TANGLES AND THE STONE-ČECH COMPACTIFICATION OF INFINITE GRAPHS

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ABSTRACT. We show that the space of $\aleph_0$-tangles of an arbitrary infinite connected graph G is homeomorphic to the quotient of the Stone-Čech remainder $G^* = \beta G \setminus G$ of G where each component is collapsed to a single point. Answering a question of Diestel, we further show that his tangle compactification of G can be obtained from βG by first declaring G to be open in βG, and then collapsing in the resulting space each component of G^* to a single point.

Our main technical result is a Šura-Bura Lemma for G^*, showing that components of the remainder of the 1-complex of a graph G can be separated pairwise by finite order separations of G.

1. Introduction

Throughout this paper, the term *graph* refers to a connected simple graph of arbitrary cardinality. When viewing a graph as a topological space, it is always endowed with the 1-complex topology. Now when G is locally finite, i.e. when every vertex is incident with only finitely many edges, then the 1-complex G will be locally compact, and possible compactifications[1] (in particular the Freudenthal compactification of G) are well-understood, both topologically and combinatorially.

Topologically, the Freudenthal compactification of a locally compact Tychonoff space is the largest Hausdorff compactification with totally disconnected remainder, i.e. without non-trivial connected components in the remainder. One particular consequence is that the end space of the Freudenthal compactification can be obtained from the Stone-Čech compactification by collapsing all connected components of the Stone-Čech remainder.

Combinatorially, the ends of a graph G are defined as equivalence classes of rays (i.e. one-way infinite paths) in G, where two rays R, R' are considered *equivalent* if for every finite vertex set $X \subseteq V(G)$, there is a path from R to R' avoiding X. Note that this definition of an end, due to Halin from 1964 [8], makes sense for all graphs, including the non-locally finite ones.

Now while ends of graphs have been known for a long time, it is only over the past two decades that it has been employed by Diestel et al. to elegantly generalise many theorems about finite graphs involving paths, cycles or spanning trees—theorems that do not generalise verbatim—to locally finite infinite graphs by replacing, in the wording of the theorems, paths and cycles and spanning trees with arcs and homeomorphic images of the unit circle and uniquely arc-connected spanning subspaces (i.e. *dendrites* containing all vertices) of the Freudenthal compactification respectively. Among these theorems are Nash-William's well-known Tree-Packing Theorem [1, Theorem 8.6.9] and the complete cycle space theory; see the survey [5] or Chapters 8.6 and 8.7 of [1] for the complete picture.

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[1]Our use of the term *compactification* does not preclude any separation properties; otherwise we say *Hausdorff compactification.*

With these recent combinatorial results about the Freudenthal compactification of locally finite graphs in mind, it is natural to wonder whether Diestel's programme can further be extended to non-locally finite graphs. While the Freudenthal compactification can be generalised topologically from locally compact to rim-compact spaces [11], this is of little help for non-locally finite graphs G (as G is rim-compact if and only if G is locally finite if and only if G is locally compact).

However, Diestel [2] recently proposed a new viewpoint of combinatorial ends of an infinite graph that might lead to a compactification allowing again to lift theorems about finite graphs to the infinite case by using suitable topological generalisations of paths and cycles. If ω is an end of G, then every finite order separation[2] $\{A, B\}$ of G gets oriented by that end towards the side $K \in \{A, B\}$ for which every ray in ω has a subray in $G[K]$. In this way, every end gives rise to its own orientation of all the finite order separations, and these orientations are *consistent* in various ways; e.g., if $\{A, B\}$ and $\{C, D\}$ are finite order separations with $A \subseteq C$ and $B \supseteq D$ and if ω orients $\{C, D\}$ towards D, then ω must also orient $\{A, B\}$ towards B. Thus, ends are a special instance of the consistent orientations of all the finite order separations: the $\aleph_0$-*tangles* of G, whose finite version was originally introduced in 1991 by Robertson and Seymour in their ground breaking work on the Graph-Minor Theorem, see [12]. Motivated by this observation, Diestel then proceeded to show that adding all $\aleph_0$-tangles to an arbitrary infinite graph (possibly disconnected and not locally finite) yields again a compactification, the so-called *tangle compactification* $|G|_\Theta$ of G.

Like the Freudenthal compactification, $|G|_\Theta$ has a totally disconnected remainder, i.e. the boundary at infinity contains no non-trivial connected components. Moreover, if G is locally finite and connected, then its $\aleph_0$-tangles turn out to be precisely its ends—and the tangle compactification coincides with the Freudenthal compactification. However, the tangle compactification in general fails to be Hausdorff, as no infinite degree vertex can be separated from tangles in the remainder.

The main objective of this paper is to investigate the question from Diestel's paper [2] about the relationship between the Stone-Čech compactification βG of a 1-complex G, and its tangle compactification $|G|_\Theta$. Let us write $G^* = \beta G \setminus G$ for the Stone-Čech remainder of the graph G, and $\Theta = \Theta(G)$ for the space of $\aleph_0$-tangles of G, equipped with the subspace topology of $|G|_\Theta$. Our main result is that by collapsing the connected components of the Stone-Čech remainder, one re-obtains precisely the $\aleph_0$-tangles of the graph, and so Diestel's tangle compactification generalises the Freudenthal compactification also in this regard. More precisely, our two main theorems are:

Theorem 1. *The tangle space Θ of any graph G is homeomorphic to the quotient $G^*/{\approx_*}$ of the Stone-Čech remainder G^* of G, where each connected component of G^* is collapsed to a single point.*

Theorem 2. *The tangle compactification $|G|_\Theta$ of any graph G is homeomorphic to the quotient $(\beta G, \tau')/{\approx_*}$ where τ' is the finer topology on βG obtained from βG by declaring G to be open in βG and then collapsing each connected component of G^* to a single point.*

[2] A *finite order separation* of a graph G is a set $\{A, B\}$ with $A \cap B$ finite and $A \cup B = V(G)$ such that G has no edge between $A \setminus B$ and $B \setminus A$.

In particular, it follows from the first theorem that $G^*/{\approx_*}$ is a compact Hausdorff space, which might be surprising, considering that G^* is generally non-compact.

The change of topology in the second theorem is necessary, since the graph G is open in its tangle compactification $|G|_\Theta$, whereas in the unmodified Stone-Čech quotient, vertices of infinite degree cannot be separated from the remainder.

To see why such a correspondence between tangles and connected components is plausible, consider a finite order separation $\{A, B\}$ of a graph G. Since topologically, $G[A]$ and $G[B]$ are two closed sets covering the 1-complex G such that $G[A] \cap G[B] = G[A \cap B]$ is a finite graph (and hence a compact subspace), it follows from standard arguments that $(\overline{G[A]} \cap G^*) \oplus (\overline{G[B]} \cap G^*)$ is a clopen bipartition of the Stone-Čech remainder G^*, see Lemma 5.8. Therefore, every connected (quasi-)component of G^* orients all the finite order separations $\{A, B\}$ to the side $K \in \{A, B\}$ with $\overline{G[K]} \cap G^*$ containing it, and these orientations are consistent—i.e., they are $\aleph_0$-tangles of G. Thus, every connected (quasi-)component of the Stone-Čech remainder induces an $\aleph_0$-tangle of the graph.

Now if G was locally finite, the converse to this observation would almost be a triviality: In this case, G^* would be compact, so components and quasi-components of G^* coincide, and to show that $\aleph_0$-tangles of G and connected components of G^* are in natural 1-1-correspondence, one would be left with the rather easy task to separate a clopen bipartition of G^* by a finite order separation of the graph (which follows readily from compactness of βG). However, since the Stone-Čech remainder X^* of a space X is compact if and only if X is locally compact, this easy route is barred for the graphs we are really interested in. Nevertheless, our crucial technical result, which we call our *Separating Lemma* (Lemma 5.12), implies that in the case where $X = G$ is a graph, any two connected components of G^* are in fact separated by some finite order separation $\{A, B\}$ of G. We remark that for general spaces X other than graphs, this result is generally false (R. Suabedissen, personal communication).

This paper is organised as follows: First, in Section 2 we provide a brief summary of Diestel's *tangle compactification* of an infinite graph. Next, in Section 3, we describe the remainder of the tangle compactification as an inverse limit of finite discrete spaces. In Section 4, we provide the necessary background on the Stone-Čech compactification, and explain how the quotient relation defining the 1-complex G can be used to describe the Stone-Čech compactification of an infinite graph as a 'fake 1-complex' on standard intervals and *non-standard* intervals (where the non-standard intervals are the standard subcontinua of the remainder of the positive half-line). In Section 5 and 6, we prove our two main theorems respectively. The *Separating Lemma*, our structural result saying that distinct components of the remainder G^* can be separated by finite graph-theoretical separations of the underlying graph G is also proved in Section 5.

We conclude this paper in Section 7 with three additional observations about the tangle compactification $|G|_\Theta$ that might be of independent interest. In particular, we show that no compactification of a non-locally finite graph can both be Hausdorff and have a totally disconnected remainder.

2. Reviewing Diestel's tangle compactification

From now on, we fix an arbitrary connected simple infinite graph $G = (V, E)$.

2.1. **The 1-complex of a graph.** In the *1-complex* of G which we denote also by G, every edge $e = xy$ is a homeomorphic copy $[x, y] := \{x\} \sqcup \mathring{e} \sqcup \{y\}$ of $\mathbb{I} = [0, 1]$ with $\mathring{e}$ corresponding to $(0, 1)$ and points in $\mathring{e}$ being called *inner edge points*. The space $[x, y]$ is called a *topological edge*, but we refer to it simply as *edge* and denote it by e as well. For each subcollection $F \subseteq E$ we write $\mathring{F}$ for the set $\bigsqcup_{e\in F} \mathring{e}$ of inner edge points of edges in F. By $E(v)$ we denote the set of edges incident with a vertex v. The point set of G is $V \sqcup \mathring{E}$, and an open neighbourhood basis of a vertex v of G is given by the unions $\bigcup_{e\in E(v)}[v, i_e)$ of half open intervals with each i_e some inner edge point of e. Note that the 1-complex of G is (locally) compact if and only if the graph G is (locally) finite, and also that the 1-complex fails to be first-countable at vertices of infinite degree. Note that if the graph G has no isolated vertices, then its 1-complex can be obtained from the disjoint sum $\bigoplus_{e\in E} \mathbb{I}_e$ of copies $\mathbb{I}_e$ of the unit interval by taking the quotient with respect to a suitable equivalence relation on $\bigoplus_{e\in E}\{0, 1\}$.

2.2. **Combinatorial ends of graphs.** Given a graph $G = (V, E)$ we write $\mathcal{X}$ for the collection of all finite subsets of its vertex set V, partially ordered and directed by inclusion. A (combinatorial) *end* of a graph is an equivalence class of rays, where a *ray* is a 1-way infinite path. Two rays are *equivalent* if for every $X \in \mathcal{X}$ both have a subray (also called *tail*) in the same component of $G - X$. In particular, for every end ω of G there is a unique component of $G - X$ in which every ray of ω has a tail, and we denote this component by $C(X, \omega)$. Whenever we say end, we mean a combinatorial one. The set of ends of a graph G is denoted by $\Omega = \Omega(G)$. Further details on ends as well as any graph-theoretic notation not explained here can be found in Diestel's book [1], especially in Chapter 8.

If ω is an end of G, then the components $C(X, \omega)$ are compatible in that they form elements of the inverse limit of the system $\{\mathscr{C}_X, \mathfrak{c}_{X',X}, \mathcal{X}\}$ where $\mathscr{C}_X$ is the set of components of $G - X$ and for $X' \supseteq X$, the bonding map $\mathfrak{c}_{X',X} : \mathscr{C}_{X'} \to \mathscr{C}_X$ sends each component of $G - X'$ to the unique component of $G - X$ including it. Clearly, the inverse limit consists precisely of the directions of the graph: choice maps f assigning to every $X \in \mathcal{X}$ a component of $G - X$ such that $f(X') \subseteq f(X)$ whenever $X' \supseteq X$. In 2010, Diestel and Kühn [4] showed that

Theorem 2.1 ([4, Theorem 2.2]). *Let G be any graph. Then there is a canonical bijection between the (combinatorial) ends of G and its directions, i.e.* $\Omega = \varprojlim \mathscr{C}_X$.

2.3. **Tangles.** Next, we formally introduce tangles for a particular type of 'separation system', referring the reader to [3] for an overview of the full theory and its applications. More precisely, we introduce a definition of $\aleph_0$-tangles provided by Diestel [2] which, as he proved, is equivalent to the original one due to Robertson and Seymour [12]. In the next subsection, however, we explain a third, equivalent viewpoint for tangles (due to Diestel), which describes $\aleph_0$-tangles as the elements of the compact Hausdorff inverse limit $\varprojlim \beta(\mathscr{C}_X)$ and which we take as our point of reference for the remainder of this paper.

A (*finite order*) *separation* of a graph G is a set $\{A, B\}$ with $A \cap B$ finite and $A \cup B = V$ such that G has no edge between $A \setminus B$ and $B \setminus A$. The collection of all finite order separations is denoted by S. The ordered pairs (A, B) and (B, A) are then called the *orientations* of the separation $\{A, B\}$, or (*oriented*) *separations*. Informally we think of A and B as the *small side* and the *big side* of (A, B), respectively. Furthermore, we think of the separation (A, B) as *pointing towards* its big side B and *away from* its small side A. We write $\vec{S}$ for the collection of all oriented separations. A subset O of $\vec{S}$ is an *orientation* if it contains precisely one of (A,B) and (B,A) for each separation $\{A,B\} \in S$.

We define a partial ordering $\leq$ on $\vec{S}$ by letting

$$(A,B) \leq (C,D) :\Leftrightarrow A \subseteq C \text{ and } B \supseteq D.$$

Here, we informally think of the oriented separation (A,B) as *pointing towards* $\{C,D\}$ and its orientations, whereas we think of (C,D) as *pointing away from* $\{A,B\}$ and its orientations. If O is an orientation and no two distinct separations (B,A) and (C,D) in O satisfy $(A,B) < (C,D)$, i.e., no two distinct separations in O point away from each other, then we call O *consistent*.

We call a set $\sigma \subseteq \vec{S}$ of oriented separations a *star* if every two distinct separations (A,B) and (D,C) in σ point towards each other, i.e. satisfy $(A,B) \leq (C,D)$. The *interior* of a star $\sigma = \{\, (A_i,B_i) \mid i \in I \,\}$ is the intersection $\bigcap_{i\in I} B_i$ of the big sides.

Definition 2.2. An $\aleph_0$-*tangle* (of G) is a consistent orientation of S that contains no finite star of finite interior as a subset. We write Θ for the set of all $\aleph_0$-tangles.

2.4. **Ends and Tangles.** If ω is an end of G, then letting

$$\tau_\omega := \{\, (A,B) \in \vec{S} \mid C(A \cap B, \omega) \subseteq G[B \setminus A] \,\}$$

defines an injection $\Omega \hookrightarrow \Theta$, $\omega \mapsto \tau_\omega$ from the ends of G into the $\aleph_0$-tangles. Therefore, we call the tangles of the form τ_ω the *end tangles* of G. By abuse of notation we write Ω for the collection of all end tangles of G, so we have $\Omega \subseteq \Theta$.

In order to understand the $\aleph_0$-tangles that are not ends, Diestel studied an inverse limit description of Θ which we introduce in a moment. First, we note that every finite order separation $\{A,B\}$ corresponds to the bipartition $\{\mathscr{C},\mathscr{C}'\}$ of the component space $\mathscr{C}_X$ with $X = A \cap B$ and

$$\{A,B\} = \big\{\, V[\mathscr{C}] \cup X \,,\, X \cup V[\mathscr{C}'] \,\big\}$$

where $V[\mathscr{C}] = \bigcup_{C\in\mathscr{C}} V(C)$, and this correspondence is bijective for fixed $X \in \mathcal{X}$. For all $\mathscr{C} \subseteq \mathscr{C}_X$ let us write $s_{X\to\mathscr{C}}$ for the separation $(V \setminus V[\mathscr{C}]\,,\, X \cup V[\mathscr{C}])$. Hence if τ is an $\aleph_0$-tangle of the graph, then for each $X \in \mathcal{X}$ it also chooses one *big side* from each bipartition $\{\mathscr{C},\mathscr{C}'\}$ of $\mathscr{C}_X$, namely the $\mathscr{K} \in \{\mathscr{C},\mathscr{C}'\}$ with $s_{X\to\mathscr{K}} \in \tau$. Since it chooses theses sides consistently, it induces an ultrafilter $U(\tau,X)$ on $\mathscr{C}_X$, one for every $X \in \mathcal{X}$, which is given by

$$U(\tau,X) = \{\, \mathscr{C} \subseteq \mathscr{C}_X \mid s_{X\to\mathscr{C}} \in \tau \,\},$$

and these ultrafilters are compatible in that they form a limit of the inverse system $\{\, \beta(\mathscr{C}_X)\,,\, \beta(\mathfrak{c}_{X',X})\,,\, \mathcal{X} \,\}$. Here, each set $\mathscr{C}_X$ is endowed with the discrete topology and $\beta(\mathscr{C}_X)$ denotes its Stone-Čech compactification. Every bonding map $\beta(\mathfrak{c}_{X',X})$ is the unique continuous extension of $\mathfrak{c}_{X',X}$ that is provided by the Stone-Čech property (see Theorem 4.1 (ii)). More explicitly, the map $\beta(\mathfrak{c}_{X',X})$ sends each ultrafilter $U' \in \beta(\mathscr{C}_{X'})$ to its *restriction*

$$U' \restriction X = \{\, \mathscr{C} \subseteq \mathscr{C}_X \mid \exists \mathscr{C}' \in U' \colon \mathscr{C} \supseteq \mathscr{C}' \restriction X \,\} \in \beta(\mathscr{C}_X)$$

where $\mathscr{C}' \restriction X = \mathfrak{c}_{X',X}[\mathscr{C}']$. As one of his main results, Diestel showed that the map

$$\tau \mapsto (\, U(\tau,X) \mid X \in \mathcal{X} \,)$$

defines a bijection between the tangle space Θ and the inverse limit $\varprojlim \beta(\mathscr{C}_X)$. From now on, we view the tangle space Θ as the compact Hausdorff space $\varprojlim \beta(\mathscr{C}_X)$.

In his paper, Diestel moreover showed that the ends of G are precisely those $\aleph_0$-tangles whose induced ultrafilters are all principal. For every $\aleph_0$-tangle τ we write $\mathcal{X}_\tau$ for the collection of all $X \in \mathcal{X}$ for which the induced ultrafilter $U(\tau, X)$ is free. The set $\mathcal{X}_\tau$ is empty if and only if τ is an end tangle; an $\aleph_0$-tangle τ with $\mathcal{X}_\tau$ non-empty is called an *ultrafilter tangle*. For every ultrafilter tangle τ the set $\mathcal{X}_\tau$ has a least element X_τ of which it is the up-closure. We characterised the sets of the form X_τ combinatorially in [10, Theorem 4.10]: they are precisely the *critical* vertex sets of G, finite sets $X \subseteq V$ whose deletion leaves some infinitely many components each with neighbourhood precisely equal to X, and they can be used together with the ends to compactify the graph, [10, Theorem 4.11].

We conclude our summary of 'Ends and tangles' with the formal construction of the tangle compactification. To obtain the tangle compactification $|G|_\Theta$ of a graph G we extend the 1-complex of G to a topological space $G \sqcup \Theta$ by declaring as open in addition to the open sets of G, for all $X \in \mathcal{X}$ and all $\mathscr{C} \subseteq \mathscr{C}_X$, the sets

$$\mathcal{O}_{|G|_\Theta}(X, \mathscr{C}) := \bigcup \mathscr{C} \cup \mathring{E}(X, \bigcup \mathscr{C}) \cup \{\, \tau \in \Theta \mid \mathscr{C} \in U(\tau, X) \,\}$$

and taking the topology this generates. Notably, $|G|_\Theta$ contains Θ as a subspace.

Theorem 2.3 ([2, Theorem 1])**.** *Let G be any graph, possibly disconnected.*

(i) *$|G|_\Theta$ is a compactification of G with totally disconnected remainder.*
(ii) *If G is locally finite and connected, then $|G|_\Theta$ coincides with the Freudenthal compactification of G.*

The tangle compactification is Hausdorff if and only if G is locally finite. However, the subspace $|G|_\Theta \setminus \mathring{E}$ is compact Hausdorff. Teegen [13] generalised the tangle compactification to topological spaces.

3. Tangles as inverse limit of finite spaces

The Stone-Čech compactification of a discrete space can be viewed as the inverse limit of all its finite partitions, where each finite partition carries the discrete topology. In this section, we extend this fact to the tangle space.

We start by choosing the point set for our directed poset:

$$\Gamma := \{\, (X, P) \mid X \in \mathcal{X} \text{ and } P \text{ is a finite partition of } \mathscr{C}_X \,\}.$$

Notation. If an element of Γ is introduced just as γ, then we write $X(\gamma)$ and $P(\gamma)$ for the sets satisfying $(X(\gamma), P(\gamma)) = \gamma$. Given $X \subseteq X' \in \mathcal{X}$ and a finite partition P of $\mathscr{C}_X$ we write $P \downharpoonright X'$ for the finite partition

$$\{\, \mathfrak{c}_{X',X}^{-1}(\mathscr{C}) \mid \mathscr{C} \in P \,\} \setminus \{\emptyset\}$$

that P induces on $\mathscr{C}_{X'}$.

Letting $(X, P) \le (Y, Q)$ whenever $X \subseteq Y$ and Q refines $P \downharpoonright Y$ defines a directed partial ordering on Γ:

Lemma 3.1. *$(\Gamma, \le)$ is a directed poset.*

Proof. Checking the poset properties is straightforward; we verify that it is directed: Given any two elements (X, P) and (Y, Q) of Γ let R be the coarsest refinement of $P \downharpoonright (X \cup Y)$ and $Q \downharpoonright (X \cup Y)$. Then $(X, P), (Y, Q) \le (X \cup Y, R) \in \Gamma$. $\square$

For a reason that will become clear in the proof of our next theorem, we consider a cofinal subset of Γ, namely

$$\Gamma' := \{\, \gamma \in \Gamma \mid \forall \mathscr{C} \in P(\gamma) \colon V[\mathscr{C}] \text{ is infinite} \,\}.$$

Lemma 3.2. Γ' *is cofinal in* Γ.

Proof. Given $(X, P) \in \Gamma$ we put

$$X' = X \cup \bigcup \{\, V[\mathscr{C}] \mid \mathscr{C} \in P \text{ with } V[\mathscr{C}] \text{ finite} \,\}.$$

Then $(X, P) \leq (X', P \downharpoonleft X') \in \Gamma'$ as desired. □

We aim to describe the tangle space as an inverse limit of finite Hausdorff spaces. For this, we choose Γ as our directed poset, and for each $\gamma \in \Gamma$ we let $\mathscr{P}_\gamma$ be the set $P(\gamma)$ endowed with the discrete topology. Our bonding maps $f_{\gamma',\gamma} \colon \mathscr{P}_{\gamma'} \to \mathscr{P}_\gamma$ send each $\mathscr{C}' \in \mathscr{P}_{\gamma'}$ to the unique $\mathscr{C} \in \mathscr{P}_\gamma$ with $\mathscr{C}' \upharpoonright X(\gamma) \subseteq \mathscr{C}$. Since the spaces $\mathscr{P}_\gamma$ are compact Hausdorff, so is their inverse limit

$$\mathscr{P} := \varprojlim \,(\, \mathscr{P}_\gamma \mid \gamma \in \Gamma \,).$$

By [6, Corollary 2.5.11] we may replace Γ with its cofinal subset Γ' without changing the inverse limit $\mathscr{P}$, so we assume without loss of generality that $\Gamma = \Gamma'$.

Notation. If τ is an $\aleph_0$-tangle and $\gamma = (X, P) \in \Gamma$ is given, then we write $\mathscr{C}(\tau, \gamma)$ for the unique partition class of P that is contained in the ultrafilter $U(\tau, X)$.

Theorem 3.3. *For any graph G, its tangle space is homeomorphic to the inverse limit $\mathscr{P}$, i.e. $\Theta \cong \mathscr{P}$.*

Proof. Letting $\varphi_\gamma \colon \Theta \to \mathscr{P}_\gamma$ assign $\mathscr{C}(\tau, \gamma)$ to each tangle $\tau \in \Theta$ defines a collection of maps that are compatible as tangles are consistent. To see that our maps are continuous, it suffices to note that for all $\gamma \in \Gamma$ and $\mathscr{C} \in \mathscr{P}_\gamma$ we have

$$\varphi_\gamma^{-1}(\mathscr{C}) = \{\, \tau \in \Theta \mid \mathscr{C} \in U(\tau, X(\gamma)) \,\}.$$

The set $V[\mathscr{C}]$ is infinite due to $\Gamma = \Gamma'$, so Diestel's [2, Lemma 3.7] ensures that the preimage $\varphi_\gamma^{-1}(\mathscr{C})$ is non-empty, i.e. that our maps are surjective. Since the tangle space Θ is compact and the inverse limit $\mathscr{P}$ is Hausdorff, the maps φ_γ combine into a continuous surjection $\varphi \colon \Theta \twoheadrightarrow \mathscr{P}$ (cf. [6, Corollary 3.2.16]). Moreover, φ is injective, so it follows from compactness that φ is a homeomorphism. □

4. Background on the Stone-Čech compactification of an infinite graph

4.1. **Stone-Čech compactification of 1-complexes.** The following characterisation of the Stone-Čech compactification is well-known:

Theorem 4.1 (Cf. [6],[7])**.** *Let X be a Tychonoff space. The following are equivalent for a Hausdorff compactification $\gamma X \supseteq X$:*

(i) $\gamma X = \beta X$,
(ii) *every continuous function $f \colon X \to T$ to a compact Hausdorff space T has a continuous extension $\hat{f} \colon \gamma X \to T$ with $\hat{f} \upharpoonright X = f$,*
(iii) *every continuous function $f \colon X \to \mathbb{I}$ has a continuous extension $\hat{f} \colon \gamma X \to \mathbb{I}$ with $\hat{f} \upharpoonright X = f$.*

Moreover, if X is normal[3]*, then we may add*

[3]In this paper, the property *normal* always includes *Hausdorff*.

(iv) *any two closed disjoint sets $Z_1, Z_2 \subseteq X$ have disjoint closures in γX,*
(v) *for any two closed sets $Z_1, Z_2 \subseteq X$ we have*

$$\overline{Z_1 \cap Z_2}^{\gamma G} = \overline{Z_1}^{\gamma G} \cap \overline{Z_2}^{\gamma G}.$$

Ultrafilter limits. Consider a compact Hausdorff space X. If $x = (\, x_i \mid i \in I\,)$ is a family of points $x_i \in X$ and U is an ultrafilter on the index set I, then there is a unique point $x_U \in \overline{\{\, x_i \mid i \in I\}} \subseteq X$ defined by

$$\{x_U\} = \bigcap_{J \in U} \overline{\{\, x_i \mid i \in J\,\}}.$$

Indeed, since U is a filter, the collection $\big\{\, \{x_i \colon i \in J\} \;\big|\; J \in U \,\big\}$ has the finite intersection property, and so by compactness of X, the intersection over their closures is non-empty; and it follows from Hausdorffness of X that the intersection can contain at most one point. We also write

$$x_U = U\text{-}\lim x = U\text{-}\lim\,(\, x_i \mid i \in I\,)$$

and call x_U the limit of $(\, x_i \mid i \in I\,)$ along U, or U-limit of x. Note that if U is the principal ultrafilter generated by $i \in I$, then $x_U = x_i$.

For an alternative description, put $T = \overline{\{\, x_i \mid i \in I\,\}} \subseteq X$ and view I as a discrete space, so that the index function

$$\tilde{x} \colon I \to \{\, x_i \mid i \in I\,\} \subseteq T,\; i \mapsto x_i$$

is continuous and βI is given by the space of ultrafilters on I. Then the Stone-Čech extension $\beta\tilde{x} \colon \beta I \to T$ of the index function $\tilde{x}$ maps each ultrafilter $U \in \beta I$ to x_U.

More generally, if $(\, X_i \mid i \in I\,)$ is a family of subsets of a compact Hausdorff space X and U is an ultrafilter on the index set I, then we write

$$X_U = U\text{-}\lim\,(\, X_i \mid i \in I\,) := \bigcap_{J \in U} \overline{\bigcup_{i \in J} X_i} \subseteq X$$

and call X_U the U-limit of $(\, X_i \mid i \in I\,)$.

Two facts about continua. We shall need the following two simple lemmas about continua. Recall that a continuum is a non-empty compact connected Hausdorff space.

Lemma 4.2. *Let X be a compact Hausdorff space, and $C \subseteq X$ a connected subspace. Then $\overline{C} \subseteq X$ is a continuum.* □

A family $(\, C_i \mid i \in I\,)$ of subcontinua of some topological space is said to be *directed* if for any $i, j \in I$ there exists a $k \in I$ such that $C_k \subseteq C_i \cap C_j$.

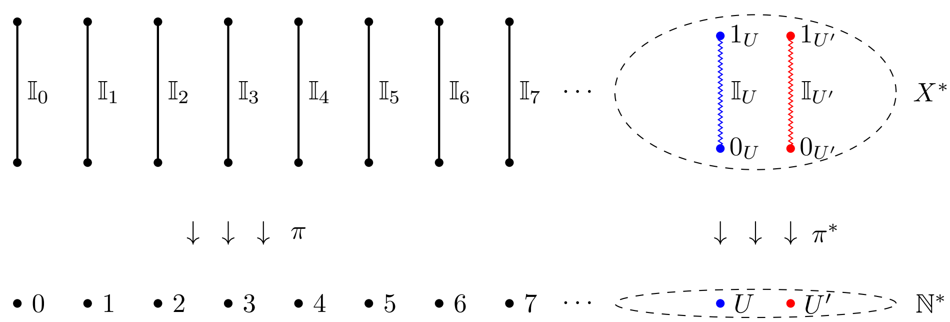
Lemma 4.3 ([6, Theorem 6.1.18])**.** *The intersection of any directed family of continua is again a continuum.* □

The Stone-Čech compactification of a disjoint sum of intervals. Recall that the 1-complex of a connected graph G can be obtained from the topological sum of disjoint unit intervals (one for each edge) by identifying suitable endpoints, and using the quotient topology. To formalise this, consider the topological space $\mathbb{M}_E = \mathbb{I} \times E$ where $E = E(G)$ carries the discrete topology. Then $G = \mathbb{M}_E/{\sim}$ for some suitable equivalence relation identifying endpoints. Write $\mathbb{I}_e$ for $\mathbb{I} \times \{e\} \subseteq \mathbb{M}_E$, and x_e for $(x, e) \in \mathbb{I}_e$, so $\mathbb{M}_E = \bigoplus_{e \in E} \mathbb{I}_e$.

Our next results, and in particular Theorem 4.8, say that the Stone-Čech compactification of a 1-complex G (which to our knowledge hasn't been studied at all) can be understood through the Stone-Čech compactification $\beta\mathbb{M}_E$ of $\mathbb{M}_E$ (which has been studied extensively over the past decades, see e.g. the survey [9]).

Lemma 4.4 ([9, Corollary 2.2]). *Let $X = \bigoplus_{i\in I} K_i$ be a topological sum of continua, and view I as a discrete space. Consider the continuous projection $\pi\colon X \to I$, sending K_i to $i \in I$. The components of βX are the fibres of the map $\beta\pi\colon \beta X \to \beta I$.*

Suppose for a moment that $X = \bigoplus_{i\in I} K_i$ has only countably many components, i.e. that $I = \mathbb{N}$. Write $X^* = \beta X \setminus X$ for the Stone-Čech remainder. In the lemma, $\beta\pi$ denotes the Stone-Čech extension of π, where we interpret π as a continuous map from X into the compact Hausdorff space $\beta\mathbb{N} \supseteq \mathbb{N}$. And since π has compact fibres (also called *perfect map*), the extension $\beta\pi$ restricts to a continuous map $\pi^* = \beta\pi \restriction X^*\colon X^* \to \mathbb{N}^*$, i.e. it maps the remainder of βX to the remainder of $\beta\mathbb{N}$, [6, Theorem 3.7.16]. The figure below illustrates this for $X = \mathbb{M}_\mathbb{N}$:



Now, for every ultrafilter $U \in \beta\mathbb{N}$ the fibre $\beta\pi^{-1}(U)$ is a connected component of βX, which is also denoted by K_U. This is well-defined, as it is not hard to check that in fact we have

$$\beta\pi^{-1}(U) = K_U = U\text{-lim}\,(\, K_i \mid i \in I \,) = \bigcap_{J\in U} \overline{\bigcup_{i\in J} K_i}^{\beta X}.$$

Also, if $(\, x_i \mid i \in I \,)$ is a family of points with $x_i \in K_i$, then x_U is the unique point of $K_U \cap \overline{\{\, x_i \mid i \in I \,\}}^{\beta X}$. If the spaces K_i are homeomorphic copies of a single space and the points $x_i \in K_i$ correspond to the same point ξ of the original space, then we write ξ_U for x_U. For example, if each K_i is a copy of the unit interval and x_i corresponds to 0 for all $i \in I$, then $x_U = 0_U$.

We shall also need the following lemma plus corollary:

Lemma 4.5 ([9, Lemma 2.3]). *For a family $(\, x_i \mid i \in I \,)$ of points $x_i \in K_i$, the point x_U is a cut-point of K_U if and only if $\{\, i \mid x_i$ is a cut-point of $K_i \,\} \in U$.*

Notation. In the context of $X = \mathbb{M}_E$ we write $\check{\mathbb{I}}_U$ for $\mathbb{I}_U \setminus \{0_U, 1_U\}$.

Corollary 4.6. *The spaces $\mathbb{I}_U \setminus \{0_U\}$, $\mathbb{I}_U \setminus \{1_U\}$ and $\check{\mathbb{I}}_U$ are connected.*

Proof. The non-standard interval $[0_U, (\frac{1}{2})_U]$ is homeomorphic to $\mathbb{I}_U$ (cf. [9, Proposition 2.8]). Thus $(0_U, (\frac{1}{2})_U]$ is connected by Lemma 4.5. So is $[(\frac{1}{2})_U, 1_U)$. Since both meet in $(\frac{1}{2})_U$, so is their union $\check{\mathbb{I}}_U$. □

Quotients. As we are interested in 1-complexes, i.e. in quotients of $\mathbb{M}_E$, we provide a theorem how the quotient operation relates to the Stone-Čech functor. We need the following lemma, which is easily verified.

Lemma 4.7. *Let V be a closed discrete subset of a normal space X, and suppose that $\sim$ is an equivalence relation on V. Then $X/{\sim}$ is again normal.* □

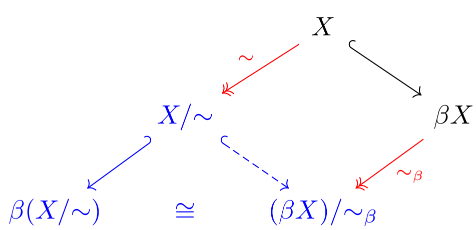
Theorem 4.8. *Let V be a closed discrete subset of a normal space X, and suppose that $\sim$ is an equivalence relation on V. Let $\{ V_i \mid i \in I \}$ be the collection of all non-trivial $\sim$-classes. Consider the equivalence relation $\sim_\beta$ on $\overline{V}^{\beta X}$ where each (non-trivial) equivalence class is of the form*

$$V_U = U\text{-}\lim (V_i \mid i \in I) = \bigcap_{J \in U} \overline{\bigcup_{i \in J} V_i}^{\beta X},$$

one for each ultrafilter U on I. Then $X/{\sim}$ is again normal and

$$\beta(X/{\sim}) = (\beta X)/{\sim_\beta} ,$$

i.e. the diagram



commutes.

Proof. The quotient $X/{\sim}$ is normal by Lemma 4.7, so its Stone-Čech compactification exists. Clearly, the quotient $(\beta X)/{\sim_\beta}$ is compact.

To show that $(\beta X)/{\sim_\beta}$ is Hausdorff, since $\overline{V}^{\beta X}$ is a closed subspace of the normal space βX, it suffices to show that for any two distinct points x, y of $\overline{V}^{\beta X}/{\sim_\beta}$ the subsets $\cup x$ and $\cup y$ of $\overline{V}^{\beta X}$ can be separated by a closed and $\sim_\beta$-respecting bipartition of $\overline{V}^{\beta X}$, which follows readily from V being discrete and Theorem 4.1 (v) with $\overline{V}^{\beta X} = \beta V$.

Next, we observe that every trivial $\sim$-class remains untouched by $\sim_\beta$, every non-trivial $\sim$-class V_i is contained in the $\sim_\beta$-class $V_U = \overline{V_i}^{\beta X}$ with $V_U \cap X = V_i$ where U is the principal ultrafilter at i, and every $\sim_\beta$-class V_U with U free is contained in the remainder X^*. Hence, we have an injection from $X/{\sim}$ into $(\beta X)/{\sim_\beta}$ such that the image will be a dense subspace. To see that this is an embedding, and hence that $(\beta X)/{\sim_\beta}$ is a Hausdorff compactification of $X/{\sim}$, it remains to show that the subspace topology of $(\beta X)/{\sim_\beta}$ agrees with the quotient topology on $X/{\sim}$. Consider a non-trivial equivalence class V_j, and, without loss of generality, an open subset $W \subseteq X$ such that $W \cap V = V_j$. By Theorem 4.1 (iv), the disjoint closed sets V_j and $C := X \setminus W$ have disjoint closures in βX. But then $\tilde{W} := \beta X \setminus \overline{C}$ is an open subset of βX such that $\tilde{W} \cap \overline{V}^{\beta X} = \overline{V_j}$, and so $V_j \subseteq \tilde{W} \cap X \subseteq W$ witnesses that the embedding is open onto its image.

Conversely, take any open neighbourhood $\tilde{W} \subseteq \beta X$ respecting the equivalence relation $\sim_\beta$. Then, by definition, $W := \tilde{W} \cap X$ is an open subset of X respecting the equivalence relation $\sim$, and hence W induces an open subset of $X/{\sim}$, showing that the embedding is continuous.

Finally, we check for the extension property of continuous functions into the unit interval, cf. Theorem 4.1. But a continuous function $\tilde{f}\colon X/{\sim} \to \mathbb{I}$ corresponds, by the quotient topology, to a continuous function $f\colon X \to \mathbb{I}$ such that

$$f[V_i] =: \{x_i\} \subseteq \mathbb{I}$$

is constant on each equivalence class V_i for all $i \in I$. By the opposite direction of Theorem 4.1, $f\colon X \to \mathbb{I}$ extends to a continuous function $\beta f\colon \beta X \to \mathbb{I}$ with $\beta f \restriction X = f$. Now we claim that

$$\beta f[V_U] = \{x_U\} \subseteq \mathbb{I}$$

is also constant on the equivalence classes of $\sim_\beta$ for all ultrafilters U on I. Indeed,

$$\begin{aligned}\beta f\left[\bigcap_{J\in U}\overline{\bigcup_{i\in J} V_i}^{\beta X}\right] &\subseteq \bigcap_{J\in U}\beta f\left[\overline{\bigcup_{i\in J} V_i}^{\beta X}\right] \subseteq \bigcap_{J\in U}\overline{\beta f\left[\bigcup_{i\in J} V_i\right]}^{\mathbb{I}}\\ &= \bigcap_{J\in U}\overline{\bigcup_{i\in J}\beta f[V_i]}^{\mathbb{I}} = \bigcap_{J\in U}\overline{\bigcup_{i\in J} f[V_i]}^{\mathbb{I}}\\ &= \bigcap_{J\in U}\overline{\bigcup_{i\in J}\{x_i\}}^{\mathbb{I}} = \{x_U\} \subseteq \mathbb{I}.\end{aligned}$$

Here, the first inclusion and the first equality sign are standard facts about images of intersections and unions respectively. The second inclusion follows from continuity of βf. The second equality follows from the fact that $\beta f \restriction X = f$.

Thus, $\beta f\colon \beta X \to \mathbb{I}$ induces a well-defined continuous function $(\beta X)/{\sim_\beta} \to \mathbb{I}$ that extends the continuous function $\tilde{f}\colon X/{\sim} \to \mathbb{I}$. □

Corollary 4.9. *Let X be a normal space and $V \subseteq X$ a closed discrete subset. Then X/V is again normal and*

$$\beta(X/V) = \beta X/\big(\overline{V}^{\beta X}\big). \qquad \square$$

Corollary 4.10. *Let X and Y be two disjoint normal spaces, and suppose that $A = \{\, a_i \mid i \in I \,\} \subseteq X$ and $B = \{\, b_i \mid i \in I \,\} \subseteq Y$ are infinite closed discrete subspaces. Consider the quotient $Z = (X \oplus Y)/{\sim}$ where we identify pairs $\{a_i, b_i\}$ for all $i \in I$. Then*

$$\beta Z = (\beta X \oplus \beta Y)/{\sim_\beta}$$

where we identify pairs $\{a_U, b_U\}$ for all ultrafilters U on I. □

4.2. **Three examples.** Before turning towards the proof of our main result, we illustrate the above topological lemmas by three representative examples: We discuss the Stone-Čech compactification of the infinite ray R, the infinite star S_λ of degree λ, and the dominated ray D, and compare it side by side with the $\aleph_0$ tangles of these examples.

The infinite ray. Consider the infinite ray R with vertex set $V = \{\, v_n \mid n \in \mathbb{N} \,\}$ and edge set $E = \{\, v_n v_{n+1} \mid n \in \mathbb{N} \,\}$. Since R is locally finite, the space of $\aleph_0$-tangles consists solely of the single end of R, by Theorem 2.3 (ii). Moreover, the 1-complex R is homeomorphic to the positive half line $\mathbb{H} = [0, \infty)$, so they have the same Stone-Čech remainder $R^* = \mathbb{H}^*$. The space $\mathbb{H}^*$ has been extensively investigated, see e.g. [9] for a survey. At this point, however, we are content to provide the standard argument showing that the Stone-Čech remainder of the infinite ray is indeed connected, confirming the connection between components in the remainder of the Stone-Čech compactification and the $\aleph_0$-tangles.

Example 4.11. The infinite ray has a connected Stone-Čech remainder.

Proof. Deleting a vertex v_n from R leaves behind exactly one infinite component $C_n = R[v_{n+1}, v_{n+2}, \ldots]$. Then $\bigcap_{n\in\mathbb{N}} \overline{C_n}^{\beta R}$ is a continuum by Lemmas 4.2 and 4.3. We claim that

$$R^* = \bigcap_{n\in\mathbb{N}} \overline{C_n}^{\beta R}.$$

Indeed, "$\supseteq$" holds as any vertex and edge of R is removed eventually by the intersection. For "$\subseteq$" note that for any $n \in \mathbb{N}$ we have $R = R[v_0, \ldots, v_{n+1}] \cup C_n$, and hence

$$R^* \subseteq \overline{R[v_0, \ldots, v_{n+1}]}^{\beta R} \cup \overline{C_n}^{\beta R},$$

since the closure operator distributes over finite unions. But $R[v_0, \ldots, v_{n+1}]$ is compact, and hence closed in the Hausdorff space βR, implying

$$\overline{R[v_0, \ldots, v_{n+1}]}^{\beta R} = R[v_0, \ldots, v_{n+1}] \subseteq R.$$

It follows $R^* \subseteq \overline{C_n}^{\beta R}$ for all $n \in \mathbb{N}$ as desired. □

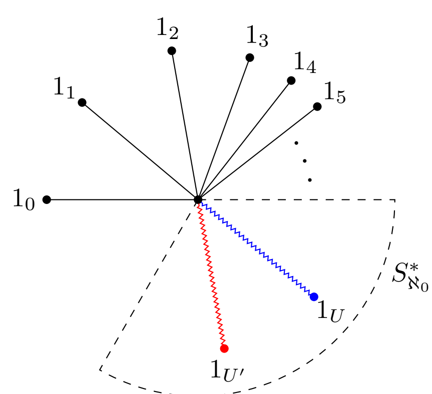


FIGURE 1. The Stone-Čech compactification of the countable infinite star

The infinite star. For any cardinal λ we denote by S_λ the star of degree λ. Clearly, this star has no end, so all $\aleph_0$-tangles are ultrafilter tangles. As a consequence of [10, Theorem 4.10], the ultrafilter tangles correspond precisely to the free ultrafilters on λ. The 1-complex of S_λ is obtained from $\mathbb{M}_E$ (with E a discrete space of cardinality λ) via

$$S_\lambda = \mathbb{M}_E / \{\, 0_e \mid e \in E\}.$$

Example 4.12. The Stone-Čech remainder of an infinite star S_λ is homeomorphic to $\mathbb{M}^*_E \setminus \{\, 0_U \mid U \in E^*\}$. Each connected component of S^*_λ is homeomorphic to $\mathbb{I}_U \setminus \{0_U\}$ for some free ultrafilter $U \in E^*$.

Proof. Since $S_\lambda = \mathbb{M}_E/\{\, 0_e \mid e \in E \,\}$, it follows immediately from Corollary 4.9 that $\beta S_\lambda = \beta\mathbb{M}_E/\overline{\{\, 0_e \mid e \in E \,\}}^{\beta\mathbb{M}_E}$. Since the equivalence class $\overline{\{\, 0_e \mid e \in E \,\}}^{\beta\mathbb{M}_E}$ corresponds to the center vertex of S_λ, it follows for the remainder of βS_λ that

$$S^*_\lambda = \mathbb{M}^*_E \setminus \overline{\{\, 0_e \mid e \in E \,\}}^{\beta\mathbb{M}_E} = \mathbb{M}^*_E \setminus \{\, 0_U \mid U \in E^* \,\}.$$

By Lemma 4.4 and Corollary 4.6, the connected components of the remainder $\mathbb{M}^*_E \setminus \{\, 0_U \mid U \in E^* \,\}$ are given by $\mathbb{I}_U \setminus \{0_U\}$ for each free ultrafilter U on E. □

The dominated ray. The dominated ray D is the quotient of an infinite star $S_{\aleph_0}$ and a ray R where the leaves of $S_{\aleph_0}$, denoted as in the previous example by $\{\, 1_n \mid n \in \mathbb{N} \,\}$, are identified pairwise with vertices of the ray, denoted by $\{\, v_n \mid n \in \mathbb{N} \,\}$ (see Fig. 2). Since deleting any finite set of vertices from D leaves only one infinite component, the sole end of D is the one and only $\aleph_0$-tangle.

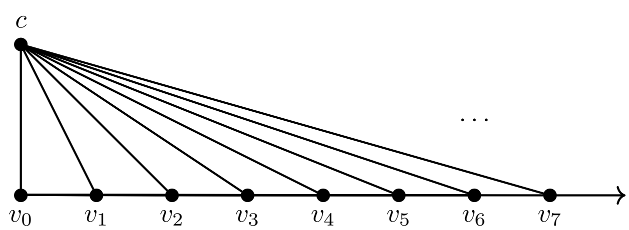


FIGURE 2. The dominated ray with dominating vertex c

Example 4.13. The dominated ray D has a connected Stone-Čech remainder.

Proof. By Corollary 4.10, the Stone-Čech remainder of D is homeomorphic to the quotient $\big(S^*_{\aleph_0} \oplus R^*\big)/{\sim_\beta}$ where $1_U \sim_\beta v_U$ for every ultrafilter $U \in \mathbb{N}^*$ and $1_U \in \mathbb{I}_U$ and $v_U \in R^*$. It follows that every connected component $\mathbb{I}_U \setminus \{0_U\}$ of $S^*_{\aleph_0}$ (see Example 4.12) is, via the identified points $1_U \sim_\beta v_U$, attached to the connected remainder R^* (see Example 4.11) of βR, and so D^* is indeed connected. □

5. Comparing the Stone-Čech remainder with the tangle space

5.1. **The Stone-Čech remainder of the vertex set.** Due to $\beta G = (\beta\mathbb{M}_E)/{\sim_\beta}$ for any representation $\mathbb{M}_E/{\sim}$ of G (Theorem 4.8) we may view $\beta V = \overline{V}^{\beta G} \subseteq \beta G$ as the closure of $\{\, [0_e]_{\sim_\beta}, [1_e]_{\sim_\beta} \mid e \in E \,\}$ in the quotient $(\beta\mathbb{M}_E)/{\sim_\beta}$. In particular, the non-standard intervals $\mathbb{I}_U$ (with $U \in E^*$) may interact with V or its Stone-Čech remainder V^*. In this subsection, we have a closer look at this interaction.

In the next lemma, we write $V^* = G^* \cap \overline{V}^{\beta G}$. Since $\beta V = \overline{V}^{\beta G}$, this potential double meaning does no harm.

Lemma 5.1. *Let $\mathbb{M}_E/{\sim}$ be a representation of G, and let $U \in E^*$ be any free ultrafilter. Then at most one of the endpoints 0_U and 1_U of $\mathbb{I}_U$ is contained in some ${\sim_\beta}$-class that belongs to V, and at least one of them is contained in some ${\sim_\beta}$-class that belongs to V^*.*

Proof. First, we use $\beta G = (\beta\mathbb{M}_E)/{\sim_\beta}$ (Theorem 4.8) to deduce from

$$\{0_U, 1_U\} \subseteq \overline{\{\, 0_e, 1_e \mid e \in E \,\}}^{\beta\mathbb{M}_E}$$

that

$$\{\,[0_U]_{\sim_\beta}\,,\,[1_U]_{\sim_\beta}\,\} \subseteq \overline{\{\,[0_e]_{\sim_\beta}\,,\,[1_e]_{\sim_\beta} \mid e \in E\,\}}^{(\beta\mathbb{M}_E)/\sim_\beta} = \overline{V}^{\beta G} = \beta V$$

(formally, it follows from the continuity of the quotient map $\beta\mathbb{M}_E \to (\beta\mathbb{M}_E)_{\sim_\beta}$ and the simple fact that a continuous function h satisfies $h[\,\overline{A}\,] \subseteq \overline{h[A]}$ for each subset A of its domain, see [14, Theorem 7.2]). Now if for both 0_U and 1_U there is a vertex v_0 and v_1 of G with $0_U \in v_0$ and $1_U \in v_1$ (with the v_i viewed as $\sim_\beta$-classes of $\beta\mathbb{M}_E$), then $i_U \in \overline{v_i}^{\beta\mathbb{M}_E}$ for the infinite $\sim$-classes v_i of $\mathbb{M}_E$ (for $i = 1, 2$). This means that there is one infinite $F \in U$ with $\{\, i_e \mid e \in F\,\} \subseteq v_i \subseteq \mathbb{M}_E$ for both $i = 1$ and $i = 2$. But then F is a collection of infinitely many parallel edges between the vertices v_0 and v_1 of G, contradicting that G was simple. □

Lemma 5.2. *Let G be a graph, and let C be a connected component of the Stone-Čech remainder G^*. Then $C \cap V^* \neq \emptyset$. In particular, the connected components of G^* induce a closed partition of V^*.*

Proof. Consider a representation $G = \mathbb{M}_E/\!\sim$ of G, and recall that by Corollary 4.6, every non-standard component $\mathbb{I}_U$ of $\mathbb{M}^*_E$ remains connected upon deleting one or both of the endpoints 0_U and 1_U.

Consider some connected component C of G^*. Then for some $\mathbb{I}_U \subseteq \mathbb{M}^*_E$ we have $\check{\mathbb{I}}_U \subseteq C$. Therefore, it suffices to show that for every free ultrafilter $U \in E^*$ at least one of $[0_U]_{\sim_\beta}$ and $[1_U]_{\sim_\beta}$ is in V^*. This is the content of Lemma 5.1. □

5.2. **An auxiliary remainder.** The remainder G^* not being compact prevents us from using topological machinery, so we study a nice subspace $G^\times \subseteq G^*$ first. As usual, we start with some new notation.

Notation. For a vertex v of G, write $O(v)$ for its open neighbourhood $\mathring{E}(v) \sqcup \{v\}$ in G consisting of all half-open incident edges at v, and write

$$O_{\beta G}(v) := \overline{\bigcup E(v)}^{\beta G} \setminus \overline{N(v)}^{\beta G}.$$

Due to $\beta G = \overline{\bigcup E(v)}^{\beta G} \cup \overline{G \setminus O(v)}^{\beta G}$ and $\overline{\bigcup E(v)}^{\beta G} \cap \overline{G \setminus O(v)}^{\beta G} = \overline{N(v)}^{\beta G}$ the set $O_{\beta G}(v)$ is open in βG, and it meets G precisely in $O(v)$.

Observation 5.3. *Put $F = E(v)$ and write H for the subspace $\bigcup F \subseteq G$. Since H is the 1-complex of a star, the set $O_{\beta G}(v)$ is homeomorphic to the space from Example 4.12 without the "endpoints" (also see Fig. 1):*

$$\begin{aligned} O_{\beta G}(v) = \overline{H}^{\beta G} \setminus \overline{N(v)}^{\beta G} &\cong \beta H \setminus \overline{N(v)}^{\beta H} \\ &\cong (\beta\mathbb{M}_F/\{\, 0_U \mid U \in \beta F\,\}) \setminus \{\, 1_U \mid U \in \beta F\,\} \end{aligned}$$

Definition 5.4. The auxiliary remainder of G is the space

$$G^\times := \beta G \setminus O_{\beta G}[V] \subseteq G^*$$

where we write $O_{\beta G}[W] = \bigcup_{v \in W} O_{\beta G}(v)$ for all $W \subseteq V$.

Fact 5.5. Since βG is compact Hausdorff, so is $G^\times$.

Lemma 5.6. *If G is a graph, then $V^* = \overline{V}^{\beta G} \setminus G \subseteq G^*$ satisfies $V^* \subseteq G^\times$.*

Proof. We show that, for every vertex $v \in V$, the set $O_{\beta G}(v)$ avoids V^*:

$$\overline{\bigcup E(v)}^{\beta G} \cap V^* = \Big(\overline{\bigcup E(v)}^{\beta G} \cap \overline{V}^{\beta G}\Big) \setminus G = \overline{\{v\} \sqcup N(v)}^{\beta G} \setminus G$$
$$= \Big(\{v\} \sqcup \overline{N(v)}^{\beta G}\Big) \setminus G = N(v)^* \subseteq \overline{N(v)}^{\beta G} \qquad \square$$

5.3. **The components of the remainder can be distinguished by finite separators.** Our next target is to prove that any two components of the remainder of a graph are—just as the $\aleph_0$-tangles—distinguished by a finite order separation.

For the tangle compactification it is true that every open set $\mathcal{O}_{|G|_\Theta}(X, \mathscr{C})$ gives rise to a clopen bipartition of the tangle space, namely

$$\big(\,\mathcal{O}_{|G|_\Theta}(X, \mathscr{C}) \cap \Theta\,\big) \oplus \big(\,\mathcal{O}_{|G|_\Theta}(X, \mathscr{C}_X \setminus \mathscr{C}) \cap \Theta\,\big),$$
$$\text{i.e. } \{\, \tau \in \Theta \mid \mathscr{C} \in U(\tau, X)\,\} \oplus \{\, \tau \in \Theta \mid \mathscr{C} \notin U(\tau, X)\,\}.$$

In fact, for every two distinct $\aleph_0$-tangles there exists such a clopen bipartition of the tangle space separating the two. That is why we start by studying a possible analogue $\mathcal{O}_{\beta G}(X, \mathscr{C})$ of $\mathcal{O}_{|G|_\Theta}(X, \mathscr{C})$ for βG.

Notation. Given $X \in \mathcal{X}$ and $\mathscr{C} \subseteq \mathscr{C}_X$ we write $G[X, \mathscr{C}]$ for $G[X \cup V[\mathscr{C}]]$. If τ is an $\aleph_0$-tangle of G and γ is an element of Γ, then we write $G[\tau, \gamma]$ for $G[X(\gamma), \mathscr{C}(\tau, \gamma)]$.

For every $X \in \mathcal{X}$ and $\mathscr{C} \subseteq \mathscr{C}_X$ we let

$$\mathcal{O}_{\beta G}(X, \mathscr{C}) := \overline{G[X, \mathscr{C}]}^{\beta G} \setminus G[X]$$

which is open in βG as a consequence of $\beta G = \overline{G[X, \mathscr{C}]} \cup \overline{G[X, \mathscr{C}_X \setminus \mathscr{C}]}$ and $\overline{G[X, \mathscr{C}]} \cap \overline{G[X, \mathscr{C}_X \setminus \mathscr{C}]} = \overline{G[X]} = G[X]$ (see Theorem 4.1 (v)). Before we check that $\mathcal{O}_{\beta G}(X, \mathscr{C})$ gives rise to clopen bipartitions of G^* and $G^\times$, we prove a lemma:

Lemma 5.7. *For all $X \in \mathcal{X}$ and $\mathscr{C} \subseteq \mathscr{C}_X$ we have*

$$\overline{G[X, \mathscr{C}]}^{\beta G} \subseteq O_{\beta G}[X] \sqcup \overline{\bigcup \mathscr{C}}^{\beta G}.$$

In particular, for all $\gamma \in \Gamma$ we have

$$\beta G = O_{\beta G}[X(\gamma)] \sqcup \bigsqcup_{\mathscr{C} \in P(\gamma)} \overline{\bigcup \mathscr{C}}^{\beta G}.$$

Proof. Due to $\beta G = \bigcup_{\mathscr{C} \in P(\gamma)} \overline{G[X(\gamma), \mathscr{C}]}^{\beta G}$ it suffices to show the first statement:

$$\overline{G[X, \mathscr{C}]} = G[X] \cup \bigcup_{x \in X} \overline{\bigcup E(x, \bigcup \mathscr{C})} \cup \overline{\bigcup \mathscr{C}}$$
$$\subseteq \bigcup_{x \in X} \big(O_{\beta G}(x) \sqcup \overline{N(x) \cap \bigcup \mathscr{C}}\,\big) \cup \overline{\bigcup \mathscr{C}} = O_{\beta G}[X] \sqcup \overline{\bigcup \mathscr{C}}$$

where at the "⊆" we used Theorem 4.1 (v) for

$$\overline{\bigcup E(x, \bigcup \mathscr{C})} = \Big(\overline{\bigcup E(x, \bigcup \mathscr{C})} \setminus \overline{N(x)}\Big) \sqcup \Big(\overline{\bigcup E(x, \bigcup \mathscr{C})} \cap \overline{N(x)}\Big)$$
$$\subseteq O_{\beta G}(x) \sqcup \overline{(\bigcup E(x, \bigcup \mathscr{C})) \cap N(x)} = O_{\beta G}(x) \sqcup \overline{N(x) \cap \bigcup \mathscr{C}}. \qquad \square$$

Lemma and Definition 5.8. *Let any $(X, P) \in \Gamma$ be given. Then*

(i) $P_* := \big\{\, \overline{G[X, \mathscr{C}]}^{\beta G} \cap G^* \;\big|\; \mathscr{C} \in P \,\big\}$ *and*
(ii) $P_\times := \big\{\, \overline{\bigcup \mathscr{C}}^{\beta G} \cap G^\times \;\big|\; \mathscr{C} \in P \,\big\}$

are finite separations of G^* and $G^\times$ into clopen subsets.

Proof. (i). First observe that

$$\beta G = \overline{G} = \overline{\bigcup_{\mathscr{C}\in P} G[X,\mathscr{C}]} = \bigcup_{\mathscr{C}\in P} \overline{G[X,\mathscr{C}]}.$$

At the same time, however, since every $G[X,\mathscr{C}]$ is a subgraph, and hence a closed subset of G, for all $\mathscr{C} \neq \mathscr{C}' \in P$ it follows from Theorem 4.1 (v) that

$$\overline{G[X,\mathscr{C}]} \cap \overline{G[X,\mathscr{C}']} = \overline{G[X,\mathscr{C}] \cap G[X,\mathscr{C}']} = \overline{G[X]} = G[X] \subseteq G$$

where the last equality follows from the fact that compact subsets of Hausdorff spaces are closed. Hence, we see that G^* is a disjoint union of finitely many closed sets $G^* = \bigsqcup_{\mathscr{C}\in P} \big(\overline{G[X,\mathscr{C}]} \cap G^*\big)$.

(ii) follows from (i) with Lemma 5.7. □

Notation. We write $\approx_*$ and $\approx_\times$ for the equivalence relations on G^* and $G^\times$ whose classes are precisely the connected components of G^* and $G^\times$ respectively. If C is a component of $G^\times$ we write $\hat{C}$ for the unique component of G^* including it.

Our next lemma, the so-called *Separating Lemma*, can be considered as our main technical result of this paper, yielding that distinct components of G^* can be distinguished by a finite order separation of the graph G, see Corollaries 5.13 and 5.15 below. However, we state the lemma in a slightly more general form, so that we can also apply it in Section 6 when proving Theorem 2. For this, we shall need the following notion of "tame":

Definition 5.9. We call a subset $A \subseteq \beta G$ *tame* if it is $\approx_\times$-closed and for every component C of G^* meeting A in a point of $O_{\beta G}[V]$ (cf. Def. 5.4) we have $C \subseteq A$.

Example 5.10. All $\approx_*$-closed subsets of βG and all $\approx_\times$-closed subsets of $G^\times$ are tame, but both $G^* \setminus G^\times$ and $O_{\beta G}[V]$ are not tame as soon as G is not locally finite.

Lemma 5.11 (Šura-Bura Lemma [6, Theorem 6.1.23])**.** *If C_1 and C_2 are distinct components of a compact Hausdorff space X, there is a clopen bipartition $A \oplus B$ of X with $C_1 \subseteq A$ and $C_2 \subseteq B$.*

Lemma 5.12 (Separating Lemma)**.** *Let $A, B \subseteq \beta G$ be two disjoint closed and tame subsets. Then there is a finite $X \subseteq V(G)$ and a bipartition $\{\mathscr{C}_1, \mathscr{C}_2\}$ of $\mathscr{C}_X$ with $A \subseteq \overline{G[X,\mathscr{C}_1]}^{\beta G}$ and $B \subseteq \overline{G[X,\mathscr{C}_2]}^{\beta G}$.*

Proof. Given A and B we use normality of $G^\times$ and a compactness argument to deduce from Lemma 5.11 that there is a clopen bipartition $K_A \oplus K_B$ of $G^\times$ with $A \cap G^\times \subseteq K_A$ and $B \cap G^\times \subseteq K_B$. Put $A' = A \cup K_A$ and $B' = B \cup K_B$ so A' and B' are closed and disjoint subsets of βG. Using that βG is normal we find disjoint open sets $O_A, O_B \subseteq \beta G$ with $A' \subseteq O_A$ and $B' \subseteq O_B$. Next, since

$$\bigcap_{v\in V} (\beta G \setminus O_{\beta G}(v)) = G^\times = K_A \oplus K_B \subseteq O_A \sqcup O_B$$

is an intersection of closed sets which is contained in the open set $O_A \sqcup O_B$, it follows from compactness that there are finitely many vertices $v_1, \dots, v_n$ such that

$$\bigcap_{i=1}^{n} (\beta G \setminus O_{\beta G}(v_i)) \subseteq O_A \sqcup O_B.$$

Put $\Xi = \{v_1, \dots, v_n\}$. Then $O_A \sqcup O_B$ induces a clopen bipartition $K'_A \oplus K'_B$ of the closed subspace $\beta G \setminus O_{\beta G}[\Xi]$ of βG which in turn induces a bipartition $Q = \{\mathcal{A}, \mathcal{B}\}$ of $\mathscr{C}_\Xi$ via

$$\mathcal{A} = \{\, C \in \mathscr{C}_\Xi \mid C \subseteq K'_A \,\} \quad \text{and} \quad \mathcal{B} = \{\, C \in \mathscr{C}_\Xi \mid C \subseteq K'_B \,\}.$$

In particular, we have

$$\overline{\bigcup \mathcal{A}}^{\beta G} \subseteq K'_A \quad \text{and} \quad \overline{\bigcup \mathcal{B}}^{\beta G} \subseteq K'_B. \tag{1}$$

Moreover, by Lemma 5.8, $Q_\times$ must be the clopen bipartition $K_A \oplus K_B$ of $G^\times$.

Now we want that

$$A \subseteq \overline{G[\Xi, \mathcal{A}]}^{\beta G} \quad \text{and} \quad B \subseteq \overline{G[\Xi, \mathcal{B}]}^{\beta G}, \tag{2}$$

but with the help of Lemma 5.7 and (1) we only get

$$\begin{aligned} A &\subseteq \beta G \setminus \overline{\bigcup \mathcal{B}}^{\beta G} = \overline{G[\Xi, \mathcal{A}]}^{\beta G} \cup O_{\beta G}[\Xi] \\ \text{and} \quad B &\subseteq \beta G \setminus \overline{\bigcup \mathcal{A}}^{\beta G} = \overline{G[\Xi, \mathcal{B}]}^{\beta G} \cup O_{\beta G}[\Xi] \end{aligned}$$

with A and B possibly meeting $O_{\beta G}[\Xi]$. To resolve this issue, we will find a way to widen Ξ by adding only finitely many vertices, and adjusting $\mathcal{A}$ and $\mathcal{B}$ accordingly so as to make (2) true.

For this, we note first that

$$A \cap G^* \subseteq \overline{G[\Xi, \mathcal{A}]}^{\beta G}. \tag{3}$$

Indeed, we know that A is tame, that each component of G^* meets $V^* \subseteq G^\times$ (see Lemmas 5.2 and 5.6), and that

$$A \cap G^\times \subseteq K_A = \overline{\bigcup \mathcal{A}}^{\beta G} \cap G^\times \subseteq \overline{G[\Xi, \mathcal{A}]}^{\beta G}$$

where $\overline{G[\Xi, \mathcal{A}]}^{\beta G} \cap G^*$ is clopen (Lemma 5.8); combining these facts yields (3).

Second, we show that there exists a finite set F_A of edges of G with

$$A \cap G \subseteq G[\Xi, \mathcal{A}] \cup \bigcup F_A. \tag{4}$$

Indeed, by $A \subseteq \overline{G[\Xi, \mathcal{A}]}^{\beta G} \cup O_{\beta G}[\Xi]$, it suffices to show that

$$F_A := \{\, e \in E(X, \bigcup \mathcal{B}) \mid \mathring{e} \text{ meets } A \,\}$$

is finite. Suppose for a contradiction that F_A is infinite, and for every edge $e \in F_A$ pick some $i_e \in \mathring{e} \cap A$. Then $\overline{\{\, i_e \mid e \in F_A \,\}}^{\beta G} \subseteq A$ meets $G^* \cap \overline{G[X, \mathcal{B}]}^{\beta G} \cap O_{\beta G}[\Xi]$ in some component C of G^*. But we noted earlier that each component of G^* meets $V^* \subseteq G^\times$, so the tame set A meeting C means that $\emptyset \neq C \cap G^\times \subseteq A \cap K_B$, a contradiction. Of course, corresponding versions of (3) and (4) hold for B.

Finally, we use (3) and (4) to yield a true version of (2). For this, we let X be the finite vertex set obtained from Ξ by adding the endvertices of the edges in $F_A \cup F_B$, and we put $\mathscr{C}_1 = \mathfrak{c}^{-1}_{X,\Xi}(\mathcal{A})$ and $\mathscr{C}_2 = \mathfrak{c}^{-1}_{X,\Xi}(\mathcal{B})$. Due to $G[X, \mathscr{C}_1] \supseteq G[\Xi, \mathcal{A}] \cup \bigcup F_A$ and $G[X, \mathscr{C}_2] \supseteq G[\Xi, \mathcal{B}] \cup \bigcup F_B$, we may use (3) and (4) to deduce that

$$A \subseteq \overline{G[X, \mathscr{C}_1]}^{\beta G} \quad \text{and} \quad B \subseteq \overline{G[X, \mathscr{C}_2]}^{\beta G}. \qquad \square$$

Using Lemma 5.7 we obtain the following corollary:

Corollary 5.13. *For every pair of distinct components C_1, C_2 of $G^\times$ there is a finite $X \subseteq V(G)$ and a bipartition $P = \{\mathscr{C}_1, \mathscr{C}_2\}$ of $\mathscr{C}_X$ such that the components $\hat{C}_1 \supseteq C_1$ and $\hat{C}_2 \supseteq C_2$ of G^* are separated by the clopen bipartition P_* of G^*.* □

Lemma 5.14. *The map $C \mapsto \hat{C}$ defines a bijection between $G^\times/{\approx_\times}$ and $G^*/{\approx_*}$.*

Proof. Each component of G^* meets $V^* \subseteq G^\times$ (see Lemmas 5.2 and 5.6), so the map $C \mapsto \hat{C}$ is onto. It is injective by Corollary 5.13. □

Corollary 5.13 and Lemma 5.14 yield another important result:

Corollary 5.15. *For every pair of distinct components C_1, C_2 of G^* there is a finite $X \subseteq V(G)$ and a bipartition P of $\mathscr{C}_X$ such that the clopen bipartition P_* of G^* separates C_1 and C_2.* □

Corollary 5.16. *The quotients $G^\times/{\approx_\times}$ and $G^*/{\approx_*}$ are Hausdorff.* □

Theorem 5.17. *For any graph G, we have $G^\times/{\approx_\times} \cong G^*/{\approx_*}$.*

Proof. Let $\hat{\iota}\colon G^\times/{\approx_\times} \to G^*/{\approx_*}$ map C to $\hat{C}$. By Lemma 5.14 this is a bijection. Denote the quotient map $G^* \to G^*/{\approx_*}$ by q_*. Clearly, the diagram

$$\begin{array}{ccc} G^\times & \xhookrightarrow{\ \iota\ } & G^* \\ \big\downarrow & & \big\downarrow q_* \\ G^\times/{\approx_\times} & \xrightarrow{\ \hat{\iota}\ } & G^*/{\approx_*} \end{array}$$

commutes. Since $G^\times/{\approx_\times}$ is compact and $G^*/{\approx_*}$ is Hausdorff (Corollary 5.16), to show that $\hat{\iota}$ is a homeomorphism it suffices to verify continuity. But note that by the quotient topology, $\hat{\iota}$ is continuous if and only if $q_* \circ \iota$ is continuous. □

5.4. **Comparing $\mathscr{P}$ with $G^\times$.** Now that we are able to distinguish distinct components of the remainder by some $\gamma \in \Gamma$, the next step is to use this to show $\Theta \cong G^*/{\approx_*}$. Technically, we will achieve this by showing $\mathscr{P} \cong G^\times/{\approx_\times}$ instead.

For every $\gamma \in \Gamma$ let $\sigma_\gamma\colon G^\times \to \mathscr{P}_\gamma$ map every point $x \in G^\times$ to the $\mathscr{C} \in \mathscr{P}_\gamma$ whose induced clopen partition class $\overline{\bigcup \mathscr{C}}^{\beta G} \cap G^\times \in P(\gamma)_\times$ containing x, i.e. including the connected component of $G^\times$ containing x.

Lemma 5.18. *The maps σ_γ are continuous surjections.*

Proof. To see that σ_γ is continuous, observe that

$$\sigma_\gamma^{-1}(\mathscr{C}) = \overline{\bigcup \mathscr{C}}^{\beta G} \cap G^\times \in P(\gamma)_\times$$

and recall that partition classes of $P(\gamma)_\times$ are clopen in $G^\times$.

The map σ_γ is surjective: since every $\mathscr{C} \in \mathscr{P}_\gamma$ is such that $V[\mathscr{C}]$ is infinite, Lemma 5.6 ensures that $\overline{\bigcup \mathscr{C}}^{\beta G} \cap G^\times$ is non-empty. □

Lemma 5.19. *The maps σ_γ are compatible.*

Proof. For this assertion it suffices to show that whenever $(X, P) \le (X', P')$, then we have $P_\times \preceq P'_\times$, i.e. the finite clopen partition $P'_\times$ refines that partition of $G^\times$ induced by $P_\times$. To see this, consider any $\mathscr{C}' \in P'$. Since P' refines $P \downharpoonright X'$, there is a unique $\mathscr{C} \in P$ with $\mathscr{C}' \upharpoonright X \subseteq \mathscr{C}$. Thus $\overline{\bigcup \mathscr{C}'}^{\beta G} \cap G^\times \subseteq \overline{\bigcup \mathscr{C}}^{\beta G} \cap G^\times$ follows. □

We put $\sigma = \varprojlim \sigma_\gamma \colon G^\times \to \mathscr{P}$, and we aim to show that σ gives rise to a homeomorphism between $G^\times/{\approx_\times}$ and $\mathscr{P}$.

Lemma 5.20. *The map $\sigma \colon G^\times \to \mathscr{P}$ is a continuous surjection.*

Proof. We combine Lemmas 5.18 and 5.19 with the fact that compatible continuous surjections from a compact space onto Hausdorff spaces combine into one continuous surjection onto the inverse limit of their image spaces (cf. [6, Corollary 3.2.16]). □

Lemma 5.21. *The fibres of σ are precisely the connected components of $G^\times$.*

Proof. First, it is clear by the definition of the σ_γ that every σ_γ is constant on connected components of $G^\times$. Conversely, we need to argue that for any pair of distinct components C_1 and C_2 of $G^\times$ there is some σ_γ with $\sigma_\gamma \restriction C_1 \neq \sigma_\gamma \restriction C_1$. Such a σ_γ is provided by Corollary 5.13. □

Proposition 5.22. *$G^\times/{\approx_\times} \cong \mathscr{P}$.*

Proof. It is well-known that every continuous surjection $f \colon X \twoheadrightarrow Y$ from a compact space X onto a Hausdorff space Y gives rise to a homeomorphism between the quotient $X/\{\, f^{-1}(y) \mid y \in Y \,\}$ over the fibres of f, and the space Y. Thus, it follows from Lemmas 5.20 and 5.21, that

$$G^\times/{\approx_\times} = G^\times/\{\, \sigma^{-1}(\xi) \mid \xi \in \mathscr{P} \,\} \cong \mathscr{P}. \qquad \square$$

We now have all ingredients to prove our first main theorem:

Proof of Theorem 1. Theorem 5.17, Proposition 5.22 and Theorem 3.3 yield

$$G^*/{\approx_*} \cong G^\times/{\approx_\times} \cong \mathscr{P} \cong \Theta. \qquad \square$$

We write τ_* for the component of G^* corresponding to τ and $\tau_\times$ for the component $\tau_* \cap G^\times$ of $G^\times$ corresponding to τ (cf. Theorem 1 and Lemma 5.14).

Theorem 5.23. *If τ is an $\aleph_0$-tangle of G, then*

(i) $\tau_* = G^* \cap \bigcap_{\gamma\in\Gamma} \overline{G[\tau,\gamma]}^{\beta G}$ *and*

(ii) $\tau_\times = \bigcap_{\gamma\in\Gamma} \overline{\bigcup \mathscr{C}(\tau,\gamma)}^{\beta G} = G^\times \cap \bigcap_{\gamma\in\Gamma} \overline{G[\tau,\gamma]}^{\beta G} = \tau_* \cap G^\times$

are the components of G^* and $G^\times$ corresponding to τ respectively.

In statement (i) of the theorem, the intersection with G^* is really necessary—we will see the reason for this in Proposition 7.3.

Proof of Theorem 5.23. We show (ii) first. The first equality is evident from the definition of σ, and the centre equality follows from Lemma 5.7 with

$$G^\times = \bigcap_{\gamma\in\Gamma} \big(\beta G \setminus O_{\beta G}[X(\gamma)]\big).$$

(i). By Corollary 5.15, the right-hand side contains at most one connected component of G^*. We have $\tau_\times \subseteq \overline{G[\tau,\gamma]}$ for all $\gamma \in \Gamma$ by (ii), so $\tau_* = \hat{\tau}_\times \subseteq \overline{G[\tau,\gamma]}$ holds for all γ as well (see Lemma 5.8), finishing the proof. □

6. Obtaining the tangle compactification from the Stone-Čech compactification

Now that we know $\Theta \cong G^*/{\approx_*}$, our next target is the proof of our second main result, Theorem 2. For this, recall that $\mathcal{O}_{\beta G}(X,\mathscr{C}) = \overline{G[X,\mathscr{C}]}^{\beta G} \setminus G[X]$, and that Lemma 5.8 and Theorem 5.23 ensure that $\mathcal{O}_{\beta G}(X,\mathscr{C})$ is $\approx_*$-closed and includes precisely the components τ_* of G^* with $\mathscr{C} \in U(\tau, X)$.

Lemma 6.1. *Let $A \subseteq \beta G$ be closed and $\approx_*$-closed, and let τ be an $\aleph_0$-tangle of G. If A avoids τ_*, then there are $X \in \mathcal{X}$ and $\mathscr{C} \subseteq \mathscr{C}_X$ with $\tau_* \subseteq \mathcal{O}_{\beta G}(X,\mathscr{C}) \subseteq \beta G \setminus A$.*

Proof. By the Separating Lemma 5.12 there is $X \in \mathcal{X}$ and a bipartition $\{\mathscr{C}_1, \mathscr{C}_2\}$ of $\mathscr{C}_X$ with $A \subseteq \overline{G[X,\mathscr{C}_1]}$ and $\overline{\tau_*} \subseteq \overline{G[X,\mathscr{C}_2]}$. Then $\tau_* \subseteq \mathcal{O}_{\beta G}(X,\mathscr{C}_2) \subseteq \beta G \setminus A$. □

We write $\widehat{\beta G}$ for the topological space obtained from βG by declaring G to be open, and we write $\hat{G}$ for the quotient $\widehat{\beta G}/{\approx_*}$. Since βG contains G as a subspace, all the open sets of G are open in $\widehat{\beta G}$ as well; and since $\approx_*$ does not affect G, all the open sets of G are also open in $\hat{G}$. As a consequence, the open sets of βG plus the open sets of G form a basis for the topology of $\widehat{\beta G}$, yielding that

Lemma 6.2. *The open sets of $(\beta G)/{\approx_*}$ plus the open sets of G form a basis for the topology of $\hat{G}$.* □

We define a bijection $\Psi\colon \hat{G} \to |G|_\Theta$ by letting it be the identity on G and letting it send each $\approx_*$-class τ_* to its corresponding $\aleph_0$-tangle τ.

Lemma 6.3. *The map Ψ is continuous.*

Proof. Since the open sets of G are open in both $|G|_\Theta$ and $\hat{G}$, it suffices to show that the preimage of any $\mathcal{O}_{|G|_\Theta}(X,\mathscr{C})$ is open in $\hat{G}$, and it is:

$$\Psi^{-1}\big(\mathcal{O}_{|G|_\Theta}(X,\mathscr{C})\big) = \mathcal{O}_{\beta G}(X,\mathscr{C})/{\approx_*}.$$ □

Lemma 6.4. *The map Ψ is closed.*

Proof. Let A be any closed subset of $\hat{G}$; we show that $\Psi[A]$ is closed in $|G|_\Theta$. For this, let ξ be any point of $|G|_\Theta \setminus \Psi[A]$, and let $\mathcal{B}$ be the basis for the topology of $\hat{G}$ provided by Lemma 6.2.

If ξ is a point of G, then we find an open neighbourhood O of ξ in G avoiding A since A is closed in $\hat{G}$. Then O witnesses $\xi \notin \overline{\Psi[A]}$ as well.

Otherwise ξ is an $\aleph_0$-tangle $\tau \in \Theta \setminus \Psi[A]$. The set A is closed in $\hat{G}$, but it need not be closed in $(\beta G)/{\approx_*}$. Let us consider the closure B of A in $(\beta G)/{\approx_*}$ and show $B \setminus A \subseteq G$ (actually, one can even show that B adds only some vertices of infinite degree to A, but $B \setminus A \subseteq G$ suffices for our cause). Each point of $\hat{G} \setminus G$ that is not contained in A has an open neighbourhood from the basis $\mathcal{B}$ avoiding A. Since all these neighbourhoods are not included in G, they must be open sets of $(\beta G)/{\approx_*}$, yielding $B \setminus A \subseteq G$. Therefore, the closed set $B' = \bigcup B$ of βG avoids the component τ_* of G^* corresponding to τ, and since B' is also $\approx_*$-closed our Lemma 6.1 yields $X \in \mathcal{X}$ and $\mathscr{C} \subseteq \mathscr{C}_X$ such that $\tau_* \subseteq \mathcal{O}_{\beta G}(X,\mathscr{C}) \subseteq \beta G \setminus B'$. Therefore, the open neighbourhood $\mathcal{O}_{|G|_\Theta}(X,\mathscr{C})$ of τ avoids $\Psi[A]$. □

Proof of Theorem 2. Lemma 6.3 and Lemma 6.4 yield a homeomorphism. □

7. Three observations about the Stone-Čech compactification

Given $\mathbb{M}_E$ and an ultrafilter $U \in \beta E$ we write P_U for the collection of all points of $\mathbb{I}_U$ that are of the form x_U for some family $(\, x_e \mid e \in E\,)$ of points $x_e \in \mathbb{I}_e$. By [9, Proposition 2.6], the set $P_U \setminus \{0_U, 1_U\}$ is dense in $\mathbb{I}_U$.

Theorem 7.1. *If G is an infinite graph that is not locally finite, then no compactification of G can both be Hausdorff and have a totally disconnected remainder.*

Proof. Suppose for a contradiction that αG is a Hausdorff compactification of G with totally disconnected remainder, and let v be a vertex of G of infinite degree. Consider a representation $\mathbb{M}_E/{\sim}$ of G, so Theorem 4.8 yields $\beta G = (\beta\mathbb{M}_E)/{\sim_\beta}$ and we find a free ultrafilter $U \in E^*$ with $[0_U]_{\sim_\beta} = v$, say. The set $P_U \setminus \{0_U, 1_U\}$ is dense in $\mathbb{I}_U$, so every open neighbourhood of v in βG meets $P_U \setminus \{0_U, 1_U\}$. In order to use this to derive a contradiction, we need to know more about αG first.

The Hausdorff compactification αG can be obtained from βG as a quotient $\beta G/{\approx}$ where $\approx$ is an equivalence relation on G^*. Since αG has a totally disconnected remainder and since the (continuous) restriction of the quotient map to components of G^* preserves connectedness, we deduce that the equivalence relation $\approx$ must refine $\approx_*$. Consequently, the connected subspace $\check{\mathbb{I}}_U$ of G^* (cf. Corollary 4.6) is included in a single $\approx$-class x, say. To yield a contradiction, it suffices to show that every open neighbourhood O of v in αG contains x. And indeed: if we view αG as the quotient $(\beta G)/{\approx}$ of βG, then $\bigcup O$ is open in βG and $\approx$-closed. Using that $\bigcup O$ meets $P_U \setminus \{0_U, 1_U\}$ and $\approx$ refines $\approx_*$ we deduce that $x \subseteq \bigcup O$, i.e. $x \in O$. $\square$

For our the second observation we need a short lemma and some notation: Since G is dense in βG, so is the locally compact subspace formed by the inner edge points and the vertices of finite degree, and hence [6, Theorem 3.3.9] yields:

Lemma 7.2. *If G is a graph, then $\mathring{E} \subseteq G$ is open in βG.* $\square$

Given an end ω of G write $\Delta(\omega)$ for the set of those vertices dominating it.

Proposition 7.3. *Let G be any graph, and let $\mathbb{M}_E/{\sim}$ be a representation of G.*

(i) *If τ is an ultrafilter tangle of G, then $\overline{\tau_*}^{\beta G} = \tau_* \sqcup X_\tau$, and for each $x \in X_\tau$ there is an ultrafilter $U \in E^*$ with $[0_U]_{\sim_\beta} = x$, say, and with $\check{\mathbb{I}}_U \subseteq \tau_*$.*

(ii) *If ω is an end of G, then $\overline{\omega_*}^{\beta G} = \omega_* \sqcup \Delta(\omega)$, and for each $x \in \Delta(\omega)$ there is an ultrafilter $U \in E^*$ with $[0_U]_{\sim_\beta} = x$, say, and with $\check{\mathbb{I}}_U \subseteq \omega_*$.*

Proof. (i). First, we show that $\overline{\tau_*}^{\beta G}$ avoids $G \setminus X_\tau$ (where G is the 1-complex). Since $\mathring{E}$ is open in βG (Lemma 7.2) we may assume that $\overline{\tau_*}^{\beta G} \cap G \subseteq V$. Let v be any vertex of G that is not in X_τ, and let C be the (graph) component of $G - X_\tau$ with $v \in C$. Then $v \notin \overline{G[X_\tau, \mathscr{C}_{X_\tau} \setminus \{C\}]}$ implies $v \notin \overline{\tau_*}^{\beta G}$ by Theorem 5.23 as desired. Therefore, $\overline{\tau_*}^{\beta G} \cap G \subseteq X_\tau$, and X_τ is non-empty since G is connected.

Now let x be any vertex in X_τ. Write Γ_x for the set of all $\gamma \in \Gamma$ with $x \in X(\gamma)$, and given $\gamma \in \Gamma_x$ put $F_\gamma = E(x, \bigcup \mathscr{C}(\tau, \gamma))$. The sets F_γ are infinite due to [10, Lemma 4.4]. We consider the filter on $E(x)$ that is given by the up-closure of the collection $\{\, F_\gamma \mid \gamma \in \Gamma_x \,\} \subseteq 2^{E(x)}$ (from the directedness of Γ_x it follows that this collection is directed by reverse inclusion, which is enough to ensure that we get a filter). Next, we extend this filter to an ultrafilter U on $E(G)$, and note that U must be free. Due to $E(x) \in U$ we may assume without loss of generality that there is some $F \in U$ with $F \subseteq E(x)$ and $\{\, 0_e \mid e \in F \,\} \subseteq x$ where we view x as a

$\sim$-class of $\mathbb{M}_E$. Then $0_U \in \overline{\{\, 0_e \mid e \in F \,\}}^{\beta\mathbb{M}_E}$ implies $[0_U]_{\sim_\beta} = x$ as a consequence of $\beta G = (\beta\mathbb{M}_E)/{\sim_\beta}$, Theorem 4.8. If we can show that $\mathbb{I}_U$ is included in $\overline{G[\tau,\gamma]}$ for all $\gamma \in \Gamma_x$, then we are done since $P_U \setminus \{0_U, 1_U\}$ is dense in $\mathbb{I}_U$. For this, let any $\gamma \in \Gamma_x$ be given. Note that

$$(\tfrac{1}{2})_U \in \overline{\{\, (\tfrac{1}{2})_e \mid e \in F_\gamma \,\}}^{\beta\mathbb{M}_E}$$

holds, so $(\frac{1}{2})_U \in \overline{G[\tau,\gamma]}^{\beta G}$ follows. Since $\check{\mathbb{I}}_U \subseteq G^*$ is connected (cf. Corollary 4.6), this yields $\check{\mathbb{I}}_U \subseteq \overline{G[\tau,\gamma]}^{\beta G}$ as a consequence of Lemma 5.8, as desired.

(ii). This is proved similar to (i), where to show $\overline{\omega_*}^{\beta G} \cap G \subseteq \Delta(\omega)$ we use that for every vertex v of G not dominating ω there is $X \in \mathcal{X}$ separating v from $C(X,\omega)$ in that $v \notin X \cup C(X,\omega)$ so in particular $v \notin \overline{G[X,\{C(X,\omega)\}]}^{\beta G} \supseteq \overline{\omega_*}^{\beta G}$. □

For the study of locally finite connected graphs, the so-called *Jumping Arc Lemma* (cf. [1, Lemma 8.5.3]) plays an important role. By considering subcontinua of the Stone-Čech compactification instead of arcs in the Freudenthal compactification, we obtain the following quite strong generalisation of this lemma:

Lemma 7.4 (Jumping 'Arc' Lemma for the Stone-Čech compactification)**.** *Let $F \subseteq E$ be a cut of G with sides V_1, V_2.*

- (i) *If F is finite, then $\overline{G[V_1]} \oplus \overline{G[V_2]}$ is a clopen bipartition of $(\beta G) \setminus \mathring{F}$, and there is no subcontinuum of $(\beta G) \setminus \mathring{F}$ meeting both V_1 and V_2.*
- (ii) *If F is infinite, then $(\beta G) \setminus \mathring{F}$ might contain a subcontinuum meeting both V_1 and V_2. This is the case, e.g., if both $G[V_1]$ and $G[V_2]$ are connected.*

Moreover, two vertices of G lie in the same component (subcontinuum) of $(\beta G) \setminus \mathring{E}$ if and only if they lie on the same side of every finite cut of the graph G.

Proof. (i) is immediate from Theorem 4.1 (v). For (ii), note that if both $G[V_1]$ and $G[V_2]$ are connected, then $(\beta G) \setminus \mathring{F}' = \mathrm{cl}_{\beta G}\,(\, G \setminus \mathring{F}'\,)$ is a continuum for every finite $F' \subseteq F$ by Lemmas 4.2, 7.2 and Theorem 4.1 (v), so $(\beta G) \setminus \mathring{F}$ is also a continuum as directed intersection of the continua $(\beta G) \setminus \mathring{F}'$, see Lemma 4.3.

Finally, note that, by (i), for the 'moreover' part it suffices to show the backward direction. For this, find infinitely many edge-disjoint paths $P_0, P_1, \ldots$ between the two vertices inductively, and note that by Lemmas 4.2, 4.3 and 7.2 the intersection

$$\bigcap_{n\in\mathbb{N}} \overline{\bigcup_{m>n} P_m}^{\beta G} \subseteq (\beta G) \setminus \mathring{E}$$

is a continuum containing the two vertices as desired. □

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