elements. We then consider the discrete regularized problem corresponding to $(\mathcal{P})$, i.e. the following minimization problem

$$\min_{f \in F_{ad}^h} J^h(f), \quad J^h(f) := \frac{1}{2}\|u^h(f) - z\|_\Gamma^2 + \rho TV(f), \tag{$\mathcal{P}^h$}$$

where $F_{ad}^h := F_{ad} \cap \mathcal{V}_1^h$. Likewise $(\mathcal{P})$, we can show that the problem $\left(\mathcal{P}^h\right)$ attains a minimizer f^h for each $h > 0$. However, due to the lack of strict convexity of the cost functional, a solution of $\left(\mathcal{P}^h\right)$ may be non-unique.

As ρ is fixed, we show that every solution sequence $(f^h)_h$ of the problems $\left(\mathcal{P}^h\right)_h$ has a subsequence which converges to a solution of the problem $(\mathcal{P})$, in $L^s(\Omega)$-norm as well as in the total variation, where $\forall s \in [1,\infty)$ (see §3, Theorem 3.1). Furthermore, in case $h \to 0$, $\|g^\dagger - z\|_\Gamma \le \delta_z \to 0$ and $\rho = \rho(h, \delta_z)$ is chosen in a suitable way, a sequence of solutions to $\left(\mathcal{P}^h\right)_h$ converges to a sought source which has the total variation-minimizing property (see §3, Theorem 3.5).

Basing on the above derived results, and following a series of Bartels' very interesting works [7, 8, 9] we would like to present an algorithm to approximate solutions of $\left(\mathcal{P}^h\right)$. For this purpose we first introduce the piecewise constant finite element space which is denoted by $\mathcal{V}_0^h := \left\{ v^h \in L^1(\Omega) \mid {v^h}_{|T} = \text{ constant},\ \forall T \in \mathcal{T}^h \right\}$ and an initial iteration $\mu_0^h := (f_0^h, p_0^h) \in F_{ad}^h \times (\mathcal{V}_0^h)^d$. Then the iteration process is generated as follows

$$f_{n+1}^h = \arg\min_{f^h \in F_{ad}^h} \left\{ \left(f^h, u_*^h(f_n^h)\right)_\Omega + \rho\left(\nabla f^h, p_n^h\right)_\Omega + \frac{1}{2\tau}\|f^h - f_n^h\|_\Omega^2 \right\} \tag{1.7}$$

$$\widetilde{f}_{n+1}^h = 2f_{n+1}^h - f_n^h \tag{1.8}$$

$$p_{n+1}^h = \arg\max_{p^h \in (\mathcal{V}_0^h)^d} \left\{ \rho\left(\nabla \widetilde{f}_{n+1}^h, p^h\right)_\Omega - I_{\mathcal{B}_1\left((\mathcal{V}_0^h)^d\right)}(p^h) - \frac{\theta}{2\tau}\|p^h - p_n^h\|_\Omega^2 \right\} \tag{1.9}$$

where the parameters τ, θ are chosen suitably, and u_*^h is the discrete adjoint state of u^h. We mention that the solution p_{n+1}^h to the sub-problem (1.9) is given by the explicit form (see, e.g., [7])

$$p_{n+1}^h = \frac{p_n^h + \frac{\tau\rho}{\theta}\nabla \widetilde{f}_{n+1}^h}{\max\{1, |p_n^h + \frac{\tau\rho}{\theta}\nabla \widetilde{f}_{n+1}^h|\}}.$$

while the sub-problem (1.7) attains a solution

$$f_{n+1}^h = \mathcal{P}_{F_{ad}^h}^{L^2}\left(f_n^h - \tau(u_*^h(f_n^h) - \rho \mathrm{div}\ p_n^h)\right),$$

where $\mathcal{P}_{F_{ad}^h}^{L^2} : L^2(\Omega) \to F_{ad}^h$ denotes the L^2-projection on the set F_{ad}^h and $-\mathrm{div}$ is the adjoint operator of ∇ defined by

$$(-\mathrm{div}\ q^h, g^h) = (q^h, \nabla g^h)_\Omega := \sum_{T \in \mathcal{T}^h} \sum_{i=1}^d |T| {q_i^h}_{|T} \frac{\partial g^h}{x_i}_{|T}$$

for all $q^h \in (\mathcal{V}_0^h)^d$ and $g^h \in \mathcal{V}_1^h$. We then show in §4 that the iteration sequence $\left(\mu_n^h\right)_n := \left(f_n^h, p_n^h\right)_n$ has a subsequence $\left(\mu_{n_k}^h\right)_k$ converges in the finite dimensional space $\mathcal{V}_1^h \times (\mathcal{V}_0^h)^d$ to an element $\mu_*^h := (f_*^h, p_*^h) \in F_{ad}^h \times \partial TV(f_*^h)$, where μ_*^h satisfies the first-order optimality condition for the considered minimization problem and f_*^h is therefore a minimizer of $(\mathcal{P}^h)$. Moreover, we derive the identity

$$\|\mu_{n+1}^h - \mu_n^h\|_{\mathcal{V}_1^h \times (\mathcal{V}_0^h)^d} = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right).$$

The source identification problem in PDEs is a mathematical model in different topics of applied sciences such as electroencephalography, geophysical prospecting and pollutant detection which attracted great attention of many scientists in the last decades. For a review on this subject we may consult in [2, 6, 12, 15, 18, 22, 32] and the references therein. Let us briefly refer to related publications that have been concerned with the identification of the source term. In [16, 25, 26] authors have used the dual reciprocity boundary element methods to simulate numerically for some source identification problems. In case some priori knowledge of the identified source is available, such as a point source, a characteristic function or a harmonic function, numerical methods treating the problem have been obtained in [4, 5, 10, 23, 30, 34]. Recently, by using a so-called energy function combining with the Tikhonov regularization we in [19] investigated a numerical method for the source identification problem from a single noisy measurement couple of the Neumann and Dirichlet data. A survey of the problem of simultaneously identifying the source term and coefficients in elliptic systems from *distributed* observations can be found in [29].

Finally, for the sake of completeness we briefly introduce the space of functions with bounded total variation; for more details one may consult [3, 17]. A scalar function $f \in L^1(\Omega)$ is said to be of bounded total variation if

$$TV(f) := \int_\Omega |\nabla f| := \sup \left\{ \int_\Omega f \mathrm{div}\, \Xi \ \Big|\ \Xi \in C_c^1(\Omega)^d,\ |\Xi(x)|_\infty \le 1,\ x \in \Omega \right\} < \infty.$$

Here $|\cdot|_\infty$ denotes the ℓ_∞-norm on $\mathbb{R}^d$ defined by $|x|_\infty = \max\limits_{1\le i\le d} |x_i|$ and $C_c^1(\Omega)$ the space of continuously differentiable functions with compact support in Ω. The space of all functions in $L^1(\Omega)$ with bounded total variation is denoted by

$$BV(\Omega) = \left\{ f \in L^1(\Omega) \ \Big|\ \int_\Omega |\nabla f| < \infty \right\}$$

which is a *non-reflexive* Banach space with the norm

$$\|f\|_{BV(\Omega)} := \|f\|_{L^1(\Omega)} + \int_\Omega |\nabla f|.$$

Furthermore, if Ω is an open bounded set with Lipschitz boundary, then $W^{1,1}(\Omega) \subsetneq BV(\Omega)$.

2 Preliminaries

Now we summarize some useful properties of the coefficient-to-solution operator $u = u(f)$. First, we note that the decomposition

$$u(f) = \overline{u}(f) + \widehat{u} \tag{2.1}$$

holds, where $\overline{u}(f)$ and $\widehat{u}$ are the solutions to the variational equations

$$a(\overline{u}(f), v) = l_0^f(v) \quad \text{and} \quad a(\widehat{u}, v) = l_j^0(v) \tag{2.2}$$

for all $v \in H^1(\Omega)$, respectively. Thus, the operator $u : L^2(\Omega) \to H^1(\Omega)$ is continuously Fréchet differentiable on $L^2(\Omega)$. For each $f \in L^2(\Omega)$ the action of the Fréchet derivative in the direction $\xi \in L^2(\Omega)$ satisfies the equation

$$u'(f)\xi = \overline{u}(\xi). \tag{2.3}$$

Next, along with (1.1) we consider the *adjoint* problem

$$\begin{aligned} -\nabla \cdot \big(\alpha \nabla u_*\big) + \beta u_* &= 0 \quad \text{in} \quad \Omega, \\ \alpha \nabla u_* \cdot \vec{n} + \sigma u_* &= u(f) - z \quad \text{on} \quad \Gamma, \\ \alpha \nabla u_* \cdot \vec{n} + \sigma u_* &= 0 \quad \text{on} \quad \partial\Omega \setminus \Gamma \end{aligned} \tag{2.4}$$

which also attains a unique weak solution $u_*(f) \in H^1(\Omega)$ defined via the variational equation

$$a(u_*(f), v) = (u(f) - z, v)_\Gamma \tag{2.5}$$

for all $v \in H^1(\Omega)$. Furthermore, for f, $\xi \in L^2(\Omega)$ the Fréchet differential $u_*'(f)\xi$ is the unique weak solution in $H^1(\Omega)$ of the variational equation

$$a\left(u_*'(f)\xi, v\right) = (\overline{u}(\xi), v)_\Gamma \tag{2.6}$$

for all $v \in H^1(\Omega)$.

Now, for all $f \in L^2(\Omega)$ letting

$$L(f) := \frac{1}{2}\|u(f) - z\|_\Gamma^2,$$

a computation for all $\xi \in L^2(\Omega)$ shows, by (2.5) and (2.3),

$$L'(f)\xi = (u(f) - z, u'(f)\xi)_\Gamma = a(u_*(f), u'(f)\xi) = a(u_*(f), \overline{u}(\xi))$$

and so, by (2.6),

$$L''(f)(\xi,\xi) = a(u_*'(f)\xi, \overline{u}(\xi)) = (\overline{u}(\xi), \overline{u}(\xi))_\Gamma \geq 0. \tag{2.7}$$

This means that the cost function J of the problem $(\mathcal{P})$ is convex. However, it may be a *non-strictly convex* and *non-differentiable* function, since the regularization term TV is a semi-norm of the space $BV(\Omega)$ only.

Lemma 2.1 ([17])**.** *(i) Let $(f_n)_n$ be a bounded sequence in the $BV(\Omega)$-norm. Then a subsequence which is denoted by the same symbol and an element $f \in BV(\Omega)$ exist such that $(f_n)_n$ converges to f in the $L^1(\Omega)$-norm.*

(ii) Let $(f_n)_n$ be a sequence in $BV(\Omega)$ converging to f in the $L^1(\Omega)$-norm. Then $f \in BV(\Omega)$ and

$$TV(f) \leq \liminf_{n\to\infty} TV(f_n). \tag{2.8}$$

Lemma 2.2. *Assume that the sequence $(f_n)_n \subset F_{ad}$ converges to an element f in the $L^1(\Omega)$-norm. Then $f \in F_{ad}$, the sequence $((u(f_n))_n$ converges to $u(f)$ in the $H^1(\Omega)$-norm and $\left((u(f_n)_{|\Gamma}\right)_n$ converges to $u(f)_{|\Gamma}$ in the $L^2(\Gamma)$-norm as well.*

Proof. First, by Lemma 2.1, we have $f \in BV(\Omega)$. Furthermore, a subsequence $(f_{n_m})_m$ exists such that

$$\lim_{m\to\infty} |f_{n_m}(x) - f(x)| = 0 \quad \text{for a.e. in} \quad \Omega. \tag{2.9}$$

Since

$$\underline{f} \leq f_{n_m}(x) \leq \overline{f} \quad \text{for all} \quad m \in \mathbb{N} \quad \text{and a.e. in} \quad \Omega, \tag{2.10}$$

sending m to ∞, we get $\underline{f} \leq f(x) \leq \overline{f}$ for a.e. in Ω which implies that $f \in F_{ad}$. Now, for all $v \in H^1(\Omega)$ and $m \in \mathbb{N}$ we have from (1.4) that

$$a\left(u(f_{n_m}) - u(f), v\right) = (f_{n_m} - f, v)_\Omega$$

which together with (1.3) yield

$$c_1 \left\|u(f_{n_m}) - u(f)\right\|_{1,\Omega} \leq \|f_{n_m} - f\|_\Omega \leq \sqrt{2\overline{f}}\|f_{n_m} - f\|_{L^1(\Omega)}^{1/2}$$

which follows $\lim_{m\to\infty} \left\|u(f_{n_m}) - u(f)\right\|_{1,\Omega} = 0$. Since the limit is unique, the whole sequence $((u(f_n))_n$ converges to $u(f)$ in the $H^1(\Omega)$-norm also. This completes the proof. □

Lemma 2.3. *The minimization problem $(\mathcal{P})$ attains a minimizer.*

Proof. The assertion follows directly from Lemma 2.1 and Lemma 2.2, therefore omitted here (see [1] for details). □

3 Discretization and convergence

Let $\left(\mathcal{T}^h\right)_{0<h<1}$ be a family of regular and quasi-uniform triangulations of the domain $\overline{\Omega}$ with the mesh size h such that each vertex of the polygonal boundary $\partial\Omega$ is a node of $\mathcal{T}^h$ (cf. [11, 14]). For the definition of the discretization space of the state functions let us denote the piecewise affine, globally continuous finite element space by

$$\mathcal{V}_1^h := \left\{v^h \in C(\overline{\Omega}) \mid {v^h}_{|T} \in \mathcal{P}_1(T),\ \forall T \in \mathcal{T}^h\right\},$$

where $\mathcal{P}_1$ consists all polynomial functions of degree less than or equal to 1. The piecewise constant finite element space is denoted by

$$\mathcal{V}_0^h := \left\{v^h \in L^1(\Omega) \mid {v^h}_{|T} = \text{ constant},\ \forall T \in \mathcal{T}^h\right\}.$$

The admissible set is discretized as

$$F_{ad}^h := F_{ad} \cap \mathcal{V}_1^h.$$

Similar to the continuous case, we for each $f \in L^2(\Omega)$ have that the equation

$$a(u^h, v^h) = l_j^f(v^h) \quad \text{for all} \quad v^h \in \mathcal{V}_1^h \tag{3.1}$$

admits a unique solution $u^h := u_j^h(f) := u^h(f) \in \mathcal{V}_1^h$ which further satisfies the estimate

$$\|u^h\|_{H^1(\Omega)} \le C\left(\|f\|_\Omega + \|j\|_{H^{-1/2}(\partial\Omega)}\right), \tag{3.2}$$

where the constant C is independent of the mesh size h.

Also, the adjoint state $u_* = u_*(f)$ in (2.4) has the discrete version $u_*^h = u_*^h(f)$ which is defined via the equation

$$a(u_*^h, v^h) = (u^h(f) - z, v^h)_\Gamma \tag{3.3}$$

for all $v^h \in \mathcal{V}_1^h$.

Then the minimization $(\mathcal{P})$ can be discretized by

$$\min_{f \in F_{ad}^h} J^h(f), \quad J^h(f) := \frac{1}{2}\|u^h(f) - z\|_\Gamma^2 + \rho TV(f). \tag{$\mathcal{P}^h$}$$

which attains a minimizer f^h for each $h > 0$.

We now state the following result on the stability of finite element approximations. Here and in the sequel, unless otherwise stated, we indicate by C a generic positive constant which is independent of the mesh size h, the regularization parameter ρ and the observation data z.

Theorem 3.1. *Let $(h_n)_n$ be a sequence with $\lim_{n\to\infty} h_n = 0$ and ρ be a fixed regularization parameter. Let $f^{h_n} \in F_{ad}^{h_n}$ be a minimizer of $\left(\mathcal{P}^{h_n}\right)$ for each $n \in \mathbb{N}$. Then a subsequence of $\left(f^{h_n}\right)_n$ not relabelled and an element $f \in F_{ad}$ exist such that*

$$\lim_{n\to\infty} \left\|f^{h_n} - f\right\|_{L^1(\Omega)} = 0 \quad \text{and} \quad \lim_{n\to\infty} TV(f^{h_n}) = TV(f). \tag{3.4}$$

Furthermore, f is a solution to $(\mathcal{P})$. If the uniqueness of the solution to $(\mathcal{P})$ is satisfied, then convergence (3.4) holds for the whole sequence.

We mention that (3.4) yields the convergence in the $L^s(\Omega)$-norm for all $1 \le s < \infty$. In fact, this assertion follows directly from the following estimate

$$\|f^{h_n} - f\|_{L^s(\Omega)}^s = \int_\Omega |f^{h_n} - f| \cdot |f^{h_n} - f|^{s-1} \le \int_\Omega |f^{h_n} - f| \cdot \left(|f^{h_n}| + |f|\right)^{s-1} \le \left(2\overline{f}\right)^{s-1} \|f^{h_n} - f\|_{L^1(\Omega)}.$$

To prove Theorem 3.1 we need the following crucial result.

Lemma 3.2. *[20, Lemma 4.6.] For any fixed $\widehat{f} \in F_{ad}$ an element $\widehat{f}^h \in F^h_{ad}$ exists such that*

$$\left\|\widehat{f}^h - \widehat{f}\right\|_{L^1(\Omega)} \le Ch|\log h| \tag{3.5}$$

and

$$\lim_{h\to 0} TV(\widehat{f}^h) = TV(\widehat{f}). \tag{3.6}$$

Lemma 3.3. *Assume that $(f_n)_n \subset F_{ad}$ converges to $f \in F_{ad}$ in the $L^1(\Omega)$-norm. Then the limit*

$$\lim_{n\to\infty} \|u^{h_n}(f_n) - z\|_\Gamma = \|u(f) - z\|_\Gamma \tag{3.7}$$

holds true.

Proof. The proof is based on standard arguments, it is therefore omitted here. □

Proof of Theorem 3.1. Let $\widehat{f} \in F_{ad}$ be arbitrary and $\widehat{f}^{h_n}$ be generated from $\widehat{f}$ due to Lemma 3.2. The optimality of f^{h_n}, yields

$$\frac{1}{2}\|u^{h_n}(f^{h_n}) - z\|^2_\Gamma + \rho TV(f^{h_n}) \le \frac{1}{2}\|u^{h_n}(\widehat{f}^{h_n}) - z\|^2_\Gamma + \rho TV(\widehat{f}^{h_n}).$$

By (3.2) and (3.6), we deduce from the last inequality that the sequence $\left(f^{h_n}\right)_n$ is bounded in the $BV(\Omega)$-norm. Then, due to Lemma 2.1, a subsequence not relabelled and an element $f \in F_{ad}$ exist such that

$$\lim_{n\to\infty} \left\|f^{h_n} - f\right\|_{L^1(\Omega)} = 0 \quad \text{and} \quad TV(f) \le \liminf_{n\to\infty} TV(f^{h_n}). \tag{3.8}$$

Combining this with Lemma 3.3, we arrive at

$$\begin{aligned}
\frac{1}{2}\|u(f) - z\|^2_\Gamma + \rho TV(f) &\le \lim_{n\to\infty} \frac{1}{2}\|u^{h_n}(f^{h_n}) - z\|^2_\Gamma + \liminf_{n\to\infty} \rho TV(f^{h_n}) \\
&= \liminf_{n\to\infty} \left(\frac{1}{2}\|u^{h_n}(f^{h_n}) - z\|^2_\Gamma + \rho TV(f^{h_n})\right) \\
&\le \limsup_{n\to\infty} \left(\frac{1}{2}\|u^{h_n}(f^{h_n}) - z\|^2_\Gamma + \rho TV(f^{h_n})\right) \\
&\le \limsup_{n\to\infty} \left(\frac{1}{2}\|u^{h_n}(\widehat{f}^{h_n}) - z\|^2_\Gamma + \rho TV(\widehat{f}^{h_n})\right) \\
&= \frac{1}{2}\|u(\widehat{f}) - z\|^2_\Gamma + \rho TV(\widehat{f}),
\end{aligned} \tag{3.9}$$

here we used Lemma 3.2 (with noting $\lim_{n\to\infty} h_n|\log h_n| = 0$) and Lemma 3.3 in the last equation, again. Thus, f is a solution to $(\mathcal{P})$. Furthermore, replacing $\widehat{f}$ in (3.9) by f, we obtain

$$\begin{aligned}
\frac{1}{2}\|u(f) - z\|^2_\Gamma + \rho TV(f) &= \limsup_{n\to\infty} \left(\frac{1}{2}\|u^{h_n}(f^{h_n}) - z\|^2_\Gamma + \rho TV(f^{h_n})\right) \\
&= \lim_{n\to\infty} \frac{1}{2}\|u^{h_n}(f^{h_n}) - z\|^2_\Gamma + \limsup_{n\to\infty} \rho TV(f^{h_n}) \\
&= \frac{1}{2}\|u(f) - z\|^2_\Gamma + \limsup_{n\to\infty} \rho TV(f^{h_n})
\end{aligned}$$

and, together with (3.8), arrive at

$$TV(f) \le \liminf_{n\to\infty} TV(f^{h_n}) \le \limsup_{n\to\infty} TV(f^{h_n}) = TV(f),$$

and so that $TV(f) = \lim_{n\to\infty} TV(f^{h_n})$. The proof is completed. □

In the remain part of this section we investigate the convergence of discrete regularized approximations to an identified source as ρ is chosen in a suitable way depending on the mesh size and the error level of observations. Before doing so, we introduce the notion of the total variation-minimizing solution of the identification problem.

Lemma 3.4. *The problem*

$$\min_{f\in\mathcal{I}(g^\dagger):=\left\{f\in F_{ad}\;\middle|\;u(f)_{|\Gamma}=g^\dagger\right\}} TV(f) \tag{$\mathcal{IP}$}$$

attains a solution, called the total variation-minimizing solution of the identification problem.

Proof. The assertion follows by stand arguments, therefore omitted here. □

Let $f \in L^2(\Omega)$ be fixed, we denote by

$$\chi_f^h := \left\|u^h(f) - u(f)\right\|_{1,\Omega}. \tag{3.10}$$

Then (cf. [11, 14]),

$$\lim_{h\to 0}\chi_f^h = 0 \quad \text{and} \quad 0 \le \chi_f^h \le Ch \quad \text{if} \quad u(f)\in H^2(\Omega). \tag{3.11}$$

We now show the convergence of finite element approximations to the identification problem.

Theorem 3.5. *Assume that* $\lim_{n\to\infty} h_n = 0$ *and* (δ_n) *and* (ρ_n) *be any positive sequences such that*

$$\rho_n \to 0, \quad \frac{\delta_n}{\sqrt{\rho_n}} \to 0, \quad \frac{h_n|\log h_n|}{\rho_n} \to 0 \quad \textit{and} \quad \frac{\chi_{\widehat{f}}^{h_n}}{\sqrt{\rho_n}} \to 0 \quad \textit{as} \quad n\to\infty, \tag{3.12}$$

where $\widehat{f} \in \mathcal{I}\left(g^\dagger\right)$ *is arbitrary and* $\chi_{\widehat{f}}^{h_n}$ *is defined by* (3.10)*. Furthermore, assume that* $(z_n) \subset L^2(\Gamma)$ *is a sequence satisfying*

$$\left\|z_n - g^\dagger\right\|_\Gamma \le \delta_n$$

and f_n *is an arbitrary minimizer of the problem*

$$\min_{f\in F_{ad}^{h_n}} \frac{1}{2}\|u^{h_n}(f) - z_n\|_\Gamma^2 + \rho_n TV(f)$$

for each $n \in N$*. Then, a subsequence of* $(f_n)_n$ *not relabelled and a total variation-minimizing solution* f_* *of the identification problem* $(\mathcal{IP})$ *exist such that*

$$\lim_{n\to\infty}\|f_n - f_*\|_{L^1(\Omega)} = 0 \quad \textit{and} \quad \lim_{n\to\infty} TV(f_n) = TV(f_*). \tag{3.13}$$

Furthermore, the discrete state sequence $\left(u^{h_n}(f_n)\right)_n$ *converges in the* $H^1(\Omega)$*-norm to the unique weak solution* $u(f_*)$ *of the boundary value problem* (1.1)*.*

Proof. By the definition of f_n, we have

$$\frac{1}{2}\left\|u^{h_n}(f_n) - z_n\right\|_\Gamma^2 + \rho_n TV(f_n) \le \frac{1}{2}\left\|u^{h_n}(\widehat{f}^{h_n}) - z_n\right\|_\Gamma^2 + \rho_n TV(\widehat{f}^{h_n}), \tag{3.14}$$

where $\widehat{f}^{h_n} \in F_{ad}^{h_n}$ generates from $\widehat{f}$, due to Lemma 3.2. We bound

$$\begin{aligned}\frac{1}{2}\left\|u^{h_n}(\widehat{f}^{h_n}) - z_n\right\|_\Gamma^2 &\le \left\|u^{h_n}(\widehat{f}^{h_n}) - u(\widehat{f})\right\|_\Gamma^2 + \left\|u(\widehat{f}) - z_n\right\|_\Gamma^2 \\ &\le C\left\|u^{h_n}(\widehat{f}^{h_n}) - u(\widehat{f})\right\|_{1,\Omega}^2 + \delta_n^2. \end{aligned}\tag{3.15}$$

Using (1.4) and (3.1), we get for all $v^{h_n} \in \mathcal{V}_1^{h_n}$ that

$$a\left(u(\widehat{f}) - u^{h_n}(\widehat{f}^{h_n}), v^{h_n}\right) = \left(\widehat{f} - \widehat{f}^{h_n}, v^{h_n}\right)_\Omega$$

and so that

$$a\left(u^{h_n}(\widehat{f}) - u^{h_n}(\widehat{f}^{h_n}), v^{h_n}\right) = \left(\widehat{f} - \widehat{f}^{h_n}, v^{h_n}\right)_\Omega + a\left(u^{h_n}(\widehat{f}) - u(\widehat{f}), v^{h_n}\right).$$

Taking $v^{h_n} = u^{h_n}(\widehat{f}) - u^{h_n}(\widehat{f}^{h_n})$, by (1.3), (3.5) and (3.10), we deduce

$$\begin{aligned} c_1\big\|u^{h_n}(\widehat{f}) - u^{h_n}(\widehat{f}^{h_n})\big\|_{1,\Omega} &\le \big\|\widehat{f} - \widehat{f}^{h_n}\big\|_\Omega + C\big\|u^{h_n}(\widehat{f}) - u(\widehat{f})\big\|_{1,\Omega} \\ &\le C\left(h_n|\log h_n|\right)^{1/2} + C\chi^{h_n}_{\widehat{f}} \end{aligned}$$

and then

$$\big\|u^{h_n}(\widehat{f}^{h_n}) - u(\widehat{f})\big\|_{1,\Omega} \le \big\|u(\widehat{f}) - u^{h_n}(\widehat{f})\big\|_{1,\Omega} + \big\|u^{h_n}(\widehat{f}) - u^{h_n}(\widehat{f}^{h_n})\big\|_{1,\Omega} \le C\left(\left(h_n|\log h_n|\right)^{1/2} + \chi^{h_n}_{\widehat{f}}\right).$$

Combining this with (3.15)–(3.14), we get

$$\frac{1}{2}\big\|u^{h_n}(f_n) - z_n\big\|^2_\Gamma + \rho_n TV(f_n) \le C\left(\delta_n^2 + h_n|\log h_n| + \left(\chi^{h_n}_{\widehat{f}}\right)^2\right) + \rho_n TV(\widehat{f}^{h_n}). \tag{3.16}$$

It follows from the last inequality and (3.12), (3.6) that

$$\lim_{n\to\infty}\big\|u^{h_n}(f_n) - z_n\big\|_\Gamma = 0 \tag{3.17}$$

and

$$\limsup_{n\to\infty} TV(f_n) \le TV(\widehat{f}). \tag{3.18}$$

Due to Lemma 2.1 and Lemma 3.3, a subsequence of $(f_n)_n$ not relabelled and an element $f_* \in F_{ad}$ exist such that

$$\begin{aligned} &f_n \to f_* \quad \text{in the } L^1(\Omega)\text{-norm}, \\ &TV(f_*) \le \liminf_{n\to\infty} TV(f_n), \\ &u^{h_n}(f_n) \to u(f_*) \quad \text{in the } L^2(\Gamma)\text{-norm}. \end{aligned} \tag{3.19}$$

Thus, it follows from (3.17) that

$$\|u(f_*) - g^\dagger\|_\Gamma \le \lim_{n\to\infty}\left(\|u(f_*) - u^{h_n}(f_n)\|_\Gamma + \|u^{h_n}(f_n) - z_n\|_\Gamma + \|z_n - g^\dagger\|_\Gamma\right) = 0$$

and so that $f_* \in \mathcal{I}\left(g^\dagger\right)$. Furthermore, by (3.18) and (3.19), we also get

$$TV(f_*) \le \liminf_{n\to\infty} TV(f_n) \le \limsup_{n\to\infty} TV(f_n) \le TV(\widehat{f})$$

for any $\widehat{f} \in \mathcal{I}\left(g^\dagger\right)$. Therefore, f_* is a total variation-minimizing of the identification problem. Furthermore, by setting $\widehat{f} = f_*$, it implies that

$$TV(f_*) = \lim_{n\to\infty} TV(f_n).$$

On the other hand, due to the standard theory of the finite element method (cf. [11, 14]), we get

$$\lim_{n\to\infty}\|u^{h_n}(f_*) - u(f_*)\|_{1,\Omega} = 0.$$

Furthermore, in view of (3.1) and (1.3) we arrive at

$$c_1\|u^{h_n}(f_n) - u^{h_n}(f_*)\|_{1,\Omega} \le \|f_n - f_*\|_\Omega \le \left(2\overline{f}\|f_n - f_*\|_{L^1(\Omega)}\right)^{1/2} \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, we conclude that $\lim_{n\to\infty}\big\|u^{h_n}(f_n) - u(f_*)\big\|_{1,\Omega} = 0$, which finishes the proof. □

4 An iterated total variation algorithm

The aim of this section proposes an algorithm to reach minimizers of the problem $(\mathcal{P}^h)$. We start with the following note.

Remark 4.1. (i) Any $\phi \in \mathcal{V}_1^h$ can be considered as an element in $(\mathcal{V}_1^h)^*$, the dual space of $\mathcal{V}_1^h$, by

$$(\phi, \xi^h)_{((\mathcal{V}_1^h)^*, \mathcal{V}_1^h)} := (\phi, \xi^h)_\Omega. \tag{4.1}$$

(ii) The inclusion $(\mathcal{V}_0^h)^d \subset (\mathcal{V}_1^h)^*$ holds via the identity

$$(p^*, \xi^h)_{((\mathcal{V}_1^h)^*, \mathcal{V}_1^h)} := (\nabla \xi^h, p^*)_\Omega \quad \text{for all} \quad p^* \in (\mathcal{V}_0^h)^d. \tag{4.2}$$

Lemma 4.2. *For each $f^h \in \mathcal{V}_1^h$ the relation*

$$\partial TV(f^h) = \left\{ p^h \in \mathcal{B}_1\left((\mathcal{V}_0^h)^d\right) \subset (\mathcal{V}_0^h)^d \mid (\nabla f^h, p^h)_\Omega = \int_\Omega |\nabla f^h| \right\} \tag{4.3}$$

holds, where

$$\mathcal{B}_1\left((\mathcal{V}_0^h)^d\right) := \left\{ p^h \in (\mathcal{V}_0^h)^d \mid \|p^h\|_\infty \le 1 \right\}.$$

Proof. Let $f^h \in \mathcal{V}_1^h$ be fixed. As shown in [7] that

$$TV(f^h) = \int_\Omega |\nabla f^h| = \max_{p^h \in (\mathcal{V}_0^h)^d,\ \|p^h\|_\infty \le 1} (\nabla f^h, p^h)_\Omega. \tag{4.4}$$

Assume that $\mathcal{T}^h := \{T_1, T_2, ..., T_{n^h}\}$ with $n^h \in \mathbb{N}$ and $\overline{\Omega} = \cup_{i=1}^{n^h} \overline{T_i}$. We then consider $p^* := (p^*_{ij})_{n^h \times d} \in \mathcal{B}_1\left((\mathcal{V}_0^h)^d\right)$ with, for all $i = 1, ..., n^h$ and $j = 1, ..., d$,

$$p^*_{ij} := \begin{cases} +1 & \text{if} \quad \frac{\partial f^h{}_{|T_i}}{x_j} > 0, \\ -1 & \text{if} \quad \frac{\partial f^h{}_{|T_i}}{x_j} < 0, \\ \in [-1, +1] & \text{if} \quad \frac{\partial f^h{}_{|T_i}}{x_j} = 0. \end{cases}$$

and deduce from (4.2), (4.4) that

$$TV(f^h) = (p^*, f^h)_{((\mathcal{V}_1^h)^*, \mathcal{V}_1^h)}.$$

The element p^* is called an *active* function of TV at f^h. The subdifferential $\partial TV(f^h)$ is then the convex hull of union of all active functions at f^h (cf. [31])

$$\partial TV(f^h) = \mathbf{Co}\left(\bigcup \left\{ p^* \mid (p^*, f^h)_{((\mathcal{V}_1^h)^*, \mathcal{V}_1^h)} = TV(f^h) \quad \text{and} \quad \|p^*\|_\infty \le 1 \right\} \right),$$

and so that (4.3) follows, which finishes the proof. □

We mention that, similar to the decomposition (2.1), for all $f \in L^2(\Omega)$ we have

$$u^h(f) = \overline{u^h}(f) + \widehat{u^h}, \tag{4.5}$$

where

$$a\left(\overline{u^h}(f), v^h\right) = l_0^f(v^h) \quad \text{and} \quad a\left(\widehat{u^h}, v^h\right) = l_j^0(v^h) \tag{4.6}$$

for all $v^h \in \mathcal{V}_1^h$. Furthermore, the identity

$$u^{h\prime}(f)\xi = \overline{u^h}(\xi) \tag{4.7}$$

holds for all $f, \xi \in L^2(\Omega)$.

Lemma 4.3. *The function $f^h \in F_{ad}^h$ is a solution of $(\mathcal{P}^h)$ if and only if there exists $p^h \in \partial TV(f^h)$ such that*

$$\left(g^h - f^h, u_*^h(f^h)\right)_\Omega + \rho\left(\nabla(g^h - f^h), p^h\right)_\Omega \geq 0 \quad \textit{for all} \quad g^h \in F_{ad}^h, \tag{4.8}$$

$$\left(\nabla f^h, q^h - p^h\right)_\Omega \leq 0 \quad \textit{for all} \quad q^h \in (\mathcal{V}_0^h)^d, \quad \|q^h\|_\infty \leq 1 \tag{4.9}$$

where u_*^h is the discrete adjoint state defined by (3.3).

Proof. In view of (2.7), the functions $\|u^h(\cdot) - z\|_\Gamma$ and $TV(\cdot)$ are both convex on the convex set F_{ad}^h. Thus, the function $f^h \in F_{ad}^h$ is a solution of $(\mathcal{P}^h)$ if and only if

$$\left(u^h(f^h) - z, u^{h\prime}(f^h)(g^h - f^h)\right)_\Gamma + \rho\left(TV(g^h) - TV(f^h)\right) \geq 0 \tag{4.10}$$

for all $g^h \in F_{ad}^h$. Using (3.3) and (4.7)–(4.6), we get

$$\begin{aligned}
\left(u^h(f^h) - z, u^{h\prime}(f^h)(g^h - f^h)\right)_\Gamma &= a\left(u_*^h(f^h), u^{h\prime}(f^h)(g^h - f^h)\right) \\
&= a\left(u_*^h(f^h), \overline{u^h}(g^h - f^h)\right) \\
&= \left(g^h - f^h, u_*^h(f^h)\right)_\Omega
\end{aligned}$$

and thus deduce from (4.10) that

$$\left(g^h, u_*^h(f^h)\right)_\Omega + \rho TV(g^h) \geq \left(f^h, u_*^h(f^h)\right)_\Omega + \rho TV(f^h), \quad \forall g^h \in F_{ad}^h. \tag{4.11}$$

We here consider the convex function

$$\Psi : \mathcal{V}_1^h \to \mathbb{R} \cup \{\infty\} \quad \text{with} \quad \xi^h \mapsto \Psi(\xi^h) := \left(\xi^h, u_*^h(f^h)\right)_\Omega + \rho TV(\xi^h) + I_{F_{ad}^h}(\xi^h),$$

where $I_{F_{ad}^h}$ is the indicator function of the set F_{ad}^h. Then, the inequality (4.11) is equivalent to the relation

$$0 \in \partial\Psi(f^h). \tag{4.12}$$

Since, for all $\xi^h \in \mathcal{V}_1^h$

$$\partial I_{F_{ad}^h}(\xi^h) = \left\{p^* \in (\mathcal{V}_1^h)^* \mid (p^*, g^h - \xi^h)_{((\mathcal{V}_1^h)^*, \mathcal{V}_1^h)} \leq 0 \quad \text{for all} \quad g^h \in F_{ad}^h\right\},$$

the relation (4.12) means that there exists $p^h \in \partial TV(f^h)$ such that

$$\left(-u_*^h(f^h) - \rho p^h, g^h - f^h\right)_{((\mathcal{V}_1^h)^*, \mathcal{V}_1^h)} \leq 0$$

for all $g^h \in F_{ad}^h$. This together with (4.1)–(4.2) yields (4.8). Now, for all $q^h \in (\mathcal{V}_0^h)^d$ with $\|q^h\|_\infty \leq 1$ we have from the fact $p^h \in \partial TV(f^h)$ defined by the equation (4.3) that

$$\left(\nabla f^h, q^h\right)_\Omega = \int_\Omega \nabla f^h \cdot q^h \leq \int_\Omega |\nabla f^h| \cdot |q^h| \leq \int_\Omega |\nabla f^h| = \left(\nabla f^h, p^h\right)_\Omega,$$

which finishes the proof. □

Remark 4.4. The system (4.8)–(4.9) is equivalent to the following inequality

$$\left(g^h - f^h, u_*^h(f^h)\right)_\Omega + \rho\left(\nabla(g^h - f^h), p^h\right)_\Omega - \left(\nabla f^h, q^h - p^h\right)_\Omega \geq 0 \tag{4.13}$$

for all $(g^h, q^h) \in F_{ad}^h \times (\mathcal{V}_0^h)^d$ with $\|q^h\|_\infty \leq 1$.

In the sequel, we make use the following notation (cf. [7, 9])

$$\|\nabla\| := \sup_{0 \neq v^h \in \mathcal{V}_1^h} \frac{\|\nabla v^h\|_\Omega}{\|v^h\|_\Omega}. \tag{4.14}$$

By the inverse inequality (cf. [11, 14]) $\|v^h\|_{1,\Omega} \le Ch^{-1}\|v^h\|_\Omega$ for all $v^h \in \mathcal{V}_1^h$, we obtain that $\|\nabla\| \le Ch^{-1}$.

Algorithm 1: Minimizing of $(\mathcal{P}^h)$

Input : Let a desired number of iterations N and a regularization parameter $\rho > 0$. Choose an initial iteration $\mu_0^h := (f_0^h, p_0^h) \in F_{ad}^h \times (\mathcal{V}_0^h)^d$ and parameters $\tau, \theta > 0$ such that

$$\left(\frac{1}{\tau} - \frac{c_\gamma^2}{c_1^2}\right)\frac{\theta}{\tau} > \rho^2\|\nabla\|^2. \tag{4.15}$$

Set $n = 0$.

Output: An approximation of a solution of $(\mathcal{P}^h)$

if $n \le N$ **then**

$$f_{n+1}^h = \arg\min_{f^h \in F_{ad}^h} \left\{ \left(f^h, u_*^h(f_n^h)\right)_\Omega + \rho\left(\nabla f^h, p_n^h\right)_\Omega + \frac{1}{2\tau}\|f^h - f_n^h\|_\Omega^2 \right\} \tag{4.16}$$

$$\widetilde{f}_{n+1}^h = 2f_{n+1}^h - f_n^h \tag{4.17}$$

$$p_{n+1}^h = \arg\max_{p^h \in (\mathcal{V}_0^h)^d} \left\{ \rho\left(\nabla \widetilde{f}_{n+1}^h, p^h\right)_\Omega - I_{\mathcal{B}_1((\mathcal{V}_0^h)^d)}(p^h) - \frac{\theta}{2\tau}\|p^h - p_n^h\|_\Omega^2 \right\} \tag{4.18}$$

end

Remark 4.5. We mention that (cf. [7, 33]), the solution p_{n+1}^h to the sub-problem (4.18) is given by the explicit form

$$p_{n+1}^h = \frac{p_n^h + \frac{\tau\rho}{\theta}\nabla\widetilde{f}_{n+1}^h}{\max\{1, |p_n^h + \frac{\tau\rho}{\theta}\nabla\widetilde{f}_{n+1}^h|\}}. \tag{4.19}$$

Let $-\mathrm{div}$ denote the adjoint operator of ∇ defined by

$$(-\mathrm{div}\ q^h, g^h) = (q^h, \nabla g^h)_\Omega := \sum_{T \in \mathcal{T}^h} \sum_{i=1}^d |T| q_i^h{}_{|T} \frac{\partial g^h}{x_i}{}_{|T} \tag{4.20}$$

for all $q^h \in (\mathcal{V}_0^h)^d$ and $g^h \in \mathcal{V}_1^h$. Then, due to the optimality of f_{n+1}^h, we have from (4.16) that

$$\left(u_*^h(f_n^h), g^h - f_{n+1}^h\right)_\Omega + \rho\left(\nabla(g^h - f_{n+1}^h), p_n^h\right)_\Omega + \frac{1}{\tau}\left(f_{n+1}^h - f_n^h, g^h - f_{n+1}^h\right)_\Omega \ge 0$$

for all $g^h \in F_{ad}^h$ which can be rewritten as

$$\left(f_n^h + \tau(\rho\mathrm{div}\ p_n^h - u_*^h(f_n^h)) - f_{n+1}^h, g^h - f_{n+1}^h\right)_\Omega \le 0 \quad \text{for all} \quad g^h \in F_{ad}^h$$

or

$$f_{n+1}^h = \mathcal{P}_{F_{ad}^h}^{L^2}\left(f_n^h - \tau(u_*^h(f_n^h) - \rho\mathrm{div}\ p_n^h)\right), \tag{4.21}$$

where $\mathcal{P}_{F_{ad}^h}^{L^2} : L^2(\Omega) \to F_{ad}^h$ denotes the L^2-projection on the set F_{ad}^h characterized by

$$\left(\phi - \mathcal{P}_{F_{ad}^h}^{L^2}\phi, g^h - \mathcal{P}_{F_{ad}^h}^{L^2}\phi\right)_\Omega \le 0 \quad \text{for all} \quad g^h \in F_{ad}^h$$

for each $\phi \in L^2(\Omega)$.

Remark 4.6. For all $f \in L^2(\Omega)$ let $\overline{u_*^h}(f) \in \mathcal{V}_1^h$ be the unique weak solution of the following variational equation

$$a\left(\overline{u_*^h}(f), v^h\right) = \left(\overline{u^h}(f), v^h\right)_\Gamma \tag{4.22}$$

for all $v^h \in \mathcal{V}_1^h$, where $\overline{u^h}(f)$ is defined by (4.6). Then we have for all $v^h \in \mathcal{V}_1^h$ and $f, g \in L^2(\Omega)$ that

$$\begin{aligned}
a\left(u_*^h(f) - u_*^h(g), v^h\right) &= a\left(u_*^h(f), v^h\right) - a\left(u_*^h(g), v^h\right) \\
&= \left(u^h(f) - u^h(g), v^h\right)_\Gamma, \quad \text{by (3.3)} \\
&= \left(\overline{u^h}(f) - \overline{u^h}(g), v^h\right)_\Gamma, \quad \text{by (4.5)} \\
&= \left(\overline{u^h}(f-g), v^h\right)_\Gamma \\
&= a\left(\overline{u_*^h}(f-g), v^h\right), \quad \text{by (4.22)}.
\end{aligned}$$

Thus, by (1.3), taking $v^h = u_*^h(f) - u_*^h(g) - \overline{u_*^h}(f-g) \in \mathcal{V}_1^h$, we obtain

$$u_*^h(f) - u_*^h(g) = \overline{u_*^h}(f-g) \tag{4.23}$$

for all $f, g \in L^2(\Omega)$. Furthermore, by (1.3), (1.6) and (4.6), it holds the estimate

$$\|\overline{u_*^h}(f)\|_\Omega \le \|\overline{u_*^h}(f)\|_{1,\Omega} \le \frac{c_\gamma}{c_1}\|\overline{u^h}(f)\|_\Gamma \le \frac{c_\gamma^2}{c_1}\|\overline{u^h}(f)\|_{1,\Omega} \le \frac{c_\gamma^2}{c_1^2}\|f\|_\Omega. \tag{4.24}$$

Remark 4.7. Let

$$A := \begin{pmatrix} -\rho\mathrm{div} + u_*^h \\ -\rho\nabla \end{pmatrix}, \quad \mu^h := \left(f^h, p^h\right) \quad \text{and} \quad \nu^h := \left(g^h, q^h\right).$$

Then the system (4.8)–(4.9) can be rewritten in the abbreviation form

$$\left(A(\mu^h), \nu^h - \mu^h\right) \ge 0. \tag{4.25}$$

Furthermore, the optimality of f_{n+1}^h and p_{n+1}^h due to Algorithm 1 yields that

$$\begin{aligned}
&\left(\frac{1}{\tau}(f_{n+1}^h - f_n^h) + u_*^h(f_n^h), g^h - f_{n+1}^h\right)_\Omega + \rho\left(p_n^h, \nabla(g^h - f_{n+1}^h)\right)_\Omega \ge 0, \\
&\left(\frac{\theta}{\tau}(p_{n+1}^h - p_n^h) - \rho\nabla\widetilde{f}_{n+1}^h, q^h - p_{n+1}^h\right) \ge 0
\end{aligned} \tag{4.26}$$

for all $\nu^h = (g^h, q^h) \in F_{ad}^h \times \mathcal{B}_1\left((\mathcal{V}_0^h)^d\right)$. By the identity (4.23), the above system (4.26) is rewritten in the form

$$\left(A(\mu_{n+1}^h) + B(\mu_{n+1}^h - \mu_n^h), \nu^h - \mu_{n+1}^h\right) \ge 0, \tag{4.27}$$

where

$$B := \begin{pmatrix} \frac{1}{\tau}I - \overline{u_*^h} & \rho\mathrm{div} \\ -\rho\nabla & \frac{\theta}{\tau}I \end{pmatrix} \quad \text{and} \quad \mu_{n+1}^h := \left(f_{n+1}^h, p_{n+1}^h\right). \tag{4.28}$$

The condition (4.15) then shows that B is a symmetric, positive definite matrix.

Theorem 4.8. *For any fixed $h > 0$ let $\left(\mu_n^h\right)_n := \left(f_n^h, p_n^h\right)_n$ be the sequence generated by Algorithm 1.*

(i) Then a subsequence $\left(\mu_{n_k}^h\right)_k$ and an element $\mu_*^h := (f_*^h, p_*^h) \in F_{ad}^h \times \partial TV(f_*^h)$ exist such that $\left(\mu_{n_k}^h\right)_k$ converges to μ_*^h in the finite dimensional space $\mathcal{V}_1^h \times (\mathcal{V}_0^h)^d$. Furthermore, μ_*^h satisfies the condition (4.25), *i.e. $\left(A(\mu_*^h), \nu^h - \mu_*^h\right) \ge 0$ for all $\nu^h \in F_{ad}^h \times \mathcal{B}_1\left((\mathcal{V}_0^h)^d\right)$ and f_*^h is thus a minimizer of $(\mathcal{P}^h)$.*

(ii) The equation

$$\|\mu_{n+1}^h - \mu_n^h\| = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \tag{4.29}$$

holds true, where $\|\cdot\|$ is some norm of the finite dimensional space $\mathcal{V}_1^h \times (\mathcal{V}_0^h)^d$.

Proof. (i) Denote $\mu^h := (f^h, p^h)$, where $f^h \in F^h_{ad}$ is an arbitrary solution to $(\mathcal{P}^h)$ and $p^h \in \partial TV(f^h)$ generates from f^h, due to Lemma 4.3. Using the notations (4.25) and (4.28), we have that

$$\begin{aligned}\|\mu^h_{n+1} - \mu^h\|^2_B &= \|\mu^h_n - \mu^h + \mu^h_{n+1} - \mu^h_n\|^2_B \\ &= \|\mu^h_n - \mu^h\|^2_B + \|\mu^h_{n+1} - \mu^h_n\|^2_B - 2\left(B(\mu^h_{n+1} - \mu^h_n), \mu^h - \mu^h_n\right).\end{aligned} \tag{4.30}$$

Talking $\nu^h = \mu^h$ in (4.27), we get

$$\begin{aligned}\left(B(\mu^h_{n+1} - \mu^h_n), \mu^h - \mu^h_n\right) &= \left(B(\mu^h_{n+1} - \mu^h_n), \mu^h - \mu^h_{n+1}\right) + \left(B(\mu^h_{n+1} - \mu^h_n), \mu^h_{n+1} - \mu^h_n\right) \\ &\geq \left(A(\mu^h_{n+1}), \mu^h_{n+1} - \mu^h\right) + \left(B(\mu^h_{n+1} - \mu^h_n), \mu^h_{n+1} - \mu^h_n\right) \\ &= \left(A(\mu^h), \mu^h_{n+1} - \mu^h\right) + \left(A(\mu^h_{n+1}) - A(\mu^h), \mu^h_{n+1} - \mu^h\right) + \|\mu^h_{n+1} - \mu^h_n\|^2_B\end{aligned}$$

and so that (4.30) yields

$$\begin{aligned}\|\mu^h_{n+1} - \mu^h\|^2_B &\leq \|\mu^h_n - \mu^h\|^2_B - \|\mu^h_{n+1} - \mu^h_n\|^2_B \\ &\quad - 2\left(A(\mu^h), \mu^h_{n+1} - \mu^h\right) - 2\left(A(\mu^h_{n+1}) - A(\mu^h), \mu^h_{n+1} - \mu^h\right).\end{aligned} \tag{4.31}$$

Talking $\nu^h = \mu^h_{n+1}$ in (4.25), we have

$$\left(A(\mu^h), \mu^h_{n+1} - \mu^h\right) \geq 0. \tag{4.32}$$

On the other hand, by (4.20), we get

$$\begin{aligned}&\left(A(\mu^h_{n+1}) - A(\mu^h), \mu^h_{n+1} - \mu^h\right) \\ &\quad = -\rho\left(\mathrm{div}(p^h_{n+1} - p^h), f^h_{n+1} - f^h\right) + \left(u^h_*(f^h_{n+1}) - u^h_*(f^h), f^h_{n+1} - f^h\right)_\Omega - \rho\left(\nabla(f^h_{n+1} - f^h), p^h_{n+1} - p^h\right)_\Omega \\ &\quad = \left(u^h_*(f^h_{n+1}) - u^h_*(f^h), f^h_{n+1} - f^h\right)_\Omega \\ &\quad = \left(\overline{u^h_*}(f^h_{n+1} - f^h), f^h_{n+1} - f^h\right)_\Omega, \quad \text{by (4.23)} \\ &\quad = a\left(\overline{u^h}(f^h_{n+1} - f^h), \overline{u^h_*}(f^h_{n+1} - f^h)\right), \quad \text{by (4.6)} \\ &\quad = \left(\overline{u^h}(f^h_{n+1} - f^h), \overline{u^h}(f^h_{n+1} - f^h)\right)_\Gamma, \quad \text{by (4.22)} \\ &\quad \geq 0.\end{aligned} \tag{4.33}$$

Combining this with (4.32)–(4.31), we arrive at

$$\|\mu^h_{n+1} - \mu^h\|^2_B \leq \|\mu^h_n - \mu^h\|^2_B - \|\mu^h_{n+1} - \mu^h_n\|^2_B. \tag{4.34}$$

Then for all $n \in \mathbb{N}$ it follows from (4.34) that

$$\|\mu^h_{n+1} - \mu^h\|^2_B + \sum_{m=0}^{n} \|\mu^h_{m+1} - \mu^h_m\|^2_B \leq \|\mu^h_0 - \mu^h\|^2_B. \tag{4.35}$$

Therefore, the sequence $(\mu^h_n)_n$ is bounded while the series $\sum_{n=0}^\infty \|\mu^h_{n+1} - \mu^h_n\|^2_B$ is convergent. A subsequence of $(\mu^h_n)_n$ not relabeled and an element $\mu^h_* := (f^h_*, p^h_*) \in F^h_{ad} \times \mathcal{B}_1\left((\mathcal{V}^h_0)^d\right)$ then exist such that

$$\lim_{n\to\infty} \mu^h_{n+1} = \lim_{n\to\infty} \mu^h_n = \mu^h_*.$$

Thus sending n to ∞ in (4.27), we obtain

$$\left(A(\mu^h_*), \nu^h - \mu^h_*\right) \geq 0 \tag{4.36}$$

for all $\nu^h \in F^h_{ad} \times \mathcal{B}_1\left((\mathcal{V}^h_0)^d\right)$. As a direct consequence of (4.36) (cf. Remark 4.4) we get

$$\left(\nabla f^h_*, q^h - p^h_*\right)_\Omega \leq 0 \quad \text{for all} \quad q^h \in (\mathcal{V}^h_0)^d, \quad \|q^h\|_\infty \leq 1$$

and so that

$$(\nabla f^h_*, p^h_*)_\Omega = \max_{q^h \in (\mathcal{V}^h_0)^d,\ \|q^h\|_\infty \leq 1} (\nabla f^h_*, q^h)_\Omega = \int_\Omega |\nabla f^h_*| = TV(f^h_*).$$

This together with Lemma 4.2 yields $p^h_* \in \partial TV(f^h_*)$. Consequently, f^h_* is then a minimizer to $(\mathcal{P}^h)$, due to (4.36) and Lemma 4.3.

(ii) It remains to show (4.29). For all $n \in \mathbb{N}$ we get from (4.35) that

$$\sum_{m=0}^{n} \|\mu^h_{m+1} - \mu^h_m\|^2_B \le \|\mu^h_0 - \mu^h\|^2_B. \tag{4.37}$$

On the other hand, in view of (4.27) we have

$$\begin{aligned}\left(A(\mu^h_{n+1}) + B(\mu^h_{n+1} - \mu^h_n), \mu^h_{n+2} - \mu^h_{n+1}\right) \ge 0\\ \left(A(\mu^h_{n+2}) + B(\mu^h_{n+2} - \mu^h_{n+1}), \mu^h_{n+1} - \mu^h_{n+2}\right) \ge 0\end{aligned}$$

which implies

$$\left(A(\mu^h_{n+2}) - A(\mu^h_{n+1}) + B(\mu^h_{n+2} - \mu^h_{n+1}) - B(\mu^h_{n+1} - \mu^h_n), \mu^h_{n+1} - \mu^h_{n+2}\right) \ge 0$$

or

$$\left(B(\mu^h_n - \mu^h_{n+1}) - B(\mu^h_{n+1} - \mu^h_{n+2}), \mu^h_{n+1} - \mu^h_{n+2}\right) \ge \left(A(\mu^h_{n+2}) - A(\mu^h_{n+1}), \mu^h_{n+2} - \mu^h_{n+1}\right). \tag{4.38}$$

In view of (4.33) we get

$$\left(A(\mu^h_{n+2}) - A(\mu^h_{n+1}), \mu^h_{n+2} - \mu^h_{n+1}\right) \ge \left(\overline{u^h}(f^h_{n+2} - f^h_{n+1}), \overline{u^h}(f^h_{n+2} - f^h_{n+1})\right)_\Gamma \ge 0 \tag{4.39}$$

which together with

$$\left(B(\mu^h_n - \mu^h_{n+1}) - B(\mu^h_{n+1} - \mu^h_{n+2}), (\mu^h_n - \mu^h_{n+1}) - (\mu^h_{n+1} - \mu^h_{n+2})\right) = \|\mu^h_n - 2\mu^h_{n+1} + \mu^h_{n+2}\|^2_B$$

implie that

$$\begin{aligned}
&\|\mu^h_n - 2\mu^h_{n+1} + \mu^h_{n+2}\|^2_B\\
&= \left(B(\mu^h_n - \mu^h_{n+1}) - B(\mu^h_{n+1} - \mu^h_{n+2}), \mu^h_n - \mu^h_{n+1}\right) - \left(B(\mu^h_n - \mu^h_{n+1}) - B(\mu^h_{n+1} - \mu^h_{n+2}), (\mu^h_{n+1} - \mu^h_{n+2})\right)\\
&\le \left(B(\mu^h_n - \mu^h_{n+1}) - B(\mu^h_{n+1} - \mu^h_{n+2}), \mu^h_n - \mu^h_{n+1}\right), \quad \text{by } (4.38) - (4.39)\\
&= \frac{1}{2}\left(B(\mu^h_n - \mu^h_{n+1}) - B(\mu^h_{n+1} - \mu^h_{n+2}), \left((\mu^h_n - \mu^h_{n+1}) + (\mu^h_{n+1} - \mu^h_{n+2})\right) + \left((\mu^h_n - \mu^h_{n+1}) - (\mu^h_{n+1} - \mu^h_{n+2})\right)\right)\\
&= \frac{1}{2}\left(B(\mu^h_n - \mu^h_{n+1}) - B(\mu^h_{n+1} - \mu^h_{n+2}), (\mu^h_n - \mu^h_{n+1}) + (\mu^h_{n+1} - \mu^h_{n+2})\right)\\
&\quad + \frac{1}{2}\left(B(\mu^h_n - \mu^h_{n+1}) - B(\mu^h_{n+1} - \mu^h_{n+2}), (\mu^h_n - \mu^h_{n+1}) - (\mu^h_{n+1} - \mu^h_{n+2})\right)\\
&= \frac{1}{2}\left(\|\mu^h_n - \mu^h_{n+1}\|^2_B - \|\mu^h_{n+1} - \mu^h_{n+2}\|^2_B + \|\mu^h_n - 2\mu^h_{n+1} + \mu^h_{n+2}\|^2_B\right)
\end{aligned}$$

and so that

$$\|\mu^h_{n+2} - \mu^h_{n+1}\|^2_B \le \|\mu^h_{n+1} - \mu^h_n\|^2_B$$

for all $n \in \mathbb{N}$. Therefore, we arrive at

$$\begin{aligned}
(n+1)\|\mu^h_{n+1} - \mu^h_n\|^2_B &= \underbrace{\|\mu^h_{n+1} - \mu^h_n\|^2_B + ... + \|\mu^h_{n+1} - \mu^h_n\|^2_B}_{(n+1)-\text{times}}\\
&\le \|\mu^h_{n+1} - \mu^h_n\|^2_B + \|\mu^h_n - \mu^h_{n-1}\|^2_B + ... + \|\mu^h_1 - \mu^h_0\|^2_B\\
&= \sum_{m=0}^{n} \|\mu^h_{m+1} - \mu^h_m\|^2_B\\
&\le \|\mu^h_0 - \mu^h\|^2_B,
\end{aligned}$$

by (4.37). The proof is completed. ☐

5 Numerical test

We now illustrate the theoretical result with numerical examples. For reduction of computations, we assume that $\beta = \sigma = 0$ in the system (1.1), and thus consider the Neumann problem

$$-\nabla \cdot \big(\alpha \nabla \Phi\big) = f^\dagger \text{ in } \Omega, \tag{5.1}$$

$$\alpha \nabla \Phi \cdot \vec{n} = j^\dagger \text{ on } \partial\Omega, \tag{5.2}$$

with $\Omega = \{x = (x_1, x_2) \in R^2 \mid -1 < x_1, x_2 < 1\}$. Let χ_D denote the characteristic function of a Lebesgue measurable set $D \subset \mathbb{R}^2$. We assume that entries of the known symmetric diffusion matrix α are defined as

$$\alpha_{11} = 3\chi_{\Omega_{11}} + \chi_{\Omega\setminus\Omega_{11}},\ \alpha_{12} = \chi_{\Omega_{12}},\ \alpha_{22} = 4\chi_{\Omega_{22}} + 2\chi_{\Omega\setminus\Omega_{22}},$$

where

$$\Omega_{11} := \left\{(x_1,x_2) \in \Omega \;\middle|\; |x_1| \le 1/2 \text{ and } |x_2| \le 1/2\right\},\ \Omega_{12} := \left\{(x_1,x_2) \in \Omega \;\middle|\; |x_1| + |x_2| \le 1/2\right\} \text{ and}$$
$$\Omega_{22} := \left\{(x_1,x_2) \in \Omega \;\middle|\; x_1^2 + x_2^2 \le 1/4\right\}.$$

The sought source function $f^\dagger$ in (5.1) is assumed to be discontinuous and defined as

$$f^\dagger = \left(2 - \frac{\pi}{8}\right)\chi_{\Omega_{22}} - \frac{\pi}{8}\chi_{\Omega\setminus\Omega_{22}}.$$

For the discretization we divide the interval $(-1,1)$ into ℓ equal segments and so that the domain $\Omega = (-1,1)^2$ is divided into $2\ell^2$ triangles, where the diameter of each triangle is $h = h_\ell = \frac{\sqrt{8}}{\ell}$. The Neumann boundary data $j^\dagger$ in the equation (5.2) is chosen as

$$\begin{aligned} j^\dagger = {}& \chi_{(0,1]\times\{-1\}} - \chi_{[-1,0]\times\{1\}} + 2\chi_{(0,1]\times\{1\}} - 2\chi_{[-1,0]\times\{-1\}} \\ & + 3\chi_{\{-1\}\times(-1,0]} - 3\chi_{\{1\}\times(0,1)} + 4\chi_{\{1\}\times(-1,0]} - 4\chi_{\{-1\}\times(0,1)}. \end{aligned} \tag{5.3}$$

We note that (5.1)–(5.2) is the pure Neumann boundary value problem, so $f^\dagger$ and $j^\dagger$ were taken such that the compatibility condition $\int_\Omega f^\dagger + \int_{\partial\Omega} j^\dagger = 0$ is satisfied. Let $\mathcal{N}_{f^\dagger} j^\dagger$ denote the unique weak solution of (5.1)–(5.2). The Dirichlet boundary data of the problem (5.1)–(5.2) is computed by $\big(\mathcal{N}_{f^\dagger} j^\dagger\big)_{|\partial\Omega}$. We use Algorithm 1 which is described in Section 4 for computing the numerical solution of the problem $\big(\mathcal{P}^h\big)$. Before doing so, we discuss about the constants c_γ and c_1 appearing in (4.15). First, we can show the inequality (see, e.g., [21, §7])

$$\frac{\underline{\alpha}}{1 + \left(\sqrt{\frac{3}{2}}\right)^{(d+2)/2} |\Omega|^{1/d}} \|u\|_{1,\Omega} \le a(u,u) \tag{5.4}$$

for all $u \in H^1(\Omega)$. Therefore, the constant c_1 can be chosen by $\frac{\underline{\alpha}}{1 + \left(\sqrt{\frac{3}{2}}\right)^{(d+2)/2} |\Omega|^{1/d}}$. The following result provides an estimate for c_γ.

Lemma 5.1. *If $\Omega := (a_1,b_1) \times (a_2,b_2) \times \ldots \times (a_d,b_d) \ni 0$, then c_γ can be chosen by $\sqrt{\frac{d + |\overline{\hbar}|^2}{|\underline{\hbar}|}}$, i.e. for all $u \in H^1(\Omega)$, the estimate*

$$\|\gamma u\|_{L^2(\partial\Omega)} \le \sqrt{\frac{d + |\overline{\hbar}|^2}{|\underline{\hbar}|}} \|u\|_{H^1(\Omega)} \tag{5.5}$$

holds true, where $\underline{\hbar} = \min\{|a_1|, |b_1|, \ldots, |a_d|, |b_d|\}$ and $\overline{\hbar} = \max\{|a_1|, |b_1|, \ldots, |a_d|, |b_d|\}$.

Proof. Denote by $\underline{S}_d := \left\{\big(x', \varphi(x')\big) \in \overline{\Omega} \;\middle|\; x' := (x_1, \ldots, x_{d-1}) \text{ and } \underline{\varphi}(x') \equiv a_d\right\}$. First, we assume that $u \in C^1(\overline{\Omega})$. Then, we get

$$u(x', a_d) = -\frac{1}{a_d}\int_{a_d}^0 \frac{\partial\big(x_d u(x', x_d)\big)}{\partial x_d} dx_d = -\frac{1}{a_d}\int_{a_d}^0 \left(u(x', x_d) + x_d \frac{\partial u(x', x_d)}{\partial x_d}\right) dx_d$$

which implies that

$$
\begin{aligned}
|a_d|^2|u(x',a_d)|^2 &\le \left(\int_{a_d}^0 \left|u(x',x_d) + x_d\frac{\partial u(x',x_d)}{\partial x_d}\right| dx_d\right)^2\\
&\le |a_d|\int_{a_d}^0 \left|\sqrt{d}\frac{u(x',x_d)}{\sqrt{d}} + x_d\frac{\partial u(x',x_d)}{\partial x_d}\right|^2 dx_d\\
&\le |a_d|\big(d+|a_d|^2\big)\int_{a_d}^0 \left(\frac{u^2(x',x_d)}{d} + \left(\frac{\partial u(x',x_d)}{\partial x_d}\right)^2\right) dx_d.
\end{aligned}
$$

Multiplying this inequality by $\sqrt{1+\left(\frac{\partial\underline{\varphi}}{x_1}\right)^2+\ldots+\left(\frac{\partial\underline{\varphi}}{x_{d-1}}\right)^2}\equiv 1$, integrating over $Q := (a_1,b_1)\times(a_2,b_2)\times\ldots\times(a_{d-1},b_{d-1})$, we obtain

$$
\|u\|^2_{L^2(\underline{S}_d)} \le \frac{d+|\overline{h}|^2}{|\underline{h}|}\int_Q\left(\int_{a_d}^0\left(\frac{u^2(x',x_d)}{d}+\left(\frac{\partial u(x',x_d)}{\partial x_d}\right)^2\right)dx_d\right)dx'.
$$

Likewise, denoting by $\overline{S}_d := \big\{\big(x',\varphi(x')\big)\in\overline{\Omega} \;\big|\; x' := (x_1,\ldots,x_{d-1}) \text{ and } \overline{\varphi}(x')\equiv b_d\big\}$, we also get

$$
\|u\|^2_{L^2(\overline{S}_d)} \le \frac{d+|\overline{h}|^2}{|\underline{h}|}\int_Q\left(\int_0^{b_d}\left(\frac{u^2(x',x_d)}{d}+\left(\frac{\partial u(x',x_d)}{\partial x_d}\right)^2\right)dx_d\right)dx'
$$

and arrive at

$$
\|u\|^2_{L^2(\underline{S}_d\cup\overline{S}_d)} \le \frac{d+|\overline{h}|^2}{|\underline{h}|}\int_\Omega\left(\frac{u^2(x)}{d}+\left(\frac{\partial u(x)}{\partial x_d}\right)^2\right)dx.
$$

This implies that (5.5) is satisfied for all $u\in C^1(\overline{\Omega})$. By the everywhere dense property of the set $C^1(\overline{\Omega})$ in $H^1(\Omega)$, we conclude that (5.5) is also satisfied for all $u\in H^1(\Omega)$, which finishes the proof. □

The especial constant $\underline{\alpha}$ in (5.4) is taken by 0.1. With $d=2$, $\Omega=(-1,1)^2$ we then can take $c_1=0.025$ and $c_\gamma=\sqrt{3}$. As an initial approximation we choose $f_0^{h_\ell}(x)\equiv 1\in\mathcal{V}_1^{h_\ell}$ and $p_0^{h_\ell}(x)\equiv(\frac{1}{2},\ldots,\frac{1}{2})\in(\mathcal{V}_0^{h_\ell})^d$. In computations below we choose $\tau=2.10^{-4}$ and $\theta=5.10^{-2}$.

For observations with noise we assume that

$$
z_{\delta_\ell} = g^\dagger + \theta_\ell\cdot R_{g^\dagger} \quad \text{for some} \quad \theta_\ell>0 \quad \text{depending on} \quad \ell, \tag{5.6}
$$

where $R_{g^\dagger}$ are $\partial M^{h_\ell}\times 1$-matrices of random numbers on the interval $(-1,1)$ which are generated by the MATLAB function "rand", and ∂M^{h_ℓ} is the number of boundary nodes of the triangulation $\mathcal{T}^{h_\ell}$ which belong to $\Gamma\subset\partial\Omega$. The measurement error is then computed as $\delta_\ell=\big\|z_{\delta_\ell}-g^\dagger\big\|_{L^2(\Gamma)}$. To satisfy the condition (3.12) in Theorem 3.5 we take $h=h_\ell$, $\rho=\rho_\ell=10^{-3}h_\ell^{1/2}$ and $\theta_\ell=h_\ell\rho_\ell^{1/2}$. We start with the coarsest level $\ell=4$. At each iteration k we compute

$$
\text{Tolerance} := \big\|\nabla J^{h_\ell}\big(f_k^{h_\ell}\big)\big\|_{L^2(\Omega)} - \tau_1 - \tau_2\big\|\nabla J^{h_\ell}\big(f_0^{h_\ell}\big)\big\|_{L^2(\Omega)},
$$

where $\tau_1 := 10^{-5}h_\ell^{1/2}$ and $\tau_2 := 10^{-4}h_\ell^{1/2}$. Then the iteration was stopped if Tolerance ≤ 0 or the number of iterations reached the maximum iteration count of $N=500$. After obtaining the numerical solution of the first iteration process with respect to the coarsest level $\ell=4$, we use its interpolation on the next finer mesh $\ell=8$ as an initial approximation $(f_0^{h_\ell},p_0^{h_\ell})$ for the algorithm on this finer mesh, and proceed similarly on the preceding refinement levels.

Let f_ℓ denote the function which is obtained at *the final iterate* of Algorithm 1 corresponding to the refinement level ℓ. Furthermore, we denote by v_ℓ *the computed numerical solution* to the Dirichlet problem

$$
-\nabla\cdot(\alpha\nabla v) = f_\ell \quad \text{in} \quad \Omega \quad \text{and} \quad v = \begin{cases}\big(\mathcal{N}_{f^\dagger}j^\dagger\big)_{|\partial\Omega\setminus\Gamma} & \text{on} \quad \partial\Omega\setminus\Gamma,\\ \big(\mathcal{N}_{f_\ell}j^\dagger\big)_{|\Gamma} & \text{on} \quad \Gamma.\end{cases}
$$

We use the following abbreviations for the errors

$$L^2_f = \left\|f_\ell - f^\dagger\right\|_{L^2(\Omega)}, \quad L^2_{\mathcal{D}} = \left\|v^\dagger - v_\ell\right\|_{L^2(\Omega)} \quad \text{and} \quad H^1_{\mathcal{D}} = \left\|v^\dagger - v_\ell\right\|_{H^1(\Omega)},$$

where $v^\dagger$ denotes the solution of the equation (5.1) supplemented with the Dirichlet boundary condition $\Phi_{|\partial\Omega} = \left(\mathcal{N}_{f^\dagger} j^\dagger\right)_{|\partial\Omega}$.

First, we consider the case Γ is the bottom surface of the domain Ω, i.e. $\Gamma = \{x = (x_1, x_2) \in \overline{\Omega} \mid x_2 = -1\}$. The numerical results of this case are summarized in Table 1 and Table 2, where we present the refinement level ℓ, mesh size h_ℓ of the triangulation, regularization parameter ρ_ℓ, measured noise δ_ℓ, number of iterations, value of tolerances and errors L^2_f, $L^2_{\mathcal{D}}$ and $H^1_{\mathcal{D}}$. All figures presented correspond to $\ell = 64$. Figure 1 on the left shows the interpolation of the exact $I_1^{h_\ell} f^\dagger(x_1, 0)$ (the blue color line) as well as the computed numerical solution f_ℓ of the algorithm at the final iteration and with respect to $x_2 = 0$ (the red color line); in the middle it performs the differences $I_1^{h_\ell} f^\dagger(x_1, 0) - f_\ell(x_1, 0)$; and on the right it shows the difference $v^\dagger - v_\ell$.

ℓ	h_ℓ	ρ_ℓ	δ_ℓ	Iterate	Tolerance
4	0.7071	8.4090e-4	2.2298e-2	146	-2.4166e-6
8	0.3536	5.9460e-4	8.8694e-3	263	-8.2494e-7
16	0.1766	4.2045e-4	2.5488e-3	337	-5.2736e-8
32	8.8388e-2	2.9730e-4	1.1764e-3	464	-2.0707e-8
64	4.4194e-2	2.1022e-4	5.5882e-4	500	1.8725e-5

Table 1: Refinement level ℓ, mesh size h_ℓ of the triangulation, regularization parameter ρ_ℓ, measurement noise δ_ℓ, number of iterates and value of tolerances.

ℓ	L^2_f	$L^2_{\mathcal{D}}$	$H^1_{\mathcal{D}}$
4	0.6823	1.4944e-2	3.9036e-2
8	0.3497	8.5693e-3	3.3397e-2
16	0.1689	3.2264e-3	2.2841e-2
32	0.1071	1.2829e-3	1.5022e-2
64	0.7345e-2	5.9989e-4	1.2209e-2

Table 2: Errors L^2_f, $L^2_{\mathcal{D}}$ and $H^1_{\mathcal{D}}$.

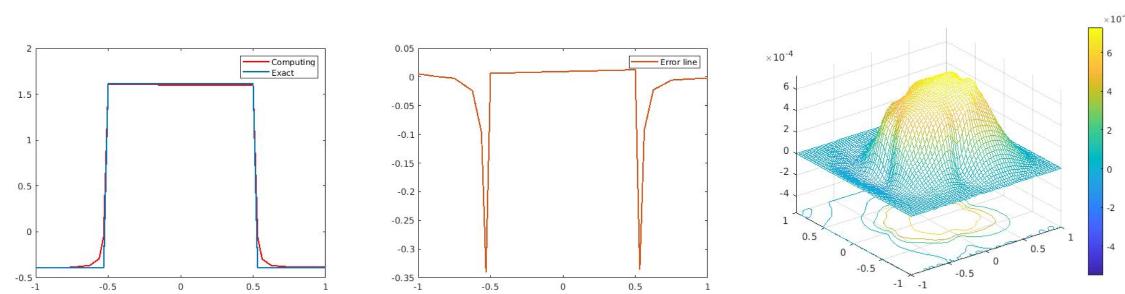

Figure 1: Observations taking at the bottom surface: Interpolation $I_1^{h_\ell} f^\dagger(x_1, 0)$ as well as computed numerical solution $f_\ell(x_1, 0)$ of the algorithm at the final iteration (Left), and the differences $I_1^{h_\ell} f^\dagger(x_1, 0) - f_\ell(x_1, 0)$ (Middle), $v^\dagger - v_\ell$ (Right).

We now consider the case Γ includes the bottom surface and the left surface of the domain Ω, i.e. $\Gamma = \{x = (x_1, x_2) \in \overline{\Omega} \mid x_2 = -1\} \cup \{x = (x_1, x_2) \in \overline{\Omega} \mid x_1 = -1\}$. In this case θ_ℓ in (5.6) is changed to $\theta_\ell = \frac{1}{2} h_\ell \rho_\ell^{1/2}$. Computations show that the measurement noise $\delta_\ell = 4.6039.10^{-4}$, Tolerance $= -1.8493.10^{-8}$, the iteration stops at $n = 477$. Errors $L^2_f = 6.1605.10^{-2}$, $L^2_{\mathcal{D}} = 5.8110.10^{-4}$ and $H^1_{\mathcal{D}} = 1.1164.10^{-2}$. Finally, in Figure 2 from left to right we show the interpolation $I_1^{h_\ell} f^\dagger$ of the exact source as well as the computed numerical solution f_ℓ of the algorithm at the final 477^{th}-iteration and differences $I_1^{h_\ell} f^\dagger - f_\ell$ $v^\dagger - v_\ell$, respectively.

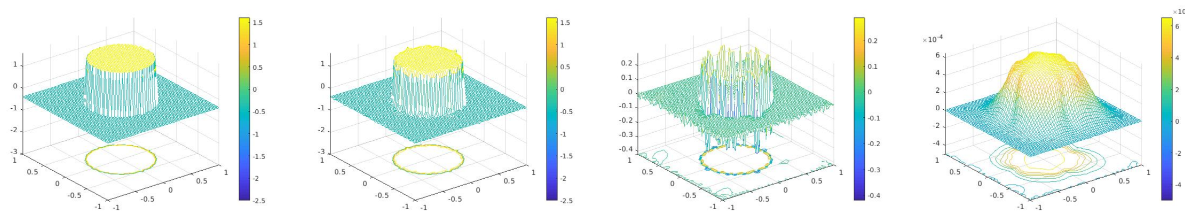

Figure 2: Observations taking at the bottom surface and the left surface: Interpolation $I_1^{h_\ell} f^\dagger$, computed numerical solution f_ℓ of the algorithm at the final 477^{th}-iteration and differences $I_1^{h_\ell} f^\dagger - f_\ell$, $v^\dagger - v_\ell$, where $\ell = 64$, measurement noise $\delta_\ell = 4.6039.10^{-4}$ and errors $L^2_f = 6.1605.10^{-2}$, $L^2_{\mathcal{D}} = 5.8110.10^{-4}$, $H^1_{\mathcal{D}} = 1.1164.10^{-2}$.

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