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A short derivation of the structure theorem for graphs with excluded topological minors

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Abstract

As a major step in their proof of Wagner's conjecture, Robertson and Seymour showed that every graph not containing a fixed graph H as a minor has a tree-decomposition in which each torso is almost embeddable in a surface of bounded genus. Recently, Grohe and Marx proved a similar result for graphs not containing H as a topological minor. They showed that every graph which does not contain H as a topological minor has a tree-decomposition in which every torso is either almost embeddable in a surface of bounded genus, or has a bounded number of vertices of high degree. We give a short proof of the theorem of Grohe and Marx, improving their bounds on a number of the parameters involved.

1 Introduction

A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. In a series of 23 papers, published between 1983 and 2012, Robertson and Seymour developed a deep theory of graph minors which culminated in the proof of *Wagner's Conjecture* [18], which asserts that in any infinite set of finite graphs there is one which is a minor of another. One of the landmark results proved along the way, and indeed a fundamental step in resolving Wagner's Conjecture, is a structure theorem for graphs excluding a fixed graph as a minor [17]. It is easy to see that G cannot contain H as a minor if there is a surface into which G can be embedded but H cannot. Loosely speaking, the structure theorem of Robertson and Seymour asserts an approximate converse to this, thereby revealing the deep connection between topological graph theory and the theory of graph minors:

Theorem 1 ([17] (informal))**.** *For any $n \in \mathbb{N}$, every graph excluding the complete graph K_n as a minor has a tree-decomposition in which every torso is almost embeddable into a surface into which K_n is not embeddable.*

A graph H is a *topological minor* of a graph G if G contains a subdivision of H as a subgraph. It is easy to see that G then also contains H as a minor. The converse is not true, as there exist cubic graphs with arbitrarily large complete minors. For topological minors, we thus have an additional degree-based obstruction, which is fundamentally different from the topological obstruction of surface-embeddings for graph minors. Grohe and Marx [10] proved a result in a similar spirit to Theorem 1 for graphs excluding a fixed graph as a topological minor:

Theorem 2 ([10] (informal))**.** *For any $n \in \mathbb{N}$, every graph excluding K_n as a topological minor has a tree-decomposition in which every torso either*

(i) *has a bounded number of vertices of high degree, or*

(ii) *is almost embeddable into a surface of bounded genus.*

More recently, Dvořák [8] refined the embeddability condition of this theorem to reflect more closely the topology of embeddings of an arbitrary graph H which is to be excluded as a topological minor.

The proof given in [10], which uses Theorem 2 as a block-box, is algorithmic and explicitly provides a construction of the desired tree-decomposition, however as a result the proof is quite technical in parts. In this paper, we give a short proof of Theorem 2 which also provides a good heuristic for the structure of graphs without a large complete topological minor, as well as improving the implicit bounds given in [10] on many of the parameters in their theorem. Our proof is non-constructive, but we note that it can easily be adapted to give an algorithm to find either a subdivision of K_r or an appropriate tree-decomposition. However, the run time of this algorithm will be much slower than that of the algorithm given in [10].

One of the fundamental structures we consider are k-blocks. A *k-block* in a graph G is a set B of at least k vertices which is inclusion-maximal with the property that for every separation (U, W) of order $< k$, we either have $B \subseteq U$ or $B \subseteq W$. The notion of a k-block, which was first studied by Mader [14, 13], has previously been considered in the study of graph decompositions [4, 3, 5].

It is clear that a subdivision of a clique on $k+1$ vertices yields a k-block. The converse is not true for any $k \geq 4$, as there exist planar graphs with arbitrarily large blocks. The second author [20] proved a structure theorem for graphs without a k-block:

Theorem 3 ([20])**.** *Let G be a graph and $k \geq 2$. If G has no $(k+1)$-block then G has a tree-decomposition in which every torso has at most k vertices of degree at least $2k(k-1)$.*

Now, since a subdivision of a complete graph gives rise to both a complete minor and a block, there are two obvious obstructions to the existence of a large topological minor, the absence of a large complete minor or the absence of a large block. The upshot of Theorem 2 is that in a local sense these are the only obstructions, any graph without a large topological minor has a tree-decomposition into parts whose torsos either don't contain a large minor, or don't contain a large block. Furthermore, by Theorem 1 and Theorem 3, the converse should also be true: if we can decompose the graph into parts whose torsos either don't contain a large minor or don't contain a large block, then we can refine this tree-decomposition into one satisfying the requirements of Theorem 2.

The idea of our proof is as follows. Both large minors and large blocks point towards a 'big side' of every separation of low order. A subdivision of a clique simultaneously gives rise to both a complete minor and a block and, what's more, the two are hard to separate in that they choose the same 'big side' for every low-order separation. A qualitative converse to this is already implicit in previous work on graph minors and linkage problems: if a graph contains a large complete minor and a large block which cannot be separated from that minor, then the graph contains a subdivision of a complete graph.

Therefore, if we assume our graph does not contain a subdivision of K_r, then we can separate any large minor from every large block. It then follows from the *tangle tree theorem* of Robertson and Seymour [15] – or rather its extension to *profiles* [11, 7, 3] – that there exists a tree-decomposition which separates the blocks from the minors. Hence each part is either free of large minors or of large blocks.

However, in order to apply Theorems 1 and 3, we need to have control over the *torsos*, and not every tree-decomposition will provide that: it might be, for example, that separating some set of blocks created a large minor in one of the torsos. We therefore contract some parts of our tree-decomposition and use the minimality of the remaining separations to prove that this does not happen.

A second nice feature of our proof is that we avoid the difficulty of constructing such a tree-decomposition by choosing initially a tree-decomposition with certain connectivity properties, the proof of whose existence already exists in the literature, and then simply *deducing* that this tree-decomposition has the required properties.

We are going to prove the following:

Theorem 4. *Let r be a positive integer and let G be a graph containing no subdivision of K_r. Then G has a tree-decomposition of adhesion $<r^2$ such that every torso either*

(i) *has fewer than r^2 vertices of degree at least $2r^4$, or*

(ii) *has no K_{2r^2}-minor.*

Combining Theorems 1 and 4 then yields Theorem 2.

Let us briefly compare the bounds we get to the result of Grohe and Marx [10, Theorem 4.1]. It is implicit in their results that if G contains no subdivision of K_r, then G has a tree-decomposition of adhesion $O(r^6)$ such that every torso either has $O(r^6)$ vertices of degree $\Omega(r^7)$, has no $K_{\Omega(r^6)}$ minor or has size at most $O(r^6)$. In this way, Theorem 4 gives an improvement on the bounds for each of the parameters. Recently Liu and Thomas [12] also proved an extension of the work of Dvořák [8], with the aim to more closely control the bound on the degrees of the vertices in (i). Their results, however, only give this structure 'relative' to some tangle.

2 Notation and background material

All graphs considered here are finite and undirected and contain neither loops nor parallel edges. Our notation and terminology mostly follow that of [6].

Given a tree T and $s, t \in V(T)$, we write sTt for the unique s-t-path in T. A *separation* of a graph $G = (V, E)$ is a pair (A, B) with $V = A \cup B$ such that there are no edges between $A \setminus B$ and $B \setminus A$. The *order* of (A, B) is the number of vertices in $A \cap B$. We call the separation (A, B) *tight* if for all $x, y \in A \cap B$, both $G[A]$ and $G[B]$ contain an x-y-path with no internal vertices in $A \cap B$.

The set of all separations of G of order $< k$ will be denoted by $S_k(G)$. An *orientation* of $S_k(G)$ is a subset of $S_k(G)$ containing precisely one element from each pair $\{(A, B), (B, A)\} \subseteq S_k(G)$. The orientation is *consistent* if it does not contain two separations $(A,B),(C,D)$ with $B \subseteq C$ and $D \subseteq A$. A separation *distinguishes* two orientations O_1, O_2 of $S_k(G)$ if precisely one of O_1, O_2 contains it. It does so *efficiently* if it has minimum order among all separations distinguishing them.

Recall that, given an integer k, a set B of at least k vertices of G is a *k-block* if it is inclusion-maximal with the property that for every separation (U,W) of order $<k$, either $B \subseteq U$ or $B \subseteq W$. Observe that B induces a consistent orientation $O_B := \{(U,W) \colon B \subseteq W\}$ of $S_k(G)$.

Given an integer m, a *model of K_m* is a family $\mathcal{X}$ of m pairwise disjoint sets of vertices of G such that $G[X]$ is connected for every $X \in \mathcal{X}$ and G has an edge between X and Y for any two $X, Y \in \mathcal{X}$. The elements of $\mathcal{X}$ are called *branch sets*. Note that, if (U,W) is a separation of order $< m$, then exactly one of $U \setminus W$ and $W \setminus U$ contains some branch set. In this way, $\mathcal{X}$ induces a consistent orientation $O_{\mathcal{X}}$ of $S_k(G)$, where $(U,W) \in O_{\mathcal{X}}$ if and only if some branch set of $\mathcal{X}$ is contained in W.

A *tree-decomposition of* G is a pair $(T, \mathcal{V})$, where T is a tree and $\mathcal{V} = (V_t)_{t \in T}$ is a family of sets of vertices of G such that:

- for every $v \in V(G)$, the set of $t \in V(T)$ with $v \in V_t$ induces a non-empty subtree of T;

- for every edge $vw \in E(G)$ there is a $t \in V(T)$ with $v, w \in V_t$.

If $(T, \mathcal{V})$ is a tree-decomposition of G, then every $st \in E(T)$ induces a separation

$$(U_s, W_t) := \Big(\bigcup_{t \notin uTs} V_u, \bigcup_{s \notin vTt} V_v \Big).$$

Note that $U_s \cap W_t = V_s \cap V_t$. In this way, every edge $e \in E(T)$ has an *order* given by the order of the separation it induces, which we will write as $|e|$. Similarly, an edge of T *(efficiently) distinguishes* two orientations if the separation it induces does. We say that $(T, \mathcal{V})$ *(efficiently) distinguishes* two orientations O and P if some edge of T does. We call $(T, \mathcal{V})$ *tight* if every separation induced by an edge of T is tight.

The *adhesion* of $(T, \mathcal{V})$ is the maximum order of an edge. If the adhesion of $(T, \mathcal{V})$ is less than k and O is an orientation of $S_k(G)$, then O induces an orientation of the edges of T by orienting an edge st towards t if $(U_s, W_t) \in O$. If O is consistent, then all edges will be directed towards some node $t \in V(T)$, which we denote by t_O and call the *home node* of O. When O is induced by a block B or model $\mathcal{X}$, we abbreviate $t_B := t_{O_B}$ and $t_{\mathcal{X}} := t_{O_{\mathcal{X}}}$, respectively. Observe that and edge $e \in E(T)$ distinguishes two orientations O and P if and only if $e \in E(t_O T t_P)$.

Given $t \in V(T)$, the *torso at t* is the graph obtained from $G[V_t]$ by adding, for every neighbor s of t, an edge between any two non-adjacent vertices in $V_s \cap V_t$. More generally, given a subtree $S \subseteq T$, the *torso at S* is the graph obtained from $G\left[\bigcup_{s \in S} V_s\right]$ by adding, for every edge $st \in E(T)$ with $S \cap \{s,t\} = \{s\}$, an edge between any two non-adjacent vertices in $V_s \cap V_t$.

We also define contractions on tree-decompositions: Given $(T, \mathcal{V})$ and an edge $st \in E(T)$, to contract the edge st we form a tree-decomposition $(T', \mathcal{V}')$ where

- T' is obtained by contracting st in T to a new vertex x;

- Let $V'_x := V_s \cup V_t$ and $V'_u := V_u$ for all $u \in V(T) \setminus \{s,t\}$.

It is simple to check that $(T', \mathcal{V}')$ is a tree-decomposition. We note that the separations induced by an edge in $E(T) \setminus \{st\}$ remain the same, as do the torsos of parts V_u for $u \neq s, t$.

We say a tree-decomposition $(T, \mathcal{V})$ is *k-lean* if it has adhesion $<k$ and the following holds for all $p \in [k]$ and $s, t \in T$: If sTt contains no edge of order $<p$, then every separation (A, B) with $|A \cap V_s| \geq p$ and $|B \cap V_t| \geq p$ has order at least p.

Let $n := |G|$. The *fatness* of $(T, \mathcal{V})$ is the sequence $(a_0, \ldots, a_n)$, where a_i denotes the number of parts of order $n - i$. A tree-decomposition of lexicographically minimum fatness among all tree-decompositions of adhesion smaller than k is called *k-atomic*. These tree-decompositions play a pivotal role in our proof, but we actually only require two properties that follow from this definition. It was observed by Carmesin, Diestel, Hamann and Hundertmark [2] that the short proof of Thomas' Theorem [19] given by Bellenbaum and Diestel in [1] also shows that k-atomic tree-decompositions are k-lean (see also [9]).

Lemma 5 ([1])**.** *Every k-atomic tree-decomposition is k-lean.*

It is also not hard to see that k-atomic tree-decompositions are tight. In [20], the second author used k-atomic tree-decompositions to prove a structure theorem for graphs without a k-block. In fact, the proof given there yields the following:

Lemma 6 ([20])**.** *Let G be a graph and k a positive integer. Let $(T, \mathcal{V})$ be a k-atomic tree-decomposition of G and $t \in V(T)$ such that V_t contains no k-block of G. Then the torso at t contains fewer than k vertices of degree at least $2k^2$.*

Let G be a graph and $Z \subseteq V(G)$. We denote by G^Z the graph obtained from G by making the vertices of Z pairwise adjacent. A *Z-based model* is a model $\mathcal{X}$ of $K_{|Z|}$ such that $X \cap Z$ consists of a single vertex for every $X \in \mathcal{X}$.

The following lemma of Robertson and Seymour [16] is crucial to our proof.

Lemma 7 ([16])**.** *Let G be a graph, $Z \subseteq V(G)$ and $p := |Z|$. Let $q \geq 2p - 1$ and let $\mathcal{X}$ be a model of K_q in G^Z. If $\mathcal{X}$ and Z induce the same orientation of $S_p(G^Z)$, then G has a Z-based model.*

3 The proof

Let us fix throughout this section a graph G with no subdivision of K_r, let $k := r(r-1)$, $m := 2k$, and let $(T, \mathcal{V})$ be a k-atomic tree-decomposition of G.

First, we will show that $(T, \mathcal{V})$ efficiently distinguishes every k-block from every model of K_m in G. This allows us to split T into two types of sub-trees, those containing a k-block and those containing a model of K_m. Lemma 6 allows us to bound the number of high degree degree vertices in the torsos in the latter components. We will then show that if we choose these sub-trees in a sensible way then we can also bound the order of a complete minor contained in the torsos of the former. Hence, by contracting each of these sub-trees in $(T, \mathcal{V})$ we will have our desired tree-decomposition.

To show that $(T, \mathcal{V})$ distinguishes every k-block from every model of K_m in G, we must first show that they are distinguishable, that is, no k-block and K_m

induce the same orientation. The following lemma, as well as its proof, is similar to Lemma 6.11 in [10].

Lemma 8. *Let B be a k-block and $\mathcal{X}$ a model of K_m in G. If B and $\mathcal{X}$ induce the same orientation of S_k, then G contains a subdivision of K_r with arbitrarily prescribed branch vertices in B.*

Proof. Suppose B and $\mathcal{X}$ induce the same orientation and let B_0 be an arbitrary subset of B of size r. Let H be the graph obtained from G by replacing every $b \in B_0$ by an independent set J_b of order $(r-1)$, where every vertex of J_b is adjacent to every neighbor of b in G and to every vertex of J_c if b, c are adjacent. Let $J := \bigcup_b J_b$ and note that $|J| = k$. We regard G as a subgraph of H by identifying each $b \in B$ with one arbitrary vertex in J_b. In this way we can regard $\mathcal{X}$ as a model of K_m in H.

Assume for a contradiction that there was a separation (U, W) of H such that $|U \cap W| < |J|$, $J \subseteq U$ and $X \subseteq W \setminus U$ for some $X \in \mathcal{X}$. We may assume without loss of generality that for every $b \in B_0$, either $J_b \subseteq U \cap W$ or $J_b \cap (U \cap W) = \emptyset$. Indeed, if there is a $z \in J_b \setminus (U \cap W)$, then $z \in U \setminus W$, and we can delete any $z' \in J_b \cap W$ from W and maintain a separation (because $N(z) = N(z'))$ with the desired properties. In particular, for every $b \in B_0$ we find $b \in W$ if and only if $J_b \subseteq W$. Since $|U \cap W| < |J|$, it follows that there is at least one $b_0 \in B_0$ with $J_{b_0} \subseteq (U \setminus W)$. Let $(U', W') := (U \cap V(G), W \cap V(G))$ be the induced separation of G. Then $X \subseteq W' \setminus U'$ and $b_0 \in U' \setminus W'$. Since $|U' \cap W'| \leq |U \cap W| < k$ and B is a k-block, we have $B \subseteq U'$. But then (U', W') distinguishes B and $\mathcal{X}$, which is a contradiction to our initial assumption.

We can now apply Lemma 7 to H and find a J-based model $\mathcal{Y} = (Y_j)_{j \in J}$ in H. For each $b \in B_0$, label the vertices of J_b as $(v_c^b)_{c \in B_0 \setminus \{b\}}$. For $b \neq c$, H has a path $P'_{b,c} \subseteq Y_{v_c^b} \cup Y_{v_b^c}$ and the paths obtained like this are pairwise disjoint, because the Y_j are, and $P'_{b,c} \cap J = \{v_c^b, v_b^c\}$. For each such path $P'_{b,c}$, obtain $P_{b,c} \subseteq G$ by replacing v_c^b by b and v_b^c by c. The collection of these paths $(P_{b,c})_{b,c \in B_0}$ gives a subdivision of K_r with branch vertices in B_0. □

Now we can show that $(T, \mathcal{V})$ efficiently distinguishes every k-block from every model of K_m in G.

Lemma 9. *$(T, \mathcal{V})$ efficiently distinguishes all orientations of $S_k(G)$ induced by k-blocks or models of K_m.*

Proof. Let us call a consistent orientation O of $S_k(G)$ *anchored* if for every $(U, W) \in O$, there are at least k vertices in $W \cap V_{t_O}$.

Note that every orientation $O = O_B$ induced by a k-block B is trivially anchored, since $B \subseteq V_{t_B}$. But the same is true for the orientation $O = O_{\mathcal{X}}$ induced by a model $\mathcal{X}$ of K_m. Indeed, let $(U, W) \in O_{\mathcal{X}}$. Then every set in $\mathcal{X}$ meets $V_{t_{\mathcal{X}}}$. At least k branch sets of $\mathcal{X}$ are disjoint from $U \cap W$, say $X_1, \ldots, X_k$, and they all lie in $W \setminus U$. For $1 \leq i \leq k$, let $x_i \in X_i \cap V_{t_{\mathcal{X}}}$ and note that $R := \{x_1, \ldots, x_k\} \subseteq W \cap V_{t_{\mathcal{X}}}$.

We now show that $(T, \mathcal{V})$ efficiently distinguishes all anchored orientations of $S_k(G)$. Let O_1, O_2 be anchored orientations of $S_k(G)$ and let their home nodes be t_1 and t_2 respectively. If $t_1 \neq t_2$, let p be the minimum order of an edge along $t_1 T t_2$, and put $p := k$ otherwise. Choose some $(U, W) \in O_2 \setminus O_1$ of minimum order. Since O_1 and O_2 are anchored, we have $|U \cap V_{t_1}| \geq k$ and

$|W \cap V_{t_2}| \geq k$. As $(T, \mathcal{V})$ is k-lean, it follows that $|U \cap W| \geq p$. Hence $t_1 \neq t_2$ and $(T, \mathcal{V})$ efficiently distinguishes O_1 and O_2. □

Let us call a node $t \in V(T)$ a *block-node* if it is the home node of some k-block and *model-node* if it is the home node of a model of K_m.

Let $F \subseteq E(T)$ be inclusion-minimal such that every k-block is efficiently distinguished from every model of K_m by some separation induced by an edge in F. We now define a red/blue colouring $c : V(T) \to \{r, b\}$ by letting $c(t) = b$ if the component of $T - F$ containing t contains a block-node and letting $c(t) = r$ if it contains a model-node. Let us first show that this is in fact a colouring of $V(T)$.

Lemma 10. *Every node receives exactly one colour.*

Proof. Suppose first that $t \in V(T)$ is such that the component of $T - F$ containing t contains both a block node and a model node. Then there is a k-block B and a K_m-minor $\mathcal{X}$ such that $t_B T t$ and $t_{\mathcal{X}} T t$ both contain no edges of F. But then B and $\mathcal{X}$ are not separated by the separations induced by F, a contradiction.

Suppose now that $t \in V(T)$ is such that the component S of $T - F$ containing t contains neither a block nor a minor. Let $f_1, \ldots, f_n$ be the edges of T between S and $T \setminus S$, ordered such that $|f_1| \geq |f_i|$ for all $i \leq n$. By minimality of F, there is a block-node t_B and a model-node $t_{\mathcal{X}}$ such that f_1 is the only edge of F that efficiently distinguishes B and $\mathcal{X}$. Since $t_B, t_{\mathcal{X}} \notin S$, there is a $j \geq 2$ such that $f_j \in E(t_B T t_{\mathcal{X}})$, and so f_j distinguishes B and $\mathcal{X}$ as well, and since $|f_1| \geq |f_j|$, it does so efficiently, contradicting our choice of B and $\mathcal{X}$ □

Lemma 11. *Let $st \in E(T)$ and suppose s is blue and t is red. Then $G[W_t]$ has a $(V_s \cap V_t)$-based model.*

Proof. Let $Q := V_s \cap V_t$. Let t_B be a block-node in the same component of $T - F$ as s and let $t_{\mathcal{X}}$ be a model-node in the same component as t. Since the separations induced by F efficiently distinguish B and $\mathcal{X}$, it must be that $st \in F$ and (U_s, W_t) efficiently distinguishes B and $\mathcal{X}$.

Let $\mathcal{Y} := (X \cap W_t)_{X \in \mathcal{X}}$. Since $(U_s, W_t) \in O_{\mathcal{X}}$, $\mathcal{Y}$ is a model of K_m in $G[W_t]^Q$. We wish to apply Lemma 7 to Q and $\mathcal{Y}$ in the graph $G[W_t]$. Suppose Q and $\mathcal{Y}$ do not induce the same orientation of $S_{|Q|}(G[W_t]^Q)$. That is, there is a separation (U, W) of $G[W_t]^Q$ with $|U \cap W| < |Q|$ and $Q \subseteq U$ such that $Y \cap U = \emptyset$ for some $Y \in \mathcal{Y}$. There is an $X \in \mathcal{X}$ so that $Y = X \cap G[W_t]$. Note that $X \cap U$ is empty as well. Now $(U', W') := (U \cup U_s, W)$ is a separation of G. Note that

$$X \cap U' = X \cap U_s = \emptyset,$$

because X is connected, meets W_t and does not meet Q. Therefore $X \subseteq W' \setminus U'$ and $B \subseteq U_s \subseteq U'$. But $|U' \cap W'| = |U \cap W| < |Q|$, which contradicts the fact that (U_s, W_t) efficiently distinguishes B and $\mathcal{X}$. Therefore, by Lemma 7, $G[W_t]$ has a Q-based model. □

Using the above we can bound the size of a complete minor in the torso of a blue component. The next lemma plays a similar role to Lemma 6.9 in [10].

Lemma 12. *Let $S \subseteq T$ be a maximal subtree consisting of blue nodes. Then the torso of S has no K_m-minor.*

Proof. Let $F_S := \{(s,t)\colon st \in E(T), s \in S, t \notin S\}$. For every $(s,t) \in F_S$, the node s is blue and t is red. By Lemma 11, G_t has a $(V_s \cap V_t)$-based complete minor $\mathcal{Y}^{s,t}$. Contract each of its branch sets onto the single vertex of $V_s \cap V_t$ that it contains. Do this for every $(s,t) \in F_S$. After deleting any vertices outside of $V_S := \bigcup_{s \in S} V_s$, we obtain the torso of S as a minor of the graph G.

Suppose the torso of S contained a K_m-minor. Then G has a K_m-minor $\mathcal{X}$ such that every $X \in \mathcal{X}$ meets V_S. Therefore $\mathcal{X}$ orients every edge $st \in E(T)$ with $(s,t) \in F_S$ towards s. But then $t_{\mathcal{X}} \in S$, contradicting the assumption that S contains no red nodes. □

We can now finish the proof. Let $(T', \mathcal{V}')$ be obtained from $(T, \mathcal{V})$ by contracting every maximal subtree consisting of blue nodes and let the vertices of T' inherit the colouring from $V(T)$. We claim that $(T', \mathcal{V}')$ satisfies the conditions of Theorem 4.

Indeed, firstly, the adhesion of $(T', \mathcal{V}')$ is at most that of $(T, \mathcal{V})$, and hence is at most k. Secondly, the torso of every red node in $(T', \mathcal{V}')$ is the torso of some red node in $(T, \mathcal{V})$, which by Lemma 6 has fewer than k vertices of degree at least $2k^2$. Finally, by Lemma 12 the torso of every blue node in $(T', \mathcal{V}')$ has no K_m minor. Since $k = r(r-1)$ and $m = 2k$, the theorem follows.

As claimed in the introduction, it is not hard to turn this proof into an algorithm to find either a subdivision of K_r or an appropriate tree-decomposition. Indeed, the proof of Lemma 5 can easily be adapted to give an algorithm to find a tight k-lean tree-decomposition. Similarly, in order to colour the vertices of the tree red or blue we must check for the existence of a K_m minor or a k-block having this vertex as a home node, both of which can be done algorithmically (see [16] and [2]). However, we note that the running time of such an algorithm, or at least a naive implementation of one, would have run time $\sim |V(G)|^{f(r)}$ for some function of the size of the topological minor K_r we are excluding, whereas the algorithm of Grohe and Marx has run time $g(r)|V(G)|^{O(1)}$, which should be much better for large values of r.

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