On partitioning the edges of an infinite digraph into directed cycles

Attila Joó*

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Abstract

Nash-Williams proved in [2] that for an undirected graph G the set $E(G)$ can be partitioned into cycles if and only if every cut has either even or infinite number of edges. We prove the following directed analogue of his theorem: the edge-set of a digraph can be partitioned into directed cycles if and only if for each subset of the vertices the cardinality of the ingoing and the outgoing edges are equal.

1 Introduction

Nash-Williams was one of the greatest researchers of the theory of finite and infinite graphs in the 20th century. One of his famous result is that the edges of a $2k$-edge-connected graph can be directed to obtain a k-edge-connected digraph. He proved it for finite graphs and claimed to be true for infinite ones in [3]. For the infinite case there is still no published proof. The best partial result is due to C. Thomassen in [5], he showed that the edges of a $8k$-edge-connected infinite graph can be directed to obtain a k-edge-connected digraph. An other famous result of Nash-Williams is the following.

Theorem 1 (Nash-Williams, [2] (p. 235 Theorem 3))**.** *If G is an undirected graph, then $E(G)$ can be partitioned into cycles if and only if every cut has either even or infinite number of edges.*

At the and of his article ([2] page 237 Theorem 3') he claimed without proof the following directed analogue of his theorem:

Theorem 2. *If $D = (V, A)$ is a directed graph, then A can be partitioned into directed cycles if and only if for all $X \subseteq V$ the cardinalities of the ingoing and the outgoing edges of X are equal.*

As far as we know there is no published (even partial) result about the directed version. L. Soukup gave a new shorter proof to the undirected version (Theorem 5.1 of [4]) based on elementary submodels but nothing about the directed case. The main difficulty in contrast to the undirected case, applying elementary submodel approach, is that in the undirected case one can find a finite witness for the violation of the condition (an odd cut) but in the directed case we do not necessarily have a finite witness. Our main result is a proof of Theorem 2 by handling the additional difficulties of the elementary submodel approach.

*MTA-ELTE Egerváry Research Group, Department of Operations Research, Eötvös University, Budapest, Hungary. E-mail: joapaat@cs.elte.hu.

2 Notation

Let $D = (V, A)$ be a digraph. We denote by $\mathbf{out_D(X)}$ and by $\mathbf{in_D(X)}$ the set of outgoing and ingoing edges of X in D respectively. For an $X \subseteq V$ let $\boldsymbol{D[X]}$ the subgraph of D induced by X. The **weakly connected** components of a digraph are the connected components of its undirected underlying graph with the original orientations. We call a digraph weakly connected if its undirected underlying graph is connected. If x, y are vertices of the path P, then we denote by $\boldsymbol{P[x, y]}$ the segment of P between x and y. We will also use some basic standard notation from set theory and model theory.

We call $X \subseteq V$ **overloaded** (with respect to $D = (V, A)$) if $|\mathsf{out}_D(X)| < |\mathsf{in}_D(X)|$ and we call D **balanced** if there is no such an X. If M is an arbitrary set, then let $\boldsymbol{D(M)} := (V \cap M, A \cap M)$ and let $\boldsymbol{D \setminus\setminus M} := (V, A \setminus M)$.

3 Preparations

3.1 An observation about overloaded sets

We need the following basic observation to find overloaded sets in an unbalanced digraph in a special form.

Lemma 3. *If $D = (V, A)$ is an unbalanced digraph, then it has a weakly connected component with vertex set Z and an $X \cup Y$ partition of Z such that $D[X]$ and $D[Y]$ are weakly connected and X is overloaded in D.*

Proof: Let $X' \subseteq V$ be overloaded and let X_i $(i \in I)$ be the vertex sets of the weakly connected components of $D[X']$. Then

$$\sum_{i \in I} |\mathsf{out}_D(X_i)| = |\mathsf{out}_D(X')| < |\mathsf{in}_D(X')| = \sum_{i \in I} |\mathsf{in}_D(X_i)|$$

therefore there is an $i_0 \in I$ such that $|\mathsf{out}_D(X_{i_0})| < |\mathsf{in}_D(X_{i_0})|$. Let Y_j $(j \in J)$ be the vertex sets of the weakly connected components of $D[V \setminus X_{i_0}]$ then

$$\sum_{j \in J} |\mathsf{in}_D(Y_j)| = |\mathsf{out}_D(X_{i_0})| < |\mathsf{in}_D(X_{i_0})| = \sum_{i \in J} |\mathsf{out}_D(Y_j)|$$

thus there is an $j_0 \in J$ such that $|\mathsf{in}_D(Y_{j_0})| < |\mathsf{out}_D(Y_{j_0})|$. Denote by Z the vertex set of the weakly connected component of D that contains Y_{j_0} then $X := Z \setminus Y_{j_0}$ is appropriate and Y_{j_0} will be the desired Y. ■

3.2 Elementary submodels

We give here a quick survey about elementary submodel techniques that we use to prove the main result of this chapter. One can find a more detailed survey with many combinatorial applications in [4].

All the formulas and models in this chapter are in the first order language of set theory and the models are $\in$-models i.e. the "element of" relation in them is the real "$\in$". A model M_0 is an **elementary submodel of** M_1 if $M_0 \subseteq M_1$ and for each formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in M_0 : M_0 \models \varphi(a_1, \ldots, a_n)$ if and only if $M_1 \models \varphi(a_1, \ldots, a_n)$. Let $\Sigma = \{\varphi_1, \ldots, \varphi_n\}$

be a finite set of formulas where the free variables of φ_i are $x_{i,1}, \dots, x_{i,n_i}$. We call a set M a **Σ-elementary submodel** if the formulas in Σ are absolute between M and the universe i.e.

$$\bigwedge_{i=1}^{n} [\forall a_1, \dots, a_{n_i} \in M[(M \models \varphi_i(a_1, \dots, a_{n_i})) \iff \varphi_i(a_1, \dots, a_{n_i})]] .$$

The common practice by elementary submodel techniques is to fix a large enough finite set Σ of formulas at the beginning and do not say explicitly what it is. After that, during the proof the author refers finitely many times that this and that formula is in Σ. If it is not satisfactory for someone, then he or she may consider Σ as the set of those formulas that have length at most 10^{10} and contains at most the variables: $v_1, \dots, v_{10^{10}}$. Anyway, from now on Σ is a fixed, large enough set of formulas.

Our next goal is to create Σ-elementary submodels. We will use the following two well-known theorems. One can find them in [1] as well as in other textbooks in the topic.

Theorem 4 (Levy's Reflection Theorem)**.** *For any ordinal α there is an ordinal $\beta \geq \alpha$ such that V_β is a Σ-elementary submodel.*

Theorem 5 (Downward Löwenheim–Skolem-Tarski Theorem)**.** *Let $\mathfrak{A}$ be a first order structure for language $\mathcal{L}$ with basic set A. Denote the set of $\mathcal{L}$-formulas by $\mathsf{Form}(\mathcal{L})$. Assume that $|\mathsf{Form}(\mathcal{L})| \leq |A|$. Then for all $B \subseteq A$ there exists an elementary submodel $\mathfrak{C}$ of $\mathfrak{A}$ with basic set C such that $B \subseteq C$ and $|C| = |\mathsf{Form}(\mathcal{L})| + |B|$.*

Remark 6. In the case of set theory $|\mathsf{Form}(\mathcal{L})| = \aleph_0$ so if B is infinite, then we may write $|C| = |B|$ in Theorem 5.

Now we can prove a fundamental fact about Σ-elementary submodels.

Proposition 7. *For all infinite set B there is a Σ-elementary submodel M such that $B \subseteq M$ and $|M| = |B|$.*

Proof: By Theorem 4 there is a $\beta \geq \mathsf{rank}(B)$ such that V_β is a Σ-elementary submodel. Then $B \subseteq V_\beta$ since $\beta \geq \mathsf{rank}(B)$. Thus by using Theorem 5 with $\mathfrak{A} = V_\beta$ and with B we get an elementary submodel M of V_β such that $|M| = |B|$ and $B \subseteq M$. Finally M is a Σ-elementary submodel because it is an elementary submodel of a Σ-elementary submodel. ∎

4 Main result

Proof of Theorem 2. A directed cycle has the same number of ingoing and outgoing edges for an $X \subseteq V$ thus if A can be partitioned into directed cycles, then D must be balanced. Next we deal with the nontrivial direction of the equivalence.

Observe that the weakly connected components of a balanced digraph are strongly connected thus each of their edges are in some directed cycle. Furthermore, a balanced digraph remains balanced after the deletion of the edges of a directed cycle. If a balanced digraph is at most countable and its edges are: $e_1, e_2, \dots$, then we can create a desired partition by the following recursion: in the n-th step delete the edges of a directed cycle which contains e_n from the remaining digraph if it still contains e_n, otherwise do nothing.

In the uncountable case the naive recursive method above does not work because in a transfinite recursion one can not ensure that after the first limit step the remaining digraph is still balanced.

Lemma 8. *For all infinite set B there is a Σ-elementary submodel M such that $B \subseteq M$, $|M| = |B|$, and for any balanced digraph $D \in M$ the edge-set $A(D) \cap M$ can be partitioned into directed cycles.*

Theorem 2 follows directly from Lemma 8: let D be an arbitrary balanced digraph and use Lemma 8 with $B = \{D\} \cup V(D) \cup A(D)$. Then $D = D(M)$ and hence we get a desired partition. *Proof:* We prove Lemma 8 by transfinite induction on $|B|$. Consider first the case $|B| = \aleph_0$ and let M be a Σ-elementary submodel such that $B \subseteq M$ and $|M| = \aleph_0$. It exists by Proposition 7. Assume that $D \in M$ is a digraph such that $A(D) \cap M$ can not be partitioned into directed cycles. We have to show that D is unbalanced. We know that $D(M)$ must be unbalanced because it is countable and we have already proved Theorem 2 for countable digraphs. Let $X \subseteq V \cap M$ be an overloaded set in $D(M)$. Then $\left|\mathsf{out}_{D(M)}(X)\right|$ is finite because $\left|\mathsf{out}_{D(M)}(X)\right| < \left|\mathsf{in}_{D(M)}(X)\right| \leq |M| = \aleph_0$. Let S be the set whose elements are the tails of the edges in $\mathsf{out}_{D(M)}(X)$ and the heads of $\left|\mathsf{out}_{D(M)}(X)\right| + 1$ many edges of $\mathsf{in}_{D(M)}(X)$. Consider the set X' of vertices that are reachable from S in $D(M)$ without using the edges in $\mathsf{out}_{D(M)}(X)$. Note that X' is definable in M as a certain subset of V using finitely many parameters from $A \cap M$. We may assume that Σ contains the appropriate instances of the subset axiom of ZFC hence $X' \in M$. Furthermore $\mathsf{out}_{D(M)}(X) = \mathsf{out}_{D(M)}(X')$ and X' has at least $\left|\mathsf{in}_{D(M)}(X)\right| + 1$ ingoing edges hence it is true in the model M that X' is an overloaded set in D. We also assume that the formula $\varphi(x)$ that says: "x is an unbalanced digraph" is in Σ thus from $M \models \varphi(D)$ we may conclude D is really unbalanced.

Let $\lambda > \aleph_0$ be a cardinal and assume that Lemma 8 is true for sets with size lesser than λ. Let $B = \{b_\alpha : \alpha < \lambda\}$ be arbitrary and let $B_\alpha = \{b_\gamma : \gamma < \alpha\}$. We define a chain of Σ-elementary submodels $\langle M_\alpha : \omega \leq \alpha < \lambda \rangle$ by transfinite recursion such that for all $\omega \leq \alpha < \lambda$:

1. $\alpha, B_\alpha \subseteq M_\alpha$,
2. $|M_\alpha| = |\alpha|$,
3. $M_\gamma \in M_\alpha$ and $M_\gamma \subseteq M_\alpha$ if $\gamma < \alpha$,
4. if $D \in M_{\alpha+1}$ is a balanced digraph, then the edge-set of $D(M_{\alpha+1})$ (i.e. $A \cap M_{\alpha+1}$) can be partitioned into directed cycles,
5. $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ if α is a limit ordinal.

M_ω can be an arbitrary countable Σ-elementary submodel with $B_\omega \subseteq M_\omega$. Suppose that M_γ is already defined if $\omega \leq \gamma < \alpha$ for some $\omega < \alpha < \lambda$ and satisfies the conditions above. If α is a limit ordinal, then let $M_\alpha = \bigcup\{M_\gamma : \gamma < \alpha\}$. If $\alpha = \delta + 1$, then do the following. Let $S_\alpha = \alpha \cup B_\alpha \cup M_\delta \cup \{M_\delta\}$ thus $|S_\alpha| \leq |\alpha| + |\alpha| + |\alpha| = |\alpha| < \lambda$. By the induction hypothesis there is a Σ-elementary submodel M_α such that $S_\alpha \subseteq M_\alpha, |M_\alpha| = |S_\alpha| = |\alpha|$, and for all balanced digraph $D \in M_\alpha$ the edge-set $A(D) \cap M$ can be partitioned into directed cycles. The recursion is done.

Let $M = \bigcup\{M_\alpha : \omega \leq \alpha < \lambda\}$. Then $B \subseteq M$ and $|M| = \lambda = |B|$. Clearly M is Σ-elementary submodel since M is the union of an increasing chain of Σ-elementary submodels. Let $D \in M$ be balanced and let $\beta + 1 < \lambda$ be the smallest ordinal such that $D \in M_{\beta+1}$. Let $D_\beta = D(M_{\beta+1})$ and for $\beta < \alpha < \lambda$ let $D_\alpha = (D \setminus\setminus M_\alpha)(M_{\alpha+1})$. These are edge-disjoint subdigraphs of $D(M)$ moreover $A(D_\alpha)$ $(\beta \leq \alpha < \lambda)$ is a partition of $A(D) \cap M$. Since $D, M_\alpha \in M_{\alpha+1}$ we get (by using Σ-elementarity with an appropriate formula) $(D \setminus\setminus M_\alpha) \in M_{\alpha+1}$.

Claim 9. *If M is a Σ-elementary submodel with $|M| \subseteq M$ and $D \in M$ is a balanced digraph, then $D \setminus\setminus M$ is also balanced.*

If we prove Claim 9, then we are done with the proof of Lemma 8 as well. Indeed, by Claim 9, the digraphs $D \setminus\setminus M_\alpha$ are balanced and therefore by using the 4th property of the recursion with $D \setminus\setminus M_\alpha$ and with $M_{\alpha+1}$ we can partition $A(D_\alpha)$ into directed cycles for all $\beta \leq \alpha < \lambda$ thus we get a desired partition of $A \cap M$ by uniting the partitions of the edge sets $A(D_\alpha)$.

Before the proof of Claim 9 we need some preparations.

Proposition 10. *Let G be an undirected graph an let M be a Σ-elementary submodel such that $G \in M$ and $|M| \subseteq M$. Assume that $\lambda_{G \setminus\setminus M}(u,v) > 0$ for some $u \neq v \in V(G) \cap M$. Then $\lambda_G(u,v) > |M|$.*

Proof: We assume that Σ contains the formulas that expressing the followings:

1. $\lambda_G(u,v) = \kappa$, $(\forall G)(\forall u \neq v \in V(G))\exists \kappa(\lambda_G(u,v) = \kappa)$
2. $E' \subseteq E(G)$ separates the vertices u and v in graph G,
3. f is a bijection between the sets X and Y.

Let $u \neq v \in V(G) \cap M$ arbitrary and suppose that $\lambda_G(u,v) = \kappa \leq |M|$. We have to show that $\lambda_{G \setminus\setminus M}(u,v) = 0$. Since $G, u, v \in M$ and κ is definable from them by a formula in Σ (see 1 above) we know that $\kappa \in M$ and $M \models \lambda_G(u,v) = \kappa$. Then there is some E' such that $M \models$ "$E' \subseteq E(G)$ separates the vertices u and v in graph G and f is a bijection between κ and E'". Formulas 2 and 3 ensures that $E' \subseteq E(G)$ separates the vertices u and v in graph G and f is a bijection between κ and E'. Since $f \in M$ and $\kappa \leq |M| \subseteq M$ the range of f is a subset of M i.e. $E' \subseteq M$ therefore $\lambda_{G \setminus\setminus M}(u,v) = 0$ since E' separates v and u. ■

We need the following result of L. Soukup (see [4] Lemma 5.3 on p. 16):

Proposition 11. *Let G be an undirected graph and let M be a Σ-elementary submodel such that $G \in M$ and $|M| \subseteq M$. Assume that $x \neq y \in V(G)$ are in the same component of $G \setminus\setminus M$ and $F \subseteq E(G \setminus\setminus M)$ separates them where $|F| \leq |M|$. Then F separates x and y in the whole G.*

Proof: Assume (reductio ad absurdum) that it is false and G, F, x, y, M witness it. Take a path P between x and y in $G \setminus\setminus F$. Denote by x' and by y' the first and the last intersection of P with $V \cap M$ with respect to some direction of P. The vertices x' and y' are well-defined and distinct since P necessarily uses some edge from $E(G) \cap M$. Fix also a path Q between x and y in $G \setminus\setminus M$. The paths $P[x', x]$, Q, $P[y, y']$ shows that x' and y' are in the same component of $G \setminus\setminus M$. Thus by Proposition 10 $\lambda_G(x', y') > |M|$. We may fix a path R between x' and y' in $G \setminus\setminus F$ since $\lambda_G(x', y') > |M| \geq |F|$. But then $P[x, x']$, R, $P[y', y]$ shows that F does not separate x and y in $G \setminus\setminus M$ which is a contradiction. ■

Now we turn to the proof of Claim 9. Assume, seeking for contradiction, that $D \setminus\setminus M$ is unbalanced. Then by Lemma 3 there is a weakly connected component of $D \setminus\setminus M$ with vertex set Z and an $X \cup Y$ partition of Z such that $(D \setminus\setminus M)[X]$ and $(D \setminus\setminus M)[Y]$ are weakly connected and X is overloaded in $D \setminus\setminus M$. Let $F = \mathsf{cut}_{D \setminus\setminus M}(X)$. We want to show that $|F| \leq |M|$. We may suppose that F is infinite and thus $\mathsf{cut}_D(X)$ as well since $F \subseteq \mathsf{cut}_D(X)$. Thus $\aleph_0 \leq |\mathsf{out}_D(X)| = |\mathsf{in}_D(X)|$. The inequality $\left|\mathsf{out}_{D \setminus\setminus M}(X)\right| < |\mathsf{out}_D(X)|$ holds because otherwise

$$\left|\mathsf{out}_{D \setminus\setminus M}(X)\right| = |\mathsf{out}_D(X)| = |\mathsf{in}_D(X)| \geq \left|\mathsf{in}_{D \setminus\setminus M}(X)\right|$$

which contradicts to the choice of X. Hence M contains $|\mathsf{out}_D(X)|$ elements of $\mathsf{out}_D(X)$ and thus $|\mathsf{out}_D(X)| \leq |M|$. Then

$$|F| = \left|\mathsf{in}_{D \setminus\setminus M}(X)\right| + \left|\mathsf{out}_{D \setminus\setminus M}(X)\right| \leq |\mathsf{in}_D(X)| + |\mathsf{out}_D(X)| = |\mathsf{out}_D(X)| \leq |M|\,.$$

By using Proposition 11 to the undirected underlying graph of D with F and with arbitrary $x \in X$ and $y \in Y$ vertices we conclude that X and Y belongs to distinct weakly connected components of $D \setminus\setminus F$. Let us denote by X' and Y' the vertex set of these components. We claim that $\mathsf{cut}_{D\setminus\setminus M}(X) = \mathsf{cut}_D(X')$. Indeed, $\mathsf{cut}_D(X')$ might not have element that not in F by the definition of X' and the elements of F goes between X and Y and therefore between X' and Y'. But then $\left|\mathsf{out}_{D\setminus\setminus M}(X)\right| = |\mathsf{out}_D(X')|$ and $\left|\mathsf{in}_{D\setminus\setminus M}(X)\right| = |\mathsf{in}_D(X')|$ thus

$$|\mathsf{out}_D(X')| = \left|\mathsf{out}_{D\setminus\setminus M}(X)\right| < \left|\mathsf{in}_{D\setminus\setminus M}(X)\right| = |\mathsf{in}_D(X')|$$

therefore X' is overloaded in D which is a contradiction. ■ ■ □

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