# Gomory-Hu trees of infinite graphs with finite total weight

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**Abstract**

Gomory and Hu proved in [3] their well-known theorem, which states that if $G$ is a finite graph with non-negative weights on its edges, then there exists a tree $T$ (now called a Gomory-Hu tree) on $V(G)$ such that for all $u \neq v \in V(G)$ there is an $e \in E(T)$ such that the two components of $T - e$ determine an optimal (minimal valued) cut between $u$ an $v$ in $G$. In this paper we extend their result to infinite weighted graphs with finite total weight. Furthermore, we show by an example that one cannot omit the condition of the finiteness of the total weight.

## 1 Introduction

Let $G = (V, E)$ be a countable connected simple graph and let $c : E \to \mathbb{R}_+ \setminus \{0\}$ be a weight-function, then $(V, E, c)$ is a **weighted graph**. We call the subsets $X$ of $V$ cuts and we write $\boldsymbol{\delta_G(X)}$ for the set of the edges with exactly one end in $X$. We say that $X$ is an $\boldsymbol{u - v}$ **cut** for some $u \neq v \in V$ if $u \in X$ and $v \notin X$. A cut $X$ separates $u$ and $v$ if $X$ is either a $u - v$ or a $v - u$ cut. Let $\boldsymbol{d_c(X)} = \sum_{e \in \delta_G(X)} c(e)$ and let $\boldsymbol{\lambda_c(u, v)} := \inf\{d_c(X) :\ X \text{ is a } u - v \text{ cut } \}$ for $u \neq v \in V$. A cut $X$ is an **optimal** $\boldsymbol{u - v}$ **cut** if it is a $u - v$ cut with $d_c(X) = \lambda_c(u, v)$. A cut $X$ is **optimal** if it is an optimal $u - v$ cut for some $u \neq v \in V$. The weighted graph $(V, E, c)$ is **finitely separable** if $\lambda_c$ has just finite values. A tree $T = (V, F)$ is a **Gomory-Hu tree** for $(V, E, c)$ if for all $u \neq v \in V$ there is an $e \in F$ such that the **fundamental cuts corresponding to** $\boldsymbol{e}$ (i.e. the vertex sets of the components of $T - e$) separate optimally $u$ and $v$ in $(V, E, c)$. Gomory and Hu proved in [3] that for all finite weighted graph there exists a Gomory-Hu tree. It has several interesting consequences. For example the function $\lambda_c$ may have at most $n - 1$ different values instead of $\binom{n}{2}$ (where $n$ is the number of the vertices) and there are at least two optimal cuts that consist of a single vertex, namely the leafs of the Gomory-Hu tree (unless the graph is trivial).

In this paper we extend their theorem for infinite weighted graphs with finite total weight. Note that the strict positivity of $c$ and the connectedness of $G$ are not real restrictions since throwing away edges $e$ with $c(e) = 0$ has no effect on the values of the cuts and one can construct Gomory-Hu trees component-wise and join them to a Gomory-Hu tree. Furthermore, if the sum of the weights is finite, then the weighted graph must be countable.

The cut structure of infinite graphs has been already investigated in some other perspectives (see for example [1] and [2]) where the authors only allow cuts with finitely many outgoing edges.

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As it seems from the definitions above we are focusing on the literal generalization of Gomory-Hu trees.

In a more abstract folklore version of the Gomory-Hu theorem there is a finite set $V$ and a function $b : \mathcal{P}(V) \to \mathbb{R}_+$ which is symmetric ($b(X) = b(V \setminus X)$) and submodular i.e.

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y) \text{ for } X, Y \subseteq V.$$

Let $\boldsymbol{\lambda_b(u,v)} = \inf\{b(X) : X \text{ is a } u-v \text{ cut}\}$ (we write infimum to make it well-defined for infinite $V$ as well). In this case, there exists an **abstract Gomory-Hu tree** with respect to $b$ in the following sense: There is a tree $T$ on the vertex set $V$ in such a way that for every $u \neq v \in V$ there is some $e \in E(T)$ such that for a fundamental cut $X$ corresponding to $e$, we have $b(X) = \lambda_b(u,v)$.

## 2 Preparations

Let $(V, E, c)$ be a weighted graph.

**Proposition 1.** *$d_c(X) + d_c(Y) \geq d_c(X \cup Y) + d_c(X \cap Y)$ for all $X, Y \subseteq V$.*

*Proof:* If edge $e$ is between $X \setminus Y$ and $Y \setminus X$, then it contributes $2c(e)$ to the left side and 0 to the right side of the inequality. The contribution of any other type of edge is the same for both sides. ■

For a sequence $(X_n)$ let

$$\textbf{lim inf}\, \boldsymbol{X_n} = \bigcup_{m=0}^{\infty} \bigcap_{n=m}^{\infty} X_n$$
$$\textbf{lim sup}\, \boldsymbol{X_n} = \bigcap_{m=0}^{\infty} \bigcup_{n=m}^{\infty} X_n.$$

If $\liminf X_n = \limsup X_n$, then we denote this set by $\textbf{lim}\, \boldsymbol{X_n}$ and we say that $(X_n)$ is convergent. If $\mathcal{L}$ is a family of sets, then $\bigcup \mathcal{L}$ is defined to be the union of its elements. We define $\bigcap \mathcal{L}$ similarly except letting $\bigcap \varnothing := V$.

**Claim 2.**

1. *If $(X_n)$ is a convergent sequence of cuts, then $d_c(\lim X_n) \leq \liminf d_c(X_n)$.*
2. *In addition, if $\sum_{e \in E} c(e) < \infty$, then $\lim d_c(X_n)$ exists and $\lim d_c(X_n) = d_c(\lim X_n)$ holds.*

*Proof:* Let $\varepsilon > 0$ be arbitrary and fix a finite $F \subseteq \delta_G(\lim X_n)$ with $d_c(\lim X_n) \leq \sum_{e \in F} c(e) + \varepsilon$. Since $F \subseteq \delta_G(X_n)$ for all large enough $n$ we have $d_c(\lim X_n) \leq d_c(X_n) + \varepsilon$ and therefore $d_c(\lim X_n) \leq \liminf d_c(X_n) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary we are done with 1.

To show 2., suppose, to the contrary, that $d_c(\lim X_n) < \limsup d_c(X_n)$ and let

$$\varepsilon := \limsup d_c(X_n) - d_c(\lim X_n).$$

Pick an $n_0$ with $d_c(X_{n_0}) - d_c(\lim X_n) \geq \varepsilon/2$. Then there is a set $F_0 \subseteq \delta_G(X_{n_0}) \setminus \delta_G(\lim X_n)$ with total weigh at least $\varepsilon/2$. Let $F_0'$ be a finite subset of $F_0$ with total weight at least $\varepsilon/4$. Note that for all large enough $n > n_0$ we have $F_0' \cap \delta_G(X_n) = \varnothing$ since $F_0' \cap \delta_G(\lim X_n) = \varnothing$. Hence we can pick an $n_1 > n_0$ such that $F'_0 \cap \delta_G(X_{n_1}) = \varnothing$ and $d_c(X_{n_1}) - d_c(\lim X_n) \geq \varepsilon/2$. We define $F'_1$ similarly as $F'_0$. Continuing the process recursively, the $F'_n$ are pairwise disjoint edge sets with total weight at least $\varepsilon/4$ for each, which contradicts the assumption $\sum_{e \in E} c(e) < \infty$. Combining this with 1. we obtain

$$\limsup d_c(X_n) \leq d_c(\lim X_n) \leq \liminf d_c(X_n),$$

thus $\lim d_c(X_n)$ exists and equals to $d_c(\lim X_n)$. ∎

The main result of the paper as follows.

**Theorem 3.** *Let $V$ be a nonempty countable set and let $b : \mathcal{P}(V) \to \mathbb{R}_+ \cup \{\infty\}$ such that*

0. *$b(X) = 0 \iff X \in \{\varnothing, V\}$, (b is connected)*
1. *$b(X) = b(V \setminus X)$ for $X \subseteq V$, (b is symmetric)*
2. *$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y)$ for $X, Y \subseteq V$, (b is submodular)*
3. *If $(X_n)$ is a nested sequence of cuts, then $b(\lim X_n) = \lim b(X_n)$, (b is monotone-continuous)*
4. *For any $u \neq v$ there is an $u - v$ cut $X$ with $b(X) < \infty$. (b is finitely separating)*

*Then there exists an abstract Gomory-Hu tree with respect to $b$.*

*Remark* 4. Properties 1 and 2 imply that for any $X, Y \subseteq V$ we also have

$$b(X) + b(Y) \geq b(X \setminus Y) + b(Y \setminus X).$$

If $\sum_{e \in E} c(e) < \infty$ holds, then $b := d_c$ satisfies the properties above (see Proposition 1 and Claim 2). Hence as a special case of Theorem 3 we obtain:

**Corollary 5.** *Every weighted graph with $\sum_{e \in E} c(e) < \infty$ admits a Gomory-Hu tree.*

Consider the following weakening of 3.

3' if $(X_n)$ is a nested sequence of cuts, then $b(\lim X_n) \leq \liminf b(X_n)$.

If we do not assume $\sum_{e \in E} c(e) < \infty$, but $(V, E, c)$ is finitely separable, then Claim 2 ensures that $b := d_c$ still satisfy this weaker condition (see Claim 2.1). We will see via a counterexample that in this case one can not guarantee the existence of a Gomory-Hu tree. Even so, the next theorem provides something similar but weaker. A system of sets is called **laminar** if any two members of it are either disjoint or $\subseteq$-comparable.

**Theorem 6.** *If $b$ satisfies conditions 0,1,2,3',4, then there is a laminar system $\mathcal{L}^*$ of optimal cuts such that any pair from $V$ is separated optimally by some element of $\mathcal{L}^*$.*

*Proof:*

**Claim 7.** *For any $u \neq v \in V$ there exists an $u - v$ cut $X^*$ with $b(X^*) = \lambda_b(u, v)$.*

*Proof:* Let $u, v$ be fixed. The **error** of the sequence $(X_n)$ of $u - v$ cuts is defined to be

$$\sum_{n=0}^{\infty}(b(X_n) - \lambda_b(u,v)).$$

It is enough to prove the existence of a nested sequence $(Y_n)$ of $u-v$ cuts with finite error. Indeed, from the finiteness of the error it follows that $\lim b(Y_n) = \lambda_b(u,v)$, hence by property 3'

$$\lambda_b(u,v) \leq b\left(\bigcap_{n=0}^{\infty} Y_n\right) \leq \liminf b(Y_n) = \lim b(Y_n) = \lambda_b(u,v).$$

**Proposition 8.** *For each sequence $(X_n)$ with finite error there is another sequence $(Z_n)$ with less or equal error such that $Z_0 \supseteq \bigcup_{n=1}^{\infty} Z_n$.*

*Proof:* In the sequence $(X_n)$ replace the member $X_0$ by $X_0 \cup X_1$ and the member $X_1$ by $X_1 \cap X_0$. By submodularity the error of the new sequence $(X_n^1)$ is less or equal. Then replace $X_0^1 = X_0 \cup X_1$ by $X_0^2 := X_0^1 \cup X_2^1 = X_0 \cup X_1 \cup X_2$ and replace $X_2^1$ by $X_2^2 := X_2^1 \cap X_0^1 = X_2 \cap (X_0 \cup X_1)$. In general let

$$X_n^{m+1} = \begin{cases} X_0^m \cup X_{m+1}^m & \text{if } n = 0 \\ X_0^m \cap X_{m+1}^m & \text{if } n = m+1 \\ X_n^m & \text{otherwise.} \end{cases}$$

Finally we claim that the following "limit" of these sequences is appropriate.

$$Z_0 := \bigcup_{n=0}^{\infty} X_n$$

$$Z_{n+1} := X_{n+1} \cap \left(\bigcup_{i=0}^{n} X_n\right).$$

For

$$S_m := \sum_{n=0}^{\infty}(b(X_n^m) - \lambda_b(u,v)),$$

$(S_m)$ is a non-negative decreasing sequence thus it has a limit $S$ i.e.

$$S := \lim_{m\to\infty} \sum_{n=0}^{\infty}(b(X_n^m) - \lambda_b(u,v)).$$

Consider the counting measure on $\mathbb{N}$ and apply Fatou's lemma and property 3':

$$
\begin{aligned}
S &= \liminf_m \sum_{n=0}^{\infty} (b(X_n^m) - \lambda_b(u,v)) \\
&\geq \sum_{n=0}^{\infty} (\liminf_m b(X_n^m) - \lambda_b(u,v)) \\
&= \liminf_m b\left(\bigcup_{i=0}^{m} X_i\right) - \lambda_b(u,v) + \sum_{n=1}^{\infty} (b(Z_n) - \lambda_b(u,v)) \\
&\geq \sum_{n=0}^{\infty} (b(Z_n) - \lambda_b(u,v)).
\end{aligned}
$$

Hence the error of $(Z_n)$ is less than or equal to the error of the earlier sequences. ■

For $n \in \mathbb{N}$, let $X_n$ be a $u-v$ cut with $b(X_n) - \lambda_b(u,v) \leq 1/2^{n+1}$. Then the error of $(X_n)$ is at most 1. Apply Proposition 8 with $(X_n)$ to obtain $(Z_n)$ and let $Y_0 := Z_0$. Use Proposition 8 on the terminal segment of $(Z_n)$ consists of all but the 0-th element (this sequence has error at most $1 - (b(Y_0) - \lambda_b(u,v)))$ to obtain $(Z_n^1)$ and let $Y_1 := Z_0^1$. By continuing the process recursively we build a nested sequence $(Y_n)$ of $u-v$ cuts with error at most 1 and hence we are done by the second sentence of the current proof (proof of Claim 7). ■ ■

**Proposition 9.** *The intersection and the union of (even infinitely many) optimal $u-v$ cuts is an optimal $u-v$ cut.*

*Proof:* Let $X$ and $Y$ be optimal $u-v$ cuts. On the one hand, $b(X) \leq b(X \cup Y)$ and $b(Y) \leq b(X \cap Y)$ hold since $X \cup Y$ and $X \cap Y$ are $u-v$ cuts. Thus

$$b(X) + b(Y) \leq b(X \cup Y) + b(X \cap Y).$$

On the other hand,

$$b(X) + b(Y) \geq b(X \cup Y) + b(X \cap Y)$$

by submodularity. Thus equality holds and by $b(X), b(Y) < \infty$ (see property 4 in Theorem 3), we may conclude $b(X) = b(X \cup Y)$ and $b(Y) = b(X \cap Y)$. By induction we know the statement for finitely many optimal $u-v$ cuts. Consider an infinite family $\mathcal{X}$ of optimal $u-v$ cuts. Let $V = \{v_n : n \in \mathbb{N}\}$ and let $X_n' \in \mathcal{X}$ with $v_n \notin X_n'$ if $v_n \notin \bigcap \mathcal{X}$ and an arbitrary element of $\mathcal{X}$ otherwise. Then $X_n := \bigcap_{m=0}^{n} X_m'$ is an optimal $u-v$ cut again and $\bigcap_{n=0}^{\infty} X_n = \bigcap \mathcal{X}$ as well by property 3'. ■

**Corollary 10.** *There is a $\subseteq$-smallest (largest) optimal $u-v$ cut $\boldsymbol{X_{u,v}}$ $(Y_{u,v})$ which is the intersection (union) of all optimal $u-v$ cuts.*

**Claim 11.** *Let $X$ be an optimal $s-t$ cut and let $Y$ be an optimal $u-v$ cut.*

1. *Assume $X$ is a $u-v$ cut. Then $Y \cap X$ is an optimal $u-v$ cut if $t \notin Y$ and $Y \cup X$ is an optimal $u-v$ cut if $t \in Y$.*

2. *Assume $X$ is a $v-u$ cut. Then $Y \setminus X$ is an optimal $u-v$ cut if $s \notin Y$ and $Y \cup (V \setminus X)$ is an optimal $u-v$ cut if $s \in Y$.*

3. *Assume $u, v \in X$. Then $Y \cap X$ is an optimal $u-v$ cut if $t \notin Y$ and $Y \cup (V \setminus X)$ is an optimal $u-v$ cut if $t \in Y$.*

4. *Assume $u, v \notin X$. Then $Y \setminus X$ is an optimal $u-v$ cut if $s \notin Y$ and $Y \cup X$ is an optimal $u-v$ cut if $s \in Y$.*

*Proof:* It is enough to prove 1. and 3. since by replacing $X$ with the optimal $t-s$ cut $V \setminus X$ in them we obtain 2. and 4. respectively. To prove 1. assume first that $t \notin Y$. Since $X \cup Y$ is an $s-t$ cut and $X \cap Y$ is a $u-v$ cut we have $b(X \cup Y) \geq b(X)$ and $b(X \cap Y) \geq b(Y)$. Combining this with submodularity we get

$$b(X) + b(Y) \geq b(X \cup Y) + b(X \cap Y) \geq b(X) + b(Y).$$

Since $b(X), b(Y) < \infty$, it implies $b(X \cup Y) = b(X)$ and $b(X \cap Y) = b(Y)$ thus $Y \cap X$ is an optimal $u-v$ cut.

If $t \in Y$ and $s \in Y$, then $X \cup Y$ is a $u-v$ cut and $X \cap Y$ is an $s-t$ cut; therefore by arguing similarly as above we obtain that $X \cup Y$ must be an optimal $u-v$ cut. Finally if $t \in Y$ and $s \notin Y$, then on the one hand, $Y$ separates $t$ and $s$ and $X$ does this optimally therefore $b(X) \leq b(Y)$. On the other hand, $Y$ is an optimal $u-v$ cut and $X$ is an $u-v$ cut hence $b(Y) \leq b(X)$. Thus $b(X) = b(Y)$ and therefore $X$ and $Y$ both are optimal $u-v$ cuts hence by Proposition 9 $X \cup Y$ and $X \cap Y$ as well. The proof of 3 is similar. ■

**Corollary 12.** *If $X$ is an optimal cut and $u \neq v \in X$, then either $X_{u,v} \subseteq X$ or $X_{v,u} \subseteq X$ (where $X_{x,y}$ stands for the $\subseteq$-smallest optimal $x-y$ cut).*

*Proof:* If $X_{u,v} \subseteq X$, then we are done. Assume $X_{u,v} \nsubseteq X$. By the definition of $X_{u,v}$ the $u-v$ cut $X_{u,v} \cap X$ cannot be optimal therefore by Claim 11.3 $X_{u,v} \cup (V \setminus X)$ is an optimal $u-v$ cut. But then $V \setminus [X_{u,v} \cup (V \setminus X)] = X \setminus X_{u,v}$ is an optimal $v-u$ cut therefore $X_{v,u} \subseteq (X \setminus X_{u,v}) \subseteq X$. ■

Theorem 6 follows immediately from the next lemma (actually we need the lemma just with finite $\mathcal{L}$).

**Lemma 13.** *If $\mathcal{L}$ is a laminar system of optimal cuts and $u \neq v \in V$, then there is a cut $X^*$ for which $\mathcal{L} \cup \{X^*\}$ is laminar and $X^*$ separates optimally $u$ and $v$.*

*Proof:* Let us partition $\mathcal{L}$ into four parts $\boldsymbol{\mathcal{L}_{u,\overline{v}}} := \{X \in \mathcal{L} :\ u \in X \wedge v \notin X\}$, we define $\boldsymbol{\mathcal{L}_{\overline{u},v}}, \boldsymbol{\mathcal{L}_{u,v}}$ and $\boldsymbol{\mathcal{L}_{\overline{u},\overline{v}}}$ similarly. If $X_{u,v} \subseteq \widehat{X}$ for some $\widehat{X} \in \mathcal{L}_{u,\overline{v}}$, then $\{X_{u,v}\} \cup \mathcal{L}_{u,v} \cup \mathcal{L}_{\overline{u},v}$ is laminar. Suppose that we have no such an $\widehat{X}$ not even if we interchange $u$ and $v$. By Corollary 12 we know that for all $W \in \mathcal{L}_{u,v}$ either $X_{u,v} \subseteq W$ or $X_{v,u} \subseteq W$. Hence by symmetry we may assume that $X_{u,v} \subseteq \bigcap \mathcal{L}_{u,v}$. We show that $\{X_{u,v}\} \cup \mathcal{L}_{u,v} \cup \mathcal{L}_{\overline{u},v}$ is laminar in this case as well. Let $X \in \mathcal{L}_{\overline{u},v}$ be arbitrary. Then $X_{v,u} \nsubseteq X$ otherwise $\widehat{X} := X$ would be a bound. But then $X_{v,u} \cap X$ cannot be an optimal $v-u$ cut by the definition of $X_{v,u}$. Therefore by Claim 11.1 we know that $X_{v,u} \cup X$ is an optimal $v-u$ cut and hence $V \setminus (X_{v,u} \cup X)$ is an optimal $u-v$ cut. Thus $V \setminus (X_{v,u} \cup X) \supseteq X_{u,v}$ from which $X \cap X_{u,v} = \varnothing$ follows.

Thus we may suppose that $\{X_{u,v}\} \cup \mathcal{L}_{u,v} \cup \mathcal{L}_{\overline{u},v}$ is laminar. If for some $Y \in \mathcal{L}_{u,\overline{v}}$ the set $\{X_{u,v}, Y\}$ is not laminar, then the cut $X_{u,v} \cap Y$ may not be an optimal $u-v$ cut by the definition of $X_{u,v}$. But then $X_{u,v} \cup Y$ is an optimal $u-v$ cut by Claim 11.1. Let

$$\mathcal{Y} := \{Y : Y \in \mathcal{L}_{u,\overline{v}} \wedge \{X_{u,v}, Y\} \text{ is not laminar}\}.$$

The set $\{X_{u,v} \cup Y : Y \in \mathcal{Y}\}$ consists of optimal $u-v$ cuts and totally ordered by $\subseteq$. By taking a cofinal sequence and applying 3' we obtain that $X_0 := X_{u,v} \cup \bigcup \mathcal{Y}$ is an optimal $u-v$ cut. Note that $\{X_0\} \cup (\mathcal{L} \setminus \mathcal{L}_{\overline{u},\overline{v}})$ is laminar.

**Proposition 14.** *If for a $Z \in \mathcal{L}_{\overline{u},\overline{v}}$ the pair $\{X_0, Z\}$ is not laminar, then $\{X_{u,v}, Z\}$ is not laminar.*

*Proof:* Suppose that $\{X_{u,v}, Z\}$ is laminar. Since $u \in X_{u,v}$ and $Z \in \mathcal{L}_{\overline{u},\overline{v}}$ we know that $X_{u,v} \nsubseteq Z$. If $Z \subseteq X_{u,v}$, then by $X_{u,v} \subseteq X_0$ we have $Z \subseteq X_0$. Finally let $X_{u,v} \cap Z = \varnothing$. An $Y \in \mathcal{Y}$ is either disjoint from $Z$ or $Z \subseteq Y$ because $\mathcal{L}$ is laminar and $u \in Y \setminus Z$. But then $X_0$ is either disjoint from $Z$ or $Z \subseteq X_0$ as well. ■

Let

$$\mathcal{Z} := \{Z \in \mathcal{L}_{\overline{u},\overline{v}} : \{X_0, Z\} \text{ is not laminar}\}.$$

We know by Proposition 14 that for $Z \in \mathcal{Z}$ the set $\{X_{u,v}, Z\}$ is not laminar. For $Z \in \mathcal{Z}$ fix some $s_Z, t_Z$ such that $Z$ is an optimal $s_Z - t_Z$ cut. By the definition of $X_{u,v}$, the cut $X_{u,v} \setminus Z$ may not be an optimal $u-v$ cut hence by Claim 11.4 it follows, that $s_Z \in X_{u,v} (\subseteq X_0)$. We can build the desired $X^* := X_0 \cup \bigcup \mathcal{Z}$ by joining countably many elements of $\mathcal{Z}$ with union $\bigcup \mathcal{Z}$ one by one to $X_0$ applying Claim 11.4 and $s_Z \in X_0$ repeatedly and taking limit using property 3'. ■ ■

# 3 A counterexample

In the previous section we obtained (as a special case of Theorem 6) the existence of a laminar system $\mathcal{L}$ of optimal cuts for finitely separable weighted graphs such that the elements of $\mathcal{L}$ separate any vertex pair optimally. In this section we provide an example which shows that one cannot guarantee the existence of a Gomory-Hu tree as well without further assumptions. Let $G = (V, E)$ where $V = \{v_n : n \in \mathbb{N}\}$ and $E = \{v_\infty v_n : n \in \mathbb{N}\} \cup \{v_n v_{n+1} : n \in \mathbb{N}\}$. Finally $c(v_\infty v_n) := 1$ for all $n \in \mathbb{N}$ and with the notation $e_n := v_n v_{n+1}$

$$c(e_n) := \begin{cases} 2 & \text{if } n = 0 \\ c(e_{n-1}) + n + 1 & \text{if } n > 0. \end{cases}$$

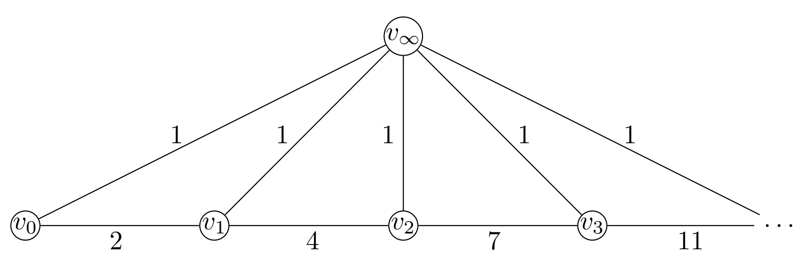


Figure 1: A finitely separable weighted graph without Gomory-Hu tree

**Claim 15.** *If $n < m$, then $\{v_0, v_1, \ldots, v_n\} =: V_n$ is the only optimal $v_n - v_m$ cut.*

*Proof:* Pick an optimal $v_n - v_m$ cut $X$. Since $d_c(V_n) < c(e_k)$ whenever $k > n$, a cut $X$ may not separate the end vertices of such an $e_k$. Then $v_\infty \notin X$ otherwise $d_c(X) = \infty$. Thus we have

$X \subseteq V_n$. Suppose, for a contradiction, that $v_l \notin X$ for some $l < n$ and $l$ is the largest such an index. Then

$$d_c(X) - d_c(V_n) \geq c(e_l) - l - 1 > 0,$$

which contradicts the optimality of $X$. ■

**Claim 16.** *$(G, c)$ has no Gomory-Hu tree.*

*Proof:* Assume, to the contrary, that $T$ is a Gomory-Hu tree of $(G, c)$. For all $e \in E(T)$, pick the fundamental cut $X_e$ that corresponds to $e$ and does not contain $v_\infty$. On the one hand, $\mathcal{L} := \{X_e\}_{e \in E(T)}$ is a laminar system of optimal cuts that contains at least one $\subseteq$-maximal element (if $e$ is incident to $v_\infty$ in $T$, then $X_e$ is a $\subseteq$-maximal element). On the other hand, $\mathcal{L} = \{V_n : n \in \mathbb{N}\}$ since the optimal cuts are unique up to complementation and the additional condition "does not contain $v_\infty$" makes them unique. This is a contradiction since $(V_n)$ is a strictly $\subseteq$-increasing sequence. ■

*Remark* 17. One can obtain also a locally finite counterexample by some easy modification of our counterexample above.

## 4 Existence of an abstract Gomory-Hu tree

In this section we prove our main result Theorem 3. It will be convenient to use the following equivalent but formally weaker definition of Gomory-Hu trees.

**Claim 18.** *$T = (V, F)$ is a Gomory-Hu tree with respect to $b$ if for all $uv \in F$ the fundamental cuts corresponding to $uv$ in $T$ separate optimally $u$ and $v$.*

*Proof:* Let $u \neq v \in V$ be arbitrary and let $v_1, v_2, \dots, v_m$ be the vertices of the unique $u - v$ path in $T$ numbered in the path order.

**Proposition 19.** *For all pairwise distinct $u, v, w \in V$, we have:*

$$\lambda_b(u, w) \geq \min\{\lambda_b(u, v), \lambda_b(v, w)\}.$$

*Proof:* It follows from the fact that if a cut separates $u$ and $w$, then it separates either $u$ and $v$ or $v$ and $w$ as well. ■

On the one hand, by applying the proposition above repeatedly we obtain

$$\lambda_b(u, v) \geq \min\{\lambda_b(v_i, v_{i+1}) : \ 1 \leq i < m\} =: \lambda_b(v_{i_0}, v_{i_0+1}) \text{ for some } 1 \leq i_0 < m.$$

On the other hand, the fundamental cuts corresponding to the edge $v_{i_0}v_{i_0+1}$ separates $u$ and $v$ and have value $\lambda_b(v_{i_0}, v_{i_0+1})$ by assumption. Thus

$$\lambda_b(u, v) = \lambda_b(v_{i_0}, v_{i_0+1}),$$

hence the fundamental cuts corresponding to $v_{i_0}v_{i_0+1} \in F$ are optimal cuts between $u$ and $v$. ■■

A sequence of optimal cuts is defined to be **essential** if all of its members separate optimally a vertex pair that the earlier members do not.

**Lemma 20.** *For an essential and $\subseteq$-monotone sequence $(X_n)$ of cuts,* $\lim X_n \in \{\varnothing, V\}$.

*Proof:*

**Proposition 21.** *For every $\varepsilon > 0$ there are only finitely many vertex pairs $\{u, v\}$ with $\lambda_b(u, v) \geq \varepsilon$.*

*Proof:* Suppose, to the contrary, that $\lambda_b(u_n, v_n) \geq \varepsilon$ for all $n \in \mathbb{N}$ and $u_n \neq u_m$ if $n \neq m$. We can assume by trimming the sequence of vertex pairs that $u_n \neq v_m$ for $n, m \in \mathbb{N}$. The cuts $U_n := \{u_m\}_{m \geq n}$ converges to $\varnothing$ monotonously and hence $b(U_n)$ tends to 0 by property 3. Since $U_n$ is a $u_n - v_n$ cut for each $n \in \mathbb{N}$, it contradicts $\lambda_b(u_n, v_n) \geq \varepsilon$. ∎

Let $(X_n)$ be an essential and $\subseteq$-monotone sequence of cuts. By Proposition 21 $\lim b(X_n) = 0$. Thus by property 3 we have $0 = \lim b(X_n) = b(\lim X_n)$ from which we may conclude $\lim X_n \in \{\varnothing, V\}$ by property 0. ∎ ∎

Take an optimal cut $X$. For $u \neq v \in X$, let $\boldsymbol{u \prec_X v}$ if $X_{u,v} \not\subseteq X$.

**Claim 22.** *The relation $\prec_X$ is a strict partial order on $X$.*

*Proof:* It is irreflexive by definition. To show transitivity assume $u \prec_X v \prec_X w$. If $u = w$, then we have $u \prec_X v$ and $v \prec_X u$ which contradicts Corollary 12. Thus $u, v, w$ are pairwise distinct. Suppose, to the contrary, that $u \prec_X w$ does not hold i.e. $X_{u,w} \subseteq X$. Assume first that $v \in X_{u,w}$. By Corollary 12, either $X_{u,v} \subseteq X_{u,w}$ or $X_{v,u} \subseteq X_{u,w}$. Since $u \prec_X v$, necessarily $X_{v,u} \subseteq X_{u,w}$. But then $X_{u,w}$ and $X_{v,u}$ are both $v - w$ cuts and

$$\lambda_b(v, w) \geq \min\{\lambda_b(v, u),\ \lambda_b(u, w)\} = \min\{b(X_{v,u}),\ b(X_{u,w})\}$$

shows that one of them is an optimal $v - w$ cut which contradicts $v \prec_X w$.

Hence necessarily $v \notin X_{u,w}$. We know that $X_{u,w}$ is not an optimal $u - v$ cut since $u \prec_X v$ therefore $b(X_{u,w}) > b(X_{v,u})$. Note that $X_{v,u} \subseteq X$ by $u \prec_X v$ and by Corollary 12. On the one hand, $w \notin X_{v,u}$ otherwise $X_{v,u}$ would be a better cut between $w$ and $u$ than an optimal. On the other hand, $X_{v,u}$ is not an optimal $v - w$ cut since $v \prec_X w$ hence $X_{w,v} \subseteq X$ and $b(X_{w,v}) < b(X_{v,u})$ hold. Necessarily $u \in X_{w,v}$, otherwise $X_{w,v}$ separates better $w$ and $u$ than $X_{u,w}$, but then $X_{w,v}$ separates better $u$ and $v$ than $X_{v,u}$, which is a contradiction. ∎

**Lemma 23.** *If $X$ is an optimal $s - t$ cut, then $X$ has a $\prec_X$-minimal element $s'$. For all such an $s'$, cut $X$ is an optimal $s' - t$ cut.*

*Proof:* Let $A = \{x \in X : \lambda_b(x, t) = \lambda_b(s, t)\}$ and $B := \{y \in X : \lambda_b(y, t) < \lambda_b(s, t)\}$. Then $A \cup B$ is a partition of $X$. Note that $A \neq \varnothing$ since $s \in A$.

**Proposition 24.** *For all $x \in A$ and $y \in B$ : $x \prec_X y$ holds.*

*Proof:* If $x \in A$ and $y \in B$, then $\lambda_b(x, y) < \lambda_b(s, t) (= \lambda_b(x, t))$, otherwise

$$\lambda_b(y, t) \geq \min\{\lambda_b(x, y), \lambda_b(x, t)\} = \lambda_b(x, t) = \lambda_b(s, t)$$

contradicts $y \in B$. Therefore if $X_{x,y} \subseteq X$ hold, then we would have (since $X_{x,y}$ is a $x - t$ cut)

$$\lambda_b(x, t) \leq b(X_{x,y}) = \lambda_b(x, y) < \lambda_b(s, t)$$

which is impossible since $x \in A$. ∎

By Proposition 24, it is enough to find a $\prec_X$-minimal element of the subposet $(A, \prec_X)$. The existence of such an element follows immediately from the following proposition.

**Proposition 25.** *Set $A$ is finite.*

*Proof:* Assume, for a contradiction, that $A$ is infinite. Pick a nested sequence $(A_n)$ of nonempty subsets of $A$ with $\bigcap_{n=0}^{\infty} A_n = \varnothing$. On the one hand, $b(A_n) \to 0$ by property 3. On the other hand, every $A_n$ separates an $x \in A$ from $t$ and hence

$$b(A_n) \geq \lambda_b(x,t) = \lambda_b(s,t) > 0,$$

which is a contradiction. ∎

For the second part of Lemma 23, let $s'$ be a $\prec_X$-minimal element of $X$. Then by Proposition 24 $s' \in A$ thus $\lambda_b(s',t) = \lambda_b(s,t)$ by the definition of $A$. ∎ ∎

**Claim 26.** *For any $s \in V$, the family $\boldsymbol{\mathcal{C}_s} := \{X_{u,s} : u \in V \setminus \{s\}\}$ of optimal cuts is laminar.*

*Proof:* Let $X_{u,s},\ X_{v,s} \in \mathcal{C}_s$. If $u \in X_{v,s}$, then $X_{u,s} \cap X_{v,s}$ is an $u - s$ cut and $X_{u,s} \cup X_{v,s}$ is a $v - s$ cut. By submodularity $X_{u,s} \cap X_{v,s}$ is an optimal $u - s$ cut (and $X_{u,s} \cup X_{v,s}$ is an optimal $v - s$ cut). By the definition of $X_{u,s}$ it implies $X_{u,s} = X_{u,s} \cap X_{v,s}$ i.e. $X_{u,s} \subseteq X_{v,s}$. If $v \in X_{u,s}$, then we obtain $X_{v,s} \subseteq X_{u,s}$ by symmetry. Finally if $u \in X_{u,s} \setminus X_{v,s}$ and $v \in X_{v,s} \setminus X_{u,s}$, then $X_{u,s} \setminus X_{v,s}$ is a $u - s$ cut and $X_{v,s} \setminus X_{u,s}$ is a $v - s$ cut. Applying Remark 4 follows that they are also optimal such a cuts, thus by $\subseteq$-minimality $X_{u,s} = X_{u,s} \setminus X_{v,s}$ i.e. $X_{u,s} \cap X_{v,s} = \varnothing$. ∎

Let $\prec_V$ be the trivial partial order on $V$ (i.e. under which there are no comparable elements).

**Lemma 27.** *Let $X$ be either an optimal cut or $V$. Pick an $\prec_X$-minimal element $s$ of $X$ (see Lemma 23). Then the $\subseteq$-maximal elements of the system $\boldsymbol{\mathcal{C}_{s,X}} := \{X_{u,s} : u \in X \setminus \{s\}\}$ forms a partition of $X \setminus \{s\}$.*

*Proof:* By the choice of $s$ we know that $\bigcup \mathcal{C}_{s,X} \subseteq X \setminus \{s\}$ (see Corollary 12). By the laminarity of $\mathcal{C}_{s,X} \subseteq \mathcal{C}_s$ (see Claim 26) it is enough to show that for any $u \in X \setminus \{s\}$ the system $\mathcal{C}_{s,X}$ has a $\subseteq$-maximal element that contains $u$. Assume, for a contradiction, that it is false, then there is a strictly $\subseteq$-increasing sequence $(X_{u_n,s})$ which shows this. On the one hand, this sequence is essential because $X_{u_m,s}$ may not be an optimal $u_n - s$ cut for $m < n$ since $X_{u_n,s}$ is the $\subseteq$-smallest such a cut. On the other hand, $u_0 \in \lim X_{u_n,s} \subseteq V \setminus \{s\}$ which contradicts Lemma 20. ∎

We build the desired abstract Gomory-Hu tree for $b$ by applying Lemma 27 repeatedly. Pick an arbitrary $r \in V$ for root. It is possible to define a unique fundamental cut for each edge $e$ of the tree $T$ we build, namely the vertex set of the component of $T$ after the deletion of $e$ which does not contain $r$. Let $\{X_i\}_{i \in I_0}$ consists of the $\subseteq$-maximal elements of the laminar system $\mathcal{C}_r$. Let $x_i$ be a $\prec_{X_i}$-minimal element of $X_i$ and draw the tree-edges $rx_i$ for $i \in I_0$. Note that Lemma 23 ensures that the fundamental cut corresponding to $rx_i$ will separate optimally $r$ and $x_i$, assuming that $X_i$ will be the vertex set of the subtree rooted at $x_i$. For each $i \in I_0$, take the $\subseteq$-maximal elements $\{X_{i,j}\}_{j \in I_1}$ of $\mathcal{C}_{x_i,X_i}$ and choose a $\prec_{X_{i,j}}$-minimal element $x_{i,j}$ of $X_{i,j}$. Draw the tree-edges $x_i x_{i,j}$ for all $i \in I_0$ and $j \in I_1$. By continuing the process recursively we claim that every $v \in V$ has to appear in the tree. Indeed, if some $v$ does not, then we would obtain a nested essential sequence of optimal cuts such that its limit contains $v$ (and does not contain $r$), which contradicts Lemma 20.

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