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SPLITTINGS AND CALCULATIONAL TECHNIQUES FOR HIGHER THH

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Abstract. Tensoring finite pointed simplicial sets X with commutative ring spectra R yields important homology theories such as (higher) topological Hochschild homology and torus homology. We prove several structural properties of these constructions relating $X \otimes (-)$ to $\Sigma X \otimes (-)$ and we establish splitting results. This allows us, among other important examples, to determine $\mathsf{THH}_*^{[n]}(\mathbb{Z}/p^m; \mathbb{Z}/p)$ for all $n \geqslant 1$ and for all $m \geqslant 2$.

Introduction

For any (finite) pointed simplicial set X one can define the tensor product of X with a commutative ring spectrum A, $X \otimes A$, where the case $X = S^1$ gives topological Hochschild homology of A. More generally, for any sequence of maps of commutative ring spectra $R \to A \to C$ we can define $\mathcal{L}_X^R(A; C)$, the Loday construction with respect to X of A over R with coefficients in C. Important examples are $X = S^n$ or X a torus. The construction specializes to $X \otimes A$ in the case $\mathcal{L}_X^S(A; A)$. For details see Definition 1.1.

An important question about the Loday construction concerns the dependence on X: Given $X, Y \in s\mathbf{Set}_*$, with $\Sigma X \simeq \Sigma Y$, does that imply that $\mathcal{L}_X^R(A) \simeq \mathcal{L}_Y^R(A)$? If it does, the Loday construction would be a "stable invariant". Positive cases arise from the work of Berest, Ramadoss and Yeung [5, Theorem 5.2]: They identify the homotopy groups of the Loday construction over an $X \in s\mathbf{Set}_*$ of a Hopf algebra over a field with representation homology of the Hopf algebra with respect to $\Sigma(X_+)$. Unpublished work of Ausoni and Dundas shows the equivalence of $\mathcal{L}_{S^2 \vee S^1 \vee S^1}(H\mathbb{F}_p)$ and $\mathcal{L}_{S^1 \times S^1}(H\mathbb{F}_p)$ and in [15] Dundas and Tenti prove that stable invariance holds if A is a smooth algebra over a commutative ring k. However in [15] they also provide a counterexample: $\mathcal{L}^{H\mathbb{Q}}_{S^2 \vee S^1 \vee S^1}(H\mathbb{Q}[t]/t^2)$ is *not* equivalent to $\mathcal{L}^{H\mathbb{Q}}_{S^1 \times S^1}(H\mathbb{Q}[t]/t^2)$ even though $\Sigma(S^2 \vee S^1 \vee S^1) \simeq \Sigma(S^1 \times S^1)$. Our juggling formula (Theorem 3.2) and our generalized Brun splitting (Theorem 4.1) relate the Loday construction on ΣX to that of X. One application among others of these results is to establish stable invariance in certain examples.

For commutative $\mathbb{F}_p$-algebras A one often observes a splitting of $\mathsf{THH}(A)$ as $\mathsf{THH}(\mathbb{F}_p) \wedge_{H\mathbb{F}_p} \mathsf{THH}^{H\mathbb{F}_p}(HA)$, so $\mathsf{THH}(A)$ splits as topological Hochschild homology of $\mathbb{F}_p$ tensored with the Hochschild homology of A [24]. It is natural to ask in which generality such splittings occur. If one replaces $\mathbb{F}_p$ by $\mathbb{Z}$, then there are many counterexamples. For instance if $A = \mathcal{O}_K$ is a

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number ring then $\mathsf{THH}_*(\mathcal{O}_K)$ is known by [26, Theorem 1.1] and is far from being equivalent to $\pi_*(\mathsf{THH}(\mathbb{Z}) \wedge^L_{H\mathbb{Z}} \mathsf{THH}^{H\mathbb{Z}}(H\mathcal{O}_K))$ in general (see [18, Remark 4.12.] for a concrete example). We prove several splitting results for higher THH and use one of them to determine higher THH of $\mathbb{Z}/p^m$ with $\mathbb{Z}/p$-coefficients for all $m \geqslant 2$.

Content. We start with a brief recollection of the Loday construction in Section 1.

In [6] we determined the higher Hochschild homology of $R = \mathbb{F}_p[x]$ and of $R = \mathbb{F}_p[x]/x^{p^\ell}$, both Hopf algebras. In Section 2 we generalize our results to the cases $R = \mathbb{F}_p[x]/x^m$ if p divides m, which is not a Hopf algebra unless $m = p^\ell$ for some ℓ.

Building on work in [18] we prove a juggling formula (see Theorem 3.2): For every sequence of cofibrations of commutative ring spectra $S \to R \to A \to B \to C$ there is an equivalence

$$\mathcal{L}^R_{\Sigma X}(B; C) \simeq \mathcal{L}^R_{\Sigma X}(A; C) \wedge^L_{\mathcal{L}^A_X(C)} \mathcal{L}^B_X(C).$$

We explain in 3.1 what happens if one oversimplifies this formula.

Using a geometric argument, Brun [9] constructs a spectral sequence for calculating THH_*-groups. We prove a generalization of his splitting and show that for any sequence of cofibrations of commutative ring spectra $S \to R \to A \to B$ we obtain a generalized spectrum-level Brun splitting (see Theorem 4.1) *i.e.*, an equivalence of commutative B-algebra spectra

$$\mathcal{L}^R_{\Sigma X}(A; B) \simeq B \wedge^L_{\mathcal{L}^R_X(B)} \mathcal{L}^A_X(B).$$

Note that B, which only appears at the basepoint on the left, now appears almost everywhere on the right. This splitting also gives rise to associated spectral sequences for calculating higher THH_*-groups.

We apply our results to prove a generalization of Greenlees' splitting formula [17, Remark 7.2]: For an augmented commutative k-algebra A we obtain in Corollary 4.6 that

$$\mathcal{L}_{\Sigma X}(HA; Hk) \simeq \mathcal{L}_{\Sigma X}(Hk) \wedge^L_{Hk} \mathcal{L}^{HA}_X(Hk)$$

and if A is flat as a k-module then this can also be written as

$$\mathcal{L}_{\Sigma X}(HA; Hk) \simeq \mathcal{L}_{\Sigma X}(Hk) \wedge^L_{Hk} \mathcal{L}^{Hk}_{\Sigma X}(HA; Hk),$$

where all the Loday constructions are over the same simplicial set. For $X = S^n$, for example, and A a flat augmented commutative k-algebra this yields Theorem 5.10,

$$\mathsf{THH}^{[n]}(A; k) \simeq \mathsf{THH}^{[n]}(k) \wedge^L_{Hk} \mathsf{THH}^{[n],k}(A; k)$$

where $\mathsf{THH}^{[n]} = \mathcal{L}_{S^n}$.

Shukla homology is a derived version of Hochschild homology. We define higher order Shukla homology in Section 5 and calculate some examples that will be used in subsequent results. We prove that the Shukla homology of order n of a ground ring k over a flat augmented k-algebra is isomorphic to the reduced Hochschild homology of order $n+1$ of the flat augmented algebra (Proposition 5.9).

Tate shows [36] how to control Tor-groups for certain quotients of regular local rings. We use this to develop a splitting on the level of homotopy groups for $\mathsf{THH}(R/(a_1, \ldots, a_r); R/\mathfrak{m})$ if R is regular local, $\mathfrak{m}$ is the maximal ideal and the a_i's are in $\mathfrak{m}^2$.

We prove a splitting result for $\mathsf{THH}^{[n]}(R/a, R/p)$ in Section 7, where R is a commutative ring and $p, a \in R$ are not zero divisors, (p) is a maximal ideal, and $a \in (p)^2$. In this situation,

$$\mathsf{THH}^{[n]}(R/a, R/p) \simeq \mathsf{THH}^{[n]}(R, R/p) \wedge^L_{HR/p} \mathsf{THH}^{[n];R}(R/a, R/p).$$

In many cases the homotopy groups of the factors on the right hand side can be completely determined. Among other important examples we get explicit formulas for $\mathsf{THH}^{[n]}(\mathbb{Z}/p^m, \mathbb{Z}/p)$ for all $n \geqslant 1$ and all $m \geqslant 2$ (compare Theorem 9.1):

$$\mathsf{THH}^{[n]}_*(\mathbb{Z}/p^m, \mathbb{Z}/p) \cong \mathsf{THH}^{[n]}_*(\mathbb{Z}, \mathbb{Z}/p) \otimes_{\mathbb{Z}/p} \mathsf{THH}^{[n],\mathbb{Z}}_*(\mathbb{Z}/p^m, \mathbb{Z}/p).$$

We know $\mathsf{THH}^{[n]}_*(\mathbb{Z}, \mathbb{Z}/p)$ from [12] and we determine $\mathsf{THH}^{[n],\mathbb{Z}}_*(\mathbb{Z}/p^m, \mathbb{Z}/p)$ explicitly for all n.

This generalizes previous results by Pirashvili [32], Brun [9], and Angeltveit [1] from $n = 1$ to all n.

In Section 8 we provide a splitting result for commutative ring spectra of the form $A \times B$: we show in Proposition 8.2 that for any connected simplicial set X, we have

$$\mathcal{L}_X(A \times B) \xrightarrow{\simeq} \mathcal{L}_X(A) \times \mathcal{L}_X(B).$$

We present some sample applications of our splitting results in Section 9: a splitting of higher THH of ramified number rings with reduced coefficients (9.2), a version of Galois descent for higher THH (9.3) and a calculation of higher THH of function fields over $\mathbb{F}_p$ (9.4). We close with a discussion of the case of higher THH of $\mathbb{Z}/p^m$ (with unreduced coefficients) (9.5).

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1. The Loday construction: basic features

We recall some definitions concerning the Loday construction and we fix notation.

For our work we can use any good symmetric monoidal category of spectra whose category of commutative monoids is Quillen equivalent to the category of E_∞-ring spectra, such as symmetric spectra [22], orthogonal spectra [28] or S-modules [16]. As parts of the paper require to work with a specific model category we chose to work with the category of S-modules.

Let X be a finite pointed simplicial set and let $R \to A \to C$ be a sequence of maps of commutative ring spectra.

Definition 1.1. The Loday construction with respect to X of A over R with coefficients in C is the simplicial commutative augmented C-algebra spectrum $\mathcal{L}^R_X(A; C)$ whose p-simplices are

$$C \wedge \bigwedge_{x \in X_p \setminus *} A$$

where the smash products are taken over R. Here, $*$ denotes the basepoint of X and we place a copy of C at the basepoint. As the smash product over R is the coproduct in the category of commutative R-algebra spectra, the simplicial structure is straightforward: Face maps d_i on X induce multiplication in A or the A-action on C if the basepoint is involved. Degeneracies s_i on X correspond to the insertion of the unit maps $\eta_A \colon R \to A$ over all n-simplices which are not hit by $s_i \colon X_{n-1} \to X_n$.

As defined above, $\mathcal{L}^R_X(A;C)$ is a simplicial commutative augmented C-algebra spectrum. If M is an A-module spectrum, then $\mathcal{L}^R_X(A;M)$ is defined. By slight abuse of notation we won't distinguish $\mathcal{L}^R_X(A;C)$ or $\mathcal{L}^R_X(A;M)$ from their geometric realization.

If X is an arbitrary pointed simplicial set, then we can write it as the colimit of its finite pointed subcomplexes and the Loday construction with respect to X can then also be expressed as the colimit of the Loday construction for the finite subcomplexes.

An important case is $X = S^n$. In this case we write $\mathsf{THH}^{[n],R}(A;C)$ for $\mathcal{L}^R_{S^n}(A;C)$; this is the *higher order topological Hochschild homology of order n of A over R with coefficients in C*.

Let k be a commutative ring, A be a commutative k-algebra, and M be an A-module. Then we define

$$\mathsf{THH}^{[n],k}(A;M) := \mathcal{L}^{Hk}_{S^n}(HA;HM).$$

If A is flat over k, then $\pi_*\mathsf{THH}^k(A;M) \cong \mathsf{HH}^k_*(HA;HM)$ [16, Theorem IX.1.7] and this also holds for higher order Hochschild homology in the sense of Pirashvili [33]: $\pi_*\mathsf{THH}^{[n],k}(A;M) \cong HH^{[n],k}_*(A;M)$ if A is k-flat [6, Proposition 7.2].

To avoid visual clutter, given a commutative ring A and an element $a \in A$, we write A/a instead of $A/(a)$.

2. Higher THH of truncated polynomial algebras

When the Loday construction is viewed as a functor on pointed simplicial sets, it transforms homotopy pushouts of pointed simplicial sets into homotopy pushouts of Loday constructions. In [37, Section 3] Veen uses this to express higher THH as a "topological Tor" of a lower THH, that is: for any commutative S-algebra A,

$$\mathsf{THH}^{[n]}(A) \simeq A \wedge^L_{\mathsf{THH}^{[n-1]}(A)} A.$$

This yields a spectral sequence

$$E^2_{s,*} = \mathrm{Tor}_s^{\mathsf{THH}^{[n-1]}_*(A)}(A_*, A_*) \Rightarrow \mathsf{THH}^{[n]}_*(A).$$

In particular cases, this spectral sequence collapses for all $n \geqslant 1$ making it possible to calculate $\mathsf{THH}^{[n]}_*(A)$ as iterated Tor's of A_*. In [6, Figures 1 and 2] we had a flow chart showing the results of iterated Tor of $\mathbb{F}_p$ over some $\mathbb{F}_p$-algebras with a particularly convenient form. We can do similar calculations over any field:

Proposition 2.1. *If F is a field of characteristic p and $|\omega|$ is even, there is a flow chart as in Figure 1 showing the calculation of iterated* Tor*'s of F: If $\mathcal{A}$ is a term in the nth generation in the flow chart, then $\mathrm{Tor}^{\mathcal{A}}(F,F)$ is the tensor product of all the terms in the $(n+1)$st generation that arrows from $\mathcal{A}$ point to. Here $|\rho^0 y| = |y|+1$, $|\epsilon z| = |z|+1$ and $|\varphi^0 z| = 2 + p|z|$.*

$$\nearrow \bigotimes_{k\geqslant 0} \Lambda(\epsilon\rho^k\epsilon\omega) \longrightarrow \cdots$$

$$F[\omega] \to \Lambda(\epsilon\omega) \to \Gamma(\rho^0\epsilon\omega) \cong \bigotimes_{k\geqslant 0} F[\rho^k\epsilon\omega]/(\rho^k\epsilon\omega)^p$$

$$\searrow \bigotimes_{k\geqslant 0} \Gamma(\varphi^0\rho^k\epsilon\omega) \cong \bigotimes_{k,i\geqslant 0} F[\varphi^i\rho^k\epsilon\omega]/(\varphi^i\rho^k\epsilon\omega)^p \rightrightarrows \cdots$$

FIGURE 1. Iterated Tor flow chart

If F a field of characteristic zero, the analogous flow chart for $|x|$ even is

$$F[x] \longrightarrow \Lambda(\epsilon x) \longrightarrow F[\rho^0\epsilon x] \to \Lambda(\epsilon\rho^0\epsilon x) \longrightarrow \cdots$$

Proof. In characteristic p, all divided power algebras split as tensor products of truncated polynomial algebras. This allows us to use the resolutions of [6, Section 2], where the tensor products in the respective bar constructions are all taken to be over F, to pass from each stage to the next.

In characteristic 0, the Tor dual of an exterior algebra is a divided power algebra, but this is isomorphic to a polynomial algebra. Thus the resolutions of [6, Section 2], with the tensor products in the bar constructions again taken to be over F, can be used analogously to get the alternation between exterior and polynomial algebras. □

Let x be a generator of even non-negative degree. In [6, Theorem 8.8] we calculated higher HH of truncated polynomial rings of the form $\mathbb{F}_p[x]/x^{p^\ell}$ for any prime p. The decomposition due to Bökstedt which is described before the statement of the theorem there does not work for $\mathbb{F}_p[x]/x^m$ when m is not a power of p, but we can nevertheless use a similar kind of argument to determine higher HH of $\mathbb{F}_p[x]/x^m$ as long as p divides m. This generalization of [6, Theorem 8.8] is interesting because if m is not a power of p, $\mathbb{F}_p[x]/x^m$ is no longer a Hopf algebra, which the cases we discussed in [6] were.

In the following $\mathsf{HH}_*^{[n]}$ will denote Hochschild homology groups of order n whereas $\mathsf{HH}^{[n]}$ denotes the corresponding simplicial object whose homotopy groups are $\mathsf{HH}_*^{[n]}$.

Theorem 2.2. *Let x be of even degree and let m be a positive integer divisible by p. Then for all $n \geqslant 1$*

$$\mathsf{HH}_*^{[n],\mathbb{F}_p}(\mathbb{F}_p[x]/x^m) \cong \mathbb{F}_p[x]/x^m \otimes B_n''(\mathbb{F}_p[x]/x^m),$$

where $B''_1(\mathbb{F}_p[x]/x^m) \cong \Lambda_{\mathbb{F}_p}(\varepsilon x) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0 x)$, *with* $|\varepsilon x| = |x| + 1$, $|\varphi^0 x| = 2 + m|x|$ *and*

$$B''_n(\mathbb{F}_p[x]/x^m) \cong \mathrm{Tor}_{*,*}^{B''_{n-1}(\mathbb{F}_p[x]/x^m)}(\mathbb{F}_p, \mathbb{F}_p).$$

Since $\mathbb{F}_p[x]/x^m$ is monoidal over $\mathbb{F}_p$, this gives a higher THH calculation,

$$\begin{aligned} \mathsf{THH}_*^{[n]}(\mathbb{F}_p[x]/x^m) &\cong \mathsf{THH}_*^{[n]}(\mathbb{F}_p) \otimes \mathsf{HH}_*^{[n],\mathbb{F}_p}(\mathbb{F}_p[x]/x^m) \\ &\cong B^n_{\mathbb{F}_p}(\mu) \otimes \mathbb{F}_p[x]/x^m \otimes B''_n(\mathbb{F}_p[x]/x^m) \end{aligned} \tag{2.3}$$

where $B^1_{\mathbb{F}_p}(\mu) \cong \mathbb{F}_p[\mu]$ with $|\mu| = 2$ and $B^n_{\mathbb{F}_p}(\mu) = \mathrm{Tor}_{*,*}^{B^{n-1}_{\mathbb{F}_p}(\mu)}(\mathbb{F}_p, \mathbb{F}_p)$ for $n > 1$ (see [12, Remark 3.6]).

Proof. We use the standard resolution [23, (1.6.1)] and get that the Hochschild homology of $\mathbb{F}_p[x]/x^m$ is the homology of the complex

$$\ldots \xrightarrow{0} \Sigma^{m|x|}\mathbb{F}_p[x]/x^m \xrightarrow{\Delta(x,x)} \Sigma^{|x|}\mathbb{F}_p[x]/x^m \xrightarrow{0} \mathbb{F}_p[x]/x^m.$$

Since p divides m, we have $\Delta(x, x) = mx^{m-1} \equiv 0$ so the above differentials are all trivial and

$$\mathsf{HH}_*^{\mathbb{F}_p}(\mathbb{F}_p[x]/x^m) \cong \mathbb{F}_p[x]/x^m \otimes \Lambda_{\mathbb{F}_p}(\varepsilon x) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0 x)$$

at least as an $\mathbb{F}_p[x]/x^m$-module, with $|\varepsilon x| = |x| + 1$, $|\varphi^0 x| = 2 + m|x|$. The map $G.$ from [23, equation (1.8.6)] embeds this small complex quasi-isomorphically with its stated multiplicative structure into the standard Hochschild complex for $\mathbb{F}_p[x]/x^m$. It sends $\mathbb{F}_p[x]/x^m$ to itself in degree zero of the Hochschild complex and

$$\varepsilon x \mapsto 1 \otimes x - x \otimes 1, \quad \varphi^0 x \mapsto \sum_{i_0=1}^{m} \sum_{j_0=0}^{1} (-1)^{1+j_0} x^{i_0-j_0} \otimes x^{m-i_0} \otimes x^{j_0}$$

which generate an exterior and divided power subalgebra, respectively, inside the standard Hochschild complex equipped with the shuffle product. The map $G.$ is shown in [23] to be half of a chain homotopy equivalence between the small complex and the standard Hochschild complex. So we get that $\mathsf{HH}_*^{\mathbb{F}_p}(\mathbb{F}_p[x]/x^m) = \mathsf{HH}_*^{[1],\mathbb{F}_p}(\mathbb{F}_p[x]/x^m)$ has the desired form as an algebra and sits as a deformation retract inside the standard complex calculating it.

For the higher HH_*-computation we use that the E^2-term of the spectral sequence for $\mathsf{HH}_*^{[2]}$ is

$$E^2_{*,*} = \mathrm{Tor}^{\mathbb{F}_p[x]/x^m \otimes \Lambda_{\mathbb{F}_p}(\varepsilon x) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0 x)}(\mathbb{F}_p[x]/x^m, \mathbb{F}_p[x]/x^m)$$

and as the generators εx and $\varphi^0 x$ come from homological degree one and two, the module structure of $\mathbb{F}_p[x]/x^m$ over $\Lambda_{\mathbb{F}_p}(\varepsilon x)$ and $\Gamma_{\mathbb{F}_p}(\varphi^0 x)$ factors over the augmentation to $\mathbb{F}_p$. Therefore the above Tor-term splits as

$$\mathbb{F}_p[x]/x^m \otimes \mathrm{Tor}_{*,*}^{\Lambda_{\mathbb{F}_p}(\varepsilon x)}(\mathbb{F}_p, \mathbb{F}_p) \otimes \mathrm{Tor}_{*,*}^{\Gamma_{\mathbb{F}_p}(\varphi^0 x)}(\mathbb{F}_p, \mathbb{F}_p). \tag{2.4}$$

Now we can argue as in [6] to show that there cannot be any differentials or extensions in this spectral sequence: although we are calculating the homology of the total complex of the bisimplicial $\mathbb{F}_p$-vector space of the bar construction $B(\mathbb{F}_p[x]/x^m, \mathsf{HH}^{\mathbb{F}_p}(\mathbb{F}_p[x]/x^m), \mathbb{F}_p[x]/x^m)$ which involves both vertical and horizontal boundary maps, we can map the bar construction

$$B(\mathbb{F}_p[x]/x^m, \mathbb{F}_p[x]/x^m \otimes \Lambda_{\mathbb{F}_p}(\varepsilon x) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0 x), \mathbb{F}_p[x]/x^m)$$

quasi-isomorphically into it, and the latter complex involves only non-trivial horizontal maps (all vertical differentials vanish) and has homology exactly equal to the algebra in equation (2.4). When all the vertical differentials in the original double complex are zero, there can be no nontrivial spectral sequence differentials d^r for $r \geqslant 2$. Also, the trivial vertical differentials mean that there can be no nontrivial extensions involving anything but the ith and $(i+1)$st filtration, but since we can produce explicit generators whose pth powers (in the even dimensional case) or squares (in the odd dimensional case) actually vanish, we do not need to worry about extensions at all. Thus we obtain the claim about $\mathsf{HH}_*^{[2],\mathbb{F}_p}(\mathbb{F}_p[x]/x^m)$.

An iteration of this argument yields the result for higher Hochschild homology, since now we only have exterior algebras on odd-dimensional classes and truncated algebras, truncated at the p'th power, on even-dimensional ones where the powers that vanish do so for combinatorial reasons not relating to the power of x that was truncated at in the original algebra. At each stage, the tensor factor $\mathbb{F}_p[x]/x^m$ will split off the E^2-term for degree reasons. What remains will be the Tor of $\mathbb{F}_p$ with itself over a differential graded algebra that can be chosen up to chain homotopy equivalence to be a graded algebra A with a zero differential which is moreover guaranteed by the flow chart to have the property that $B(\mathbb{F}_p, A, \mathbb{F}_p)$ is chain homotopy equivalent to its homology embedded as a subcomplex with trivial differential inside it.

The splitting for higher THH follows from the splitting for higher HH by arguing as in [6, 6.1] (following [19, Theorem 7.1]) for the abelian pointed monoid $\{1, x, \ldots, x^{m-1}, x^m = 0\}$. □

Reducing the coefficients via the augmentation simplifies things even further. Here, the result does not depend on the p-valuation of m, because x augments to zero and therefore Hochschild homology of $\mathbb{F}_p[x]/x^m$ with coefficients in $\mathbb{F}_p$ is the homology of the complex

$$\ldots \xrightarrow{0} \Sigma^{m|x|}\mathbb{F}_p \xrightarrow{\Delta(x,x)=0} \Sigma^{|x|}\mathbb{F}_p \xrightarrow{0} \mathbb{F}_p.$$

Thus we obtain the following result.

Proposition 2.5. *For all primes p and for all $m > 1$*

$$\mathsf{HH}_*^{[n],\mathbb{F}_p}(\mathbb{F}_p[x]/x^m; \mathbb{F}_p) \cong B_n''(\mathbb{F}_p[x]/x^m)$$

where $B_1''(\mathbb{F}_p[x]/x^m) \cong \Lambda_{\mathbb{F}_p}(\varepsilon x) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0 x)$ and $B_n''(\mathbb{F}_p[x]/x^m) = \mathrm{Tor}_{*,*}^{B_{n-1}''(\mathbb{F}_p[x]/x^m)}(\mathbb{F}_p, \mathbb{F}_p)$ for $n > 1$. Therefore we obtain

$$\mathsf{THH}_*^{[n]}(\mathbb{F}_p[x]/x^m; \mathbb{F}_p) \cong \mathsf{THH}_*^{[n]}(\mathbb{F}_p) \otimes B_n''(\mathbb{F}_p[x]/x^m). \tag{2.6}$$

This is shown as in the proof of the previous theorem using the method of [6], embedding

$$\varepsilon x \mapsto 1 \otimes x \otimes 1, \quad \varphi^0 x \mapsto 1 \otimes x^{m-1} \otimes x \otimes 1$$

inside the bar complex $B(\mathbb{F}_p, \mathbb{F}_p[x]/x^m, \mathbb{F}_p)$, where they generate exterior and divided power algebras, respectively, regardless of the divisibility of m.

Remark 2.7. Note that the calculation becomes drastically different if $(p, m) = 1$ and we look at the full $\mathsf{HH}_*^{[n],\mathbb{F}_p}(\mathbb{F}_p[x]/x^m)$ rather than reducing coefficients to get $\mathsf{HH}_*^{[n],\mathbb{F}_p}(\mathbb{F}_p[x]/x^m; \mathbb{F}_p)$. Then multiplication by m is an isomorphism on $\mathbb{F}_p[x]/x^m$-modules and hence

$$\mathsf{HH}_*^{\mathbb{F}_p}(\mathbb{F}_p[x]/x^m) \cong \begin{cases} \mathbb{F}_p[x]/x^m, & \text{for } * = 0, \\ (\Sigma^{|x|(km+1)}\mathbb{F}_p[x]/x^m)/x^{m-1}, & \text{for } * = 2k+1, \\ \Sigma^{km|x|}\ker(\cdot x^{m-1}), & \text{for } * = 2k, k > 0. \end{cases}$$

3. A juggling formula

In this section we generalize juggling formulas from [18, §2 and §3]: we allow working relative to a ring spectrum R that can be different from the sphere spectrum and we relate the Loday construction on a suspension on a pointed simplicial set X to the Loday construction on X. In [18] we mainly considered the cases where X is a sphere.

Lemma 3.1. *Let X be a pointed simplicial set. For a sequence of cofibrations of commutative S-algebras*

$$S \longrightarrow R \longrightarrow A \longrightarrow B \longrightarrow C$$

there is an equivalence of augmented commutative C-algebras.

$$\mathcal{L}_X^A(B; C) \simeq C \wedge^L_{\mathcal{L}_X^R(A;C)} \mathcal{L}_X^R(B; C).$$

Proof. For the duration of this proof smash products are formed over R, not S, but we still denote them by $\wedge$ in order to simplify notation. For finite X the unit map of C, $\eta_C \colon R \to C$, induces a map of coequalizer diagrams

$$\begin{array}{ccc} C \wedge \left(\bigwedge_{X_n \setminus *} A\right) \wedge \left(\bigwedge_{X_n \setminus *} B\right) & \xrightarrow{\mathrm{id}\wedge\eta_C\wedge\eta_C} & C \wedge \left(C \wedge \bigwedge_{X_n \setminus *} A\right) \wedge \left(C \wedge \bigwedge_{X_n \setminus *} B\right) \\ \nu_L \downarrow \downarrow \nu_R & & n_L \downarrow \downarrow n_R \\ C \wedge \left(\bigwedge_{X_n \setminus *} B\right) & \xrightarrow{\mathrm{id}\wedge\eta_C} & C \wedge \left(C \wedge \bigwedge_{X_n \setminus *} B\right) \end{array}$$

Here, ν_L sends the copies of A to C first and then multiplies $C \wedge \left(\bigwedge_{X_n \setminus *} C\right)$ to C whereas ν_R sends the copies of A to B and then uses the multiplication in each coordinate separately. The map n_L sends the copies of A to C and multiplies $C \wedge \left(C \wedge \bigwedge_{X_n \setminus *} C\right)$ to C sitting at the basepoint whereas n_R sends the copies of A to B and then uses the multiplication in B in every coordinate. This morphism of coequalizer diagrams induces an isomorphism η on the corresponding coequalizers, *i.e.*,

$$\eta \colon (\mathcal{L}_X^A(B; C))_n \longrightarrow (C \wedge_{\mathcal{L}_X^R(A;C)} \mathcal{L}_X^R(B; C))_n.$$

This is an isomorphism because the diagram

$$\begin{array}{ccc} C & \xrightarrow{\mathrm{id}\wedge\eta_C\wedge\eta_C} & C\wedge C\wedge C \\ {\scriptstyle =}\downdownarrows{\scriptstyle =} & & {\scriptstyle \mathrm{id}\wedge\mu}\downdownarrows{\scriptstyle \mu\wedge\mathrm{id}} \\ C & \xrightarrow{\mathrm{id}\wedge\eta_C} & C\wedge C \end{array}$$

induces the identity map between the coequalizers. A colimit argument proves the claim for general X. □

Theorem 3.2 (Juggling Formula)**.** *Let X be a pointed simplicial set. Then for any sequence of cofibrations of commutative S-algebras $S \to R \to A \to B \to C$ we get an equivalence of augmented commutative C-algebras*

$$\mathcal{L}^R_{\Sigma X}(B;C) \simeq \mathcal{L}^R_{\Sigma X}(A;C) \wedge^L_{\mathcal{L}^A_X(C)} \mathcal{L}^B_X(C).$$

Proof. Consider the diagram

$$\begin{array}{ccccc} C & \longleftarrow & \mathcal{L}^R_X(A;C) & \longrightarrow & C \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{L}^R_X(C) & \longleftarrow & \mathcal{L}^R_X(A;C) & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{L}^R_X(C) & \longleftarrow & \mathcal{L}^R_X(B;C) & \longrightarrow & C. \end{array}$$

By Lemma 3.1, taking the homotopy pushouts of the rows produces the diagram

$$\begin{array}{c} \mathcal{L}^R_{\Sigma X}(A;C) \\ \uparrow \\ \mathcal{L}^A_X(C) \\ \downarrow \\ \mathcal{L}^B_X(C) \end{array}$$

whose homotopy pushout is

$$\mathcal{L}^R_{\Sigma X}(A;C) \wedge^L_{\mathcal{L}^A_X(C)} \mathcal{L}^B_X(C).$$

We get an equivalent result by first taking the homotopy pushouts of the columns and then of the rows. Homotopy pushouts on the columns produces

$$C \wedge^L_{\mathcal{L}^R_X(C)} \mathcal{L}^R_X(C) \longleftarrow \mathcal{L}^R_X(A;C) \wedge^L_{\mathcal{L}^R_X(A;C)} \mathcal{L}^R_X(B;C) \longrightarrow C \wedge^L_C C$$

which simplifies to

$$C \longleftarrow \mathcal{L}^R_X(B;C) \longrightarrow C$$

whose homotopy pushout is equivalent to $\mathcal{L}^R_{\Sigma X}(B;C)$. □

Restricting our attention to spheres we obtain the following result. This is a relative variant of [18, Theorem 3.6].

Corollary 3.3. *Let $S \to R \to A \to B \to C$ be a sequence of cofibrations of commutative S-algebras. Then for all $n \geqslant 0$ there is an equivalence of augmented commutative C-algebras:*

$$\mathsf{THH}^{[n+1],R}(B;C) \simeq \mathsf{THH}^{[n+1],R}(A;C) \wedge^L_{\mathsf{THH}^{[n],A}(C)} \mathsf{THH}^{[n],B}(C).$$

Remark 3.4. The previous corollary gives a splitting of the same form as [18, Theorem 3.6]. However, as the proof is different it is not obvious that the maps in the smash product are the same. Thus (unlikely as it may be) it may turn out to be the case that this gives two different but similar-looking splittings.

3.1. **Beware the phony right-module structure!** In some cases it is tempting to use the (valid) splitting of $\mathsf{THH}^{[n+1],R}(A;C)$ as $\mathsf{THH}^{[n+1],R}(A) \wedge_A C$ and oversimplify the juggling formula we got in Corollary 3.3 to the **invalid** identification of $\mathsf{THH}^{[n+1],R}(B;C)$ with $\mathsf{THH}^{[n+1],R}(A) \wedge_A (C \wedge^L_{\mathsf{THH}^{[n],A}(C)} \mathsf{THH}^{[n],B}(C))$ which in the case $B = C$ becomes

$$\mathsf{THH}^{[n+1],R}(A) \wedge_A \mathsf{THH}^{[n+1],A}(C). \tag{3.5}$$

This transformation is incorrect because it disregards the module structures, without which the maps of pushouts are not well-defined. The spectrum $\mathsf{THH}^{[n+1],R}(A;C)$ is *not* equivalent to $\mathsf{THH}^{[n+1],R}(A) \wedge_A C$ as a right-module spectrum over $\mathsf{THH}^{[n],A}(C)$. Assuming that the rearrangement that leads to (3.5) were valid, any cofibration of commutative S-algebras $S \to A \to B$ would produce an equivalence between $\mathsf{THH}^{[n]}(B)$ and $\mathsf{THH}^{[n]}(A) \wedge^L_A \mathsf{THH}^{[n],A}(B)$. But this equivalence does *not* hold in many examples, *e.g.*, for $A = H\mathbb{Z}$ and B equal to the Eilenberg Mac Lane spectrum of $\mathbb{F}_p$ or of the ring of integers in a number field.

4. A generalization of Brun's spectral sequence

In [9] Morten Brun uses the geometry of the circle to identify $\mathsf{THH}(HQ; HQ \wedge^L_{Hk} HQ)$ with $\mathsf{THH}(Hk; HQ)$ where k is a commutative ring and Q is a commutative k-algebra: $\mathsf{THH}(Hk; HQ)$ is a circle with HQ at the basepoint and Hk sitting at every non-basepoint of S^1. Homotopy invariance says that we can let the point take over half the circle, so that it covers an interval. This idea identifies $\mathsf{THH}(Hk; HQ)$ with $\mathsf{THH}(Hk; B(HQ, HQ, HQ))$ where B denotes the two-sided bar construction. Brun then shows in [9, Lemma 6.2.3] that the latter is equivalent to $\mathsf{THH}(HQ; B(HQ, Hk, HQ))$ by a shift of perspective. This idea inspired our juggling formula 3.2 and also the following result.

Theorem 4.1 (Brun Juggling)**.** *Let X be a pointed simplicial set. For any sequence of cofibrations of commutative S-algebras $S \to R \to A \to B$ we get an equivalence of commutative B-algebras*

$$\mathcal{L}^R_{\Sigma X}(A;B) \simeq B \wedge^L_{\mathcal{L}^R_X(B)} \mathcal{L}^A_X(B).$$

Note that B, which only appears at the basepoint on the left, now appears almost everywhere on the right. Thus we can think of the basepoint as having "eaten" most of ΣX.

In the following we will use the notation from [18, §2]. If Y is a pointed simplicial subset of X, then we denote by $\mathcal{L}^R_{(X,Y)}(A,B;B)$ the relative Loday construction where we attach B to every point in Y including the basepoint, A to every point in the complement and we use the structure maps to turn this into a augmented commutative B-algebra spectrum. Note that if $Y=*$, then $\mathcal{L}^R_{(X,*)}(A,B;B)=\mathcal{L}^R_X(A;B)$, so in this case we omit the $*$ from the notation as in Definition 1.1.

Proof. We consider the pair $(\Sigma X, *)$ as $(CX\cup_X CX, CX)$, with the cone sitting as the upper half of the suspension. Then, since the Loday construction is homotopy invariant,

$$\begin{aligned}\mathcal{L}^R_{\Sigma X}(A;B)=\mathcal{L}^R_{\Sigma X,*}(A,B;B)=\mathcal{L}^R_{CX\cup_X CX,*}(A,B;B)&\simeq \mathcal{L}^R_{CX\cup_X CX,CX}(A,B;B)\\ &=\mathcal{L}^R_{CX\cup_X CX,CX\cup_X X}(A,B;B).\end{aligned}$$

By [18, Proposition 2.10(b)]

$$\mathcal{L}^R_{CX\cup_X CX,CX\cup_X X}(A,B;B)\simeq \mathcal{L}^R_{CX,CX}(A,B;B)\wedge_{\mathcal{L}^R_{X,X}(A,B;B)}\mathcal{L}^R_{CX,X}(A,B;B).$$

By definition $\mathcal{L}^R_{CX,CX}(A,B;B)=\mathcal{L}^R_{CX}(B)$ and $\mathcal{L}^R_X(B)=\mathcal{L}^R_{X,X}(A,B;B)$ and by homotopy invariance $\mathcal{L}^R_{CX}(B)\simeq B$, hence

$$\mathcal{L}^R_{CX\cup_X CX,CX\cup_X X}(A,B;B)\simeq B\wedge_{\mathcal{L}^R_X(B)}\mathcal{L}^R_{CX,X}(A,B;B).$$

Using [18, (3.0.1)] we can identify $\mathcal{L}^R_{CX,X}(A,B;B)$ with

$$\mathcal{L}^R_{CX}(A;B)\wedge^L_{\mathcal{L}^R_X(A;B)}\mathcal{L}^R_X(B;B) \tag{4.2}$$

and as CX is contractible we obtain $B\simeq \mathcal{L}^R_{CX}(A;B)$ and then Lemma 3.1 yields an equivalence of (4.2) with $\mathcal{L}^A_X(B)$. □

Example 4.3. Consider the case when $X=S^0$.

$$\mathsf{THH}(A;B)\simeq B\wedge_{B\wedge B}(B\wedge_A B)=\mathsf{THH}(B;B\wedge_A B). \tag{4.4}$$

There is an Atiyah–Hirzebruch spectral sequence [16, IV.3.7]

$$E^2_{p,q}=\pi_p(E\wedge_R H\pi_q M)\Longrightarrow \pi_{p+q}(E\wedge_R M).$$

Let B be a connective A-algebra. Setting $R=B\wedge B$, $E=B$ and $M=B\wedge_A B$ we get

$$E^2_{p,q}=\pi_p(B\wedge_{B\wedge B}H\pi_q(B\wedge_A B))\Longrightarrow \pi_{p+q}(B\wedge_{B\wedge B}(B\wedge_A B)).$$

Setting $B=HQ$ and $A=Hk$ gives us

$$\begin{aligned}E^2_{p,q}=\mathsf{THH}_p(Q;\mathrm{Tor}^k_q(Q,Q))\Longrightarrow&\pi_{p+q}(HQ\wedge_{HQ\wedge HQ}(HQ\wedge_{Hk}HQ))\\ &\cong \mathsf{THH}_{p+q}(k;Q)\end{aligned}$$

by (4.4). This recovers a spectral sequence with the same E^2 page and limit as Brun's [9, Theorem 6.2.10]. A substantial generalization of Brun's spectral sequence for THH can be found in [21, Theorem 1.1].

Example 4.5. We can generalize Example 4.3 to any X. In particular, consider a commutative ring k and a commutative k-algebra Q. If we apply the Atiyah–Hirzebruch spectral sequence in the case

$$E = HQ \qquad R = \mathcal{L}_{S^n}(HQ) \qquad M = \mathcal{L}^{Hk}_{S^n}(HQ)$$

then the Brun juggling formula 4.1 gives us a spectral squence

$$E^2_{p,q} = \pi_p(HQ \wedge_{\mathsf{THH}^{[n]}(Q)} H\mathsf{THH}^{[n],k}_q(Q)) \Longrightarrow \mathsf{THH}^{[n+1]}_{p+q}(k;Q).$$

In the next section, we will see that we can identify $\mathsf{THH}^{[n],k}(Q)$ with higher order Shukla homology, $\mathsf{Sh}^{[n],k}(Q)$, so we get the simpler description

$$E^2_{p,q} = \pi_p(HQ \wedge_{\mathsf{THH}^{[n]}(Q)} H(\mathsf{Sh}^{[n],k}_q(Q)) \Longrightarrow \mathsf{THH}^{[n+1]}_{p+q}(k;Q).$$

Corollary 4.6. *Let B be an augmented commutative A-algebra spectrum. Then applying Theorem 3.2 to the sequence $S \xrightarrow{=} S \longrightarrow A \longrightarrow B \longrightarrow A$ gives*

$$\mathcal{L}_{\Sigma X}(B;A) \simeq \mathcal{L}_{\Sigma X}(A;A) \wedge^L_A \mathcal{L}^B_X(A).$$

In particular, if k is a commutative ring, $A = Hk$, and $B = HQ$ for an augmented commutative k-algebra Q, then

$$\mathcal{L}_{\Sigma X}(HQ;Hk) \simeq \mathcal{L}_{\Sigma X}(Hk;Hk) \wedge^L_{Hk} \mathcal{L}^{HQ}_X(Hk)$$

and if k is a field, then we obtain on the level of homotopy groups

$$\pi_*\mathcal{L}_{\Sigma X}(HQ;Hk) \cong \pi_*(\Sigma X \otimes Hk) \otimes_k \pi_*(\mathcal{L}^{HQ}_X(Hk)).$$

Remark 4.7. We stress that in Corollary 4.6 there is a spectrum level splitting of $\mathcal{L}_{\Sigma X}(HQ;Hk)$ into $\mathcal{L}_{\Sigma X}(Hk)$ smashed with an additional factor. In particular, for $X = S^n$ higher THH of an augmented commutative k-algebra splits as

$$\mathsf{THH}^{[n+1]}(Q;k) \simeq \mathsf{THH}^{[n+1]}(k) \wedge^L_{Hk} \mathsf{THH}^{[n],Q}(k).$$

Greenlees proposed a splitting result in [17, Remark 7.2]: If k is a field and Q is a augmented commutative k-algebra, then his results yield a splitting

$$\mathsf{THH}_*(Q;k) \simeq \mathsf{THH}_*(k) \otimes_k \mathrm{Tor}^Q_*(k,k).$$

Our result generalizes his because for $X = S^0$ the term $\mathcal{L}^{HQ}_{S^0}(Hk)$ is nothing but $Hk \wedge^L_{HQ} Hk$ whose homotopy groups are isomorphic to $\mathrm{Tor}^Q_*(k,k)$. We will revisit this splitting result later in Theorem 5.10, relating it to higher order Hochschild homology.

5. Higher Shukla homology

Let k be a commutative ring. Ordinary Shukla homology [35] of a k-algebra A with coefficients in an A-bimodule M can be identified with $\mathsf{THH}^k(A;M)$. We will define higher order Shukla homology in the context of commutative algebras as an iterated bar construction and identify it with $\mathsf{THH}^{[n],k}(A;M)$ in Proposition 5.4.

Definition 5.1. Let A be a commutative k-algebra and B be a commutative A-algebra. We define

$$\mathsf{Sh}^{[0],k}(A;B) = HA \wedge^L_{Hk} HB.$$

For $n \geqslant 1$ we define *nth order Shukla homology of A over k with coefficients in B* as

$$\mathsf{Sh}^{[n],k}(A;B) = B^S(HB, \mathsf{Sh}^{[n-1],k}(A;B), HB).$$

where the latter is the two sided bar construction with respect to HB over the sphere spectrum.

Thus for $n = 1$ we have $\mathsf{Sh}^{[1],k}(A;B) \simeq \mathsf{THH}^k(A;B)$. For example, when $k = \mathbb{Z}$, p is a prime and $a = p^m$,

$$\mathsf{Sh}^{\mathbb{Z}}_*(\mathbb{Z}/p^m;\mathbb{Z}/p) \cong \Gamma_{\mathbb{Z}/p}(x(m))$$

with $|x(m)| = 2$.

It is consistent to set $\mathsf{Sh}^{[-1],k}(A) = Hk$.

A priori, $\mathsf{THH}^k(A;B)$ is a simplicial spectrum and $\mathsf{Sh}^{[n],k}(A;B)$ is therefore an n-simplicial spectrum, but we take iterated diagonals to get a simplicial spectrum and can then use geometric realization to get an honest spectrum.

Proposition 5.2. *Let R be a commutative ring and let $a, p \in R$ be elements which are not zero divisors such that (p) is maximal and $a \in (p)^2$. Then*

$$\mathsf{Sh}^{[0],R}_*(R/p) \cong \Lambda_{R/p}(\tau_1) \cong \mathsf{Sh}^{[0],R}_*(R/a;R/p), \qquad |\tau_1| = 1$$

and for $n \geqslant 2$,

$$\mathsf{Sh}^{[n],R}_*(R/p) \cong \mathrm{Tor}_*^{\mathsf{Sh}^{[n-1],R}_*(R/p)}(R/p;R/p)$$

and

$$\mathsf{Sh}^{[n],R}_*(R/a;R/p) \cong \mathrm{Tor}_*^{\mathsf{Sh}^{[n-1],R}_*(R/a,R/p)}(R/p;R/p).$$

Warning: the reduction $R/a \to R/p$ does **not** induce an isomorphism $\mathsf{Sh}^{[n],R}_*(R/a;R/p) \to \mathsf{Sh}^{[n],R}_*(R/p)$. By considering resolutions we can see that at $n = 0$ the induced map is the map taking τ_1 to 0. In fact, in Corollary 7.4 we show that the map induced by $R/a \to R/p$ is zero on all generators other than the R/p in dimension 0.

Proof. We prove this by induction on n. At $n = 0$,

$$\mathsf{Sh}^{[0],R}(R/p;R/p) = R/p \wedge^L_R R/p.$$

There is a Künneth spectral sequence,

$$E^2_{s,t} = \mathrm{Tor}^R_{s,t}(R/p, R/p) \Longrightarrow \pi_{s+t}(R/p \wedge^L_R R/p).$$

We have a short resolution

$$R \xrightarrow{\cdot p} R \to R/p,$$

so

$$\mathrm{Tor}^R_{s,t}(R/p, R/p) \cong H_s(R/p \xrightarrow{0} R/p)_t \cong \begin{cases} R/p, & s = 0 = t, \\ R/p, & s = 1, t = 0, \\ 0, & \text{otherwise.} \end{cases}$$

For degree reasons, there cannot be any differentials or extensions in this spectral sequence, and the product of τ_1 with itself has to vanish. Thus $\mathsf{Sh}^{[0]}_*(R/p) \cong \Lambda_{R/p}(\tau_1)$, as desired. Note that this proof works (almost) verbatim for $\mathsf{Sh}^{[0],R}_*(R/a; R/p)$.

By [12, Proposition 2.1], as a augmented commutative HR/p-algebra,

$$\mathsf{Sh}^{[0],R}(R/p) \simeq HR/p \vee \Sigma HR/p \simeq \mathsf{Sh}^{[0],R}(R/a; R/p).$$

Thus

$$\mathsf{Sh}^{[1],R}(R/p) = B(HR/p, \mathsf{Sh}^{[0],R}(R/p), HR/p) \simeq HB(R/p, \Lambda_{R/p}(\tau_1), R/p).$$

In the following let $\mathbb{F}$ be R/p. By [6], if we start with $B^{\mathbb{F}}(\mathbb{F}, \Lambda_{\mathbb{F}}(\tau_1), \mathbb{F})$ for $\mathbb{F}$ a field of positive characteristic, we know that the spectral sequence $\mathrm{Tor}^{\Lambda_{\mathbb{F}}(\tau_1)}_{*,*}(\mathbb{F}, \mathbb{F}) \Rightarrow H_*(B^{\mathbb{F}}(\mathbb{F}, \Lambda_{\mathbb{F}}(\tau_1), \mathbb{F}))$ collapses at E^2, which concludes the proof of the $n = 1$ case. We have that $B^{\mathbb{F}}(\mathbb{F}, \Lambda_{\mathbb{F}}(\tau_1), \mathbb{F}) \simeq B(\mathbb{F}, \Lambda_{\mathbb{F}}(\tau_1), \mathbb{F})$ because both calculate the homology of $\mathbb{F} \otimes^L_{\Lambda_{\mathbb{F}}(\tau_1)} \mathbb{F}$. Moreover, in [6] we show that if we keep applying $B^{\mathbb{F}}(\mathbb{F}, -, \mathbb{F})$ to the result, having started with $\Lambda_{\mathbb{F}}(\tau_1)$, the spectral sequences $\mathrm{Tor}^{H_*(-)}_{*,*}(\mathbb{F}, \mathbb{F}) \Rightarrow H_*(B^{\mathbb{F}}(\mathbb{F}, -, \mathbb{F}))$ will keep collapsing. Since in the case of a characteristic zero field, a divided power algebra is isomorphic to a polynomial one, we can use the method of [6], adjusted as in the proof of Proposition 2.1, to get the same result.

Finally, in [12] we show that once we can exhibit a commutative $H\mathbb{F}$-algebra as the image of the Eilenberg Mac Lane functor on some simplicial algebra, we can continue doing that when we apply $B(\mathbb{F}, -, \mathbb{F})$ to that algebra—once we get to the algebraic setting we can stay there. This concludes the proof for the collapsing of the spectral sequences both for R/p and for R/a. □

Definition 5.3. For any commutative k-algebra A and any commutative A-algebra B we define *higher derived Hochschild homology of A over k with coefficients in B*, $\widetilde{\mathsf{HH}}^{[n],k}(A; B)$, as

$$\widetilde{\mathsf{HH}}^{[n],k}(A; B) = \mathsf{THH}^{[n],k}(A; B).$$

Note that $\mathsf{Sh}^{[1],k}(A; B) = \widetilde{\mathsf{HH}}^{[1],k}(A; B) = \mathsf{Sh}^k(A; B)$.

Proposition 5.4. *There is an isomorphism*

$$\mathsf{Sh}^{[n],k}_*(A; B) \cong \widetilde{\mathsf{HH}}^{[n],k}_*(A; B)$$

for all $n \geqslant 0$.

The idea for the following proof is due to Bjørn Dundas.

Proof. By definition, the claim is true for $n = 0$. For higher n we have to show that the two-sided bar construction $B^S(HB, \mathsf{THH}^{[n],k}(A;B), HB)$ is a model for $\mathsf{THH}^{[n+1],k}(A;B)$; then the claim follows by induction. Using the decomposition of the $(n+1)$-sphere as two hemispheres glued along the equator S^n gives a homotopy pushout diagram for $\mathsf{THH}^{[n+1],k}(A;B)$

$$\begin{array}{ccc} \mathsf{THH}^{[n],k}(A;B) & \longrightarrow & HB \\ \downarrow & & \downarrow \\ HB & \longrightarrow & \mathsf{THH}^{[n+1],k}(A;B) \end{array}$$

in the model category of commutative Hk-algebras. The latter category is equivalent to the category of commutative S-algebras under Hk. The two-sided bar construction

$$B^S(HB, \mathsf{THH}^{[n],k}(A;B), HB)$$

models the homotopy pushout

$$HB \wedge^L_{\mathsf{THH}^{[n],k}(A;B)} HB$$

in the category of S-modules but this is also the homotopy pushout in the category of commutative S-algebras and in the category of commutative S-algebras under Hk; here one can use the model structure from [16] where cofibrant commutative S-algebras give the correct homotopy type when involved in a smash product as an underlying S-module. □

Proposition 5.5. *In the special case of a sequence of cofibrations of commutative S-algebras $R = Hk \to Hk \to HA \to Hk$ with a cofibrant model Hk and an augmented commutative k-algebra A we obtain*

$$\mathcal{L}^{Hk}_{\Sigma X}(HA; Hk) \simeq \mathcal{L}^{HA}_X(Hk) \tag{5.6}$$

for any X.

Proof. The juggling formula 3.2 for the sequence $Hk \to Hk \to HA \to Hk$ gives

$$\mathcal{L}^{Hk}_{\Sigma X}(HA; Hk) \simeq \mathcal{L}^{Hk}_{\Sigma X}(Hk; Hk) \wedge^L_{\mathcal{L}^{Hk}_X(Hk)} \mathcal{L}^{HA}_X(Hk)$$

but $\mathcal{L}^{Hk}_X(Hk) \simeq Hk \simeq \mathcal{L}^{Hk}_{\Sigma X}(Hk; Hk)$. □

Remark 5.7. Note that Proposition 5.5 implies that $\mathcal{L}^{HA}_X(Hk)$ depends only on the homotopy type of ΣX, so $\mathcal{L}^{HA}_X(Hk)$ is a stable invariant of X.

Consider the case where A is flat over k and $X = S^n$. Then Equation (5.6) gives

$$\mathcal{L}^{Hk}_{S^{n+1}}(HA; Hk) \simeq \mathcal{L}^{HA}_{S^n}(Hk). \tag{5.8}$$

The term on the left hand side of (5.8) has as homotopy groups the Hochschild homology of order $n+1$ of A with coefficients in k. The right hand side simplifies to $\mathsf{THH}^{[n],A}(k)$ and this is Shukla homology of order n of k over A. Therefore we obtain:

Proposition 5.9. *Let k be a commutative ring and let A be an augmented commutative k-algebra which is flat as a k-module. Then for all $n \geqslant 0$*

$$\mathsf{HH}^{[n+1],k}_*(A;k) \cong \mathsf{Sh}^{[n],A}_*(k).$$

Note that for $n = 0$ we obtain the classical formula [10, X.2.1]

$$\mathsf{HH}^k_*(A;k) \cong \mathrm{Tor}^A_*(k,k)$$

Combining Proposition 5.9 with Corollary 4.6 we get the following splitting result for augmented commutative k-algebras.

Theorem 5.10. *Let k be a commutative ring and let A be an augmented commutative k-algebra which is flat as a k-module. Then for all $n \geqslant 0$*

$$\mathsf{THH}^{[n]}(A;k) \simeq \mathsf{THH}^{[n]}(k) \wedge_{Hk} \mathsf{THH}^{[n],k}(A;k).$$

If k is a field then we obtain the following isomorphism on the level of homotopy groups

$$\mathsf{THH}^{[n]}_*(A;k) \cong \mathsf{THH}^{[n]}_*(k) \otimes_k \mathsf{HH}^{[n],k}_*(A;k).$$

6. A weak splitting for $\mathsf{THH}(R/(a_1,\dots,a_r);R/\mathfrak{m})$

Using a Tor-calculation by Tate from the 50's we obtain a splitting on the level of homotopy groups of $\mathsf{THH}_*(R/(a_1,\dots,a_r);R/\mathfrak{m})$ in good cases. This yields an easy way of calculating $\mathsf{THH}_*(\mathbb{Z}/p^m;\mathbb{Z}/p)$ for $m \geqslant 2$. Compare [32, 9, 1] for other approaches.

Theorem 6.1. *Let R be a regular local ring with maximal ideal $\mathfrak{m}$ and let $(a_1,\dots,a_r)$ be a regular sequence in R with $a_i \in \mathfrak{m}^2$ for $1 \leqslant i \leqslant r$. Then*

$$\mathsf{THH}_*(R/(a_1,\dots,a_r);R/\mathfrak{m}) \cong \mathsf{THH}_*(R;R/\mathfrak{m}) \otimes_{R/\mathfrak{m}} \Gamma_{R/\mathfrak{m}}(S_1,\dots,S_r)$$

with $|S_i| = 2$.

Proof. Let $I = (a_1,\dots,a_r)$. Applying the juggling formula 3.2 to $X = S^0$ and to the sequence $S \to HR \to HR/I \to HR/\mathfrak{m}$ gives

$$\mathsf{THH}(R/I;R/\mathfrak{m}) \simeq \mathsf{THH}(R;R/\mathfrak{m}) \underset{HR/\mathfrak{m}\wedge^L_{HR}HR/\mathfrak{m}}{\wedge^L} (HR/\mathfrak{m} \underset{HR/I}{\wedge^L} HR/\mathfrak{m}).$$

In [36] Tate determines the algebra structure on the homotopy groups of the last term,

$$\mathrm{Tor}^{R/I}_*(R/\mathfrak{m},R/\mathfrak{m}) \cong \Lambda_{R/\mathfrak{m}}(T_1,\dots,T_d) \otimes_{R/\mathfrak{m}} \Gamma_{R/\mathfrak{m}}(S_1,\dots,S_r).$$

Here, d is the dimension of $\mathfrak{m}/\mathfrak{m}^2$ as an $R/\mathfrak{m}$-vector space. We can choose a regular system of generators $(t_1,\dots,t_d)$ for $\mathfrak{m}$ such that the module structure of $\mathrm{Tor}^{R/I}_*(R/\mathfrak{m},R/\mathfrak{m})$ over $\mathrm{Tor}^R_*(R/\mathfrak{m},R/\mathfrak{m}) \cong \Lambda_{R/\mathfrak{m}}(T_1,\dots,T_d)$ is the canonical one (see [36, p. 22]). Hence the Künneth spectral sequence for $\mathsf{THH}(R/I;R/\mathfrak{m})$ has an E^2-term isomorphic to

$$\mathsf{THH}_*(R;R/\mathfrak{m}) \otimes_{R/\mathfrak{m}} \Gamma_{R/\mathfrak{m}}(S_1,\dots,S_r)$$

which is concentrated in the zeroth column that consists of

$$\mathsf{THH}_*(R;R/\mathfrak{m}) \otimes_{\mathrm{Tor}^R_*(R/\mathfrak{m},R/\mathfrak{m})} \mathrm{Tor}^{R/I}_*(R/\mathfrak{m},R/\mathfrak{m}).$$

Therefore, there are no non-trivial differentials and extensions in this spectral sequence. □

We call the splitting of Theorem 6.1 a *weak splitting* because it is only a splitting on the level of homotopy groups. In Section 7 we develop a stronger spectrum-level splitting of a similar form.

We apply the above result in the special case where R is a principal ideal domain. Let $p \neq 0$ be an element of R, such that (p) is a maximal ideal in R and let n be bigger or equal to 2. Then we are in the situation of Theorem 6.1 because $R_{(p)}/(p^n) \cong R/(p^n)$ so we can drop the assumption that R is local. The above result immediately gives an explicit formula for $\mathsf{THH}(R/(p^n); R/(p))$.

Corollary 6.2. *For all $n > 1$:*

$$\mathsf{THH}_*(R/(p^n); R/(p)) \cong \mathsf{THH}_*(R; R/(p)) \otimes_{R/(p)} \Gamma_{R/(p)}(S_1).$$

Remark 6.3. One may try to use the same method for $\mathsf{THH}^{[n]}$. The juggling formula from Theorem 3.2 for $S \xrightarrow{=} S \longrightarrow H\mathbb{Z} \longrightarrow H\mathbb{Z}/p^m \longrightarrow H\mathbb{Z}/p$ gives us

$$\mathsf{THH}^{[n]}(\mathbb{Z}/p^m; \mathbb{Z}/p) \simeq \mathsf{THH}^{[n]}(\mathbb{Z}; \mathbb{Z}/p) \underset{\mathsf{Sh}^{[n-1],\mathbb{Z}}(\mathbb{Z}/p)}{\wedge^L} \mathsf{Sh}^{[n-1],\mathbb{Z}/p^m}(\mathbb{Z}/p).$$

Thus we must understand the structure of $\mathsf{Sh}^{[n-1],\mathbb{Z}/p^m}(\mathbb{Z}/p)$ as a $\mathsf{Sh}^{[n-1],\mathbb{Z}}(\mathbb{Z}/p)$-algebra. It is not possible to do this through direct Tor computations for all n, as the computations rapidly become intractable; even $\mathsf{Sh}^{[1],\mathbb{Z}/p^2}(\mathbb{Z}/p)$ is rather involved [4, (5.2)], but see Proposition 7.5 for a general formula.

In order to obtain calculations in this example and in related cases, we need to develop the more delicate splitting of Section 7.

7. A splitting for $\mathsf{THH}^{[n]}(R/a; R/p)$

Throughout this section, we assume that R is a commutative ring and $a, p \in R$ are elements which are not zero divisors for which (p) is a maximal ideal and $a \in (p)^2$.

Lemma 7.1. *Let R, p, and a be as above, and let $\pi : R/a \to R/p$ be the obvious reduction. Then the map induced by π,*

$$\pi_* : \mathsf{Sh}_*^{[0],R}(R/a; R/p) \longrightarrow \mathsf{Sh}_*^{[0],R}(R/p),$$

factors as

$$\mathsf{Sh}_*^{[0],R}(R/a; R/p) \xrightarrow{\epsilon} R/p \xrightarrow{\eta} \mathsf{Sh}_*^{[0],R}(R/p).$$

Proof. The assumptions on a and p ensure that there exists a $b \in R$ such that $a = bp^2$. We have the following diagram:

$$\begin{array}{ccccc} R & \xrightarrow{\cdot a} & R & \xrightarrow{\epsilon} & R/a \\ \Big\downarrow{\scriptstyle \cdot bp} & & \Big\downarrow{\scriptstyle =} & & \\ R & \xrightarrow{\cdot p} & R & \xrightarrow{\epsilon} & R/p \end{array}$$

Thus we have a map of resolutions. When we tensor up with R/p we get the following diagram:

$$\begin{array}{ccc} R \otimes_R R/p & \xrightarrow{a\otimes 1} & R \otimes_R R/p \\ {\scriptstyle bp\otimes 1 = 0}\downarrow & & \downarrow{\scriptstyle =} \\ R \otimes_R R/p & \xrightarrow{p\otimes 1} & R \otimes_R R/p \end{array}$$

We take the homology of the top and bottom row. Note that since $a, p \in (p)$, the horizontal maps are 0; thus the top and bottom row produce Tor's which are of the form $\Lambda_{R/p}(\tau_1)$. However, when we look at where τ_1 goes from the top to the bottom, it maps by multiplication by bp—which is 0 in R/p. Thus this map is 0. □

Surprisingly enough, this special case allows us to prove a spectrum-level splitting for all $n \geqslant 0$.

Definition 7.2. Let $\mathcal{A}_{HR/p}$ be the category of augmented commutative HR/p-algebras and $h\mathcal{A}_{HR/p}$ its homotopy category. Let $\mathrm{Mod}_{HR/p}$ be the category of HR/p-modules.

Lemma 7.3. *For R, p, and a as above, the map*

$$\varphi_n\colon \mathsf{THH}^{[n],R}(R/a; R/p) \to \mathsf{THH}^{[n],R}(R/p)$$

induced by $R/a \to R/p$ factors through HR/p in $h\mathcal{A}_{HR/p}$.

Proof. The key step is the $n = 0$ case.

From [12, Proposition 2.1] we know that $\mathsf{THH}^{[0],R}(R/a; R/p) \simeq HR/p \vee \Sigma HR/p$ and also $\mathsf{THH}^{[0],R}(R/p) \simeq HR/p \vee \Sigma HR/p$. So we need to understand

$$h\mathcal{A}_{HR/p}(HR/p \vee \Sigma HR/p, HR/p \vee \Sigma HR/p).$$

By [2, Proposition 3.2], we can identify this as

$$h\,\mathrm{Mod}_{HR/p}(\mathbb{L}Q\mathbb{R}I(HR/p \vee \Sigma HR/p), \Sigma HR/p).$$

Given an $A \in \mathcal{A}_{HR/p}$, we have a pullback

$$\begin{array}{ccc} IA & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle \epsilon} \\ * & \longrightarrow & HR/p \end{array}$$

Let B be a nonunital HR/p-algebra. Basterra defines $Q(B)$ to be the pushout

$$\begin{array}{ccc} B \wedge_{HR/p} B & \longrightarrow & B \\ \downarrow & & \downarrow \\ * & \longrightarrow & Q(B) \end{array}$$

We want to take the left- and right-derived versions of these functors for Basterra's result.

Let X be a fibrant replacement for $HR/p \vee \Sigma HR/p$ in $\mathcal{A}_{HR/p}$, so that $X \xrightarrow{\epsilon} HR/p$ is a fibration. Thus the square

$$\begin{array}{ccc} \mathbb{R}I(HR/p \vee \Sigma HR/p) & \longrightarrow & X \\ \downarrow & & \downarrow \epsilon \\ * & \longrightarrow & HR/p \end{array}$$

is a homotopy pullback square (since every spectrum is fibrant [20, Proposition 13.1.2]). For conciseness we write $Y = \mathbb{R}I(HR/p \vee \Sigma HR/p)$. We have a long exact sequence of homotopy groups

$$0 \to \pi_1 Y \to \pi_1 X \to \pi_1 HR/p \to \pi_0 Y \to \pi_0 X \to \pi_0 HR/p \to \pi_{-1} Y \to 0,$$

where we have used that $X \simeq HR/p \vee \Sigma HR/p$ so that its homotopy groups are concentrated in degrees 0 and 1. Note that the map $\pi_0 X \to \pi_0 HR/p$ is the identity. We thus see that $\pi_i Y \cong 0$ for $i \neq 1$ and $\pi_1 Y \cong R/p$.

We need to identify

$$h\,\mathrm{Mod}_{HR/p}(\mathbb{L}Q(\Sigma HR/p), \Sigma HR/p) \cong \pi_0 F_{HR/p}(\mathbb{L}Q(\Sigma HR/p), \Sigma HR/p),$$

where $F_{HR/p}(\cdot,\cdot)$ is the function spectrum. We use the universal coefficient spectral sequence

$$\begin{aligned} E_2^{s,t} &= \mathrm{Ext}^{s,t}_{R/p}(\pi_* \mathbb{L}Q(\Sigma HR/p), \pi_* \Sigma HR/p) \\ &\Longrightarrow \pi_{t-s} F_{HR/p}(\mathbb{L}Q(\Sigma HR/p), \Sigma HR/p). \end{aligned}$$

Note that we're working over a field, so $E_2^{s,t} = 0$ for $s \neq 0$, and $\pi_*(\Sigma HR/p)$ is zero everywhere except at π_1, so in fact the spectral sequence collapses at E_2. Since we are only interested in π_0, the only term relevant to us is

$$E_\infty^{0,0} \cong E_2^{0,0} \cong \mathrm{Hom}_{R/p}(\pi_1 \mathbb{L}Q(\Sigma HR/p), R/p).$$

Note that this group cannot be 0, since our hom-set contains at least two elements: the identity map and the 0 map. Thus it remains to compute $\pi_1 \mathbb{L}Q(\Sigma HR/p)$.

Consider the diagram

$$\begin{array}{ccc} (\Sigma HR/p)^{cof} & \xrightarrow{f} & \mathbb{L}Q(\Sigma HR/p) \\ \sim \downarrow & & \\ \Sigma HR/p & & \end{array}$$

By [3, Proposition 2.1], since $(\Sigma HR/p)^{cof}$ is 0-connected, f is 1-connected; thus $\pi_1 f$ is surjective. Since R/p is a field, we must have

$$\pi_1 \mathbb{L}Q(\Sigma HR/p) \cong 0 \text{ or } R/p.$$

Since it can't be 0, it must be R/p, with the induced map being the identity.

Let $\tau_1^{(a)}$ be the generator of the $\Lambda(\tau_1)$ obtained as $\mathsf{Sh}^{[0],R}(R/a;R/p)$ and let $\tau_1^{(p)}$ be the generator of the $\Lambda(\tau_1)$ obtained as $\mathsf{Sh}^{[0],R}(R/p)$ in the calculation of Lemma 7.1. The above calculation shows that

$$h\,\mathrm{Mod}_{HR/p}(\mathbb{L}Q\mathbb{R}I(HR/p \vee \Sigma HR/p), \Sigma HR/p) \\ \cong \mathrm{Hom}_{R/p}(R/p, R/p)$$

where the first copy of R/p is generated by $\tau_1^{(a)}$ and the second copy is generated by $\tau_1^{(p)}$. But the induced map on $\mathsf{Sh}^{[0],R}$ takes $\tau_1^{(a)}$ to 0. Thus the corresponding map in $h\mathcal{A}_{R/p}$ is also 0. This proves the $n = 0$ case.

We now turn to the induction step. We have the composition

$$\begin{array}{ccc} B(HR/p, \mathsf{THH}^{[n-1],R}(R/a;R/p), HR/p) & & \\ \downarrow{\scriptstyle B(1,\varphi_{n-1},1)} & \searrow^{\text{hyp.}} & \\ & & B(HR/p, HR/p, HR/p) \\ & \swarrow & \\ B(HR/p, \mathsf{THH}^{[n-1],R}(R/p;R/p), HR/p) & & \end{array}$$

of maps of simplicial spectra. Taking the realization gives us the composition

$$\varphi_n : \mathsf{THH}^{[n]}(R/a;R/p) \longrightarrow HR/p \longrightarrow \mathsf{THH}^{[n]}(R/p),$$

as desired. □

By applying π_* to the result of Lemma 7.3 we get the following generalization of Lemma 7.1.

Corollary 7.4. *For R, p, and a as above, for all $n \geqslant 0$ the map*

$$\mathsf{Sh}_*^{[n],R}(R/a;R/p) \longrightarrow \mathsf{Sh}_*^{[n],R}(R/p)$$

induced by $R/a \to R/p$ factors as

$$\mathsf{Sh}_*^{[n],R}(R/a;R/p) \xrightarrow{\epsilon} R/p \xrightarrow{\eta} \mathsf{Sh}_*^{[n],R}(R/p).$$

Here, R/p is considered as a graded ring concentrated in degree 0; the first map in the factorization is the augmentation and the second is the unit map induced by the inclusion of the basepoint.

By Lemma 3.1,

$$\mathsf{THH}^{[n],R/a}(R/p) \simeq HR/p \wedge_{\mathsf{THH}^{[n],R}(R/a;R/p)} \mathsf{THH}^{[n],R}(R/p).$$

However, by Lemma 7.3 the map $\mathsf{THH}^{[n],R}(R/a;R/p) \to \mathsf{THH}^{[n],R}(R/p)$ factors through HR/p. This proves the following result about higher order Shukla homology:

Proposition 7.5. *For R, p, and a as above,*

$$\begin{aligned}\mathsf{THH}^{[n],R/a}(R/p) &\simeq (HR/p \wedge_{\mathsf{THH}^{[n],R}(R/a;R/p)} \wedge HR/p) \wedge_{HR/p} \mathsf{THH}^{[n],R}(R/p)\\ &\simeq \mathsf{THH}^{[n+1],R}(R/a;R/p) \wedge_{R/p} \mathsf{THH}^{[n],R}(R/p).\end{aligned}$$

This recovers the calculation of $\mathsf{Sh}_*^{\mathbb{Z}/p^2}(\mathbb{Z}/p)$ from [4, 5.2]. It also explains why these Shukla calculations are more involved than Shukla homology calculations of the form $\mathsf{Sh}_*^R(R/x)$ where x is a regular element. In the latter case we just obtain a divided power algebra over R/x on a generator of degree two, whereas for all $m \geqslant 2$

$$\begin{aligned}\mathsf{Sh}_*^{\mathbb{Z}/p^m}(\mathbb{Z}/p) &\cong \mathsf{Sh}_*^{[2],\mathbb{Z}}(\mathbb{Z}/p^m;\mathbb{Z}/p) \otimes_{\mathbb{Z}/p} \mathsf{Sh}_*^{\mathbb{Z}}(\mathbb{Z}/p)\\ &\cong \bigotimes_{i\geqslant 0} \Lambda(\varepsilon(\varrho^i(\tau_1^{(p^m)}))) \otimes \Gamma_{\mathbb{Z}/p}(\varphi^0\varrho^i(\tau_1^{(p^m)}))) \otimes_{\mathbb{Z}/p} \Gamma_{\mathbb{Z}/p}(\varrho^0(\tau_1^{(p)})).\end{aligned}$$

We are now ready to prove the main splitting result:

Theorem 7.6. *If R is a commutative ring and if $p, a \in R$ are elements which are not zero divisors for which (p) is a maximal ideal and $a \in (p)^2$, then*

$$\mathsf{THH}^{[n]}(R/a;R/p) \simeq \mathsf{THH}^{[n]}(R;R/p) \wedge^L_{HR/p} \mathsf{THH}^{[n],R}(R/a;R/p).$$

Proof. Recall that in the category of commutative algebras, the smash product is the same as the pushout. Consider the following diagram:

$$\begin{array}{ccccc} HR/p & \longrightarrow & \mathsf{THH}^{[n-1],R}(R/p;R/p) & \longrightarrow & \mathsf{THH}^{[n]}(R;R/p)\\ \downarrow & & \downarrow & & \downarrow\\ \mathsf{THH}^{[n],R}(R/a;R/p) & \longrightarrow & \mathsf{THH}^{[n-1],R/a}(R/p;R/p) & \longrightarrow & \mathsf{THH}^{[n]}(R/a;R/p).\end{array}$$

By Proposition 7.5 the left square is a homotopy pushout square and the right square is a homotopy pushout square by the juggling formula 3.2, with the maps of those formulas. Thus the outside of the diagram also gives a homotopy pushout, producing the formula

$$\mathsf{THH}^{[n]}(R/a;R/p) \simeq \mathsf{THH}^{[n]}(R;R/p) \wedge^L_{HR/p} \mathsf{THH}^{[n],R}(R/a;R/p),$$

as desired. □

8. The Loday construction of products

We establish a splitting formula for Loday constructions of products of ring spectra. This result is probably well-known, but as we will need it later, we provide a proof. In the case of $X = S^1$ such a splitting is proved in [14] in the context of 'ring functors'. See also [11, Proposition 4.2.4.4].

For two commutative ring spectra A and B we consider their product $A \times B$ with the multiplication

$$(A \times B) \wedge (A \times B) \to A \times B$$

that is induced by the maps $(A \times B) \wedge (A \times B) \to A$ and $(A \times B) \wedge (A \times B) \to B$ that are given by the projection maps to A and B and the multiplication on A and B:

$$\begin{array}{ccccc} (A \times B) \wedge (A \times B) & \xrightarrow{\mathrm{pr}_A \wedge \mathrm{pr}_A} & A \wedge A & \xrightarrow{\mu_A} & A \\ \downarrow{\scriptstyle \mathrm{pr}_B \wedge \mathrm{pr}_B} & & & & \\ B \wedge B & & & & \\ \downarrow{\scriptstyle \mu_B} & & & & \\ B & & & & \end{array}$$

For $X = S^0$ we obtain

$$\mathcal{L}_{S^0}(A \times B) = (A \times B) \wedge (A \times B)$$

and this is equivalent to $A \wedge A \vee A \wedge B \vee B \wedge A \vee B \wedge B$ whereas $\mathcal{L}_{S^0}(A) \times \mathcal{L}_{S^0}(B)$ is equivalent to $A \wedge A \vee B \wedge B$ so in this case $\mathcal{L}_{S^0}(A \times B)$ is not equivalent to $\mathcal{L}_{S^0}(A) \vee \mathcal{L}_{S^0}(B)$. In general, if a simplicial set has finitely many connected components, say $X = X_1 \sqcup \ldots \sqcup X_n$, then

$$\mathcal{L}_X(A \times B) \simeq \mathcal{L}_{X_1}(A \times B) \wedge \ldots \wedge \mathcal{L}_{X_n}(A \times B)$$

so it suffices to study $\mathcal{L}_X(A \times B)$ for connected simplicial sets X. We will first consider the case $X = S^1$, where $\mathcal{L}_{S^1}$ with respect to the minimal simplicial model of the circle is $\mathsf{THH} = \mathsf{THH}^{[1]}$, and then use that special case to prove the result for general connected finite simplicial sets X.

By slight abuse of notation we denote by $A \times B$ a cofibrant model of the product of the commutative S-algebras A and B.

Lemma 8.1. *The projection maps $pr_A\colon A \times B$ and $pr_B\colon A \times B \to B$ give rise to a homotopy equivalence*

$$\mathsf{THH}(A \times B) \to \mathsf{THH}(A) \times \mathsf{THH}(B).$$

Proof. The commutative S-algebra $S \times S$ is separable [34, Definition 9.1.1] because in the derived category of $S \times S$-bimodules there is a section, σ, of the multiplication map

$$\mu\colon (S \times S) \wedge_S (S \times S) \to S \times S.$$

If we label the left-hand copy of S by S_L and the right-hand copy by S_R, then the section σ is given by

$$S_L \times S_R \simeq S_L \vee S_R \xrightarrow{((\eta_{S_L},*,*,*),(*,*,*,\eta_{S_R}))} (S_L \wedge_S S_L) \times (S_L \wedge_S S_R) \times (S_R \wedge_S S_L) \times (S_R \wedge_S S_R).$$

Therefore, $\mathsf{THH}^S(S \times S) \simeq S \times S$ (and more generally this holds for any commutative S-algebra C: $\mathsf{THH}^C(C \times C) \simeq C \times C$, compare [34, Lemmas 9.1.2,9.2.6]).

For a cofibrant replacement of $A \times B$ we also obtain

$$\mathsf{THH}^S(S \times S, A \times B) \simeq \mathsf{THH}^S(S \times S) \wedge_{S \times S} (A \times B) \simeq A \times B.$$

As the canonical map

$$(A \times B) \wedge_{S \times S} (A \times B) \to A \wedge_S A \times B \wedge_S B$$

is an equivalence, we get $\mathsf{THH}^{S\times S}(A \times B) \simeq \mathsf{THH}(A) \times \mathsf{THH}(B)$.

Applying Lemma 3.1 to the sequence $S = S \to S \times S \to A \times B = A \times B$ gives

$$\mathsf{THH}^{S\times S}(A \times B) \simeq (A \times B) \wedge_{\mathsf{THH}^S(S\times S, A\times B)} \mathsf{THH}^S(A \times B)$$

and hence in total this yields

$$\mathsf{THH}(A \times B) \simeq \mathsf{THH}(A) \times \mathsf{THH}(B).$$

□

In the following we denote by $A \times B$ a cofibrant model of the product of the commutative S-algebras A and B.

Proposition 8.2. *For any connected finite simplicial set X, the projection maps $pr_A\colon A \times B$ and $pr_B\colon A \times B \to B$ give rise to a homotopy equivalence*

$$\mathcal{L}_X(A \times B) \simeq \mathcal{L}_X(A) \times \mathcal{L}_X(B)$$

and in particular, for all $n \geqslant 1$,

$$\mathsf{THH}^{[n]}(A \times B) \simeq \mathsf{THH}^{[n]}(A) \times \mathsf{THH}^{[n]}(B).$$

Proof. We prove the result for all finite connected simplicial sets X by induction on the dimension n of the top non-degenerate simplex in X; the result for general simplicial sets then follows by taking a colimit. Since the only finite connected simplicial set with its only non-degenerate simplices in dimension zero is a point, the result is obvious for $n = 0$.

For higher n, the crucial observation is that if we have simplicial sets X, Y, and Z so that Z is a non-empty subset of both X and Y, then if the projection maps $A \times B \to A$ and $A \times B \to B$ induce equivalences as given in the statement of this proposition for X, Y and Z, it also gives an equivalence

$$\mathcal{L}_{X\cup_Z Y}(A \times B) \xrightarrow{\simeq} \mathcal{L}_{X\cup_Z Y}(A) \times \mathcal{L}_{X\cup_Z Y}(B). \tag{8.3}$$

This is because then

$$\begin{aligned}
\mathcal{L}_{X\cup_Z Y}(A) \times \mathcal{L}_{X\cup_Z Y}(B) &\simeq (\mathcal{L}_X(A) \wedge_{\mathcal{L}_Z(A)} \mathcal{L}_Y(A)) \times (\mathcal{L}_X(B) \wedge_{\mathcal{L}_Z(B)} \mathcal{L}_Y(B)) \\
&\simeq (\mathcal{L}_X(A) \times \mathcal{L}_X(B)) \wedge_{\mathcal{L}_Z(A)\times\mathcal{L}_Z(B)} (\mathcal{L}_Y(A) \times \mathcal{L}_Y(B)) \\
&\simeq \mathcal{L}_X(A \times B) \wedge_{\mathcal{L}_Z(A\times B)} \mathcal{L}_Y(A \times B) \\
&\simeq \mathcal{L}_{X\cup_Z Y}(A \times B).
\end{aligned}$$

Here we use for the first equivalence that the Loday construction sends pushouts to (homotopy) pushouts and that $\mathcal{L}_Z(A) \to \mathcal{L}_Y(A)$ and $\mathcal{L}_Z(B) \to \mathcal{L}_Y(B)$ are cofibrations. The second equivalence holds because multiplication by $\mathcal{L}_Z(A)$ sends $\mathcal{L}_X(B)$ and $\mathcal{L}_Y(B)$ to the terminal ring spectrum, as does multiplication by $\mathcal{L}_Z(B)$ of $\mathcal{L}_X(A)$ and $\mathcal{L}_Y(A)$. The third equivalence holds because we are assuming that the proposition holds for X, Y and Z.

So for $n = 1$, we use homotopy invariance of the Loday construction and the fact that any finite connected simplicial set with non-degenerate simplices only in dimensions 0 and 1 is homotopy equivalent to $\bigvee_{i=1}^{m} S^1$ for some $m \geqslant 0$. If $m = 0$, we deduce the proposition from the $n = 0$ case above; if $m = 1$, we use Lemma 8.1, and for $m > 1$, we use induction and Equation (8.3).

For the inductive step, assume that $n > 1$ and we know that the proposition holds for any finite connected simplicial set with non-degenerate cells in dimensions $< n$, and in particular for $\partial\Delta^n$, the boundary of the standard n-simplex. Assume that we have a simplicial set X for which the proposition holds. We then prove that the proposition also holds for $X \cup_{\partial\Delta^n} \Delta^n$, that is: X with an additional n-simplex glued to it along the boundary. Without loss of generality, we may assume that the boundary of the new n-simplex is embedded in X: if it is not, apply four-fold edgewise subdivision to everything, and then $X \cup_{\partial\Delta^n} \Delta^n$ will consist of the central small n-simplex inside the original n-simplex that was added that does not touch the boundary of the originally added n-simplex and all the rest of the subdivided complex. But the rest of the subdivided complex is homotopy equivalent to the original X, so the proposition holds for it, and the central small n-simplex does indeed have its boundary embedded in the four-fold edgewise subdivision of $X \cup_{\partial\Delta^n} \Delta^n$.

Then by assumption, the proposition holds for X, by the inductive hypothesis it holds for $\partial\Delta^n$, by homotopy invariance it holds for $\Delta^n \simeq *$, and so by Equation (8.3) it holds for $X \cup_{\partial\Delta^n} \Delta^n$. □

For later use we need a version of Proposition 8.2 with coefficients.

Corollary 8.4. *Let M be an A-module spectrum and let N be a B-module spectrum. Then for all connected pointed finite simplicial sets there is an equivalence*

$$\mathcal{L}_X(A \times B; M \times N) \to \mathcal{L}_X(A; M) \times \mathcal{L}_X(B; N).$$

Proof. The argument in the proof of Proposition 8.2 can be adapted to pointed finite simplicial sets. We know that

$$\mathcal{L}_X(A \times B; M \times N) \simeq \mathcal{L}_X(A \times B) \wedge_{A \times B} (M \times N)$$

and by the result above this in turn is equivalent to

$$(\mathcal{L}_X(A) \times \mathcal{L}_X(B)) \wedge_{A \times B} (M \times N).$$

As the action of A on N and $\mathcal{L}_X(B)$ is trivial and the action of B on M and $\mathcal{L}_X(A)$ is trivial we can identify this coequalizer with

$$(\mathcal{L}_X(A) \wedge_A M) \times (\mathcal{L}_X(B) \wedge_B N).$$

□

9. Applications

9.1. $\mathsf{THH}^{[n]}(\mathbb{Z}/p^m; \mathbb{Z}/p)$**.** This example was our original motivation for obtaining the splitting result of Theorem 7.6. We apply it to the case where $R = \mathbb{Z}$, p is a prime, and $a = p^m$ for $m \geqslant 2$. As a special case of Theorem 7.6 we obtain the following splitting.

Theorem 9.1.

$$\begin{aligned}\mathsf{THH}^{[n]}(\mathbb{Z}/p^m;\mathbb{Z}/p) &\simeq \mathsf{THH}^{[n]}(\mathbb{Z};\mathbb{Z}/p)\wedge^L_{H\mathbb{Z}/p}\mathsf{THH}^{[n],\mathbb{Z}}(\mathbb{Z}/p^m;\mathbb{Z}/p)\\ &\cong \mathsf{THH}^{[n]}(\mathbb{Z};\mathbb{Z}/p)\wedge^L_{H\mathbb{Z}/p}\mathsf{Sh}^{[n],\mathbb{Z}}(\mathbb{Z}/p^m;\mathbb{Z}/p).\end{aligned}$$

This gives a direct calculation of $\mathsf{THH}^{[n]}_*(\mathbb{Z}/p^m;\mathbb{Z}/p)$ for all n because in [12, Theorem 3.1] we determine $\mathsf{THH}^{[n]}_*(\mathbb{Z};\mathbb{Z}/p)$ as an iterated Tor-algebra $B^n_{\mathbb{F}_p}(x)\otimes_{\mathbb{F}_p}B^{n+1}_{\mathbb{F}_p}(y)$ where $|x|=2p$, $|y|=2p-2$, $B^1_{\mathbb{F}_p}(z)=\mathbb{F}_p[z]$ and $B^n_{\mathbb{F}_p}(z)=\mathrm{Tor}_*^{B^{n-1}_{\mathbb{F}_p}(z)}(\mathbb{F}_p,\mathbb{F}_p)$. We determined $\mathsf{Sh}^{[n],\mathbb{Z}}(\mathbb{Z}/p^m;\mathbb{Z}/p)$ in Proposition 5.2.

The case $n=1$ was calculated in [32], [9] and later as well in [1].

Remark 9.2. Note that we cannot use the sequence canonical projection maps

$$\ldots\longrightarrow\mathbb{Z}/p^{m+1}\mathbb{Z}\longrightarrow\mathbb{Z}/p^m\mathbb{Z}\longrightarrow\ldots\longrightarrow\mathbb{Z}/p^2\mathbb{Z}\longrightarrow\mathbb{Z}/p\mathbb{Z}$$

in order to compare the groups $\mathsf{THH}(H\mathbb{Z}/p^m\mathbb{Z};H\mathbb{Z}/p\mathbb{Z})$ for varying m because the tensor factors coming from $\mathsf{THH}_*(H\mathbb{Z};H\mathbb{Z}/p\mathbb{Z})$ are mapped isomorphically from $\mathsf{THH}(H\mathbb{Z}/p^{m+1}\mathbb{Z};H\mathbb{Z}/p\mathbb{Z})$ to $\mathsf{THH}(H\mathbb{Z}/p^m\mathbb{Z};H\mathbb{Z}/p\mathbb{Z})$ whereas the tensor factor $\mathsf{Sh}^{[n],\mathbb{Z}}(\mathbb{Z}/p^{m+1};\mathbb{Z}/p)$ is mapped via the augmentation map to $\mathsf{Sh}^{[n],\mathbb{Z}}(\mathbb{Z}/p^m;\mathbb{Z}/p)$ in each step of the sequence. This is straightforward to see with the help of the explicit resolutions used in the proof of Lemma 7.3.

9.2. **Number rings.** As a warm-up we consider $R=\mathbb{Z}[i]$, $p=1-i$ and $2\in(p)^2$. Then we get that

$$\mathsf{THH}^{[n]}(\mathbb{Z}[i]/2;\mathbb{Z}[i]/(1-i))\simeq\mathsf{THH}^{[n]}(\mathbb{Z}[i];\mathbb{Z}[i]/(1-i))\wedge_{H\mathbb{Z}[i]/(1-i)}\mathsf{Sh}^{[n],\mathbb{Z}[i]}(\mathbb{Z}[i]/2;\mathbb{Z}[i]/(1-i)).$$

Note that $\mathbb{Z}[i]/(1-i)\cong\mathbb{Z}/2$ and $\mathbb{Z}[i]/2\cong\mathbb{F}_2[x]/x^2$. Thus we can calculate

$$\mathsf{THH}^{[n]}(\mathbb{Z}[i]/2;\mathbb{Z}[i](1-i))\cong\mathsf{THH}^{[n]}(\mathbb{F}_2[x]/x^2;\mathbb{F}_2)$$

using the flow chart in [6] and we know from [12, Theorem 4.3] that $\mathsf{THH}^{[n]}_*(\mathbb{Z}[i];\mathbb{Z}[i]/(1-i))$ can also be computed using iterated Tor's. The term $\mathsf{Sh}^{[n],\mathbb{Z}[i]}(\mathbb{Z}[i]/2;\mathbb{Z}[i]/(1-i))$ can be computed as an iterated Tor by Proposition 5.2. Thus all of the components of the above expression are known. What was not known before is that $\mathsf{THH}^{[n]}(\mathbb{Z}[i]/2;\mathbb{Z}[i]/(1-i))$ splits in the above manner.

The general case is as follows: Consider $p\in\mathbb{Z}$ a prime, and let K be a number field such that p is ramified in $\mathcal{O}_K$, with $p=\mathfrak{p}_1^{e_1}\cdots\mathfrak{p}_r^{e_r}$ with one $e_i>1$. The Chinese Remainder Theorem let's us split $\mathcal{O}_K/p$ as a ring as

$$\mathcal{O}_K/p\cong\prod_{j=1}^r\mathcal{O}_K/\mathfrak{p}_j^{e_j}$$

and $\mathcal{O}_K/\mathfrak{p}_i$ as an $\mathcal{O}_K/p$-module is then isomorphic to $0\times\ldots\times\mathcal{O}_k/\mathfrak{p}_i\times\ldots\times 0$ with the non-trivial component sitting in spot number i. With Corollary 8.4 we obtain the following result.

Theorem 9.3.

$$\mathsf{THH}^{[n]}(\mathcal{O}_K/p;\mathcal{O}_K/\mathfrak{p}_i) \simeq \mathsf{THH}^{[n]}(\mathcal{O}_K;\mathcal{O}/\mathfrak{p}_i) \wedge_{H\mathcal{O}_K/\mathfrak{p}_i} \mathsf{THH}^{[n],\mathcal{O}_K}(\mathcal{O}_K/p;\mathcal{O}_K/\mathfrak{p}_i). \tag{9.4}$$

Again, $\mathcal{O}_K/p \cong (\mathcal{O}_K)_{\mathfrak{p}_i}/p$ is isomorphic to $\mathcal{O}_K/\mathfrak{p}_i[\pi]/\pi^{e_i}$ where π is the uniformizer, hence $\mathcal{O}_K/\mathfrak{p}_i[\pi]/\pi^{e_i} \cong \mathcal{O}_K/\mathfrak{p}_i[x]/x^{e_i}$ so we can calculate $\mathsf{THH}^{[n]}(\mathcal{O}_K/p;\mathcal{O}_K/\mathfrak{p}_i)$ using Proposition 2.5. We can determine $\mathsf{THH}^{[n]}(\mathcal{O}_K;\mathcal{O}/\mathfrak{p}_i)$ using [12, Theorem 4.3] and we calculated $\mathsf{THH}^{[n],\mathcal{O}_K}(\mathcal{O}_K/p;\mathcal{O}_K/\mathfrak{p}_i)$ in Proposition 5.2. Using these calculations one can deduce right away that there is a splitting on the level of homotopy groups of $\mathsf{THH}^{[n]}_*(\mathcal{O}_K/p;\mathcal{O}_K/\mathfrak{p}_i)$. But (9.4) yields a splitting of $\mathsf{THH}^{[n]}(\mathcal{O}_K/p;\mathcal{O}_K/\mathfrak{p}_i)$ on the level of augmented commutative $H\mathcal{O}_K/\mathfrak{p}_i$-algebra spectra.

9.3. **Galois descent.** In [34, Definition 9.2.1] John Rognes defines a map of commutative S-algebras $f\colon A \to B$ to be formally THH-étale, if the unit map $B \to \mathsf{THH}^A(B)$ is a weak equivalence. Note that this implies that the augmentation map $\mathsf{THH}^A(B) \to B$ that is induced by multiplying all B-entries in $\mathsf{THH}^A(B)$ together, is also a weak equivalence because the compositve $B \to \mathsf{THH}^A(B) \to B$ is the identity on B.

Therefore, applying the Brun juggling formula 4.1 in this case to $X = S^1$ we obtain

$$\mathsf{THH}^{[2]}(A) \wedge^L_A B \simeq \mathsf{THH}^{[2]}(A;B) \simeq B \wedge_{\mathsf{THH}(B)} \mathsf{THH}^A(B) \simeq B \wedge^L_{\mathsf{THH}(B)} B \simeq \mathsf{THH}^{[2]}(B).$$

We can slightly generalize this:

Definition 9.5. Let X be a pointed simplicial set. A morphism $f\colon A \to B$ is formally X-étale, if the unit map $B \to \mathcal{L}^A_X(B)$ is a weak equivalence.

For formally X-étale morphisms $f\colon A \to B$ the Brun juggling formula 4.1 for X implies

$$\mathcal{L}_{\Sigma X}(A) \wedge^L_A B \simeq \mathcal{L}_{\Sigma X}(A;B) \simeq B \wedge_{\mathcal{L}_X(B)} \mathcal{L}^A_X(B) \simeq B \wedge_{\mathcal{L}_X(B)} B \simeq \mathcal{L}_{\Sigma X}(B).$$

This statement is related to Akhil Mathew's result [29, Proposition 5.2] where he shows that $\mathcal{L}_Y(A;B) \simeq \mathcal{L}_Y(B)$ if $f\colon A \to B$ is a faithful finite G-Galois extension and if Y is a simply-connected pointed simplicial set. Such Galois extensions are formally THH-étale by [34, Lemma 9.2.6].

9.4. **Algebraic function fields over $\mathbb{F}_p$.** In several of our splitting formulas higher THH of the ground field is a tensor factor. So far we have only considered prime fields or rather simple-minded algebraic extensions of those. Topological Hochschild homology groups of algebraic function fields are an important class of examples.

Let L be an algebraic function field over $\mathbb{F}_p$. Then there is a transcendence basis $(x_1,\ldots,x_d)$ such that L is a finite separable extension of $\mathbb{F}_p(x_1,\ldots,x_d)$ [31, Theorem 9.27]. As separable extensions do not contribute anything substantial to topological Hochschild homology we obtain the following result:

Theorem 9.6. *Let L be an algebraic function field over $\mathbb{F}_p$, then*

$$\mathsf{THH}(L)_* \cong L \otimes_{\mathbb{F}_p} \mathsf{THH}_*(\mathbb{F}_p) \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(\varepsilon x_1,\ldots,\varepsilon x_d).$$

Proof. McCarthy and Minasian show in [30, 5.5, 5.6] that THH has étale descent in our case. Therefore

$$\begin{aligned}\mathsf{THH}(L) &\simeq HL \wedge^L_{H\mathbb{F}_p(x_1,\ldots,x_d)} \mathsf{THH}(\mathbb{F}_p(x_1,\ldots,x_d))\\ &\simeq HL \wedge^L_{H\mathbb{F}_p(x_1,\ldots,x_d)} H\mathbb{F}_p(x_1,\ldots,x_d) \wedge^L_{H\mathbb{F}_p[x_1,\ldots,x_d]} \mathsf{THH}(\mathbb{F}_p[x_1,\ldots,x_d])\\ &\simeq HL \wedge^L_{H\mathbb{F}_p[x_1,\ldots,x_d]} \mathsf{THH}(\mathbb{F}_p[x_1,\ldots,x_d]).\end{aligned}$$

But the topological Hochschild homology of monoid rings is known by [19, Theorem 7.1] and hence $\pi_*\mathsf{THH}(\mathbb{F}_p[x_1,\ldots,x_d]) \cong \mathsf{THH}_*(\mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathsf{HH}_*(\mathbb{F}_p[x_1,\ldots,x_d])$. As

$$\mathsf{HH}_*(\mathbb{F}_p[x_1,\ldots,x_d]) \cong \mathbb{F}_p[x_1,\ldots,x_d] \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(\varepsilon x_1,\ldots,\varepsilon x_d)$$

we get the result. □

McCarthy and Minasian actually show more in [30, 5.5,5.6], and we can adapt the above proof to a more general situation.

Theorem 9.7. *Let X be a connected pointed simplicial set X. Then*

$$\mathcal{L}_X(HL) \simeq HL \wedge^L_{H\mathbb{F}_p[x_1,\ldots,x_d]} \mathcal{L}_X(\mathbb{F}_p[x_1,\ldots,x_d]).$$

The Loday construction on pointed monoid algebras satisfies a splitting of the form

$$\mathcal{L}_X(H\mathbb{F}_p[\Pi_+]) \simeq \mathcal{L}_X(H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} \mathcal{L}_X^{H\mathbb{F}_p}(H\mathbb{F}_p[\Pi_+]),$$

see [19, Theorem 7.1]. Therefore $\mathcal{L}_X(\mathbb{F}_p[x_1,\ldots,x_d])$ splits as $\mathcal{L}_X(H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} \mathcal{L}_X^{H\mathbb{F}_p}(H\mathbb{F}_p[x_1,\ldots,x_d])$.

In particular, for $X = S^n$ we get an explicit formula for $\mathsf{THH}^{[n]}(L)$:

Corollary 9.8. *For all $n \geqslant 1$:*

$$\mathsf{THH}^{[n]}(L) \simeq HL \wedge^L_{H\mathbb{F}_p[x_1,\ldots,x_d]} (\mathsf{THH}^{[n]}(\mathbb{F}_p) \wedge_{H\mathbb{F}_p} \mathsf{THH}^{[n],\mathbb{F}_p}(\mathbb{F}_p[x_1,\ldots,x_d])).$$

Recall that we know

$$\begin{aligned}\pi_*\mathsf{THH}^{[n],\mathbb{F}_p}(\mathbb{F}_p[x_1,\ldots,x_d]) &= \mathsf{HH}_*^{[n],\mathbb{F}_p}(\mathbb{F}_p[x_1,\ldots,x_d])\\ &\cong \mathsf{HH}_*^{[n],\mathbb{F}_p}(\mathbb{F}_p[x]^{\otimes_{\mathbb{F}_p} d})\\ &\cong \mathsf{HH}_*^{[n],\mathbb{F}_p}(\mathbb{F}_p[x])^{\otimes_{\mathbb{F}_p} d}\end{aligned}$$

and we determined $\mathsf{HH}_*^{[n],\mathbb{F}_p}(\mathbb{F}_p[x])$ in [6, Theorem 8.6].

Remark 9.9. Topological Hochschild homology of L considers HL as an S-algebra and this allows us to consider L over the prime field. The Hochschild homology of an algebraic function field L over a general field K was for instance determined in [8, Corollary 5.3] and is more complicated.

Remark 9.10. Note that Theorem 9.6 contradicts the statement of [17, Remark 7.2]. In that remark, it is crucial to assume that one works in an *augmented* setting; in our example, we do not.

9.5. $\mathsf{THH}^{[n]}(\mathbb{Z}/p^m)$. We close with the open problem of computing $\mathsf{THH}^{[n]}(\mathbb{Z}/p^m)$ for higher n.

The juggling formula 3.2 applied to the sequence $S \to H\mathbb{Z} \to H\mathbb{Z}/p^m = H\mathbb{Z}/p^m$ for $m \geqslant 2$ yields the equivalence

$$\mathsf{THH}^{[n]}(\mathbb{Z}/p^m) \simeq \mathsf{THH}^{[n]}(\mathbb{Z}; \mathbb{Z}/p^m) \wedge^L_{\mathsf{THH}^{[n-1],\mathbb{Z}}(H\mathbb{Z}/p^m)} H\mathbb{Z}/p^m.$$

Up to $n = 2$ we know $\mathsf{THH}^{[n]}_*(H\mathbb{Z})$: The case $n = 1$ is Bökstedt's calculation [7] and $n = 2$ is [13, Theorem 2.1]. Therefore we can determine $\mathsf{THH}^{[n]}_*(\mathbb{Z}; \mathbb{Z}/p^m)$ up to $n = 2$. As p^m is regular in $\mathbb{Z}$,

$$\mathsf{THH}^{\mathbb{Z}}_*(\mathbb{Z}/p^m) = \mathsf{Sh}^{\mathbb{Z}}_*(\mathbb{Z}/p^m) \cong \Gamma_{\mathbb{Z}/p^m}(x_2)$$

with $|x_2| = 2$. If we could determine the right $\mathsf{Sh}^{\mathbb{Z}}_*(\mathbb{Z}/p^m)$-module structure on $\mathsf{THH}^{[2]}_*(\mathbb{Z}; \mathbb{Z}/p^m)$, then this would allow us to calculate the E^2-term of the Künneth spectral sequence for $\mathsf{THH}^{[2]}_*(\mathbb{Z}/p^m)$,

$$E^2_{p,q} = \mathrm{Tor}^{\Gamma_{\mathbb{Z}/p^m}(x_2)}_{p,q}(\mathsf{THH}^{[2]}_*(\mathbb{Z}; \mathbb{Z}/p^m), \mathbb{Z}/p^m) \Rightarrow \mathsf{THH}^{[2]}_*(\mathbb{Z}/p^m).$$

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