

Attraction of principal heteroclinic cycles

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Abstract

We provide a counterexample for a conjecture on the stability of principal heteroclinic cycles, i.e. subcycles consisting of connections tangent to the eigenspaces of the strongest expanding eigenvalues at the equilibria, which was stated in P. Ashwin and P. Chossat (1998) *Attractors for robust heteroclinic cycles with continua of connections, Journal of Nonlinear Science* **8**, 103–129. This contributes to a clear distinction between heteroclinic dynamics when the jacobian matrix at equilibria has only real eigenvalues and when some eigenvalues are complex.

Keywords: heteroclinic cycle, heteroclinic network, asymptotic stability

AMS classification: 34C37, 37C80, 37C75

1 Introduction

Heteroclinic dynamics (involving equilibria and trajectories connecting them) appear in various real-life systems ranging from fluid dynamics to Lotka-Volterra-type models in a persistent way. When 1-dimensional connecting trajectories from an equilibrium ξ_i to an equilibrium ξ_j are contained in a higher-dimensional unstable manifold of ξ_i , does the connection tangent to the eigendirection of the greatest expanding eigenvalue attract the biggest proportion of initial conditions near the cycle? Answering this type of question is crucial when it comes to identifying (the most relevant) attractors in a system with a possibly large invariant heteroclinic set. If not, some competition may arise among the various 1-dimensional connections in the unstable manifold of ξ_i . Such a situation is described by Kirk and Silber [10] who construct a heteroclinic network (a connected union of finitely many heteroclinic cycles), consisting of two heteroclinic cycles. They show that, depending on a combination of the magnitudes of the eigenvalues, the cycles

take turns in attracting the biggest proportion of the initial conditions close to the network.

Ashwin and Chossat [1] conjecture that, under some hypotheses not satisfied by the example in [10], the answer to the question above is affirmative. The authors of [1] prove a special case of their conjecture when the cycle is homoclinic (connections are from one equilibrium to itself) and the eigenvalues are complex.

One of the simplest heteroclinic objects is a cycle with two hyperbolic equilibria and connections in both directions between these equilibria. We construct an equivariant vector field in \mathbb{R}^4 supporting such a heteroclinic cycle and satisfying the hypotheses of [1]. There is a 1-dimensional connection from one equilibrium to the other and a 2-dimensional connection in the opposite direction. Our construction is a modification of an example in Castro and Lohse [4]. We prove that the conjecture of [1] does not hold for this cycle.

Our result shows that heteroclinic dynamics involving equilibria at which the linearised dynamics are governed by complex eigenvalues is substantially different from heteroclinic dynamics when only real eigenvalues play a part. Several authors have contributed to the study of the stability of heteroclinic cycles and connections (see, for instance, Hofbauer [7], Melbourne [15], Krupa and Melbourne [12, 13] or Podvignina and Ashwin [16]) either by establishing asymptotic stability of heteroclinic objects or by defining and providing conditions to determine intermediate notions of stability. However, when two connections between a pair of equilibria are available, it is important to determine which one is the preferred one. The conjecture in [1] solves this problem in some instances. Our counterexample shows there are unanswered questions when only real eigenvalues are present in the linearization at the equilibria. These we leave for further research.

We finish this section with a brief description of the essential definitions and concepts. The following section constructs the vector field which we show to be a counterexample for the conjecture in [1]. The final section concludes and points towards relevant open questions in the context of stability in heteroclinic dynamics.

Background: We use the term heteroclinic cycle as in [1] where all the precise definitions and further detail can be found. We assume the reader is somewhat familiar with robust heteroclinic cycles in a symmetric context, for a comprehensive overview we refer to Krupa [11]. In what follows we consider dynamics induced by an ODE

$$\dot{x} = f(x), \tag{1}$$

where $x \in \mathbb{R}^n$ and f is smooth and Γ -equivariant for some finite group $\Gamma \subset O(n)$.

Given two hyperbolic equilibria ξ_i and ξ_j of system (1), a connecting trajectory between them exists in $W^u(\xi_i) \cap W^s(\xi_j)$ if this intersection is non-empty. A heteroclinic cycle is a sequence of such connecting trajectories among a set of finitely many distinct equilibria ξ_1, \dots, ξ_m such that $\xi_{m+1} = \xi_1$. The heteroclinic cycle is the union of the equilibria and the connections.

In what follows $C_{ij} = W^u(\xi_i) \cap W^s(\xi_j)$ denotes the set of trajectories connecting two equilibria ξ_i and ξ_j . At an equilibrium ξ_i the set of *principal connections* is

$$C_{ij}^p = W^{pu}(\xi_i) \cap W^s(\xi_j),$$

where $W^{pu}(\xi_i)$ is the invariant manifold of trajectories tangent to the generalized eigenspace of the strongest expanding eigenvalue at ξ_i . A *principal cycle* is comprised only of principal connections.

We use *attractor* in the sense of Milnor, as in Definition 3 of [1], that is, a Milnor attractor is a compact invariant set whose basin of attraction has positive Lebesgue measure.

2 The conjecture and its counterexample

The conjecture of [1] states that for a closed heteroclinic cycle among equilibria ξ_1, \dots, ξ_m satisfying (Ha)-(Hd) and (3) below, generically, the principal heteroclinic cycle is an attractor. The hypotheses are:

- (Ha) for any non-empty connection C_{ij} there exists an isotropy subgroup Σ such that all trajectories in C_{ij} have isotropy Σ ,
 - (Hb) for any Σ in (Ha) ξ_j is a sink for the flow restricted to $\text{Fix}(\Sigma)$,
 - (Hc) the heteroclinic cycle contains all unstable manifolds of its equilibria,
 - (Hd) the eigenspaces tangent to connections C_{ij} and C_{jk} at $\xi_j \in \text{Fix}(\Delta_j)$ lie within a single Δ_j -isotypic component of \mathbb{R}^n .
- (3) Let $-\bar{c}_j$ be the weakest contracting eigenvalue and \bar{e}_j be the strongest expanding eigenvalue at ξ_j . Then $\prod_{j=1}^m \bar{c}_j > \prod_{j=1}^m \bar{e}_j$.

Our counterexample consists in the following modification of a vector field

generating a (B_2^+, B_2^+) network¹ from [4],

$$\begin{cases} \dot{x}_1 = x_1 + \sum_{i=1}^4 b_{1i}x_i^2 + c_1x_1^3 \\ \dot{x}_2 = x_2 + x_2 \sum_{i=1}^4 b_{2i}x_i^2 + d_2x_1x_2 \\ \dot{x}_3 = x_3 + x_3 \sum_{i=1}^4 b_{3i}x_i^2 + c_3x_3^2x_4 + d_3x_1x_3 \\ \dot{x}_4 = x_4 + x_4 \sum_{i=1}^4 b_{4i}x_i^2 + c_4x_3x_4^2 + d_4x_1x_4 \end{cases}, \quad (2)$$

where all constants are real and chosen conveniently below.

This vector field is equivariant under the action of the group $\Gamma \cong \mathbb{Z}_2^2$ generated by

$$\begin{aligned} \kappa_2.(x_1, x_2, x_3, x_4) &= (x_1, -x_2, x_3, x_4), \\ \kappa_{34}.(x_1, x_2, x_3, x_4) &= (x_1, x_2, -x_3, -x_4). \end{aligned}$$

The isotypic decomposition of \mathbb{R}^4 with respect to Γ is

$$\mathbb{R}^4 = L_1 \oplus L_2 \oplus P_{34}, \quad (3)$$

where L_i is the i -th coordinate axis and $P_{ij} = L_i \oplus L_j$.

We choose coefficients in an open set such that the system (2)

- (i) possesses two equilibria $\xi_a = (x_a, 0, 0, 0), \xi_b = (x_b, 0, 0, 0) \in L_1$ such that $x_a < 0 < x_b$;
- (ii) in P_{12} , ξ_a is a saddle and ξ_b is a sink; furthermore, there is a connection $C_{ab} \subset P_{12}$;
- (iii) in $S_{123} := L_1 \oplus P_{34}$, ξ_a is a sink and ξ_b is a saddle; furthermore, there is a continuum of connections $C_{ba} \subset L_1 \oplus P_{34}$;
- (iv) the principal connection C_{ba}^p from ξ_b to ξ_a lies in P_{13} .

To achieve such a choice we proceed as follows, collecting the conditions in Table 1. Choose $b_{11} \neq 0$ and $b_{11}^2 - 4c_1 > 0$ so that $x_a \neq -x_b$. In L_1 , x_a and x_b satisfy

$$1 + b_{11}x_{a/b} + c_1x_{a/b}^2 = 0 \Leftrightarrow x_{a/b} = \frac{-b_{11} \pm \sqrt{b_{11}^2 - 4c_1}}{2c_1}.$$

Choose $c_1 < 0$ so that

$$x_a = \frac{-b_{11} + \sqrt{b_{11}^2 - 4c_1}}{2c_1} < 0 \quad \text{and} \quad x_b = \frac{-b_{11} - \sqrt{b_{11}^2 - 4c_1}}{2c_1} > 0,$$

¹In [4] another definition of heteroclinic cycle and network is used which is why the heteroclinic object is called a network. In the context of the present article it is a heteroclinic cycle.

proving (i).

To prove (ii), choose coefficients so that the flow far from the origin points inwards towards the origin. For example, $b_{22} < 0$ besides the choice already made of $c_1 < 0$. Define a region $D \subset P_{12}$ bounded by the horizontal axis and the two vertical lines $x_1 = x_{a/b}$. If we choose $b_{12} > 0$ then $\dot{x}_1 > 0$ if $x_1 \in \{\xi_a, 0, \xi_b\}$ and (see the next proof of stability below) the unstable manifold of ξ_a and the stable manifold of ξ_b are as pictured in Figure 1.

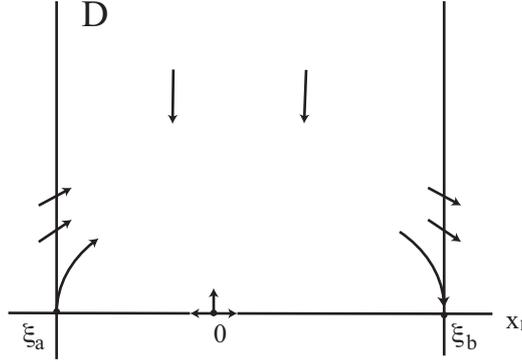


Figure 1: The flow in $D \subset P_{12}$.

Since the origin is a source, it suffices to show there are no equilibria in D to ensure the existence of C_{ab} . Equilibria in P_{12} , outside the axes, satisfy

$$\begin{cases} x_1 + b_{11}x_1^2 + b_{12}x_2^2 + c_1x_1^3 = 0 \\ 1 + b_{21}x_1^2 + b_{22}x_2^2 + d_2x_1 = 0 \end{cases} \Leftrightarrow \begin{cases} x_2^2 = -\frac{1}{b_{12}}(x_1 + b_{11}x_1^2 + c_1x_1^3) \\ x_2^2 = -\frac{1}{b_{22}}(1 + b_{21}x_1^2 + d_2x_1) \end{cases}.$$

The right-hand side of the first equation is a cubic with zeros at the origin, $\xi_{a/b}$. The right-hand side of the second equation is a parabola with a positive minimum value provided $b_{21} > 0$. There is always a solution for this system of equations but, by choosing b_{22} close to zero, the parabola is pulled upwards thus guaranteeing that it occurs outside D . See Figure 2.

To prove the stability claims in (ii) and (iii), note that at the equilibria the Jacobian matrix is diagonal with the following entries:

$$\begin{aligned} & b_{11}x_{a/b} + 2c_1x_{a/b}^2, \quad 1 + b_{21}x_{a/b}^2 + d_2x_{a/b}, \\ & 1 + b_{31}x_{a/b}^2 + d_3x_{a/b}, \quad 1 + b_{41}x_{a/b}^2 + d_4x_{a/b}. \end{aligned}$$

At ξ_a and ξ_b , the eigenvalue along L_1 is, respectively,

$$b_{11}x_a + 2c_1x_a^2 = x_a\sqrt{b_{11}^2 - 4c_1} < 0 \quad \text{and} \quad b_{11}x_b + 2c_1x_b^2 = -x_b\sqrt{b_{11}^2 - 4c_1} < 0.$$

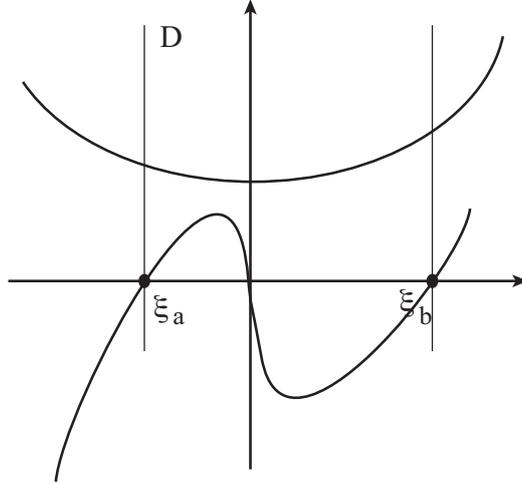


Figure 2: The location of equilibria in P_{12} .

The eigenvalue along L_2 can be written at the equilibria as

$$1 - \frac{b_{21}}{c_1} + \left(d_2 - \frac{b_{21}}{c_1} b_{11} \right) x_{a/b}.$$

Since $x_a < 0 < x_b$, by choosing $d_2 - \frac{b_{21}}{c_1} b_{11} < 0$ and large in absolute value, we achieve the desired signs:

$$1 - \frac{b_{21}}{c_1} + \left(d_2 - \frac{b_{21}}{c_1} b_{11} \right) x_a > 0 \quad \text{and} \quad 1 - \frac{b_{21}}{c_1} + \left(d_2 - \frac{b_{21}}{c_1} b_{11} \right) x_b < 0.$$

Analogous calculations for the eigenvalues along L_3 and L_4 at the equilibria show that if $d_3 - \frac{b_{31}}{c_1} b_{11} > 0$ and $d_4 - \frac{b_{41}}{c_1} b_{11} > 0$ and large in absolute value, both eigenvalues are negative at ξ_a and positive at ξ_b .

To finish proving (iii) and ensure the existence of C_{ba} we choose coefficients $b_{33}, b_{44}, c_3, c_4 < 0$ so that infinity is repelling and proceed as in the proof of the existence of C_{ab} to ensure connections in P_{13} and P_{14} .

Next, we observe that the set of points Z in S_{134} where $\dot{x}_1 = 0$ is determined by $x_1 + b_{11}x^2 + c_1x^3 = -b_{13}x_3^2 - b_{14}x_4^2$. In planes of constant x_1 this is an ellipse, see Figure 3. Outside of Z we have $\dot{x}_1 < 0$ (inside we have $\dot{x}_1 > 0$) provided $b_{13}, b_{14} < 0$, so that trajectories on $W^u(\xi_b)$, which is tangent to the affine plane spanned by x_3 and x_4 , have decreasing x_1 component close to ξ_b and therefore move in the direction of ξ_a . Since any equilibria in S_{134} lie in Z then all the trajectories in $W^u(\xi_b)$ approach ξ_a .

To prove (iv), choose coefficients so that at ξ_b the eigenvalue along x_3 is

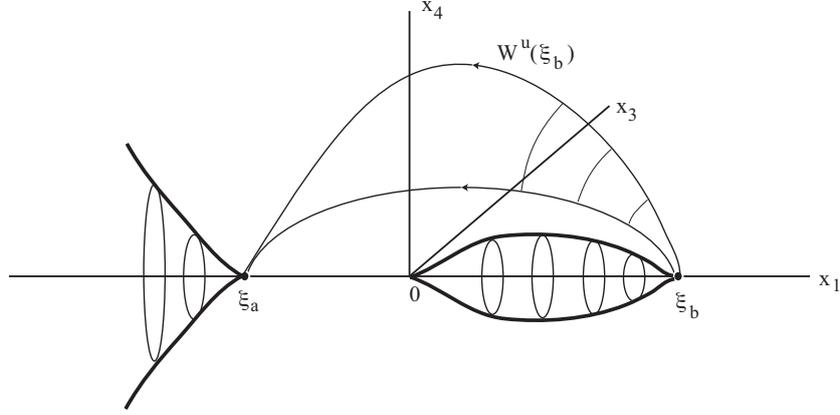


Figure 3: The set Z (thick lines) in S_{134} .

greater than that along x_4 , that is, so that

$$1 + b_{31}x_b^2 + d_3x_b > 1 + b_{41}x_b^2 + d_4x_b \Leftrightarrow x_b > \frac{d_4 - d_3}{b_{31} - b_{41}},$$

which is trivially verified if the right-hand side is negative.

The conditions imposed so far correspond to C1–C15 in Table 1. It is clear that C1–C11 are compatible and define an open set in the space of all coefficients. Writing

$$C13 - C12 = d_4 - d_3 - \frac{b_{11}}{c_1}(b_{41} - b_{31}),$$

relates these conditions to C14 and C15, ensuring the compatibility of C11–C15. In particular, $C13 - C12 > 0$.

Theorem 2.1. *The system (2) with the choice of coefficients in Table 1 satisfies (Ha)–(Hd) and (3) and the principal cycle is not an attractor.*

Proof. We verify the hypotheses:

- (Ha) The connection C_{ab} has isotropy $\{\text{Id}, \kappa_{34}\}$ and C_{ba} has isotropy $\{\text{Id}, \kappa_2\}$.
- (Hb) This is clear by (ii) in the construction above.
- (Hc) The 2-dimensional unstable manifold of ξ_b is just the set of connections C_{ba} . Analogously, the 1-dimensional unstable manifold of ξ_a is the connection C_{ab} .
- (Hd) At ξ_a and ξ_b the eigenspace tangent to C_{ab} is L_2 and that tangent to C_{ba} is P_{34} .

(3) Direct inspection shows that

$$\begin{aligned}\bar{e}_a &= 1 - \frac{b_{21}}{c_1} + \left(d_2 - \frac{b_{21}}{c_1}b_{11}\right)x_a \\ \bar{e}_b &= 1 - \frac{b_{31}}{c_1} + \left(d_3 - \frac{b_{31}}{c_1}b_{11}\right)x_b \\ \bar{c}_b &= 1 - \frac{b_{21}}{c_1} + \left(d_2 - \frac{b_{21}}{c_1}b_{11}\right)x_b,\end{aligned}$$

whereas we have

$$\bar{c}_a = 1 - \frac{b_{31}}{c_1} + \left(d_3 - \frac{b_{31}}{c_1}b_{11}\right)x_a \quad \text{or} \quad \bar{c}'_a = 1 - \frac{b_{41}}{c_1} + \left(d_4 - \frac{b_{41}}{c_1}b_{11}\right)x_a.$$

Using \bar{c}_a we obtain $\prod_{j=a,b} \bar{c}_j > \prod_{j=a,b} \bar{e}_j$ if and only if

$$(c_1 - b_{31})(d_2c_1 - b_{21}b_{11}) < (c_1 - b_{21})(d_3c_1 - b_{31}b_{11}).$$

The choice of \bar{c}'_a produces the analogous inequality by replacing the index 3 by 4.

Under these circumstances the principal cycle consists of the two equilibria together with C_{ab} and the trajectory $C_{ba}^p = C_{ba} \cap P_{13}$. In order to apply Theorem A.1(ii) in [4], we observe that $\rho, \bar{\rho} > 1$ follows from condition (3) above. We write the condition $\delta > 0$ in [4] equivalently in the coefficients of (2) as follows

$$(1 + d_3x_a + b_{31}x_a^2)(1 + d_4x_b + b_{41}x_b^2) - (1 + d_3x_b + b_{31}x_b^2)(1 + d_4x_a + b_{41}x_a^2) > 0.$$

This is equivalent to

$$d_4 - d_3 + (b_{41} - b_{31})(x_a + x_b) + (d_3b_{41} - d_4b_{31})x_ax_b > 0,$$

where the only quantities not already chosen to be positive are: $x_a + x_b$ has the sign of b_{11} and can be chosen positive, and $d_3b_{41} - d_4b_{31}$ which we choose to be negative so that its product with $x_ax_b < 0$ is positive.

Then by Theorem A.1(ii) in [4] the (non-principal) cycle consisting of the equilibria together with C_{ab} and the trajectory $C_{ba} \cap P_{14}$ attracts a set of positive measure of nearby initial conditions, while the principal cycle does not². Note that even though none of these cycles is simple³ in the sense of

²For the reader familiar with the concept of *local stability index* introduced by Podvigina and Ashwin [16] and its relation with determining the stability of compact invariant sets both in [16] and in [14], Theorem A.1(ii) in [4] establishes that the local stability indices of both connections in the principal cycle are equal to $-\infty$, whereas those of the two connections in the non-principal cycle are equal to $+\infty$ and negative but finite, respectively. Since having at least one stability index greater than $-\infty$ is equivalent to being an attractor, this proves our claim.

³The cycles are not simple because P_{13} and P_{14} are not fixed point spaces.

	Inequality	ensuring
C1	$b_{13} < 0$	$\dot{x}_1 < 0$ in S_{134}
C2	$b_{14} < 0$	
C3	$b_{12} > 0$	$\dot{x}_1 > 0$ in P_{12}
C4	$b_{22} < 0$	infinity is repelling
C5	$b_{33} < 0$	
C6	$b_{44} < 0$	
C7	$c_3 < 0$	
C8	$c_4 < 0$	
C9	$c_1 < 0$	infinity is repelling and position of $\xi_{a/b}$ is as desired
C10	$b_{21} > 0$	location of equilibria in P_{12}
C11	$d_2 - \frac{b_{21}}{c_1}b_{11} < 0$, large	the eigenvalues outside L_1 have the desired sign
C12	$d_3 - \frac{b_{31}}{c_1}b_{11} > 0$, large	
C13	$d_4 - \frac{b_{41}}{c_1}b_{11} > 0$, large	
C14	$d_4 - d_3 > 0$	$C_{ba}^p \subset P_{13}$ and not attracting, together with C16
C15	$b_{41} - b_{31} > 0$	
C16	$d_3b_{41} - d_4b_{31} < 0$	$C_{ba} \cap P_{14} \neq C_{ba}^p$ is attracting
C17	$(c_1 - b_{31})(d_2c_1 - b_{21}b_{11}) < (c_1 - b_{21})(d_3c_1 - b_{31}b_{11})$	hypothesis (3) is satisfied
C18	$b_{11} > 0$	

Table 1: List of conditions imposed on the coefficients of (2) in the construction of the vector field and the proof of Theorem 2.1. The first column assigns a label to each condition and the third column states the purpose of the restriction. The middle column contains the conditions on the coefficients.

Krupa and Melbourne [13], Theorem 3.10 in [6] guarantees that the stability properties are the same as those in Theorem A.1 in [4].

□

Note that C11 implies that in C17 we have $d_2c_1 - b_{21}b_{11} > 0$. Also, C12 implies that in C17 we have $d_3c_1 - b_{31}b_{11} < 0$. Then C17 can be satisfied by setting $c_1 - b_{31} < 0$ and $c_1 - b_{21} < 0$. Observe also that C16 is compatible with C14 and C15. Hence, the subset of coefficients satisfying the inequalities in Table 1 is open.

3 Concluding remarks

Our construction of the counterexample above illustrates the fact that dynamics ruled by real eigenvalues are quite different from those ruled by complex eigenvalues. Although this is not surprising, it is worth noting. To further explore this effect, it would be interesting to check the conjecture in examples such as that given by Kirk *et al.* [9], of a heteroclinic network with some two-dimensional connections and complex eigenvalues at one of the equilibria. Its asymptotic stability stands in contrast to a recent result in Podvignina *et al.* [17], stating that heteroclinic networks with 1-dimensional connections and only real eigenvalues cannot be asymptotically stable.

Given our counterexample, a good understanding of the dynamics near heteroclinic cycles for which only real eigenvalues exist requires more than the knowledge and comparison of the magnitude of eigenvalues. Such heteroclinic cycles appear naturally in the context of population dynamics and game theory when restricting the state space to a finite-dimensional simplex, see Hofbauer and Sigmund [8] and references therein. The flow-invariance of the edges of the simplex creates heteroclinic cycles with connections along the edges and only real eigenvalues. These are called edge cycles/networks by Field [5].

The work of Rodrigues [18] provides a first step in one of the directions opened by the fact that the conjecture of [1] does not hold true: that of finding minimal conditions such that it does. Our example is excluded from the result in [18] by their assumption that all connections are 1-dimensional. This also points towards another interesting possibility of broadening the scope of the conjecture in [1], i.e. asking when principal subcycles of heteroclinic networks (rather than cycles) are attracting. Of course, attention must then be paid to the distinction between cycle and network, which is not obvious as mentioned above. Much in this context is still open to understanding.

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