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GENERALIZED LOGARITHMIC GAUSS CURVATURE FLOW OF THE LEAVES OF A FOLIATION

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Abstract. In our paper we study a generalized logarithmic Gauss curvature flow of the leaves of a foliation of $\mathbb{R}^{n+1} \setminus \{0\}$ consisting of uniformly convex hypersurfaces. We show that there is exactly one leaf in this foliation so that its flow converges to a translating solution of the flow equation, the flows of the leaves in its open convex body shrink to a point and the flows of the leaves outside its closed convex body converge to expanding spheres.

1. Introduction and main result

Let f be a positive, smooth function on S^n and let $(M_\Theta)_{\Theta>0}$ be a foliation of $\mathbb{R}^{n+1} \setminus \{0\}$ by embedded, closed, uniformly convex (i.e. the Gauss curvature is positive) hypersurfaces M_Θ where we assume that Θ can be viewed as a smooth function with non-vanishing gradient. W.l.o.g. we assume that the monotone ordering of the associated open convex bodies C_Θ of the M_Θ with respect to inclusion is increasing, cf. Remark 3.1. We study the motion of the initial hypersurface M_Θ with normal speed given by $\log(\frac{F}{f})$ where F is a curvature function with inverse $\tilde{F}$ satisfying Assumption 1.3. We show that there exists $\Theta^* > 0$ such that if $\Theta < \Theta^*$ the flow hypersurfaces shrink to a point in finite time, if $\Theta > \Theta^*$ they expand to an asymptotic sphere, and if $\Theta = \Theta^*$ they converge to a translating solution to the flow equation.

The above scenario in the special case $F = K$, K the Gauss curvature, and $(M_\Theta)_{\Theta>0}$ being a family of homothetic transformations of an embedded, closed, uniformly convex hypersurface M_0 in $\mathbb{R}^{n+1}$, i.e. $M_\Theta = \Theta M_0$, corresponds to the logarithmic Gauss curvature flow approach to the Minkowski problem of Chou and Wang [5]. They even obtain for $\Theta = \Theta^*$ convergence to a translating hypersurface with Gauss curvature (when considered as a function of the normal) given by an expression depending explicitly on f.

We introduce the setting of our paper more detailed. Let $(X_\Theta)_{\Theta>0}$ be a family of embeddings $X_\Theta : S^n \to \mathbb{R}^{n+1}$ of M_Θ. Let F be a curvature function with inverse $\tilde{F}$ satisfying Assumption 1.3. We consider the evolution of convex hypersurfaces $M(t)$, parametrized by $X(\cdot, t)$, so that

$$\frac{\partial X}{\partial t} = -\log \frac{F}{f} \nu \tag{1.1}$$

with

$$X(p, 0) = X_\Theta(p). \tag{1.2}$$

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Here, $\nu(p,t)$ denotes the unit outer normal of $M(t)$ at $X(p,t)$, $f = f(\nu(p,t))$ is considered as a function of the normal, $F = F(\kappa_i)$ is a curvature function as above and κ_i are the principal curvatures of $M(t)$.

We obtain the following main results, compare with [5].

Theorem 1.1. *Let $(M_\Theta)_{\Theta>0}$ be as above. There exists $\Theta^* > 0$ and $\xi \in \mathbb{R}^n$ so that we have for the flow (1.1), (1.2) with initial hypersurface X_{Θ^*} that*

$$X(\cdot,t) - \xi t \to X^* \tag{1.3}$$

in $C^m(S^n)$, $m \in \mathbb{N}$, for $t \to \infty$ where X^* is the embedding of a smooth, uniformly convex hypersurface, i.e. $X(\cdot,t)$ converges to a translating solution of the flow equation (1.1).

Theorem 1.2. *Let Θ^* be as in Theorem 1.1. If $\Theta \in (0,\Theta^*)$ then the solution of (1.1), (1.2) shrinks to a point in finite time. If $\Theta \in (\Theta^*,\infty)$ then the solution expands to infinity as t goes to infinity. In the latter case, the hypersurface $X(\cdot,t)/r(t)$ where $r(t)$ is the inner radius of $X(\cdot,t)$ converges to a unit sphere uniformly.*

In the following we recall some facts about curvature functions from [7]. Let $\Gamma \subset \mathbb{R}^n$ denote a symmetric cone, (Ω,ξ^i) a coordinate chart in $\mathbb{R}^n$, (g_{ij}) a fixed positive definite $T^{0,2}(\Omega)$-tensor with inverse (g^{ij}) and $S = \operatorname{Sym}(n)$ the subset of symmetric tensors in $T^{0,2}(\Omega)$. Let S_Γ be the set of the tensors (h_{ij}) in S with eigenvalues with respect to (g_{ij}), i.e. eigenvalues of the $T^{1,1}(\Omega)$-tensor $(g^{ik}h_{kj})$, lying in Γ. In this setting we always consider a symmetric function F defined in Γ also as a function $F(\kappa_i) \equiv F(h_{ij},g_{ij}) \equiv F(\frac{1}{2}(h_{ij}+h_{ji}),g_{ij})$ where the last expression is defined for general $(h_{ij}) \in T^{0,2}(\Omega)$. Using these interpretations we denote partial derivatives by

$$F_i = \frac{\partial F}{\partial \kappa_i}, \quad F_{ij} = \frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j} \tag{1.4}$$

and

$$F^{ij} = \frac{\partial}{\partial h_{ij}} F(\frac{1}{2}(h_{ij}+h_{ji}),g_{ij}), \quad F^{ij,kl} = \frac{\partial^2}{\partial h_{ij}\partial h_{kl}} F(\frac{1}{2}(h_{ij}+h_{ji}),g_{ij}). \tag{1.5}$$

For a symmetric function F in $\Gamma_+ = \{\kappa \in \mathbb{R}^n : \kappa_i > 0\}$ we define its inverse $\tilde{F}$ by

$$\tilde{F}(\kappa_i^{-1}) = \frac{1}{F(\kappa_i)}, \quad (\kappa_i) \in \Gamma_+. \tag{1.6}$$

In the following assumption we list the properties which we need for the inverse $\tilde{F}$ of our curvature function F, especially $\tilde{F}$ of class (K^*), cf. [7, Definition 2.2.15], is feasible.

Assumption 1.3. Throughout the paper we assume that $\tilde{F}$ is a symmetric and positively homogeneous of degree d_0 function $\tilde{F} \in C^\infty(\Gamma_+) \cap C^0(\bar{\Gamma}_+)$ with

$$\tilde{F}_{|\partial\Gamma_+} = 0, \tag{1.7}$$

$$\tilde{F}_i = \frac{\partial \tilde{F}}{\partial \kappa_i} > 0 \quad \text{in } \Gamma_+ \tag{1.8}$$

and

$$\epsilon_0 \tilde{F} \operatorname{tr}(h_{ij}) \le \tilde{F}^{ij} h_{ik} h_j^k \quad \forall (h_{ij}) \in S_{\Gamma_+} \tag{1.9}$$

where $\epsilon_0 = \epsilon_0(\tilde{F}) > 0$ and where we raise and lower indices with respect to (g_{ij}).

Furthermore, we assume that (i) or (ii) hold where

(i) means that $\tilde{F}$ is concave and $d_0 = 1$ and

(ii) means that

$$\tilde{F}^{ij,kl}\eta_{ij}\eta_{kl} \le \tilde{F}^{-1}\left(\tilde{F}^{ij}\eta_{ij}\right)^2 - \tilde{F}^{ik}\tilde{h}^{jl}\eta_{ij}\eta_{kl} \quad \forall \eta \in S \tag{1.10}$$

where $(\tilde{h}^{ij})$ is the inverse of (h_{ij}).

Assumption 1.3 is independent from the chosen tensor (g_{ij}) but expressions like $\tilde{F}(h_{ij})$ depend on (g_{ij}) where the latter will always refer to the corresponding induced metric and will be suppressed in the notation.

We mention some related literature. The flow (1.1) in the case $F = K$ in [5] is a gradient flow of a certain functional and is used in [5] for a variational proof of the Minkowski problem in the smooth category which is the problem of finding a smooth convex hypersurface with Gauss curvature (when considered as a function of the normal) equal to a prescribed positive, smooth function f on S^n. The Minkowski problem in the smooth category has been solved in [9, 12, 13, 2, 14] and previously in [10, 11] in the case where one wants to find a (not necessarily smooth) convex hypersurface with area measure equal to a certain prescribed Borel measure on S^n. While in [5] the prescribed curvature is given as a function of the normal a similar problem with prescribed curvature function defined in $\mathbb{R}^{n+1}$ and a similar flow are considered in [4]. Concerning boundary problems in the non-parametric case for similar flows and equations we refer to [17, 15, 16, 18] and the references therein. In doing so we especially refer to [16] and [15] for the case of more general curvature functions. The latter studies the second boundary value problem for a generalized non-parametric Gauss curvature flow. The class of feasible curvature functions therein is similar to the one we use in our paper and similar to our paper the obtained translating speed in the limit is not given by an explicit expression of the initial data.

In the remaining part of the paper we prove Theorem 1.1 and Theorem 1.2 following the argumentation in [5] and adapting it where necessary. This uses an explicit expression (Monge-Ampère equation) for the Gauss curvature of a hypersurface in terms of the second derivatives of the restriction u of the homogeneous degree one extension of the support function of the hypersurface to a tangent plane, cf. [5, Equ. (1.2)] and the end of page 738 therein for such a representation of the Gauss curvature. To handle the fact that an explicit expression in terms of the second derivatives of u does not seem to be available for the curvature F we use instead the well-known representation (2.8) of the principal radii of a hypersurface as zeros of a determinant of a certain matrix in $\text{Sym}(n+1)$ and that we can write these zeros in special cases as eigenvalues of appropriate matrices in $\text{Sym}(n)$, see the proof of Lemma 2.4. In Section 2 we prove a priori estimates, especially the crucial estimates for the principal curvatures. In Section 3 we use the a priori estimates from Section 2 to prove our main results.

We thank Oliver Schnürer for telling us this interesting problem and the suggestion to apply his method from [15] to a derivative of the support function, which we use in (ii) of the proof of Theorem 1.1 in Section 3, to deduce convergence to a translating solution when corresponding a priori bounds are available.

2. A priori estimates

We recall some facts about the support function of a closed and convex hypersurface M in $\mathbb{R}^{n+1}$ from [5], see also [14] and [2]. The support function H of M is defined on S^n by

$$H(x) = \sup_{y \in M} x \cdot y \tag{2.1}$$

where the dot denotes the inner product in $\mathbb{R}^{n+1}$. We extend H to a homogeneous function of degree one in $\mathbb{R}^{n+1}$. So H is convex and we have

$$\sup_{S^n} |\nabla H| \le \sup_{S^n} |H| \tag{2.2}$$

since it is the supremum of linear functions. If M is strictly convex, i.e. for each x in S^n there is a unique point $p = p(x)$ on M whose unit outer normal is x, H is differentiable at x and

$$p_\alpha = \frac{\partial H}{\partial x_\alpha}, \quad \alpha = 1, ..., n+1. \tag{2.3}$$

Furthermore, given an orthonormal frame fields $e_1, ..., e_n$ on S^n and denoting covariant differentiation with respect to e_i by ∇_i the eigenvalues of $(\nabla_i \nabla_j H + H\delta_{ij})_{i,j=1,\dots,n}$, are the principal radii of curvature at $p(x)$. When H is viewed as a homogeneous function over $\mathbb{R}^{n+1}$, the principal radii of curvature of M are also equal to the non-zero eigenvalues of the Hessian

$$\left(\frac{\partial^2 H}{\partial x_\alpha \partial x_\beta}\right)_{\alpha,\beta=1,\dots,n+1} \tag{2.4}$$

on S^n.

We begin with a reformulation of Equation (1.1) locally in Euclidean space, cf. Equation (2.14). Let $H(\cdot, t) : S^n \to \mathbb{R}$ be the support function of $M(t)$ where we denote its homogeneous degree one extension to $\mathbb{R}^{n+1}$ again by $H(\cdot, t)$ and let $p(\cdot) = p(\cdot, t)$ denote the inverse of the Gauss map $M(t) \to S^n$.

Using

$$\frac{H}{\partial t}(x,t) = x \cdot \frac{\partial X}{\partial t}(p(x), t), \quad x \in S^n, \tag{2.5}$$

we rewrite problem (1.1) as the following initial value problem for H

$$\begin{aligned} \frac{\partial H}{\partial t} &= \log \frac{f}{F} = \log \tilde{F} f \\ H(x,0) &= H_\Theta(x) \end{aligned} \tag{2.6}$$

where H_Θ is the support function for M_Θ and $\tilde{F}$ a function of the principal radii $r_i = \kappa_i^{-1}$ defined by

$$F = F(\kappa_i) = \tilde{F}(\kappa_i^{-1})^{-1} = \tilde{F}(r_i)^{-1}. \tag{2.7}$$

We set $u(y,t) = H(y, -1, t)$, $y \in \mathbb{R}^n$. Then $u(\cdot, t)$ is convex and the principal radii r_i of $X(\cdot, t)$ in $p(x,t)$, $x \in S^n$, are given as nonzero zeros of the equation

$$\det B = 0 \tag{2.8}$$

where $B = (B_{\alpha\beta})_{\alpha,\beta=0,\dots,n}$ with

$$(B_{\alpha\beta}) = \begin{pmatrix} -\frac{\lambda^2}{r} & y_1 & \dots & y_n \\ y_1 & \lambda u_{11} - r & \dots & \lambda u_{1n} \\ \dots & & & \\ y_n & \lambda u_{n1} & \dots & \lambda u_{nn} - r \end{pmatrix}, \tag{2.9}$$

$\lambda = (1 + y_1^2 + \dots + y_n^2)^{\frac{1}{2}}$ and x and y are related by

$$x = (y, -1)/\sqrt{1 + |y|^2}, \tag{2.10}$$

cf. [14, page 16], and note that we have rewritten the equation therein slightly. Furthermore, we have

$$\frac{\partial u}{\partial t}(y,t) = \sqrt{1+|y|^2}\frac{\partial H}{\partial t}(x,t). \tag{2.11}$$

Extending f to be a homogeneous function of degree 0 in $\mathbb{R}^{n+1}$ we obtain the local representation of (1.1) in terms of u

$$\frac{\partial u}{\partial t} = \sqrt{1+|y|^2}\log \tilde{F} + l(y), \quad y \in \mathbb{R}^n, \tag{2.12}$$

where

$$l(y) = \sqrt{1+|y|^2}\log f(y,-1) \tag{2.13}$$

and $\tilde{F}$ is evaluated at the zeros r_i of Equation (2.8). For technical reasons we rewrite this equation slightly by using the homogeneity of $\tilde{F}$

$$\frac{\partial u}{\partial t} = \sqrt{1+|y|^2}\log \tilde{F}(\lambda^{-3} r_i) + g(y), \quad y \in \mathbb{R}^n, \tag{2.14}$$

where

$$g(y) = l(y) + 3d_0 \lambda \log \lambda. \tag{2.15}$$

From the maximum principle one gets an analogous comparison principle as [5, Lemma 2.1] which implies uniqueness of a solution of (2.6).

Lemma 2.1. *For $i = 1, 2$ let f_i be two positive C^2-functions on S^n and H_i $C^{2,1}$-solutions of*

$$\frac{\partial H_i}{\partial t} = \log \tilde{F} f_i. \tag{2.16}$$

If $H_1(x,0) \le H_2(x,0)$ and $f_1(x) \le f_2(x)$ on S^n then $H_1 \le H_2$ for all $t > 0$ and $H_1 < H_2$ unless $H_1 \equiv H_2$.

In the following we will always assume that $H \in C^\infty(S^n \times [0,T])$ is a solution of (2.6). We denote the outer and inner radii of the hypersurface $X(\cdot,t)$ determined by $H(\cdot,t)$ by $R(t)$ and $r(t)$, respectively, and set

$$R_0 = \sup\{R(t) : t \in [0,T]\} \tag{2.17}$$

and

$$r_0 = \inf\{r(t) : t \in [0,T]\}. \tag{2.18}$$

The goal of the present section is to estimate the principal radii of curvatures of $X(\cdot,t)$ from below and above in terms of r_0, R_0 and initial data.

We state two lemmas needed in the following.

Lemma 2.2. *Let r and R be the inner and outer radii of a uniformly convex hypersurface X respectively. Then there exists a dimensional constant C such that*

$$\frac{R^2}{r} \leq C \sup\{R(x,\xi) : x, \xi \in S^n\}, \tag{2.19}$$

where $R(x,\xi)$ is the principal radius of curvature of X at the point with normal x and along the direction ξ.

Proof. See [5, Lemma 2.2]. □

Lemma 2.3. *Let $a(t), b(t) \in C^1([0,T])$ and $a(t) < b(t)$ for all t. Then there exists $h(t) \in C^{0,1}([0,T])$ such that*
i) $a(t) - 2M \leq h(t) \leq b(t) + 2M$,
ii) $\sup\{\frac{|h(t_1)-h(t_2)|}{|t_1-t_2|} : t_1, t_2 \in [0,T]\} \leq 2\max\{\sup_t b'(t), \sup_t(-a'(t))\}$,
where $M = \sup_t(b(t) - a(t))$.

Proof. See [5, Lemma 2.3] □

In the following lemma we prove an upper bound for the principal radii of curvature.

Lemma 2.4. *For any $\gamma \in (1,2]$ there exists a constant c_γ which may depend on initial data such that*

$$\sup\{H_{\xi\xi}(x,t) : (x,t) \in S^n \times [0,T], \xi \in T_xS^n, |\xi| = 1\} \leq c_\gamma(1 + D^\gamma), \tag{2.20}$$

where $D = \sup\{d(t) : t \in [0,T]\}$ and $d(t)$ is the diameter of $X(\cdot,t)$.

Proof. We adapt the proof of [5, Lemma 2.4]. Applying Lemma 2.3 to the functions $-H(-e_i,t)$ and $H(e_i,t)$ where $\pm e_i$ are the intersection points of S^n with the x_i-axis, $i = 1, ..., n+1$, we obtain $p_i(t)$ so that

$$-H(-e_i,t) - 2D \leq p_i(t) \leq H(e_i,t) + 2D \tag{2.21}$$

and

$$\begin{aligned} &\sup\left\{\frac{|p_i(t_1) - p_i(t_2)|}{|t_1 - t_2|} : t_1, t_2 \in [0,T]\right\} \\ &\leq 2\sup\{H_t(x,t) : (x,t) \in S^n \times [0,T]\}. \end{aligned} \tag{2.22}$$

Henceforth

$$\left| H(x,t) - \sum_{i=1}^{n+1} p_i(t)x_i \right| \leq cD \quad \text{for } (x,t) \in S^n \times [0,T], \tag{2.23}$$

and by (1.1)

$$\sum_{i=1}^{n+1} |H_i(x,t) - p_i|^2 \leq cD^2. \tag{2.24}$$

Let

$$\Phi(x,t) = H_{\xi\xi}(x,t) + \left[1 + \sum_{i=1}^{n+1} |H_i(x,t) - p_i(t)|^2\right]^{\frac{\gamma}{2}} \tag{2.25}$$

where $\gamma \in (1,2]$. Suppose that the supremum

$$\sup\{\Phi(x,t) : (x,t) \in S^n \times [0,T], \xi \text{ tangential to } S^n, |\xi| = 1\} \tag{2.26}$$

is attained at the south pole $x = (0, ...0, -1)$ at $t = \bar{t} > 0$ and in the direction $\xi = e_i$. For any x on the south hemisphere, let

$$\xi(x) = \left(\sqrt{1 - x_1^2}, -\frac{x_1 x_2}{\sqrt{1 - x_1^2}}, ..., -\frac{x_1 x_{n+1}}{1 - x_1^2}\right). \tag{2.27}$$

Let u be the restriction of H on $x_{n+1} = -1$. Using the homogeneity of H we obtain, after a direct computation,

$$\begin{aligned} &\sum_{i=1}^{n+1} (H_i - p_i)^2(x, t) \\ &\qquad = \sum_{i=1}^{n} (u_i(y, t) - p_i(t))^2 + \left| u(y, t) + p_{n+1} - \sum_{i=1}^{n} y_i u_i(y, t) \right|^2 \end{aligned} \tag{2.28}$$

and

$$H_{\xi\xi}(x, t) = u_{11}(y, t) \frac{(1 + y_1^2 + ... + y_n^2)^{\frac{3}{2}}}{1 + y_2^2 + ... + y_n^2}, \tag{2.29}$$

where $y = -(x_1, ..., x_n)/x_{n+1}$ in $\mathbb{R}^n$. Thus the function

$$\begin{aligned} \varphi(y, t) =& u_{11} \frac{(1 + y_1^2 + ... + y_n^2)^{\frac{3}{2}}}{1 + y_2^2 + ... + y_n^2} \\ &+ \left[1 + \sum (u_i - p_i)^2 + |u + p_{n+1} - \sum y_i u_i|^2\right]^{\frac{\gamma}{2}} \end{aligned} \tag{2.30}$$

attains its maximum at $(y, t) = (0, \bar{t})$. Without loss of generality we may further assume that the Hessian of u at $(0, \bar{t})$ is diagonal. Hence at $(0, \bar{t})$ we have for each k,

$$\begin{aligned} 0 \leq \varphi_t =& u_{11t} + \gamma[(u_i - p_i)(u_{it} - p_{i;t}) + (u + p_{n+1})(u_t + p_{n+1;t})] Q^{\frac{\gamma-2}{2}}, \\ 0 =& \varphi_k = u_{11k} + \gamma(u_i - p_i) u_{ik} Q^{\frac{\gamma-2}{2}} \end{aligned} \tag{2.31}$$

and

$$\begin{aligned} 0 \geq& \varphi_{kk} = u_{kk11} + \tau_k u_{11} + \gamma[u_{kk}^2 + (u_i - p_i) u_{ikk} - (u + p_{n+1}) u_{kk}] Q^{\frac{\gamma-2}{2}} \\ &+ \gamma(\gamma - 2)(u_i - p_i)^2 u_{ik}^2 Q^{\frac{\gamma-4}{2}}, \end{aligned} \tag{2.32}$$

where $Q = 1 + \sum (u_i - p_i)^2 + (u + p_{n+1})^2$, $\tau_k = 1$ if $k > 1$, $\tau_1 = 3$ and $p_{i;t} = \frac{dp_i}{dt}$.

On the other hand, we are going to differentiate equation (2.14). In $(0, \bar{t})$ we have $y = 0$ and the Hessian (u_{ij}) is diagonal, hence B is diagonal.

Let us fix $y_i = 0$, $i = 2, ..., n$, and vary y_1 for a moment. In this case we rewrite Equation (2.8) by using the matrices $B^1 = (B_{ij})_{i,j=1,...,n}$ and $B^2 = (B_{ij})_{i,j=2,...,n}$

as follows. We have for $r \neq 0$ that

$$
\begin{aligned}
&\det B = 0 \\
&\Leftrightarrow \det \begin{pmatrix} -\frac{\lambda^2}{r} & y_1 & 0 & ... & 0 \\ y_1 & \lambda u_{11} - r & \lambda u_{12} & ... & \lambda u_{1n} \\ 0 & \lambda u_{21} & \lambda u_{22} - r & ... & \lambda u_{2n} \\ ... \\ 0 & \lambda u_{n1} & \lambda u_{n2} & ... & \lambda u_{nn} - r \end{pmatrix} = 0 \\
&\Leftrightarrow -\frac{\lambda^2}{r} \det B^1 - y_1^2 \det B^2 = 0 \\
&\Leftrightarrow \det B^1 + \frac{y_1^2 r}{\lambda^2} \det B^2 = 0 \qquad (2.33)\\
&\Leftrightarrow \det \begin{pmatrix} \lambda u_{11} - r\left(1 - \frac{y_1^2}{\lambda^2}\right) & \lambda u_{12} & ... & \lambda u_{1n} \\ \lambda u_{21} & \lambda u_{22} - r & ... & \lambda u_{2n} \\ ... \\ \lambda u_{n1} & \lambda u_{n2} & ... & \lambda u_{nn} - r \end{pmatrix} = 0 \\
&\Leftrightarrow \det \begin{pmatrix} \lambda^3 u_{11} - r & \lambda^2 u_{12} & ... & \lambda^2 u_{1n} \\ \lambda^2 u_{21} & \lambda u_{22} - r & ... & \lambda u_{2n} \\ ... \\ \lambda^2 u_{n1} & \lambda u_{n2} & ... & \lambda u_{nn} - r \end{pmatrix} = 0.
\end{aligned}
$$

Setting

$$
(a_{ij}) = \begin{pmatrix} \lambda u_{11} & ... & \lambda u_{1n} \\ ... & & \\ \lambda u_{n1} & ... & \lambda u_{nn} \end{pmatrix}, \qquad (a^1_{ij}) = \begin{pmatrix} \lambda^3 u_{11} & \lambda^2 u_{12} & ... & \lambda^2 u_{1n} \\ \lambda^2 u_{21} & \lambda u_{22} & ... & \lambda u_{2n} \\ ... \\ \lambda^2 u_{n1} & \lambda u_{n2} & ... & \lambda u_{nn} \end{pmatrix} \tag{2.34}
$$

the zeros of Equation (2.8) can be written as eigenvalues of the matrix (a^1_{ij}). Analogously, defining for $r = 1, ..., n$ the matrix (a^r_{ij}) as the matrix which is obtained by multiplying row r and column r in (a_{ij}) with λ we can write the zeros of Equation (2.8) as eigenvalues of the matrix (a^r_{ij}) in the case where we vary y_r, r fixed, and fix $y_i = 0$ for $i \neq r$.

Hence we may write $\tilde{F}$ in (2.14) as

$$
\tilde{F} = \tilde{F}(\lambda^{-3} r_i) = \tilde{F}(\tilde{a}^r_{ij}) \tag{2.35}
$$

where

$$
(\tilde{a}^r_{ij}) = \lambda^{-3}(a^r_{ij}) \tag{2.36}
$$

if $(y, t) = (0, ..., 0, y_r, 0, ..., 0, t)$. And we have in $(0, \bar{t})$ that

$$
\frac{\partial \tilde{F}}{\partial y_k} = \tilde{F}^{ii} a^k_{ii;k} \quad \wedge \quad \frac{\partial^2 \tilde{F}}{\partial {y_k}^2} = \tilde{F}^{ii} a^k_{ii;kk} + \tilde{F}^{ij,rs} a^k_{ij;k} a^k_{rs;k} \tag{2.37}
$$

where we do not sum over k and where we used [7, Lemma 2.1.9] to deduce that $\tilde{F}^{ij}$ is diagonal. Here and in the following we sometimes denote partial derivatives by indices separated by a semicolon for greater clarity of the presentation

Differentiating (2.14) gives in $(0,\bar t)$ that

$$\begin{aligned} u_{kt} =& \left(1+|y|^2\right)^{-\frac12} y_k \log\tilde F + \sqrt{1+|y|^2}\frac{1}{\tilde F}\tilde F^{ij}\tilde a^k_{ij;k} + g_k \\ u_{kkt} =& \log\tilde F - \frac{1}{\tilde F^2}\tilde F^{ij}\tilde a^k_{ij;k}\tilde F^{rs}\tilde a^k_{rs;k} + \frac{1}{\tilde F}\tilde F^{ij,rs}\tilde a^k_{ij;k}\tilde a^k_{rs;k} \\ &+ \frac{1}{\tilde F}\tilde F^{ij}\tilde a^k_{ij;kk} + g_{kk}, \end{aligned} \tag{2.38}$$

here, we do not sum over k. Hence at $(0,\bar t)$ we have

$$\begin{aligned} 0 \ge& \sum_k \frac{1}{\tilde F}\tilde F^{kl}\varphi_{kl} - \varphi_t \\ =& \sum_k \frac{1}{\tilde F}\tilde F^{kk}\varphi_{kk} - \varphi_t \\ =& \frac{1}{\tilde F}\sum_k \tilde F^{kk}u_{kk11} + \frac{1}{\tilde F}\tilde F^{kk}u_{11}\tau_k \\ &+ \gamma\{\frac{1}{\tilde F}\tilde F^{kk}u_{kk}^2[1+\frac{(\gamma-2)(u_k-p_k)^2}{1+\sum(u_i-p_i)^2+(u+p_{n+1})^2}] \\ &+ (u_i-p_i)(\frac{1}{\tilde F}\tilde F^{rs}u_{irs}-u_{it}) - \frac{1}{\tilde F}\tilde F^{kk}u_{kk}(u+p_{n+1}) \\ &- (u+p_{n+1})(u_t+p_{n+1;t}) + (u_i-p_i)p_{i;t}\}Q^{\frac{\gamma-2}{2}} - u_{11t} \\ \ge& \frac{1}{\tilde F}\tilde F^{kk}u_{11} - \log\tilde F + \frac{1}{\tilde F^2}\tilde F^{ij}\tilde a^1_{ij;1}\tilde F^{rs}\tilde a^1_{rs;1} - \frac{1}{\tilde F}\tilde F^{ij,rs}\tilde a^1_{ij;1}\tilde a^1_{rs;1} \\ &- \frac{1}{\tilde F}\tilde F^{kk}\tilde a^1_{kk;11} - g_{11} + \frac{1}{\tilde F}\tilde F^{kk}u_{kk11} \\ &+ \gamma\{(\gamma-1)\frac{1}{\tilde F}\tilde F^{kk}u_{kk}^2 - (u_i-p_i)g_i \\ &+ \frac{1}{\tilde F}\tilde F^{rs}(u_{rsi}-\tilde a^i_{rs;i})(u_i-p_i) - \frac{1}{\tilde F}\tilde F^{kk}u_{kk}(u+p_{n+1}) \\ &- (u+p_{n+1})(u_t+p_{n+1,t}) + (u_i-p_i)p_{i;t}\}Q^{\frac{\gamma-2}{2}} \\ \ge& \frac{1}{\tilde F}\tilde F^{kk}u_{11} - \log\tilde F \\ &+ \frac{1}{\tilde F}\tilde F^{kk}(u_{kk11}-\tilde a^1_{kk;11}) - g_{11} \\ &+ \gamma\{(\gamma-1)\epsilon_0\tilde H - (u_i-p_i)g_i + \frac{1}{\tilde F}\tilde F^{rs}(u_{rsi}-\tilde a^i_{rs;i})(u_i-p_i) \\ &- d_0(u+p_{n+1}) - (u+p_{n+1})(u_t+p_{n+1,t}) + (u_i-p_i)p_{i,t}\}Q^{\frac{\gamma-2}{2}} \end{aligned} \tag{2.39}$$

where we used for the last inequality (1.9) and (1.10) or the concavity of $\tilde F$, cf. Assumption 1.3, and denoted the trace of (u_{ij}) by $\tilde H$. From

$$u_{ij} = \tilde a^r_{ij} \quad\wedge\quad u_{ij;k} = \tilde a^r_{ij;k} \quad\wedge\quad \tilde a^r_{rr;11} = u^r_{rr;11} \tag{2.40}$$

and

$$\tilde a^1_{ii;11} = u_{ii;11} - 2u_{ii} \tag{2.41}$$

for $i \neq 1$ in $(0, \bar{t})$ we conclude that

$$\begin{aligned} 0 \geq & \frac{1}{\tilde{F}} \tilde{F}^{kk} u_{11} - \log \tilde{F} - g_{11} \\ & + \gamma\{(\gamma-1)\epsilon_0 u_{11} - (u_i - p_i) g_i - d_0(u + p_{n+1}) \\ & - (u + p_{n+1})(u_t + p_{n+1,t}) + (u_i - p_i) p_{i,t}\} Q^{\frac{\gamma-2}{2}}. \end{aligned} \tag{2.42}$$

From (2.23) and (2.24) we deduce that $|u + p_{n+1}| \leq cD$ and $|u_i - p_i| \leq cD$ so that

$$\gamma(\gamma-1)\epsilon_0 c D^{\gamma-2} u_{11} \leq \log \tilde{F} + c + cQ^{\frac{\gamma-2}{2}} D(1 + |u_t| + |H_t|) \tag{2.43}$$

and hence

$$u_{11} \leq cD^{2-\gamma} \log u_{11} + cD^{2-\gamma} + cD(1 + \log u_{11}) \tag{2.44}$$

which implies the claim. □

Corollary 2.5. *For any $\gamma \in (1, 2]$ there exists $\delta = \delta(\gamma) > 0$ such that*

$$r(t) \geq \frac{\delta R(t)^2}{1 + \sup_{\tau \leq t} R^\gamma(\tau)}. \tag{2.45}$$

Proof. Use Lemma 2.2 and Lemma 2.4. □

In the following lemma we estimate H_t from below. In view of Lemma 2.4 and Equation (2.6) this immediately implies a lower bound for the principal radii of curvature.

Lemma 2.6. *There exists a constant c depending only on n, r_0, R_0, f and initial data such that*

$$\inf\{H_t(x, t) : (x, t) \in S^n \times [0, T]\} \geq -c. \tag{2.46}$$

Proof. We adapt the proof of [5, Lemma 2.6]. Let

$$q(t) = \frac{1}{|S^n|} \int_{S^n} x H(x, t) d\sigma(x) \tag{2.47}$$

be the Steiner point of $X(\cdot, t)$. Then there exists a positive δ which depends only on n, r_0 and R_0 so that

$$H(x, t) - q(t) \cdot x \geq 2\delta. \tag{2.48}$$

We assume that the function

$$\psi(x, t) = \frac{H_t(x, t)}{H(x, t) - x \cdot q(t) - \delta} \tag{2.49}$$

attains its negative infimum on $S^n \times [0, T]$ at $x = (0, ..., 0, -1)$ and $\bar{t} \in (0, T]$ and that (u_{ij}) is diagonal. Let u be the restriction of H to $x_{n+1} = -1$ as before. Then

$$\psi(y, t) = \frac{u_t(y, t)}{u(y, t) - q(t) \cdot (y, -1) - \delta\sqrt{1 + |y|^2}} \tag{2.50}$$

attains its negative minimum at $(0, \bar{t})$. Hence in this point we have

$$0 \geq \psi_t = \frac{u_{tt}}{u + q_{n+1}(t) - \delta} - \frac{u_t(u_t + \frac{dq_{n+1}}{dt})}{(u + q_{n+1}(t) - \delta)^2}, \tag{2.51}$$

$$0 = \psi_k = \frac{u_{tk}}{u + q_{n+1}(t) - \delta} - \frac{u_t(u_k - q_k(t))}{(u + q_{n+1}(t) - \delta)^2} \tag{2.52}$$

and

$$0 \le \psi_{kk} = \frac{u_{tkk}}{u + q_{n+1}(t) - \delta} - \frac{u_t u_{kk}}{(u + q_{n+1}(t) - \delta)^2} + \frac{\delta u_t}{(u + q_{n+1}(t) - \delta)^2}. \tag{2.53}$$

Using the notation from the proof of Lemma 2.4 we get on the other hand by differentiating (2.14) that in $(0, \bar{t})$

$$u_{tt} = \frac{1}{\tilde{F}} \tilde{F}^{ij} u_{ijt}. \tag{2.54}$$

We have in $(0, \bar{t})$ using that $(\tilde{F}^{ij})$ is diagonal

$$\begin{aligned} 0 \le& \sum \frac{1}{\tilde{F}} \tilde{F}^{kk} \psi_{kk} - \psi_t \\ \le& \frac{\delta u_t \frac{1}{\tilde{F}} \sum \tilde{F}^{kk} - \frac{1}{\tilde{F}} \tilde{F}^{kk} u_{kk} u_t + u_t (u_t + \frac{dq_{n+1}}{dt})}{(u + q_{n+1} - \delta)^2}. \end{aligned} \tag{2.55}$$

Since u_t is negative at $(0, \bar{t})$, it follows that

$$\begin{aligned} \frac{1}{\tilde{F}} \sum_k \tilde{F}^{kk} \le& \frac{c}{\delta}(1 + |u_t|) \\ \le& \frac{c}{\delta}(1 + \log \tilde{F}^{-1}) \end{aligned} \tag{2.56}$$

where we used the homogeneity of $\tilde{F}$ and where $c = c(f, R_0)$.

Now we distinguish cases. In case (i) of Assumption 1.3 we have

$$\sum_k \tilde{F}^{kk} \ge F(1, 1..., 1) \tag{2.57}$$

in view of [7, Lemma 2.2.19]. It follows that $\tilde{F} \ge c > 0$ and

$$\begin{aligned} u_t \ge& -c + c \log \tilde{F} \\ \ge& -c \end{aligned} \tag{2.58}$$

where c depends on n, r_0, R_0, f and initial data as claimed.

In case (ii) of Assumption 1.3 we choose $i_0 \in \{1, ..., n\}$ such that

$$u_{i_0 i_0} = \min_{1 \le i \le n} u_{ii} \tag{2.59}$$

and hence

$$\tilde{F} = \frac{1}{d_0} \tilde{F}^{ii} u_{ii} \le c \tilde{F}^{i_0 i_0} u_{i_0 i_0} \tag{2.60}$$

in view of the homogeneity of $\tilde{F}$ and [7, Lemma 2.2.4]. Hence we estimate

$$\begin{aligned} \sum_k \tilde{F}^{kk} \ge& \tilde{F}^{i_0 i_0} \\ \ge& \frac{\tilde{F}}{c u_{i_0 i_0}} \end{aligned} \tag{2.61}$$

and deduce from (2.56) that

$$(u_{i_0 i_0})^{-1} \le c(1 + \log((u_{i_0 i_0})^{-1}) \tag{2.62}$$

so that $\tilde{F} \ge c > 0$ and the claim follows as in case (i). □

Using a comparison principle and comparing the flow (2.6) with the ODE

$$\frac{\partial \rho}{\partial t} = \log\left(\frac{\rho^{d_0}}{F(1, ..., 1)}\right) M, \quad \rho(0) = \rho_0, \tag{2.63}$$

where $M = \max\{f(x) : x \in S^n\}$ and ρ_0 sufficiently large, we obtain that $H(x,t)$ is bounded in any finite time interval. Furthermore, its gradient is also bounded by (2.2). From Krylov-Safonov estimates and parabolic regularity theory, cf. [8], one gets that problem (2.6) has for $H_\Theta \in C^{4+\alpha}(S^n)$ a unique $C^{4+\alpha, 2+\frac{\alpha}{2}}$ solution in a maximal interval $[0, T^*)$, $T^* \leq \infty$ and since H_Θ is even of class C^∞ in our case that this solution is also of class C^∞. For the outer radius $R(t)$ of $X(\cdot, t)$ we have

$$\lim_{t \uparrow T^*} R(t) = 0 \tag{2.64}$$

if T^* is finite.

3. Proof of the theorems

We state some elementary properties of the foliation $(M_\Theta)_{\Theta > 0}$ in the following two remarks.

Remark 3.1. For each M_Θ we denote the to M_Θ associated open convex body by C_Θ and have w.l.o.g (otherwise consider $1/\Theta$)

$$\Theta_1 < \Theta_2 \Rightarrow \overline{C_{\Theta_1}} \subset C_{\Theta_2}. \tag{3.1}$$

Furthermore, all C_Θ contain 0, otherwise

$$0 < d := \inf\{\Theta > 0 : \forall_{\tilde{\Theta} \geq \Theta}\ 0 \in C_{\tilde{\Theta}}\} < \infty \tag{3.2}$$

where the last inequality is due to the fact that for $p \in \mathbb{R}^{n+1} \setminus \{0\}$ there is $\Theta(p) > 0$ so that $p, -p \in C_{\Theta(p)}$ and hence also $0 \in C_{\Theta(p)}$. We conclude $0 \in M_d$, a contradiction.

Remark 3.2. For all $r > 0$ exist $\Theta_1, \Theta_2 > 0$ so that

$$M_{\Theta_1} \subset B_r(0) \subset C_{\Theta_2}. \tag{3.3}$$

Proof. Let $r > 0$. Existence of Θ_2 as claimed is clear in view of

$$\overline{B_r(0)} \subset \bigcup_{\Theta > 0} C_\Theta. \tag{3.4}$$

Assume there are sequences $0 < \Theta_k \to 0$, $x_k \in C_{\Theta_k}$, $x_k \notin B_r(0)$. W.l.o.g. assume $x_k \to x \in B_r(0)^c$. Let $p = \frac{x}{2}$. There is $\Theta = \Theta(p) > 0$ so that $p \in M_{\Theta(p)}$. If $[0, x]$ meets $M_{\Theta(p)}$ tangentially in p then $0 \notin C_{\Theta(p)}$ in view of the uniform convexity of $M_{\Theta(p)}$ which is a contradiction. Hence there is a neighborhood U of x so that for every $q \in U$ the segment $[0, q]$ meets $M_{\Theta(p)}$ non-tangentially. This implies

$$U \subset (C_{\Theta(p)})^c \subset (C_{\Theta_k})^c \tag{3.5}$$

for large k. On the other hand

$$x_k \in U \cap C_{\Theta_k} \tag{3.6}$$

for large k, a contradiction. □

Proof of Theorem 1.1. (i) We follow the proof of [5, Theorem A]. Let $m = \inf_{S^n} f$ and $M = \sup_{S^n} f$. If the initial hypersurface X_Θ is a sphere of radius $\rho_0 > \left(\frac{F(1,\ldots,1)}{m}\right)^{\frac{1}{d_0}}$, the solution $X(\cdot,t)$ to the equation

$$\frac{\partial X}{\partial t} = -\log \frac{F}{m}\nu, \quad X(\cdot,0) = X_\Theta, \tag{3.7}$$

remains to be spheres and the flow expands to infinity as $t \to \infty$. On the other hand, if X_Θ is a sphere of radius less than $\left(\frac{F(1,\ldots,1)}{M}\right)^{\frac{1}{d_0}}$, the solution to

$$\frac{\partial X}{\partial t} = -\log \frac{F}{M}\nu, \quad X(\cdot,0) = X_\Theta, \tag{3.8}$$

is a family of spheres which shrinks to a point in finite time. Henceforth by the comparison principle and Remark 3.2 the solution $X(x,t)$ of (1.1) will shrink to a point if Θ is small enough, and will expand to infinity if $\Theta > 0$ is large.

Hence using Corollary 2.5 we obtain that the sets

$$\begin{aligned} A =& \{\Theta > 0 : X(\cdot,t) \text{ shrinks to a point in finite time}\} \\ B =& \{\Theta > 0 : X(\cdot,t) \text{ expands to infinity as } t \to \infty\} \end{aligned} \tag{3.9}$$

are non-empty and open since the solution $X(x,t)$ of (1.1) on a fixed finite time interval $[0,T)$ depends continuously on Θ. We define

$$\Theta_* = \sup A \tag{3.10}$$

and

$$\Theta^* = \inf B. \tag{3.11}$$

and deduce $\Theta_* \leq \Theta^*$ from the comparison principle.

Using Corollary 2.5 we deduce that for any $\Theta \in [\Theta_*, \Theta^*]$ the inner radii of $X(\cdot,t)$ have a uniform positive lower bound and the outer radii are uniformly bounded from above, furthermore, $T^* = \infty$ in view of (2.64). Hence (2.6) is uniformly parabolic and we have uniform bounds for $D_t^k D_x^l X(\cdot,\cdot)$ if $k + l \geq 1$, $k \geq 0$ and $l \geq 0$ on $S^n \times [0,\infty)$.

(ii) Let $\Theta \in [\Theta_*, \Theta^*]$. We shall use a method from [15] to show that our solution that exists for all positive times converges to a translating solution. The main difference from our case to [15] is that we argue on the level of a derivative of the support function while [15] uses a graphical representation of the flow hypersurfaces.

One easily checks that a family of smoothly evolving uniformly convex hypersurfaces represented by its family of support functions $\tilde{H}(\cdot,t)$ is translating iff there is $\xi \in \mathbb{R}^{n+1}$ so that

$$\tilde{H}(x,t) = \tilde{H}(x,0) + t\xi x, \quad x \in \mathbb{R}^{n+1}. \tag{3.12}$$

Let us fix $1 \leq \gamma \leq n+1$ and let e_γ denote the corresponding standard basis vector. Differentiating the homogeneous degree one extension (not relabelled) of (3.12) with respect to x in direction e_γ we get

$$\frac{\partial}{\partial x^\gamma}\tilde{H}(x,t) = \frac{\partial}{\partial x^\gamma}\tilde{H}(x,0) + t\xi_\gamma. \tag{3.13}$$

Hence $\frac{\partial}{\partial x^\gamma}\tilde{H}(\cdot,t)$ is a scalar translating function. Conversely, if (3.13) holds then $\tilde{H}$ satisfies (3.12). Note, that $\tilde{H}(0,t) = 0$ and that $\frac{\partial}{\partial x^\gamma}\tilde{H}(\cdot,t)$ is homogeneous of degree zero.

Let H be a solution of (2.6). We denote the homogeneous degree one extension of H to $\mathbb{R}^{n+1}$ again by H and the homogeneous degree 0 extension of f to $\mathbb{R}^{n+1}\setminus\{0\}$ also by f. We recall the flow equation for H

$$\frac{\partial H}{\partial t} = \log \tilde{F} f \quad \text{in} \quad S^n \times [0,\infty), \tag{3.14}$$

where $\tilde{F} = \tilde{F}(r_i)$ and r_i, $i = 1, ..., n$, are the principal radii of $M(t)$ given as non-zero eigenvalues of the Hessian matrix $\left(\frac{\partial^2 H}{\partial x_\alpha \partial x_\beta}\right)_{\alpha,\beta=1,\ldots,n+1}$. Using the homogeneity of H this can be rewritten as a flow equation for H on $\left(\mathbb{R}^{n+1}\setminus\{0\}\right)\times[0,\infty)$

$$\begin{aligned} \frac{\partial H}{\partial t}(x,t) =& |x| \frac{\partial H}{\partial t}\left(\frac{x}{|x|}, t\right) \\ =& |x| \log \tilde{F} f \end{aligned} \tag{3.15}$$

where $\tilde{F} = \tilde{F}(r_i)$ and r_i, $i = 1, ..., n$, are the principal radii of $M(t)$ given as non-zero eigenvalues of the matrix $\left(|x|\frac{\partial^2 H}{\partial x_\alpha \partial x_\beta}\right)_{\alpha,\beta=1,\ldots,n+1}$ at (x,t) and $f = f(x)$. We will replace (formally) the curvature function $\tilde{F}$ in Equation (3.15) by a curvature function $\hat{F}$ which depends on all eigenvalues r_α, $\alpha = 1, ..., n+1$, of $\left(|x|\frac{\partial^2 H}{\partial x_\alpha \partial x_\beta}\right)_{\alpha,\beta=1,\ldots,n+1}$ at (x,t) and satisfies $\tilde{F}(r_i) = \hat{F}(r_\alpha)$ in order to be notational in the framework of the introduction.

a) In the case that $\tilde{F} \in C^\infty(\bar{\Gamma}_+)$ and $\tilde{F}_{|\partial\Gamma_+} = 0$ we define

$$\hat{F}(r_1, ..., r_{n+1}) = \sum_{\alpha_0=1}^{n+1} \tilde{F}(\hat{r}^{\alpha_0}) \tag{3.16}$$

where $\hat{r}^{\alpha_0} = (r_1, ..., r_{\alpha_0-1}, r_{\alpha_0+1}, ..., r_{n+1})$.

b) Let us consider the general case (which includes case a)). In view of our a priori estimates there are constants $b_1, b_2 > 0$ so that the non-zero eigenvalues of $\left(\frac{\partial^2 H}{\partial x_\alpha \partial x_\beta}\right)_{\alpha,\beta=1,\ldots,n+1}$ on $S^n \times [0,\infty)$ are in the interval $[b_1, b_2]$. Having the later application of the argumentation in [15, Subsection 6.2] in mind we remark that this property carries over to the Hessians of convex combinations of $H(\cdot, t_1)$ and $H(\cdot, t_2)$ with arbitrary $t_1, t_2 > 0$. Note that the vector x is a zero eigenvector of the Hessian of H at every $(x,t) \in S^n \times [0,\infty)$. We define

$$\hat{F}(r_1, ..., r_{n+1}) = \tilde{F}(\hat{r}) + \check{r} \tag{3.17}$$

on the set

$$\Omega = \bigcup_{1\le\alpha\le n+1} I_\alpha \tag{3.18}$$

where

$$I_\alpha = \left(\frac{b_1}{2},\infty\right) \times ... \times \left(\frac{b_1}{2},\infty\right) \times \left(-\frac{b_1}{2},\frac{b_1}{2}\right) \times \left(\frac{b_1}{2},\infty\right) \times ... \times \left(\frac{b_1}{2},\infty\right) \tag{3.19}$$

with factor $(-\frac{b_1}{2}, \frac{b_1}{2})$ at position α and where $\check{r} = r_{\alpha_0} = \min_{\alpha=1,\ldots,n+1} r_\alpha$, $\alpha_0 \in \{1, ..., n+1\}$ suitable, and $\hat{r} = (r_1, ...r_{\alpha_0-1}, r_{\alpha_0+1}, ..., \alpha_{n+1})$. We have $\tilde{F}(r_i) = \hat{F}(r_\alpha)$. From standard arguments we deduce that $\hat{F}$ defines in the way explained in the introduction a differentiable function on the set of symmetric matrices with eigenvalues in Ω.

Differentiating (3.15) we get the following equation for H_γ

$$\frac{\partial}{\partial t} H_\gamma(x,t) = |x|^2 \frac{1}{\hat{F}} \hat{F}^{\alpha\beta} \left(H_\gamma\right)_{\alpha\beta} + |x|_\gamma \log \hat{F} f + |x| \frac{f_\gamma}{f} + d_0 |x|_\gamma \tag{3.20}$$

where $\hat{F}^{\alpha\beta}$ is uniformly elliptic and the coefficients of the elliptic operator on the right-hand side depend on the derivative of H_γ and x and not explicitly on t or H_γ.

Applying the argumentation from [15, Subsection 6.2] more or less word by word to the function H_γ on $(B_{\rho_2}(0) \setminus B_{\rho_1}(0)) \times [0,\infty)$, $0<\rho_1<1<\rho_2$ both close to 1, where we use that H_γ is homogeneous of degree zero (instead of the compactness of the spatial domain and the boundary condition when we apply maximum principles) we obtain that H_γ converges smoothly to a translating solution of (3.20) with a translating speed $\xi = \xi(\Theta,\gamma) \in \mathbb{R}$.

(iii) We show $\Theta_* = \Theta^*$. From (ii) we know that for every $\Theta \in [\Theta_*, \Theta^*]$ the solution $X(x,t)$ of (1.1) with initial value X_Θ converges to a translating solution with a certain translating speed $\xi_\Theta \in \mathbb{R}^{n+1}$.

a) We show that there is $\xi \in \mathbb{R}^{n+1}$ so that $\xi_\Theta = \xi$ for all $\Theta \in [\Theta_*, \Theta^*]$. For this let $\Theta_* \le \Theta_1 < \Theta_2 \le \Theta^*$, differentiating (2.6) in Θ gives

$$\begin{aligned} \frac{\partial H'}{\partial t} =& A^{ij}(\nabla_i\nabla_j H' + H'\delta_{ij}) \\ H'(0) =& \frac{d}{d\Theta} H_\Theta \end{aligned} \tag{3.21}$$

where (A^{ij}) is the inverse of $(\nabla_i\nabla_j H + \delta_{ij} H)$. By the maximum principle

$$H'(x,t) \ge \min_{S^n} \frac{d}{d\Theta} H_\Theta(x). \tag{3.22}$$

Thus

$$\begin{aligned} c(x,t) + t(\xi_{\Theta_2} - \xi_{\Theta_1})x =& H_{\Theta_2}(x,t) - H_{\Theta_1}(x,t) \\ \ge& \int_{\Theta_1}^{\Theta_2} \min_{S^n} \frac{d}{d\Theta} H_\Theta > 0 \end{aligned} \tag{3.23}$$

where $c(x,t)$ is a uniformly bounded function and where we used Lemma 3.3. This implies $\xi_{\Theta_1} = \xi_{\Theta_2}$.

b) Using a) we deduce from the comparison principle that $H_* = H^*$ where H_* and H^* is the solution of $F = e^{\xi x} f$ starting from H_{Θ_*} and H_{Θ^*}, respectively. We deduce from (3.23) with $\Theta_1 = \Theta_*$ and $\Theta_2 = \Theta^*$ by using that $H_{\Theta_2}(\cdot,t) - H_{\Theta_1}(\cdot,t)$ converges uniformly to zero as $t \to \infty$ that $\Theta_* < \Theta^*$ leads to a contradiction, hence $\Theta_* = \Theta^*$.

The proof of Theorem 1.1 is finished. □

Lemma 3.3.

$$\frac{d}{d\Theta} H_\Theta > 0. \tag{3.24}$$

Proof. Let $0 < \Theta_1 < \Theta_2 < \infty$, $x \in S^n$. In view of $D\Theta \ne 0$ there is $c_0 = c_0(\Theta_1) > 0$ so that

$$\operatorname{dist}(M_{\Theta_1}, M_{\Theta_2}) \ge c_0(\Theta_2 - \Theta_1). \tag{3.25}$$

For $x \in S^n$ let $y_x \in M_{\Theta_1}$ be so that

$$H_{\Theta_1}(x) = x y_x > 0 \tag{3.26}$$

and hence also

$$c_1 = \inf_{x\in S^n} x\frac{y_x}{|y_x|} > 0. \tag{3.27}$$

Let y be the intersection of the ray starting in 0 through y_x with M_{Θ_2} then

$$x \cdot y \geq x \cdot y_x + c_0c_1(\Theta_2 - \Theta_1) \tag{3.28}$$

hence

$$\begin{aligned} H_{\Theta_2}(x) \geq& x \cdot y_x + c_0c_1(\Theta_2 - \Theta_1) \\ =& H_{\Theta_1}(x) + c_0c_1(\Theta_2 - \Theta_1) \end{aligned} \tag{3.29}$$

which implies

$$\left(\frac{d}{d\Theta}H_\Theta(x)\right)_{|\Theta=\Theta_1} > 0. \tag{3.30}$$

□

Proof of Theorem 1.2. We show that the normalized hypersurface $X(\cdot,t)/r(t)$ converges to a unit sphere in case $\Theta > \Theta^*$ and follow for it the lines of [5, Theorem B]. Since X is expanding, we may w.l.o.g. assume at $t = 0$ that it contains the ball $B_{R_1}(0)$ where $R_1 > 1 + \left(\frac{F(1,...,1)}{m}\right)^{\frac{1}{d_0}}$, $m = \inf_{S^n} f$, and that it is contained in the ball $B_{R_2}(0)$ where $R_2 > 0$ is sufficiently large. For $i = 1, 2$ let $X_i(\cdot,t)$ be the solution of (1.1) where f is replaced by m and $M = \sup_{S^n} f$ respectively and $X_i(\cdot,0) = \partial B_{R_i}$. The $X_i(\cdot,t)$ are spheres and their radii $R_i(t)$ satisfy

$$c^{-1}(1+t)\log(1+t) \leq R_1(t) \leq R_2(t) \leq c(1 + (1+t)\log^2(1+t)) \tag{3.31}$$

for some $c > 0$. We deduce from the ODEs for the R_i , $i = 1, 2$, that

$$\begin{aligned} \frac{d}{dt}(R_2(t) - R_1(t)) \leq& d_0 \log\frac{R_2(t)}{R_1(t)} + c \\ \leq& c \log\log(1+t) + c \end{aligned} \tag{3.32}$$

where the last inequality uses (3.31) and hence

$$R_2(t) - R_1(t) \leq c(1 + t\log\log(1+t)) \tag{3.33}$$

so that

$$\lim_{t\to\infty} \frac{R_2(t) - R_1(t)}{R_1(t)} = 0. \tag{3.34}$$

By the comparison principle $X(\cdot,t)$ is pinched between $X_2(\cdot,t)$ and $X_1(\cdot,t)$ and, furthermore, we deduce that $X(\cdot,t)/r(t)$ converges to the unit sphere uniformly. □

Combining the proofs of [5, Theorem A] and Theorem 1.1 we get the following Corollary.

Corollary 3.4. *In the situation of Theorem 1.1 with $F = K$ the translating speed ξ is uniquely determined by*

$$\int_{S^n} \frac{x_i}{e^{\xi\cdot x}f(x)} d\sigma(x) = 0, \quad i = 1, ..., n+1. \tag{3.35}$$

And the Gauss curvature of X^*, when regarded as a function of the normal, is equal to $e^{\xi\cdot x}f(x)$.

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