Pseudo-simple heteroclinic cycles in $\mathbb{R}^4$

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Abstract

We study *pseudo-simple* heteroclinic cycles for a Γ-equivariant system in $\mathbb{R}^4$ with finite $\Gamma \subset O(4)$, and their nearby dynamics. In particular, in a first step towards a full classification – analogous to that which exists already for the class of *simple* cycles – we identify all finite subgroups of $O(4)$ admitting pseudo-simple cycles. To this end we introduce a constructive method to build equivariant dynamical systems possessing a robust heteroclinic cycle. Extending a previous study we also investigate the existence of periodic orbits close to a pseudo-simple cycle, which depends on the symmetry groups of equilibria in the cycle. Moreover, we identify subgroups $\Gamma \subset O(4)$, $\Gamma \not\subset SO(4)$, admitting fragmentarily asymptotically stable pseudo-simple heteroclinic cycles. (It has been previously shown that for $\Gamma \subset SO(4)$ pseudo-simple cycles generically are completely unstable.) Finally, we study a generalized heteroclinic cycle, which involves a pseudo-simple cycle as a subset.

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1 Introduction

A heteroclinic cycle is an invariant set of a dynamical system comprised of equilibria $\xi_1, \ldots, \xi_M$ and heteroclinic orbits κ_i from ξ_i to ξ_{i+1}, $i = 1 \ldots M$ with the convention $M + 1 = 1$. For several decades these objects have been of keen interest to the nonlinear science community. A heteroclinic cycle is associated with intermittent dynamics, where the system alternates between states of almost stationary behaviour and phases of quick change. It is well-known that a heteroclinic cycle can exist robustly in equivariant dynamical systems, i.e. persist under generic equivariant perturbations, namely when all heteroclinic orbits are saddle-sink connections in (flow-invariant) fixed-point subspaces. Robust heteroclinic cycles, their nearby dynamics and attraction properties have been thorougly studied, especially in low dimensions. See [4, 8] for a general overview. In $\mathbb{R}^3$, there are comparatively few possibilities for heteroclinic dynamics and these are rather well-understood. In $\mathbb{R}^4$, the situation is significantly more involved. We therefore consider systems

$$\dot{x} = f(x), \tag{1}$$

where $f : \mathbb{R}^4 \to \mathbb{R}^4$ is a smooth map that is equivariant with respect to the action of a finite group $\Gamma \subset O(4)$, i.e.

$$f(\gamma x) = \gamma f(x) \quad \text{for all } x \in \mathbb{R}^4,\ \gamma \in \Gamma. \tag{2}$$

In this setting, much attention has been paid to so-called *simple* cycles, see e.g. [1, 2, 9, 10], for which (i) all connections lie in two-dimensional fixed-point spaces $P_j = \text{Fix}\,(\Sigma_j)$ with $\Sigma_j \subset \Gamma$, and (ii) the cycle intersects each connected component of $P_{j-1} \cap P_j \setminus \{0\}$ at most once. This definition was introduced by [10], who also suggested several examples of subgroups of $O(4)$ that admit such a cycle (in the sense that there is an open set of Γ-equivariant vector fields possessing such an invariant set). The classification of simple cycles was completed in [17, 18] (for homoclinic cycles) and finally in [15] by finding all groups $\Gamma \subset O(4)$ admitting such a cycle. In [15] it was also discovered that the original definition of simple cycles from [10] implicitly assumed a condition on the isotypic decomposition of $\mathbb{R}^4$ with respect to the isotropy subgroup of an equilibrium, see subsection 2.1 for details. This prompted them to define *pseudo-simple* heteroclinic cycles as those satisfying (i) and (ii) above, but not this implicit condition.

It is the primary aim of the present paper to carry out a systematic study of pseudo-simple cycles in $\mathbb{R}^4$, by establishing a complete list of all groups $\Gamma \subset O(4)$ that admit such a cycle. This is done in a similar fashion to the classification of simple cycles in [15], by using a quaternionic approach to describe finite subgroups of $O(4)$. First examples for pseudo-simple cycles were investigated in [15, 16]. The latter of those also addressed stability issues: it was shown that a pseudo-simple cycle with $\Gamma \subset SO(4)$ is generically completely unstable, while for the case $\Gamma \not\subset SO(4)$ a cycle displaying a weak form of stability, called *fragmentary asymptotic stability*, was found. A fragmentarily asymptotically stable (f.a.s.) cycle has a positive measure basin of attraction that does not necessarily include a full neighbourhood of the cycle. We extend this stability study by showing an example of group $\Gamma \not\subset SO(4)$ which admits an asymptotically stable generalized heteroclinic cycle and pseudo-simple subcycles that are f.a.s.. Moreover, we look at the dynamics near a pseudo-simple cycle and discover that asymptotically stable periodic orbits may bifurcate from it. Whether or not this happens depends on the isotropy subgroup $\mathbb{D}_k$, $k \geq 3$ of equilibria comprising the cycle. The case $k = 3$ was already considered in [16]. We illustrate our more general results by numerical simulations for an example with $\Gamma = (\mathbb{D}_4 \,|\, \mathbb{D}_2; \mathbb{D}_4 \,|\, \mathbb{D}_2)$ in the case $k = 4$.

This paper is organized as follows. Section 2 recalls background information on (pseudo-simple) heteroclinic cycles and useful properties of quaternions as a means to describe finite subgroups of $O(4)$. Then, in section 3 we give conditions that allow us to decide whether or not such a group $\Gamma \subset O(4)$ admits pseudo-simple heteroclinic cycles. Section 4 contains the statement and proofs of theorems 1 and 2, which use the previous results to list all subgroups of $O(4)$ admitting pseudo-simple heteroclinic cycles. The proof of theorem 1 relies on properties of finite subgroups of $SO(4)$ that are given in appendices A-C. In section 5 we investigate the existence of asymptotically stable periodic orbits close to a pseudo-simple cycle, depending on the symmetry groups $\mathbb{D}_k$, of equilibria. The cases $k = 3, 4$ and $k \geq 5$ are covered by theorems 3 and 4, respectively. In section 6 we employ the ideas of the previous sections to provide a numerical example of a pseudo-simple cycle with a nearby attracting periodic orbit. Finally, in section 7 for a family of subgroups $\Gamma \not\subset SO(4)$ we construct a generalized heteroclinic cycle (i.e., a cycle with multidimensional connection(s)) and prove conditions for its asymptotic stability in theorem 5. This cycle involves as a subset a pseudo-simple heteroclinic cycle, that can be fragmentarily asymptotically stable. Section 8 concludes and identifies possible continuations of this study. The appendices contain additional information on subgroups of $SO(4)$ that is relevant for the proof of theorem 1.

2 Background

Here we briefly review basic concepts and terminology for pseudo-simple heteroclinic cycles and the quaternionic approach to describing subgroups of $SO(4)$ as needed in this paper.

2.1 Pseudo-simple heteroclinic cycles

In this subsection we give the precise framework in which we investigate robust heteroclinic cycles and the associated dynamics. Given an equivariant system (1) with finite $\Gamma \subset O(4)$ recall that for $x \in \mathbb{R}^4$ the *isotropy subgroup of* x is the subgroup of all elements in Γ that fix x. On the other hand, given a subgroup $\Sigma \subset \Gamma$ we denote by $\mathrm{Fix}\,(\Sigma)$ its *fixed point space*, i.e. the space of points in $\mathbb{R}^4$ that are fixed by all elements of Σ.

Let $\xi_1, \ldots, \xi_M$ be hyperbolic equilibria of a system (1) with stable and unstable manifolds $W^s(\xi_j)$ and $W^u(\xi_j)$, respectively. Also, let $\kappa_j \subset W^u(\xi_j) \cap W^s(\xi_{j+1}) \neq \varnothing$ for $j = 1, \ldots, M$ be connections between them, where we set $\xi_{M+1} = \xi_1$. Then the union of equilibria $\{\xi_1, \ldots, \xi_M\}$ and connecting trajectories $\{\kappa_1, \ldots, \kappa_M\}$ is called a *heteroclinic cycle*. Following [9] we say it is *structurally stable* or *robust* if for all j there are subgroups $\Sigma_j \subset \Gamma$ such that ξ_{j+1} is a sink in $P_j := \mathrm{Fix}\,(\Sigma_j)$ and κ_j is contained in P_j. We also employ the established notation $L_j := P_{j-1} \cap P_j = \mathrm{Fix}\,(\Delta_j)$, with a subgroup $\Delta_j \subset \Gamma$. As usual we divide the eigenvalues of the Jacobian $df(\xi_j)$ into *radial* (eigenspace belonging to L_j), *contracting* (belonging to $P_{j-1} \ominus L_j$), *expanding* (belonging to $P_j \ominus L_j$) and *transverse* (all others), where we write $X \ominus Y$ for a complementary subspace of Y in X. In accordance with [9] our interest lies in cycles where

(H1) $\dim P_j = 2$ for all j,

(H2) the heteroclinic cycle intersects each connected component of $L_j \setminus \{0\}$ at most once.

Then there is one eigenvalue of each type and we denote the corresponding contracting, expanding and transverse eigenspaces of $df(\xi_j)$ by V_j, W_j and T_j, respectively. In [15] it is shown that under these conditions there are three possibilities for the unique Δ_j-isotypic decomposition of $\mathbb{R}^4$:

(1) $\mathbb{R}^4 = L_j \oplus V_j \oplus W_j \oplus T_j$

(2) $\mathbb{R}^4 = L_j \oplus V_j \oplus \widetilde{W}_j$, where $\widetilde{W}_j = W_j \oplus T_j$ is two-dimensional

(3) $\mathbb{R}^4 = L_j \oplus W_j \oplus \widetilde{V}_j$, where $\widetilde{V}_j = V_j \oplus T_j$ is two-dimensional

Here $\oplus$ denotes the orthogonal direct sum. This prompts the following definition.

Definition 1 ([15]) *We call a heteroclinic cycle satisfying conditions (H1) and (H2) above* simple *if case 1 holds true for all j, and* pseudo-simple *otherwise.*

Remark 1 *In case 1 the group Δ_j acts as $\mathbb{Z}_2$ on each one-dimensional component other than L_j and $\Delta_j \cong \mathbb{D}_2$ (which is always the case if $\Gamma \subset SO(4)$) or $\Delta_j \cong (\mathbb{Z}_2)^3$. In cases 2 and 3 the group acts on the two-dimensional isotypic component as a dihedral group $\mathbb{D}_k$ in $\mathbb{R}^2$ for some $k \geq 3$ and $\Delta_j \cong \mathbb{D}_k =< \rho_j, \sigma_j >$ (always for $\Gamma \subset SO(4)$) or $\Delta_j \cong \mathbb{D}_k \times \mathbb{Z}_2$. For $\Gamma \subset SO(4)$ in case 2 the element ρ_j acts as a k-fold rotation on $\widetilde{W}_j$ and trivially on $P_{j-1} = L_j \oplus V_j$, while σ_j acts as $-I$ on $V_j \oplus T_j$ and trivially on $L_j \oplus W_j$. In case 3 the element ρ_j acts as a k-fold rotation on $\widetilde{V}_j$ and trivially on $P_j = L_j \oplus W_j$, while σ_j acts as $-I$ on $W_j \oplus T_j$ and trivially on $L_j \oplus V_j$.*

Remark 2 *The existence of a two-dimensional isotypic component implies that in case 2 the contracting and transverse eigenvalues are equal ($c_j = t_j$) and the associated eigenspace is two-dimensional, while in case 3 the expanding and transverse eigenvalues are equal ($e_j = t_j$) and the associated eigenspace is two-dimensional. Hence, we say that $df(\xi_j)$ has a multiple contracting or expanding eigenvalue in cases 2 or 3, respectively.*

We are interested in identifying all subgroups of $O(4)$ that admit pseudo-simple heteroclinic cycles in the following sense. For simple cycles this task has been achieved step by step in [10, 15, 17, 18].

Definition 2 ([15]) *We say that a subgroup Γ of $O(n)$* admits *(pseudo-)simple heteroclinic cycles if there exists an open subset of the set of smooth Γ-equivariant vector fields in $\mathbb{R}^n$, such that all vector fields in this subset possess a (pseudo-)simple heteroclinic cycle.*

In order to establish the existence of a heteroclinic cycle it is sufficient to find a sequence of connections $\xi_1 \to \ldots \to \xi_{m+1} = \gamma\xi_1$ with some finite order $\gamma \in \Gamma$, that is minimal in the sense that no $i, j \in \{1, \ldots, m\}$ satisfy $\xi_i = \gamma'\xi_j$ for any $\gamma' \in \Gamma$.

Definition 3 ([16]) *Such a sequence $\xi_1 \to \ldots \to \xi_m$ together with the element $\gamma \in \Gamma$ is called a* building block *of the heteroclinic cycle.*

Remark 3 *Heteroclinic cycle in an equivariant system can be decomposed as a union of building blocks. Usually, it is tacitly assumed that all blocks in such a decomposition can be obtained from just one building block by applying the associated symmetry γ. We also make this assumption.*

2.2 Asymptotic stability

Given a heteroclinic cycle X and writing the flow of (1) as $\Phi_t(x)$, the δ-basin of attraction of X is the set

$$\mathcal{B}_\delta(X) = \{x \in \mathbb{R}^4 \ ; \ d(\Phi_t(x), X) < \delta \text{ for all } t > 0 \text{ and } \lim_{t\to+\infty} d(\Phi_t(x), X) = 0\}.$$

Definition 4 *A heteroclinic cycle X is* asymptotically stable *if for any $\delta > 0$ there exists $\varepsilon > 0$ such that $B_\varepsilon \subset \mathcal{B}_\delta(X)$, where $B_\varepsilon(X)$ denotes ε-neighbourhood of X.*

Definition 5 *A heteroclinic cycle X is* completely unstable *if there exists $\delta > 0$ such that $l(\mathcal{B}_\delta(X)) = 0$, where $l(\cdot)$ denotes Lebesgue measure on $\mathbb{R}^4$.*

Definition 6 *A heteroclinic cycle X is* fragmentarily asymptotically stable *if $l(\mathcal{B}_\delta(X)) > 0$ for any $\delta > 0$.*

2.3 Quaternions and subgroups of $SO(4)$

We briefly recall some information on quaternions and their relation to subgroups of $SO(4)$, mainly following the notation and exposition in [6, Chapter 3]. For a more detailed background on this in general, and in the context of heteroclinic cycles, we also refer the reader to [5] and [15], respectively.

A quaternion $\mathbf{q} \in \mathbb{H}$ may be described by four real numbers as $\mathbf{q} = (q_1, q_2, q_3, q_4)$. With the convention $1 = (1, 0, 0, 0)$, $i = (0, 1, 0, 0)$, $j = (0, 0, 1, 0)$ and $k = (0, 0, 0, 1)$ any $\mathbf{q} \in \mathbb{H}$ can be written as $\mathbf{q} = q_1 + q_2 i + q_3 j + q_4 k$. We denote the conjugate of $\mathbf{q}$ as $\tilde{\mathbf{q}} := q_1 - q_2 i - q_3 j - q_4 k$. Multiplication is defined in the standard way through the rules $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$, such that for $\mathbf{p}, \mathbf{q} \in \mathbb{H}$ we have

$$\begin{aligned} \mathbf{pq} = \ & (p_1q_1 - p_2q_2 - p_3q_3 - p_4q_4,\ p_1q_2 + p_2q_1 + p_3q_4 - p_4q_3, \\ & p_1q_3 - p_2q_4 + p_3q_1 + p_4q_2,\ p_1q_4 + p_2q_3 - p_3q_2 + p_4q_1). \end{aligned}$$

By $\mathcal{Q} \subset \mathbb{H}$ we denote the multiplicative group of unit quaternions, with identity element $(1, 0, 0, 0)$. There is a 2-to-1 homomorphism from $\mathcal{Q}$ to $SO(3)$, relating $\mathbf{q} \in \mathcal{Q}$ to the map $\mathbf{v} \mapsto \mathbf{qvq}^{-1}$, which is a rotation in the three-dimensional space of points $\mathbf{v} = (0, v_2, v_3, v_4) \in \mathbb{H}$. Any finite subgroup of $\mathcal{Q}$ then falls into one of the following cases, which are pre-images of the respective subgroups of $SO(3)$ under this homomorphism:

$$\begin{aligned} \mathbb{Z}_n &= \oplus_{r=0}^{n-1}(\cos 2r\pi/n, 0, 0, \sin 2r\pi/n) \\ \mathbb{D}_n &= \mathbb{Z}_{2n} \oplus \oplus_{r=0}^{2n-1}(0, \cos r\pi/n, \sin r\pi/n, 0) \\ \mathbb{V} &= ((\pm 1, 0, 0, 0)) \\ \mathbb{T} &= \mathbb{V} \oplus \left(\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}\right) \\ \mathbb{O} &= \mathbb{T} \oplus \sqrt{\tfrac{1}{2}}((\pm 1, \pm 1, 0, 0)) \\ \mathbb{I} &= \mathbb{T} \oplus \tfrac{1}{2}((\pm\tau, \pm 1, \pm\tau^{-1}, 0)), \end{aligned} \tag{3}$$

where $\tau = (\sqrt{5}+1)/2$. Double parenthesis denote all even permutations of quantities within the parenthesis.

The four numbers (q_1, q_2, q_3, q_4) can be regarded as Euclidean coordinates of a point in $\mathbb{R}^4$. For any pair of unit quaternions $(\mathbf{l}; \mathbf{r})$, the transformation $\mathbf{q} \to \mathbf{l}\mathbf{q}\mathbf{r}^{-1}$ is a rotation in $\mathbb{R}^4$, i.e. an element of the group $SO(4)$. The mapping $\Phi : \mathcal{Q} \times \mathcal{Q} \to SO(4)$ that relates the pair $(\mathbf{l}; \mathbf{r})$ with the rotation $\mathbf{q} \to \mathbf{l}\mathbf{q}\mathbf{r}^{-1}$ is a homomorphism onto, whose kernel consists of two elements, $(1; 1)$ and $(-1; -1)$; thus the homomorphism is two to one. Therefore, a finite subgroup of $SO(4)$ is a subgroup of a product of two finite subgroups of $\mathcal{Q}$. Following [6] we write $(\mathbf{L} \,|\, \mathbf{L}_K; \mathbf{R} \,|\, \mathbf{R}_K)$ for the group Γ. The isomorphism between $\mathbf{L}/\mathbf{L}_K$ and $\mathbf{R}/\mathbf{R}_K$ may not be unique and different isomorphisms give rise to different subgroups of $SO(4)$. The complete list of finite subgroups of $SO(4)$ is given in table 1, where the subscript s distinguishes subgroups obtained by different isomorphisms for $s < r/2$ and prime to r. This is explained in more detail in [6, Chapter 3] and in section 2.2 of [15].

#	group	order		#	group	order		#	group	order
1	$(\mathbb{Z}_{2nr} \,\|\, \mathbb{Z}_{2n}; \mathbb{Z}_{2kr} \,\|\, \mathbb{Z}_{2k})_s$	$2nkr$		15	$(\mathbb{D}_n \,\|\, \mathbb{D}_n; \mathbb{O} \,\|\, \mathbb{O})$	$96n$		29	$(\mathbb{O} \,\|\, \mathbb{O}; \mathbb{I} \,\|\, \mathbb{I})$	2880
2	$(\mathbb{Z}_{2n} \,\|\, \mathbb{Z}_{2n}; \mathbb{D}_k \,\|\, \mathbb{D}_k)_s$	$4nk$		16	$(\mathbb{D}_n \,\|\, \mathbb{Z}_{2n}; \mathbb{O} \,\|\, \mathbb{T})$	$48n$		30	$(\mathbb{I} \,\|\, \mathbb{I}; \mathbb{I} \,\|\, \mathbb{I})$	7200
3	$(\mathbb{Z}_{4n} \,\|\, \mathbb{Z}_{2n}; \mathbb{D}_k \,\|\, \mathbb{Z}_{2k})$	$4nk$		17	$(\mathbb{D}_{2n} \,\|\, \mathbb{D}_n; \mathbb{O} \,\|\, \mathbb{T})$	$96n$		31	$(\mathbb{I} \,\|\, \mathbb{Z}_2; \mathbb{I} \,\|\, \mathbb{Z}_2)$	120
4	$(\mathbb{Z}_{4n} \,\|\, \mathbb{Z}_{2n}; \mathbb{D}_{2k} \,\|\, \mathbb{D}_k)$	$8nk$		18	$(\mathbb{D}_{3n} \,\|\, \mathbb{Z}_{2n}; \mathbb{O} \,\|\, \mathbb{V})$	$48n$		32	$(\mathbb{I}^\dagger \,\|\, \mathbb{Z}_2; \mathbb{I} \,\|\, \mathbb{Z}_2)$	120
5	$(\mathbb{Z}_{2n} \,\|\, \mathbb{Z}_{2n}; \mathbb{T} \,\|\, \mathbb{T})$	$24n$		19	$(\mathbb{D}_n \,\|\, \mathbb{D}_n; \mathbb{I} \,\|\, \mathbb{I})$	$240n$		33	$(\mathbb{Z}_{2nr} \,\|\, \mathbb{Z}_n; \mathbb{Z}_{2kr} \,\|\, \mathbb{Z}_k)_s$ $n \equiv k \equiv 1(\mod 2)$	nkr
6	$(\mathbb{Z}_{6n} \,\|\, \mathbb{Z}_{2n}; \mathbb{T} \,\|\, \mathbb{V})$	$24n$		20	$(\mathbb{T} \,\|\, \mathbb{T}; \mathbb{T} \,\|\, \mathbb{T})$	288				
7	$(\mathbb{Z}_{2n} \,\|\, \mathbb{Z}_{2n}; \mathbb{O} \,\|\, \mathbb{O})$	$48n$		21	$(\mathbb{T} \,\|\, \mathbb{Z}_2; \mathbb{T} \,\|\, \mathbb{Z}_2)$	24		34	$(\mathbb{D}_{nr} \,\|\, \mathbb{Z}_n; \mathbb{D}_{kr} \,\|\, \mathbb{Z}_k)_s$ $n \equiv k \equiv 1 \pmod 2$	2nkr
8	$(\mathbb{Z}_{2n} \,\|\, \mathbb{Z}_{2n}; \mathbb{O} \,\|\, \mathbb{T})$	$48n$		22	$(\mathbb{T} \,\|\, \mathbb{V}; \mathbb{T} \,\|\, \mathbb{V})$	96				
9	$(\mathbb{Z}_{2n} \,\|\, \mathbb{Z}_{2n}; \mathbb{I} \,\|\, \mathbb{I})$	$120n$		23	$(\mathbb{T} \,\|\, \mathbb{T}; \mathbb{O} \,\|\, \mathbb{O})$	576		35	$(\mathbb{T} \,\|\, \mathbb{Z}_1; \mathbb{T} \,\|\, \mathbb{Z}_1)$	12
10	$(\mathbb{D}_n \,\|\, \mathbb{D}_n; \mathbb{D}_k \,\|\, \mathbb{D}_k)$	$8nk$		24	$(\mathbb{T} \,\|\, \mathbb{T}; \mathbb{I} \,\|\, \mathbb{I})$	1440		36	$(\mathbb{O} \,\|\, \mathbb{Z}_1; \mathbb{O} \,\|\, \mathbb{Z}_1)$	24
11	$(\mathbb{D}_{nr} \,\|\, \mathbb{Z}_{2n}; \mathbb{D}_{kr} \,\|\, \mathbb{Z}_{2k})_s$	$4nkr$		25	$(\mathbb{O} \,\|\, \mathbb{O}; \mathbb{O} \,\|\, \mathbb{O})$	1152		37	$(\mathbb{O} \,\|\, \mathbb{Z}_1; \mathbb{O} \,\|\, \mathbb{Z}_1)^\dagger$	24
12	$(\mathbb{D}_{2n} \,\|\, \mathbb{D}_n; \mathbb{D}_{2k} \,\|\, \mathbb{D}_k)$	$16nk$		26	$(\mathbb{O} \,\|\, \mathbb{Z}_2; \mathbb{O} \,\|\, \mathbb{Z}_2)$	48		38	$(\mathbb{I} \,\|\, \mathbb{Z}_1; \mathbb{I} \,\|\, \mathbb{Z}_1)$	60
13	$(\mathbb{D}_{2n} \,\|\, \mathbb{D}_n; \mathbb{D}_k \,\|\, \mathbb{Z}_{2k})$	$8nk$		27	$(\mathbb{O} \,\|\, \mathbb{V}; \mathbb{O} \,\|\, \mathbb{V})$	192		39	$(\mathbb{I}^\dagger \,\|\, \mathbb{Z}_1; \mathbb{I} \,\|\, \mathbb{Z}_1)$	60
14	$(\mathbb{D}_n \,\|\, \mathbb{D}_n; \mathbb{T} \,\|\, \mathbb{T})$	$48n$		28	$(\mathbb{O} \,\|\, \mathbb{T}; \mathbb{O} \,\|\, \mathbb{T})$	576				

Table 1: Finite subgroups of $SO(4)$

The superscript † is employed to denote subgroups of $SO(4)$ where the isomorphism between the quotient groups $\mathbf{L}/\mathbf{L}_K$ and $\mathbf{R}/\mathbf{R}_K \cong \mathbf{L}/\mathbf{L}_K$ is not the identity. The group $\mathbb{I}^\dagger$, isomorphic to $\mathbb{I}$, involves the elements $((\pm\tau^*, \pm 1, \pm(\tau^*)^{-1}, 0))$, where $\tau^* = (-\sqrt{5}+1)/2$. The groups 1-32 contain the central rotation $-I$, and the groups 33-39 do not.

A reflection in $\mathbb{R}^4$ can be expressed in the quaternionic presentation as $\mathbf{q} \to \mathbf{a}\tilde{\mathbf{q}}\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ is a pair of unit quaternions. We write this reflection as $(\mathbf{a}; \mathbf{b})^*$. The transformations $\mathbf{q} \mapsto \mathbf{a}\tilde{\mathbf{q}}\mathbf{a}$ and $\mathbf{q} \mapsto -\mathbf{a}\tilde{\mathbf{q}}\mathbf{a}$ are respectively the axial reflection in the $\mathbf{a}$-axis (leaving unchanged all vectors parallel to the axis $\mathbf{a}$ and reversing all those perpendicular to it) and reflection through the hyperplane orthogonal to the vector $\mathbf{a}$.

2.4 Lemmas

In this subsection we recall lemmas 1-5 from [6, 13, 15] and prove lemmas 6 and 7. They provide basic geometric information that is used to prove theorem 1 in section 4.

Lemma 1 (see proof in [13]) *Let N_1 and N_2 be two planes in $\mathbb{R}^4$ and p_j, $j = 1, 2$, be the elements of $SO(4)$ which act on N_j as identity, and on $N_j^\perp$ as $-I$, and $\Phi^{-1}p_j = (\mathbf{l}_j, \mathbf{r}_j)$, where Φ is the homomorphism defined in the previous subsection. Denote by $(\mathbf{l}_1\mathbf{l}_2)_1$ and $(\mathbf{r}_1\mathbf{r}_2)_1$ the first components of the respective quaternion products. The planes N_1 and N_2 intersect if and only if $(\mathbf{l}_1\mathbf{l}_2)_1 = (\mathbf{r}_1\mathbf{r}_2)_1 = \cos\alpha$ and α is the angle between the planes.*

Lemma 2 (see proof in [15]) *Let P_1 and P_2 be two planes in $\mathbb{R}^n$, $\dim(P_1 \cap P_2) = 1$, $\rho \in O(n)$ is a plane reflection about P_1 and $\sigma \in O(n)$ maps P_1 into P_2. Suppose that ρ and σ are elements of a finite subgroup $\Delta \subset O(n)$. Then $\Delta \supset \mathbb{D}_m$, where $m \geq 3$.*

Lemma 3 (see proof in [15]) *Consider $g \in SO(4)$, $\Phi^{-1}g = ((\cos\alpha, \sin\alpha\mathbf{v}); (\cos\beta, \sin\beta\mathbf{w}))$. Then $\dim \mathrm{Fix} < g >= 2$ if and only if $\cos\alpha = \cos\beta$.*

Lemma 4 (see proof in [15]) *Consider $g, s \in SO(4)$, where $\Phi^{-1}g = ((\cos\alpha, \sin\alpha\mathbf{v}); (\cos\alpha, \sin\alpha\mathbf{w}))$ and $\Phi^{-1}s = ((0, \mathbf{v}); (0, \mathbf{w}))$. Then $\mathrm{Fix} < g >= \mathrm{Fix} < s >$.*

Lemma 5 (see proof in [6]) *If $\mathbf{l} = (\cos\omega, \mathbf{v}\sin\omega)$ and $\mathbf{r} = (\cos\omega', \mathbf{v}'\sin\omega')$, then the transformation $\mathbf{q} \to \mathbf{l}\mathbf{q}\mathbf{r}^{-1}$ is a rotation of angles $\omega \pm \omega'$ in a pair of absolutely perpendicular planes.*

Lemma 6 *If $\Gamma \subset SO(4)$, $\Phi^{-1}\Gamma = (\mathbf{L} \,|\, \mathbf{L}_K; \mathbf{R} \,|\, \mathbf{R}_K)$, admits pseudo-simple heteroclinic cycles then $\mathbf{L} \supset \mathbb{D}_k$ and $\mathbf{R} \supset \mathbb{D}_k$, where $k \geq 3$.*

Proof: Let $\Phi^{-1}\rho_j = (\mathbf{l}^{(1)}; \mathbf{r}^{(1)})$ and $\Phi^{-1}\sigma_j = (\mathbf{l}^{(2)}; \mathbf{r}^{(2)})$, where $\Delta_j =< \rho_j, \sigma_j >\subset \Gamma$ is the group discussed in remark 1. Existence of at least one such Δ_j follows from definitions 1 and 2. Lemma 5 implies that the order of the elements $\mathbf{l}^{(1)}$ and $\mathbf{r}^{(1)}$ is k, while the order of $\mathbf{l}^{(2)}$ and $\mathbf{r}^{(2)}$ is 2. Since Φ is a homomorphism, $\mathbf{L} \supset< \mathbf{l}^{(1)}, \mathbf{l}^{(2)} >\cong \mathbb{D}_\mathbf{k}$ and $\mathbf{R} \supset< \mathbf{r}^{(1)}, \mathbf{r}^{(2)} >\cong \mathbb{D}_\mathbf{k}$.
QED

Lemma 7 *If $\Gamma \subset SO(4)$ admits pseudo-simple heteroclinic cycles, then it has a symmetry axis $L = \mathrm{Fix}\,\Sigma$, where $\Sigma \subset \Gamma$ is a maximal isotropy subgroup such that $\Sigma_L \cong \mathbb{D}_k$ with $k \geq 3$.*

The proof follows from definitions 1 and 2 and remark 1.

3 Construction of a Γ-equivariant system, possessing a heteroclinic cycle

In this section we prove the following lemma:

Lemma 8 *(i) If for a given finite subgroup $\Gamma \subset O(4)$ there exist two sequences of isotropy subgroups Σ_j, Δ_j, $j = 1, \dots, m$, and an element $\gamma \in \Gamma$ satisfying the following conditions:*

C1. *Denote $P_j = \mathrm{Fix}\,(\Sigma_j)$ and $L_j = \mathrm{Fix}\,(\Delta_j)$. Then $\dim P_j = 2$ and $\dim L_j = 1$ for all j.*

C2. *For $i \neq j$, Σ_i and Σ_j are not conjugate.*

C3. *For $j = 2, \dots, m$, $L_j = P_{j-1} \cap P_j$, and $L_1 = \gamma^{-1} P_m \gamma \cap P_1$. We set $\Delta_{m+1} = \gamma \Delta_1 \gamma^{-1}$.*

C4. *For all j, the subspaces L_j, $P_{j-1} \ominus L_j$ and $P_j \ominus L_j$ belong to different isotypic components in the isotypic decomposition of Δ_j in $\mathbb{R}^4$.*

C5. *$\Sigma_j \cong \mathbb{Z}_{k_j}$ with $k_j \geq 3$ for at least one j.*

Consider $G_j = N_\Gamma(\Sigma_j)/\Sigma_j \cong \mathbb{D}_{k_j}$, the dihedral group of order $2k_j$, where we write $k_j = 0$ for a trivial G_j or $k_j = 1$ for $G_j \cong \mathbb{Z}_2$. Let n_j be the number of isotropy types of axes $\widetilde{L}_{sj} \subset P_j$ that are not fixed by an element of G_j and $\widetilde{L}_{sj} = \mathrm{Fix}\,\widetilde{\Delta}_{sj}$, $1 \leq s \leq n_j$.

C6. *Depending on n_j one of the following takes place:*
(a) if $n_j = 0$ then either k_j is even and the groups Δ_{j-1}, Δ_j are not conjugate, or k_j is odd;
(b) if $n_j = 1$ then the groups Δ_{j-1} and Δ_j are not conjugate and one of Δ_{j-1} or Δ_j is conjugate to $\widetilde{\Delta}_{1j}$;
(c) if $n_j = 2$ then Δ_{j-1} and Δ_j are conjugate to $\widetilde{\Delta}_{1j}$ and $\widetilde{\Delta}_{2j}$.

then Γ admits *pseudo-simple heteroclinic cycles.*

(ii) If $\Gamma \subset O(4)$ admits pseudo-simple heteroclinic cycles, then there are two sequences of isotropy subgroups Σ_j, Δ_j, $j = 1, \dots, m$, where $m \geq 2$, and an element γ, satisfying conditions **C1**,**C3**,**C4**, **C5** *and* **C6***.

C6*. *If $-I \in \Gamma$, then Δ_j and Δ_i are not conjugate for any $i \neq j$, $1 \leq i, j \leq m$.*

In [15] we proved a similar lemma, stating necessary and sufficient conditions for a group $\Gamma \subset O(n)$ to admit simple heteroclinic cycles. As noted in [16], with minor modifications the proof can be used to prove sufficient conditions for a group $\Gamma \subset O(4)$ to admit pseudo-simple heteroclinic cycles. Here, our proof of lemma 8 employs a different idea. We explicitly build a Γ-equivariant dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ possessing a pseudo-simple heteroclinic cycle and prove that the cycle persists under small Γ-equivariant perturbations.

Proof: Starting with the proof of (i), we show that for any group $\Gamma \subset O(4)$ satisfying conditions **C1-C6** there is a vector field $\mathbf{f}$ such that the associated dynamics possess a heteroclinic cycle between equilibria in ΓL_j with connections in the fixed-point planes ΓP_j.

As a first step, for each plane P_j, $j = 1, \dots, m$, that contains the axes L_j and L_{j+1} (in agreement with **C3**, $L_{m+1} = \gamma L_1$), we define a two-dimensional vector field $\mathbf{h}_j$, which in the polar coordinates (r, θ) is:

$$\mathbf{h}_j(r,\theta) = \left(r(1-r),\ \sin(\theta)\prod_{i=1}^{n}\sin(\theta_{ij}-\theta) \right), \tag{4}$$

where $0 \le \theta_{ij} < \pi$ are the angles of all fixed-point axes in P_j other than L_j, and the angle of L_j is $\theta = 0$. For the flow of $(\dot{r}, \dot{\theta}) = \mathbf{h}_j(r, \theta)$ each of these axes is invariant and has an equilibrium $r = 1$ which is attracting along the direction of r. Moreover, there are heteroclinic connections between equilibria on neighbouring axes, since the sign of $\dot{\theta}$ changes when an axis is crossed.

We extend the vector fields $\mathbf{h}_j$ to $\mathbf{g}_j : \mathbb{R}^4 \to \mathbb{R}^4$ as follows: Denote by π_j and $\pi_j^{\perp}$ the projections onto the plane P_j and its orthogonal complement in $\mathbb{R}^4$, respectively. We set

$$\pi_j \mathbf{g}_j(\mathbf{x}) = \frac{\mathbf{h}_j(\pi_j \mathbf{x})}{1 + A|\pi_j^{\perp}\mathbf{x}|^2}, \quad \pi_j^{\perp}\mathbf{g}_j(\mathbf{x}) = 0, \tag{5}$$

with a positive constant A (to be chosen sufficiently large later). The vector field $\mathbf{f} : \mathbb{R}^4 \to \mathbb{R}^4$ is then defined as

$$\mathbf{f}(\mathbf{x}) = \sum_{j=1}^{m}\sum_{\gamma_{ij}\in\mathcal{G}_j} \gamma_{ij}\mathbf{g}_j(\gamma_{ij}^{-1}\mathbf{x}), \tag{6}$$

where $\mathcal{G}_j = \Gamma / N_{\Gamma}(\Sigma_j)$ and $N_{\Gamma}(\Sigma_j)$ is the normalizer of Σ_j in Γ.

As the second step, we show that the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{7}$$

possesses steady states $\xi_j \in L_j$, $j = 1, \dots, m$, and heteroclinic connections $\xi_j \to \xi'_{j+1} \subset P_j$, where $\xi'_{j+1} = \gamma'_j \xi_{j+1}$ and $\gamma'_j \in N_{\Gamma}(\Sigma_j)$. Note, that by construction the system (7) is Γ-equivariant, which implies invariance of the axes L_j and planes P_j.

The system (7) restricted to L_j is

$$\dot{x}_j = \sum_{k=1}^{m}\sum_{\gamma_{ik}\in\mathcal{G}_k} \frac{x_j\cos\beta_{ik}(1 - x_j\cos\beta_{ik})}{1 + Ax_j^2\sin^2\beta_{ik}}, \tag{8}$$

where x_j is the coordinate along L_j and β_{ik} is the angle between L_j and $\gamma_{ik}P_k$. We split the sum in (8) into two, for $L_j \subset \gamma_{ik}P_k$ and $L_j \not\subset \gamma_{ik}P_k$, and write

$$\dot{x}_j = s_j x_j(1 - x_j) + \sum_{i,k:\, L_j \not\subset \gamma_{ik}P_k} \frac{x_j\cos\beta_{ik}(1 - x_j\cos\beta_{ik})}{1 + Ax_j^2\sin^2\beta_{ik}},$$

where s_j is the number of planes $\gamma_{ik}P_k$ that contain L_j. Hence, for sufficiently large positive A there exists an equilibrium $\xi_j \in L_j$ with $x_j = c_j \approx 1$, attracting in L_j.

To prove existence of a heteroclinic connection $\xi_j \to \xi'_{j+1}$, we consider a sector in P_j between L_j and L'_{j+1}, where $L'_{j+1} = \gamma'_j L_{j+1}$ with $\gamma'_j \in N_\Gamma(\Sigma_j)$ such that there are no invariant axes between L_j and L'_{j+1}. Existence of such a sector follows from **C6**. (In case (a) the axes L_j and L'_{j+1} are invariant axes of G_j, they are the only invariant axes in P_j. In case (b) invariant axes of G_j alternate with (symmetric copies of) $\tilde{L}_{1j}$. In case (c) there is one $\tilde{L}_{1j}$ and one $\tilde{L}_{2j}$ between any two neighbouring invariant axes of G_j.) We choose a small number $a > 0$ and divide this sector into three subsets, as sketched in Figure 1:

- V_1: a strip of width a near L_j
- V_3: a strip of width a near L_{j+1}
- V_2: the rest of the sector

We now consider the dynamics of system (7) in each of these regions.

- For V_1 we distinguish three cases: (a) ξ_j is a simple equilibrium, i.e. the isotypic decomposition of $\mathbb{R}^4$ w.r.t. Δ_j has only 1D components, (b) ξ_j is a pseudo-simple equilibrium, i.e. the isotypic decomposition of $\mathbb{R}^4$ w.r.t. Δ_j has a 2D component, and the component is the contracting eigenspace, (c) ξ_j is a pseudo-simple equilibrium with 2D expanding eigenspace.

  In V_1 we employ the coordinates $(x_j, x_{j+1}) = (r\cos(\theta), r\sin(\theta))$. Choosing $a > 0$ sufficiently small and $A > 0$ sufficiently large, to approximate the dynamics near ξ_j, we take into account only leading terms in $\mathbf{h}_{j-1}$ and $\mathbf{h}_j$ and in (7) we omit the terms corresponding to the planes $\gamma_{ik}P_k$ that do not contain L_j. The condition **C4** implies that in case (a) the axis L_j belongs to planes P_{j-1} and P_j only; in case (b) it also belongs to several symmetric copies of P_{j-1}; in case (c) to P_{j-1}, P_j and several symmetric copies of P_j. In case (a) we have

  $$\dot{x}_j = x_j(c_j - x_j)\left(1 + \frac{1}{1 + Ax_{j+1}^2}\right), \quad \dot{x}_{j+1} = C_j x_{j+1}, \quad \text{where } C_j = \prod_{i=1}^{n_j} \sin\theta_{ij},$$

  and $C_j > 0$ since $0 < \theta_{ij} < \pi$. In case (b), P_j is orthogonal to P_{j-1} and its q_j symmetric copies. Hence, near ξ_j we have

  $$\dot{x}_j = x_j(c_j - x_j)\left(1 + \frac{q_j}{1 + Ax_{j+1}^2}\right), \quad \dot{x}_{j+1} = C_j x_{j+1}.$$

  In case (c), assuming that there are q_j symmetric copies of P_j containing L_j, hence the angles between neighbouring planes are π/q_j, we approximate the dynamics in P_j as

$$\dot{x}_j = x_j(c_j - x_j)\left(\frac{1}{1+Ax_{j+1}^2} + \sum_{k=0}^{q_j-1}\frac{1}{1+A\sin^2(k\pi/q_j)x_{j+1}^2}\right)$$
$$\dot{x}_{j+1} = C_j x_{j+1}\sum_{k=0}^{q_j-1}\frac{\cos^2(k\pi/q_j)}{1+A\sin^2(k\pi/q_j)x_{j+1}^2}.$$

Thus, in all cases (a)-(c) the equilibrium ξ_j is attracting in L_j and we have $\dot{x}_{j+1} > 0$ in V_1, so trajectories leave V_1 and enter V_2. At the point of entrance, $x_j \approx 1$.

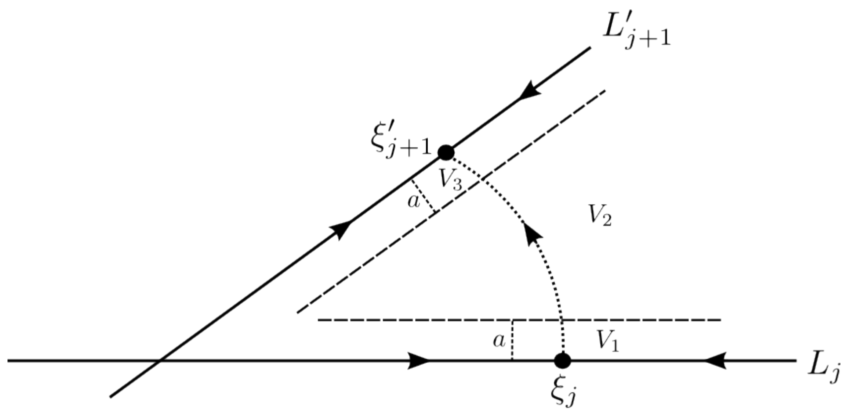


Figure 1: Division of the sector in P_j into V_1, V_2, V_3.

- The region V_2 is bounded away from the axes L_j and L'_{j+1}. Then, for any given $a > 0$ there exists $A_0 > 0$ such that for all $A > A_0$, the dynamics away from the fixed-point axes are essentially those of $\mathbf{h}_j$. Namely, the trajectories through $(x_j, x_{j+1}) \approx (1, a)$ are attracted by V_3 and at the entrance point $r \approx 1$.

- In V_3, for sufficiently small $a > 0$ and sufficiently large $A > 0$ all trajectories with $r \approx 1$ are attracted by ξ'_{j+1} by arguments that are analogous to those for V_1.

So we have shown that for the dynamics of (7) in each P_j there is a connection $\xi_j \to \xi'_{j+1} = \gamma'_j\xi_{j+1}$. Taking into account their symmetric copies we obtain a sequence of connections $\xi_1 \to \gamma'_1\xi_2 \to \gamma'_1\gamma'_2\xi_3 \to \ldots \to \gamma'_1\gamma'_2\ldots\gamma'_m\xi_{m+1} =: \gamma'\xi_1$ forming a building block of a heteroclinic cycle. The cycle is pseudo-simple because of **C5**, and robust since by construction all connections lie in fixed-point subspaces that persist under equivariant perturbations.

For (ii) we note that necessity of conditions **C1**, **C3**, **C4**, **C5** and **C6*** follows directly from the definition of pseudo-simple heteroclinic cycles. The two sequences of subgroups are found by choosing Σ_j as the isotropy subgroups of the planes P_j and Δ_j as the isotropy subgroups of the equilibria ξ_j. **QED**

In the following lemma we state sufficient conditions for a group $\Gamma \subset O(4)$ to admit pseudo-simple cycles, that are slightly different from the ones proven in lemma 8(i). The difference is that in lemma 9 the subgroups Σ_j can be conjugate in Γ. Since the proof of lemma 9 is similar to the one of lemma 8(i), it is omitted.

Lemma 9 *If for a given finite subgroup $\Gamma \subset O(4)$, with $-I \in \Gamma$, there exist two sequences of isotropy subgroups Σ_j, Δ_j, $j = 1, \ldots, m$, and an element $\gamma \in \Gamma$ satisfying conditions* **C1**, **C2'**, **C3**, **C4**, **C5** *and* **C6'**, *where*

C2'. *Δ_i and Δ_j are not conjugate for any $i \neq j$.*

C6'. *For any j, there exists a sector in P_j, bounded by L_j and L_{j+1}, that does not contain any other isotropy axes of Γ.*

then Γ admits *pseudo-simple heteroclinic cycles.*

Remark 4 *Note that lemma 9 can be generalised to $\mathbb{R}^n$ as follows:*
If for a given finite subgroup $\Gamma \subset O(n)$, with $-I \in \Gamma$, there exist two sequences of isotropy subgroups Σ_j, Δ_j, $j = 1, \ldots, m$, and an element $\gamma \in \Gamma$ satisfying conditions **C1**, **C2'**, **C3**, **C4** *and* **C6'**, *then* Γ admits *heteroclinic cycles.*

4 List of groups

4.1 The groups Γ in $SO(4)$

In this subsection we prove theorem 1 that exhibits all finite subgroups of $SO(4)$, admitting robust pseudo-simple heteroclinic cycles. The proof employs lemmas 8(i) and 9, that give sufficient conditions for $\Gamma \subset SO(4)$ to admit pseudo-simple cycles, and lemma 8(ii) that gives necessary conditions. The lemmas allow us to split subgroups of $SO(4)$ into two classes, those admitting and those not admitting pseudo-simple heteroclinic cycles. Similarly to [15], we use the quaternionic presentation for subgroups of $SO(4)$, see subsection 2.3. Appendices A-C contain detailed information on the geometry of various subgroups of $SO(4)$ which are used for proving the theorem.

Theorem 1 *A group $\Gamma \subset SO(4)$ admits pseudo-simple heteroclinic cycles, if and only if it is one of those listed in table 2.*

$(\mathbb{D}_{2K_1}\,|\,\mathbb{D}_{2K_1};\mathbb{D}_{2K_2}\,|\,\mathbb{D}_{2K_2}),\;\; \gcd(K_1,K_2)\geq 2$

$(\mathbb{D}_{K_1r}\,|\,\mathbb{Z}_{2K_1};\mathbb{D}_{K_2r}\,|\,\mathbb{Z}_{2K_2})_s,\;\; \gcd(K_1,K_2)\gcd(r,K_1-sK_2)\geq 3$

$(\mathbb{D}_{2K_1}\,|\,\mathbb{D}_{K_1};\mathbb{D}_{2K_2}\,|\,\mathbb{D}_{K_2}),\gcd(K_1,K_2)\geq 2$

$(\mathbb{D}_{2K_1}\,|\,\mathbb{D}_{K_1};\mathbb{D}_{K_2}\,|\,\mathbb{Z}_{2K_2}),\;\; \gcd(K_1,K_2)\geq 3$

$(\mathbb{D}_{K}\,|\,\mathbb{D}_{K};\mathbb{O}\,|\,\mathbb{O}),\;\; K=3m_1$ and/or $K=4m_2$

$(\mathbb{D}_{K}\,|\,\mathbb{Z}_{2K};\mathbb{O}\,|\,\mathbb{T}),\;\; K=3m$

$(\mathbb{D}_{2K}\,|\,\mathbb{D}_{K};\mathbb{O}\,|\,\mathbb{T}),\;\; K=3m_1$ and/or $K=2(2m_2+1)$

$(\mathbb{D}_{3K}\,|\,\mathbb{Z}_{2K};\mathbb{O}\,|\,\mathbb{V})$

$(\mathbb{D}_{K}\,|\,\mathbb{D}_{K};\mathbb{I}\,|\,\mathbb{I}),\;\; K=3m_1$ and/or $K=5m_2$

$(\mathbb{D}_{K_1r}\,|\,\mathbb{Z}_{K_1};\mathbb{D}_{K_2r}\,|\,\mathbb{Z}_{K_2})_s,\;\; K_1\equiv K_2\equiv 1(\mathrm{mod}\ 2),\;\; \gcd(K_1,K_2)\gcd(r,K_1-sK_2)\geq 3$

Table 2: Groups $\Gamma\subset SO(4)$ admitting pseudo-simple heteroclinic cycles

To prove the theorem, we proceed in four steps:

In step [i], using lemmas 6 and 7 we identify subgroups of $SO(4)$, that do not satisfy necessary conditions for existence of pseudo-simple heteroclinic cycles stated in the lemmas. The groups 1-9 and 33 (see table 1) do not satisfy conditions of lemma 6. The groups 14, 20-32 and 35-39 do not satisfy conditions of lemma 7. The groups 10-13, 15-19 and 34 should satisfy extra conditions on k_1, k_2, n, r and s.

In step [ii], using lemmas 1-5 and the correspondence between **L** and **R** (see section 2.3), we identify all subgroups Σ such that $\dim\mathrm{Fix}\,\Sigma=2$, which are elements of groups found in step [i]. The results are listed in appendix A.

In step [iii], using the results obtained at step [ii], we determine the (maximal) conjugacy classes of subgroups of Γ, isomorphic to $\mathbb{Z}_k$, which have two-dimensional fixed-point subspaces and (maximal) conjugacy classes of $\Delta\cong\mathbb{D}_k$ such that $\dim\mathrm{Fix}\,(\Delta)=1$. The results are listed in appendix B.

Finally, in step [iv], using the list in appendix B, we identify all groups that possess sequences of subgroups Σ_j and Δ_j satisfying conditions **C1**-**C6** of lemmas 8(i) or 9 (they are presented in appendix C). All the other groups do not have sequences satisfying conditions **C1**,**C3**, **C4**,**C5** and **C6***. In fact, the only groups satisfying the conditions of lemma 9, but not those of lemma 8(i), are $(\mathbb{D}_{15K}\,|\,\mathbb{D}_{15K};\mathbb{I}\,|\,\mathbb{I})$ and $(\mathbb{D}_{3K}\,|\,\mathbb{Z}_{6K};\mathbb{O}\,|\,\mathbb{T})$ with odd K.

Proof of the theorem

Step [i]

The groups that satisfy conditions of lemmas 6 and 7 are:

#	group
10	$(\mathbb{D}_n \,|\, \mathbb{D}_n; \mathbb{D}_k \,|\, \mathbb{D}_k)$, $\gcd(n,k) \geq 3$
11	$(\mathbb{D}_{nr} \,|\, \mathbb{Z}_{2n}; \mathbb{D}_{kr} \,|\, \mathbb{Z}_{2k})_s$ $\gcd(n,k)\gcd(r,k-sn) \geq 3$
12	$(\mathbb{D}_{2n} \,|\, \mathbb{D}_n; \mathbb{D}_{2k} \,|\, \mathbb{D}_k)$, $\gcd(n,k) \geq 2$
13	$(\mathbb{D}_{2n} \,|\, \mathbb{D}_n; \mathbb{D}_k \,|\, \mathbb{Z}_{2k})$, $\gcd(n,k) \geq 2$
15	$(\mathbb{D}_n \,|\, \mathbb{D}_n; \mathbb{O} \,|\, \mathbb{O})$, $n = 3m_1$ and/or $n = 4m_2$
16	$(\mathbb{D}_n \,|\, \mathbb{Z}_{2n}; \mathbb{O} \,|\, \mathbb{T})$, $n = 3m$
17	$(\mathbb{D}_{2n} \,|\, \mathbb{D}_n; \mathbb{O} \,|\, \mathbb{T})$, $n = 3m_1$ and/or $n = 2(2m_2+1)$
18	$(\mathbb{D}_{3n} \,|\, \mathbb{Z}_n; \mathbb{O} \,|\, \mathbb{V})$
19	$(\mathbb{D}_n \,|\, \mathbb{D}_n; \mathbb{I} \,|\, \mathbb{I})$, $n = 3m_1$ and/or $n = 5m_2$
34	$(\mathbb{D}_{nr} \,|\, \mathbb{Z}_n; \mathbb{D}_{kr} \,|\, \mathbb{Z}_k)_s$, $n \equiv k \equiv 1(\mod 1)$, $\gcd(n,k)\gcd(r,k-sn) \geq 3$

Below we show that the groups $(\mathbb{D}_{3K} \,|\, \mathbb{Z}_{6K}; \mathbb{O} \,|\, \mathbb{T})$ and $(\mathbb{D}_{15K} \,|\, \mathbb{D}_{15K}; \mathbb{I} \,|\, \mathbb{I})$ admit heteroclinic cycles, while the groups $(\mathbb{D}_{K_1} \,|\, \mathbb{D}_{K_1}; \mathbb{D}_{K_2} \,|\, \mathbb{D}_{K_2})$, where at least one of K_1 or K_2 is odd, do not. For other groups the proofs are similar and we omit them.

The group $\Gamma = (\mathbb{D}_{K_1} \,|\, \mathbb{D}_{K_1}; \mathbb{D}_{K_2} \,|\, \mathbb{D}_{K_2})$.

[ii] The group $\mathbb{D}_n$ (see (3)) is comprised of the elements

$$\rho_n(t) = (\cos t\pi/n, 0, 0, \sin t\pi/n),\ \ \sigma_n(t) = (0, \cos t\pi/n, \sin t\pi/n, 0),\ \ 0 \leq t < 2n. \tag{9}$$

The pairs $(\mathbf{l}; \mathbf{r}) \in (\mathbb{D}_{K_1} \,|\, \mathbb{D}_{K_1}; \mathbb{D}_{K_2} \,|\, \mathbb{D}_{K_2})$ satisfy $\mathbf{l} \in \mathbb{D}_{K_1}$, $\mathbf{r} \in \mathbb{D}_{K_2}$, where all possible combinations are elements of the group. If both K_1 and K_2 are odd, then the elements $\gamma \in \Gamma$ satisfying $\dim \mathrm{Fix}\,\gamma = 2$ are

$$\begin{array}{l} \kappa_1(\pm, n) = ((\cos(n\theta), 0, 0, \pm\sin(n\theta)); (\cos(n\theta), 0, 0, \sin(n\theta))) \\ \kappa_2(n_1, n_2) = ((0, \cos(n_1\theta_1), \sin(n_1\theta_1), 0); (0, \cos(n_2\theta_2), \sin(n_2\theta_2), 0)), \end{array} \tag{10}$$

where $\theta_1 = \pi/K_1$, $\theta_2 = \pi/K_2$, $\theta = \pi/m$, $m = \gcd(K_1, K_2) \geq 3$, $0 \leq n_1 < 2K_1$, $0 \leq n_2 < K_2$ and $0 \leq n < m$. The elements $\kappa_2(n_1, n_2)$ are plane reflections, while $\kappa_1(\pm, n)$ is a rotation by $2n\theta$ in the plane orthogonal to $\mathrm{Fix}\,\kappa_1(\pm, n)$. For even K_2 the group possesses an additional set of plane reflections

$$\kappa_3(n_1) = ((0, \cos(n_1\theta_1), \sin(n_1\theta_1), 0); (0, 0, 0, 1)).$$

[iii] In the group $\mathbb{D}_n$ the elements $(0, \cos(t\pi/n), \sin(t\pi/n), 0)$ split into two conjugacy classes, corresponding to odd and even t. Since $\kappa_2(n_1, n_2) = \kappa_2(n_1 + K_1, n_2 + K_2)$, in the case when both K_1 and K_2 are odd the group Γ has three maximal isotropy types of subgroups satisfying $\dim \mathrm{Fix}\,\Sigma = 2$. The subgroups are

$$\Sigma^{(1)}(\pm) =< \kappa_1(\pm, 1) >,\ \Sigma^{(2)}(n_1, n_2) =< \kappa_2(n_1, n_2) >,\ n_1 + n_2 \text{ even or odd.}$$

The subgroups $\Sigma^{(1)}(+)$ and $\Sigma^{(1)}(-)$ are conjugate, e.g. by $\sigma(n_1) = ((0, \cos(n_1\theta_1), \sin(n_1\theta_1), 0); (1, 0, 0, 0))$.

For any plane $P = \mathrm{Fix}\,\Sigma^{(2)}(n_1, n_2)$ the only symmetry axes $L \subset P$ are the intersections with $\Sigma^{(1)}(\pm)$. The axes are conjugate by $\sigma(n_1) \in N_\Gamma(\Sigma^{(2)}(n_1, n_2))$. Therefore, the group has two maximal isotropy types of subgroups satisfying $\dim \mathrm{Fix}\,(\Delta) = 1$:

$$\Delta(\pm, n_1, n_2) =< \kappa_1(\pm, 1), \kappa_2(n_1, n_2) >, \; n_1 + n_2 \text{ even or odd.}$$

Since planes $\mathrm{Fix}\,\Sigma^{(2)}(n_1, n_2)$ do not satisfy the condition **C4*** and the remaining planes $\mathrm{Fix}\,\Sigma^{(1)}(+)$ and $\mathrm{Fix}\,\Sigma^{(1)}(-)$ do not intersect, the group $(\mathbb{D}_{K_1} \,|\, \mathbb{D}_{K_1}; \mathbb{D}_{K_2} \,|\, \mathbb{D}_{K_2})$ with odd K_1K_2 does not admit heteroclinic cycles.

In the case when K_1 is odd and K_2 is even, a plane fixed by the reflection $\kappa_3(n_1)$ does not intersect with any of $\mathrm{Fix}\,\Sigma_1(\pm)$ and $\mathrm{Fix}\,\Sigma_2(n_1, n_2)$ (see lemma 1). Moreover, $\mathrm{Fix}\,\kappa_3(n_1)$ does dot intersect with $\mathrm{Fix}\,\kappa_3(n_1')$ for any $n_1 \neq n_1'$. Similar arguments apply when K_1 is even and K_2 is odd. Therefore, the group $(\mathbb{D}_{K_1} \,|\, \mathbb{D}_{K_1}; \mathbb{D}_{K_2} \,|\, \mathbb{D}_{K_2})$ does not admit heteroclinic cycles when at least one of K_1 or K_2 is odd.

The group $\Gamma = (\mathbb{D}_{3K} \,|\, \mathbb{Z}_{6K}; \mathbb{O} \,|\, \mathbb{T})$.

[ii] The group $\mathbb{O}$ can be decomposed as $\mathbb{O} = \mathbb{T} \oplus \sqrt{\frac{1}{2}}((\pm 1, \pm 1, 0, 0))$, see (3). Therefore, the group $(\mathbb{D}_{3K} \,|\, \mathbb{Z}_{6K}; \mathbb{O} \,|\, \mathbb{T})$ is comprised of the following elements:

$$\begin{array}{l} ((\cos(n\theta), 0, 0, \sin(n\theta)); \mathbb{T}) \\ ((0, \cos(n\theta), \sin(n\theta), 0); \sqrt{\frac{1}{2}}((\pm 1, \pm 1, 0, 0)) \end{array} \tag{11}$$

where $\theta = \pi/3K$ and $0 \leq n < 3K$. For odd K the elements $\gamma \in \Gamma$ satisfying $\dim \mathrm{Fix}\,\gamma = 2$ are

$$\begin{array}{l} \kappa_1(\pm, \pm, \pm, \pm) = ((1, 0, 0, \pm\sqrt{3})/2); (1, \pm 1, \pm 1, \pm 1)/2) \\ \kappa_2(n, r, \pm) = ((0, \cos(n\theta), \sin(n\theta), 0); \rho^r(0, 1, \pm 1, 0)), \end{array} \tag{12}$$

where $\rho(a, b, c, d) = (a, c, d, b)$. Here κ_2 are plane reflections and κ_1 are rotations by $2\pi/3$ in the planes orthogonal to $\mathrm{Fix}\,\kappa_1$. For even K the group possesses an additional set of plane reflections

$$\kappa_3(r, \pm) = ((0, 0, 0, 1); \rho^r(0, 0, 0, \pm 1)).$$

[iii] Since $\kappa_2(n, r, \pm) = -\kappa_2(n{+}3K, r, \pm)$ and in $\mathbb{T}$ the elements $(0, 1, \pm 1, 0)$ and $-(0, 1, \pm 1, 0)$ are conjugate, for odd K all $\kappa_2(n, r, \pm)$ are conjugate in Γ. The elements $\kappa_1(+, (-1)^{s_1}, (-1)^{s_2}, (-1)^{s_3})$ split into two conjugacy classes, depending on whether $s_1 + s_2 + s_3$ is even or odd. Hence, for odd K the group has three maximal isotropy types of subgroups satisfying $\dim \mathrm{Fix}\,\Sigma = 2$:

$$\begin{array}{l} \Sigma^{(1)}((-1)^{s_1}, (-1)^{s_2}, (-1)^{s_3})) =< \kappa_1(+, (-1)^{s_1}, (-1)^{s_2}, (-1)^{s_3}) >, \; s_1 + s_2 + s_3 \text{ even or odd} \\ \Sigma^{(2)}(n, r, \pm) =< \kappa_2(n, r, \pm) > . \end{array} \tag{13}$$

Each $\mathrm{Fix}\,\Sigma^{(1)}$ contains $3K$ isotropy axes, each of them are intersections with three Fix $<\kappa_2(n,r,\pm)>$, where $r=0,1,2$. Hence, the isotropy groups of symmetry axes can be written as

$$\Delta(n,(-1)^{s_1},(-1)^{s_2},(-1)^{s_3}) =< \kappa_1(+,(-1)^{s_1},(-1)^{s_2},(-1)^{s_3}),\kappa_2(n,0,(-1)^{s_1+s_2+1}) > . \tag{14}$$

They split into two isotropy types, depending on whether s_1+n is even or odd. Any plane $\mathrm{Fix}\,\Sigma^{(2)}$ contains four isotropy axes which are intersections with $\mathrm{Fix}\,\Sigma^{(1)}$. Since $N_\Gamma(\Sigma_2) =< \Sigma_2, -\Sigma_2 >$ (this can be checked directly using the list (11)), all four isotropy axes are of different types. Therefore, the group has four types of isotropy subgroups (14) safisfying $\dim\mathrm{Fix}\,\Delta=2$, corresponding to odd and even s_1+n and $s_1+s_2+s_3$.

In the case when K is even there exist five isotropy types of subgroups satisfying $\dim\mathrm{Fix}\,\Sigma=2$:

$$\begin{array}{l}\Sigma^{(1)}((-1)^{s_1},(-1)^{s_2},(-1)^{s_3})) =< \kappa_1(+,(-1)^{s_1},(-1)^{s_2},(-1)^{s_3}) >,\ s_1+s_2+s_3 \text{ even or odd}\\ \Sigma^{(2)}(n,r,\pm) =< \kappa_2(n,r,\pm) >,\ n \text{ even or odd}\\ \Sigma^{(3)}(r,\pm) =< \kappa_3(r,\pm) > .\end{array} \tag{15}$$

A plane $\mathrm{Fix}\,\Sigma^{(2)}(n,r,\pm)$ orthogonally intersects with the ones $\mathrm{Fix}\,\Sigma^{(2)}(n-(-1)^s3K/2,r,\mp)$ (and also with $\mathrm{Fix}\,\Sigma^{(3)}(r,(-1)^s)$), hence for odd and even $K/2$ the isotropy axes are different. Namely, for odd $K/2$ they are

$$\begin{array}{l}\Delta^{(1)}(n,(-1)^{s_1},(-1)^{s_2},(-1)^{s_3}) =\\ \quad < \kappa_1(+,(-1)^{s_1},(-1)^{s_2},(-1)^{s_3}),\kappa_2(n,0,(-1)^{s_1+s_2+1}) >,\ s_1+s_2+s_3,\ n \text{ even or odd}\\ \Delta^{(2)}(n,r,\pm,(-1)^s) =< \kappa_2(n,r,\pm),\kappa_3(r,(-1)^s) >,\ n+s \text{ even or odd,}\end{array} \tag{16}$$

while for even $K/2$ the second set of isotropy axes is

$$\Delta^{(2)}(n,r,\pm,\pm) =< \kappa_2(n,r,\pm),\kappa_3(r,\pm) >,\ n \text{ even or odd.} \tag{17}$$

[iv] According to (iii), for odd K the group $(\mathbb{D}_{3K}\,|\,\mathbb{Z}_{6K};\mathbb{O}\,|\,\mathbb{T})$ does not have isotropy subgroups satisfying conditions **C1**-**C6** of lemma 8(i). Let us show that we can find subgroups satisfying conditions of lemma 9. Set

$$\begin{array}{l}\Sigma_1 =< \kappa_1(+,+,+,+) >,\ \Sigma_2 =< \kappa_2(0,0,-) >,\ \Sigma_3 =< \kappa_1(+,+,+,-) >,\\ \Sigma_4 =< \kappa_2(1,0,-) >,\ \Delta_j =< \Sigma_{j-1},\Sigma_j >,\ j=2,3,4,\ \Delta_1 =< \Sigma_4,\Sigma_1 > \text{ and } \gamma=e.\end{array} \tag{18}$$

By construction and due to (14) and (13), the subgroups satisfy conditions **C1**,**C2'**,**C3**,**C4** and **C5**.

To show that Fix $<\kappa_2(0,0,-)>$ satisfies condition **C6'**, we recall (see (iii)) that the plane involves four isotropy axes, all non-conjugate, that are intersections with $\mathrm{Fix}\,\kappa_1(+,+,+,+)$, $\mathrm{Fix}\,\kappa_1(+,+,+,-)$, $\mathrm{Fix}\,\kappa_1(+,-,-,+)$ and $\mathrm{Fix}\,\kappa_1(+,-,-,-)$. To determine the angles between axes, we use lemmas 4 and 5. By lemma 4, $\mathrm{Fix}\,\kappa_1(+,\pm,\pm,\pm)=\mathrm{Fix}\,\kappa'$, where $\kappa'(\pm,\pm,\pm)=((0,0,0,1);(0,\pm1,\pm1,\pm1)/\sqrt{3})$ is a plane reflection about a plane that intersects with Fix $<\kappa_2(0,0,-)>$ orthogonally. Therefore, Fix $<\kappa_2(0,0,-)>$ is κ'-invariant. Composition of two reflections about axes, intersecting with the angle α, is a rotation by 2α. Since $\kappa'(+,+,+)\kappa'(+,+,-)=((1,0,0,0);(1,2,-2,0)/3)$, by lemma 5 the angle in Fix $<\kappa_2(0,0,-)>$ between the lines of intersections with $\kappa'(+,+,+)$ and $\kappa'(+,+,-)$ is $\arccos(1/3)/2$, while the lines of intersections with $\kappa'(+,+,+)$ and $\kappa'(-,-,-)$ are orthogonal. Hence, in Fix $<\kappa_2(0,0,-)>$ no other isotropy axes belong to the smaller sector bounded by $\mathrm{Fix}\,\kappa_1(+,+,+,+)$ and $\mathrm{Fix}\,\kappa_1(+,+,+,-)$. Similarly, it can be shown that the condition **C6'** holds true for $j=1,3,4$ as well.

For even K we apply lemma 8. We choose

$$\Sigma_1=<\kappa_1(+,+,+,+)>,\ \Sigma_2=<\kappa_2(0,0,-)>,\ \Sigma_3=<\kappa_3(+,0)>,\ \Sigma_4=<\kappa_2(1,0,-)>,$$

$$\Delta_1=<\kappa_2(1,0,-),\kappa_1(+,+,+,+)>,\ \Delta_2=<\kappa_1(+,+,+,+)>,\kappa_2(0,0,-)>,$$
$$\Delta_3=<\kappa_2(0,0,-),\kappa_3(+,0)>,\ \Delta_4=<\kappa_3(+,0),\kappa_2(1,0,-)>,$$

which together with $\gamma=e$ satisfy conditions **C1-C6**, as follows from (15), (16) and (17).

The group $\Gamma=(\mathbb{D}_{15K}\,|\,\mathbb{D}_{15K};\mathbb{I}\,|\,\mathbb{I})$.

[ii] The group is comprised of the pairs $(\mathbf{l};\mathbf{r})$, where $\mathbf{l}\in\mathbb{D}_{15K}$ and $\mathbf{r}\in\mathbb{I}$. Since for odd K all elements $((0,\cos(n\theta),\sin(n\theta),0);\mathbf{r})$ are conjugate, the group has the following the elements satisfying $\dim\mathrm{Fix}\,\gamma=2$:

$$\begin{array}{l}\kappa_1(\pm,\pm,\pm,\pm)=((1,0,0,\pm\sqrt{3})/2;(1,\pm1,\pm1,\pm1)/2)\\ \kappa'_1(\pm,r,\pm,\pm)=((1,0,0,\pm\sqrt{3})/2);\rho^r(1,\pm\tau^{-1},\pm\tau,0)/2)\\ \kappa_2(\pm,r,\pm,\pm)=((\tau,0,0,\pm\tau^*)/2;\rho^r(\tau,\pm1,\pm\tau^{-1},0)/2)\\ \kappa'_2(\pm,r,\pm,\pm)=((\tau^{-1},0,0,\pm\tau^{**})/2;\rho^r(\tau^{-1},\pm\tau,\pm1,0)/2)\\ \kappa_3(n,r,\pm)=((0,\cos(n\theta),\sin(n\theta),0);\rho^r(0,0,0,\pm1))\\ \kappa'_3(n,r,\pm,\pm)=((0,\cos(n\theta),\sin(n\theta),0);\rho^r(0,1,\pm\tau,\pm\tau^{-1})),\end{array}\qquad(19)$$

where $\theta=\pi/15K$, $0\le n<30K$, $\tau^*=2\sin(\pi/5)=\sqrt{5}(\tau)^{-1}$ and $\tau^{**}=2\sin(2\pi/5)=\sqrt{5}/2$. By κ_i and κ'_i we denote elements that are conjugate in Γ. Here κ_3 and κ'_3 are plane reflections, κ_1 and κ'_1 are rotations by $2\pi/3$, κ_2 is a rotation by $2\pi/5$ and κ'_2 is a rotation by $4\pi/5$. For even K the group possesses additional set of plane reflections:

$$\kappa_4(r,\pm)=((0,0,0,1);\rho^r(0,0,0,\pm1)),\ \kappa'_4(r,\pm,\pm)=((0,0,0,1);\rho^r(0,1,\pm\tau^{-1},\pm\tau)).$$

[iii] For odd K all plane reflections are conjugate in Γ. The rotations by $2\pi/3$ are conjugate, the rotations by $2\pi/5$ and $4\pi/5$ are conjugate as well. Hence, the group has

three maximal isotropy types of subgroups satisfying $\dim \mathrm{Fix}\,\Sigma = 2$:

$$\begin{array}{l} \Sigma^{(1)}(\pm,\pm,\pm) =<\kappa_1(+,\pm,\pm,\pm)>, \\ \Sigma^{(1')}(r,\pm,\pm) =<\kappa_1'(+,r,\pm,\pm)>, \\ \Sigma^{(2)}(r,\pm,\pm) =<\kappa_2(+,r,\pm,\pm)>, \\ \Sigma^{(3)}(n,r,\pm) =<\kappa_3(n,r,\pm)>, \\ \Sigma^{(3')}(n,r,\pm,\pm) =<\kappa_3'(n,r,\pm,\pm)>. \end{array} \tag{20}$$

Each $\mathrm{Fix}\,\Sigma^{(1)}$ contains $30K$ isotropy axes, each of them is an (orthogonal) intersection with three Fix $<\kappa_3>$. Each $\mathrm{Fix}\,\Sigma^{(2)}$ contains $30K$ isotropy axes, each of them in an (orthogonal) intersection with five Fix $<\kappa_2>$. Hence, the isotropy groups of symmetry axes can be written as

$$\begin{array}{l} \Delta^{(1)}(n,(-1)^{s_1},(-1)^{s_2},(-1)^{s_3}) = \\ \quad <\kappa_1(+,(-1)^{s_1},(-1)^{s_2},(-1)^{s_3}),\kappa_3'(n,0,(-1)^{s_1+s_2+1}(-1)^{s_1+s_3+1},))>,\ s_1+n \text{ even or odd}, \\ \Delta^{(1')}(n,(-1)^{s_1},(-1)^{s_2}) =<\kappa_1'(+,r,(-1)^{s_1},(-1)^{s_2}),\kappa_3(n,0,\pm)>,\ s_1+s_2+n \text{ even or odd}, \\ \Delta^{(2)}(n,(-1)^{s_1},(-1)^{s_2}) =<\kappa_2(+,(-1)^{s_1},(-1)^{s_2}),\kappa_3(n,0,\pm))>,\ s_1+s_2+n \text{ even or odd}, \\ \Delta^{(2')}(n,(-1)^{s_1},(-1)^{s_2}) = \\ \quad <\kappa_2(+,(-1)^{s_1},(-1)^{s_2}),\kappa_3'(n,0,(-1)^{s_1+s_2+1},\pm))>,\ s_1+n \text{ even or odd}. \end{array} \tag{21}$$

In the case when K is even there exist five isotropy types of subgroups satisfying $\dim \mathrm{Fix}\,\Sigma = 2$:

$$\begin{array}{l} \Sigma^{(1)},\ \Sigma^{(1')},\ \Sigma^{(2)}, \\ \Sigma^{(3)}(n,r,\pm) =<\kappa_3(n,r,\pm)>,\ n \text{ even or odd}, \\ \Sigma^{(3')}(n,r,\pm,\pm) =<\kappa_3'(n,r,\pm,\pm)>,\ n \text{ even or odd}, \\ \Sigma^{(4)}(r,\pm) =<\kappa_4(\pm,r)>, \\ \Sigma^{(4')}(r,\pm,\pm) =<\kappa_4'(r,\pm,\pm)>. \end{array} \tag{22}$$

Each of the planes $\mathrm{Fix}\,\Sigma^{(3)}$ has twelve isotropy axes. Four of them (of two isotropy types) are orthogonal intersections with $\mathrm{Fix}\,\Sigma^{(4)}$, therefore $N_\Gamma(\Sigma^{(3)})/\Sigma^{(3)} \cong \mathbb{D}_4$. The other eight axes (of two isotropy types) are intersections with $\mathrm{Fix}\,\Sigma^{(1)}$ and $\mathrm{Fix}\,\Sigma^{(2)}$. The respective isotropy subgroups are different for odd or even $K/2$, as stated in appendix B. A plane $\mathrm{Fix}\,\Sigma^{(1)}$ or $\mathrm{Fix}\,\Sigma^{(2)}$ involves two isotropy types (with odd or even n) of symmetry axes, which are intersections with $\mathrm{Fix}\,\Sigma^{(3)}$.

[iv] For odd K we show that the groups

$$\begin{array}{l} \Sigma_1 =<\kappa_1'(+,0,+,+)>,\ \Sigma_2 =<\kappa_3(0,0,+)>,\ \Sigma_3 =<\kappa_2(+,0,+,+)>, \\ \Sigma_4 =<\kappa_3(1,0,+)>,\ \Delta_j =<\Sigma_{j-1},\Sigma_j>,\ j=2,3,4,\ \Delta_1 =<\Sigma_4,\Sigma_1> \end{array} \tag{23}$$

and $\gamma = e$ satisfy conditions of lemma 9. By construction and due to (20) and (21), the subgroups satisfy conditions **C1**,**C2'**,**C3**,**C4** and **C5**.

Consider $\mathrm{Fix}\,\kappa_3(0,0,+)$. Denote by α_1, α_2 and α_3 the angles between the intersection with $\mathrm{Fix}\,\kappa'_1(+,0,+,+)$ and the following three axes: intersections with $\mathrm{Fix}\,\kappa_2(+,0,+,+)$, $\mathrm{Fix}\,\kappa'_1(+,0,+,-)$ and $\mathrm{Fix}\,\kappa_2(+,0,+,-)$, respectively. By lemmas 4 and 5,

$$\cos 2\alpha_1 = (3+\sqrt{5})/(2\sqrt{15\tau}),\ \cos 2\alpha_2 = \sqrt{5}/3,\ \cos 2\alpha_3 = (\sqrt{5}-1)/(2\sqrt{15\tau}),$$

which implies that $\alpha_1 < \alpha_2 < \alpha_3$.

Since $N_\Gamma(\Sigma^{(3)})/\Sigma^{(3)} \cong \mathbb{Z}_4$ for old K and due to (21), in $\mathrm{Fix}\,\kappa_3(0,0,+)$ the angle between the intersection with $\mathrm{Fix}\,\kappa'_1(+,0,+,+)$ and any other isotropy axes is not smaller that α_1. Therefore, $j=2$ satisfies the condition **C6'**. Similar arguments imply that this condition is satisfied for $j=1,3,4$ as well.

Since for even K the elements $\kappa_3(0,0,+)$ and $\kappa_3(1,0,+)$ are not conjugate, the set (23) satisfies conditions **C1-C6** of lemma 8.

QED

4.2 The groups Γ in $O(4)$ but not in $SO(4)$

In this subsection we prove theorem 2 that completes the list of finite subgroups of $O(4)$, admitting pseudo-simple heteroclinic cycles.

A reflection in $\mathbb{R}^4$ can be expressed in the quaternionic presentation as $\mathbf{q} \to \mathbf{a}\tilde{\mathbf{q}}\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ is a pair of unit quaternions (see [6, 15]). We write this reflection as $(\mathbf{a};\mathbf{b})^*$. The transformations $\mathbf{q} \mapsto \mathbf{a}\tilde{\mathbf{q}}\mathbf{a}$ and $\mathbf{q} \mapsto -\mathbf{a}\tilde{\mathbf{q}}\mathbf{a}$ are respectively the reflections about the axis $\mathbf{a}$ and through the hyperplane orthogonal to the vector $\mathbf{a}$.

A group $\Gamma^* \subset O(4)$, $\Gamma^* \not\subset SO(4)$, can be decomposed as $\Gamma^* = \Gamma \oplus \sigma\Gamma$, where $\Gamma \subset SO(4)$ and $\sigma = (\mathbf{a};\mathbf{b})^* \notin SO(4)$. If Γ^* is finite, then in the quaternionic form of Γ

$$\Gamma = (\mathbf{L}\,|\,\mathbf{L}_K;\mathbf{R}\,|\,\mathbf{R}_K),\ \text{where } \mathbf{L} \cong \mathbf{R} \text{ and } \mathbf{L}_K \cong \mathbf{R}_K. \qquad (24)$$

Theorem 2 *A group* $\Gamma^* \subset O(4)$,

$$\Gamma^* = \Gamma \oplus \sigma\Gamma,\ \textit{where } \Gamma \subset SO(4) \textit{ and } \sigma \notin SO(4), \qquad (25)$$

admits pseudo-simple heteroclinic cycles, if and only if Γ *and* σ *are listed in table 3.*

Proof: Lemma 8 in [15] states that if a group Γ^* admits simple heteroclinic cycles, then so does Γ. By similar arguments the same holds true for pseudo-simple heteroclinic cycles. Therefore (see lemma 8), the group Γ has two sequences of isotropy subgroups Σ_j, Δ_j, $j = 1,\ldots,m$, satisfying conditions **C1**,**C3**,**C4**,**C5** and **C6***. Let Σ_1 be the subgroup satisfying **C5**, i.e. $\Sigma_1 \cong \mathbb{Z}_{k_1}$ with $k_1 \geq 3$. An element $\sigma' \in \Gamma^*$, $\sigma' \notin SO(4)$, maps $P_1 = \mathrm{Fix}\,\Sigma_1$ either to itself, or to another $P' = \mathrm{Fix}\,\Sigma'$ with $\Sigma' \cong \mathbb{Z}_{k_1}$.

Γ	σ
$(\mathbb{D}_{Kr}\,\vert\,\mathbb{Z}_K;\mathbb{D}_{Kr}\,\vert\,\mathbb{Z}_K)_s,\ K\gcd(r,K(1-s))\geq 3$	$-((0,1,0,0);(0,1,0,0))^*$
$(\mathbb{D}_{Kr}\,\vert\,\mathbb{Z}_{2K};\mathbb{D}_{Kr}\,\vert\,\mathbb{Z}_{2K})_s,\ K\gcd(r,K(1-s))\geq 3$	$((\cos\theta_0,0,0,\sin\theta_0);(1,0,0,0))^*$, $\theta_0=\pi/(2K)$

Table 3: Groups $\Gamma\oplus\sigma\Gamma\subset O(4)$ admitting pseudo-simple heteroclinic cycles

First, we assume the existence of σ', such that $\sigma' P_1 = P_1$. Hence, there exists $\sigma\in\Gamma^*$ which is a reflection through a hyperplane that contains P_1. Let the hyperplane be spanned by $\mathbf{e}_1$, $\mathbf{e}_3$ and $\mathbf{e}_4$ and $P_1 =<\mathbf{e}_1,\mathbf{e}_4>$. The hyperplane is mapped by elements of Σ_1 to

$$<\mathbf{e}_1,\mathbf{e}_4,\cos\theta_n\mathbf{e}_2+\sin\theta_n\mathbf{e}_3>,\ 0\leq n<k_1/2,\ \theta_n=2\pi n/k_1. \tag{26}$$

Any isotropy plane of Γ, that intersects with P_1, is $P(\theta',\theta_n)=<\cos\theta'\mathbf{e}_1+\sin\theta'\mathbf{e}_4,\cos\theta_n\mathbf{e}_2+\sin\theta_n\mathbf{e}_3>$. An isotropy plane $P'=\mathrm{Fix}\,\Sigma'\neq P_1$, such that $\Sigma'\cong\mathbb{Z}_{k'}$ with $k'\geq 3$, is orthogonal to all hyperplanes (26). Therefore (if such an isotropy plane exists), it is $<\mathbf{e}_2,\mathbf{e}_3>$. Any other isotropy plane of Γ (different from P_1, P' and $P(\theta',\theta_n)$) either intersects all hyperplanes (26) orthogonally, or the line of intersection belongs to P_1 or P'. Since there is no isotropy plane that satisfies these conditions, we conclude that the only isotropy planes of Γ are P_1, P' and $P(\theta',\theta_n)$. The groups listed in table 2 satisfying these conditions and (24) are $(\mathbb{D}_{Kr}\,|\,\mathbb{Z}_K;\mathbb{D}_{Kr}\,|\,\mathbb{Z}_K)_s$, $K\gcd(r,K(1-s))\geq 3$. The element σ acting as a reflection through $<\mathbf{e}_1,\mathbf{e}_3,\mathbf{e}_4>$ is $-((0,1,0,0),(0,1,0,0))^*$.

Second, we assume that there is no $\sigma\in\Gamma^*$, $\sigma\notin SO(4)$, such that $\sigma P_1=P_1$. Therefore, $\sigma\notin SO(4)$ satisfies $\sigma P_1=P'$, where $P'=\mathrm{Fix}\,\Sigma'$ with $\Sigma'\cong\mathbb{Z}_{k_1}$, and the subgroups Σ_1 and Σ' are not conjugate in Γ. The only groups in table 2 that contain such Σ_1 and Σ' are $(\mathbb{D}_{Kr}\,|\,\mathbb{Z}_{2K};\mathbb{D}_{Kr}\,|\,\mathbb{Z}_{2K})_s$, $K\gcd(r,K(1-s))\geq 3$. Moreover, $\Sigma'=\Sigma_3$ (see appendix C). The element σ maps a symmetry axis in P_1 to a symmetry axis in $\mathrm{Fix}\,\Sigma_3$. For definiteness, we assume that σ maps $\mathrm{Fix}\,\Delta_1$ to $\mathrm{Fix}\,\Delta_3$, where according to the appendices
$\Delta_1=<\kappa_2(1,0,0),\kappa_1(+,1)>$, $\Delta_3=<\kappa_2(0,0,0),\kappa_1(-,1)>$,
$\kappa_2(n,0,0)=((0,\cos n\theta_1,\sin n\theta_1,0);(0,1,0,0))$, $\kappa_1(\pm,1)=((\cos\theta,0,0,\pm\sin\theta);(\cos\theta,0,0,\sin\theta))$, $\theta_1=\pi/K$, $\theta=\pi/m$ and $m=K\gcd(r,K(1-s))$. Such σ is $((\cos\theta_0,0,0,\sin\theta_0);(1,0,0,0))^*$.
QED

Remark 5 *A heteroclinic cycle in a Γ^*-equivariant system, where in the decomposition (25) $\Gamma=(\mathbb{D}_{Kr}\,|\,\mathbb{Z}_{2K};\mathbb{D}_{Kr}\,|\,\mathbb{Z}_{2K})_s$, $K\gcd(r,K(1-s))\geq 3$ and $\sigma=((\cos\theta_0,0,0,\sin\theta_0);(1,0,0,0))^*$, in general is completely unstable. The proof follows the same arguments as the proof of theorem 1 in [16]. Similarly, the conditions for existence of a nearby periodic orbit are the ones given in theorems 3 and 4 in section 5 below.*

Remark 6 *A heteroclinic cycle in a* Γ^**-equivariant system, where in the decomposition (25)* $\Gamma = (\mathbb{D}_{Kr}\,|\,\mathbb{Z}_K;\mathbb{D}_{Kr}\,|\,\mathbb{Z}_K)_s,\ K\gcd(r,K(1-s))\geq 3$ *and* $\sigma = -((0,1,0,0),(0,1,0,0))^*$*, can be fragmentarily asymptotically stable. The conditions for stability can be obtained by algebra, which is standard (see, e.g., theorem 3 in [16]) but tedious; we do not present it here.*

5 Existence of nearby periodic orbits when $\Gamma \subset SO(4)$

As shown in [16], despite complete instability of a pseudo-simple heteroclinic cycle in a Γ-equivariant system for $\Gamma \subset SO(4)$, trajectories staying in a small neighbourhood of a pseudo-simple cycle for all $t > 0$ can possibly exist. Namely, it was proven *ibid* that in a one-parameter dynamical system an asymptotically stable periodic orbit can bifurcate from a cycle. More specifically, in their example such an asymptotically stable periodic orbit exists as long as a double positive eigenvalue is sufficiently small. Building blocks of the considered cycles were comprised of two equilibria, whose isotropy groups were isomorphic to $\mathbb{D}_3$. One of these equilibria had a multiple expanding eigenvalue, while the other equilibrium had a multiple contracting one. In this section we prove that similar periodic orbits can bifurcate in a more general setup – we do not restrict the number of equilibria in a building block (note that building block of a pseudo-simple cycle in $\mathbb{R}^4$ is comprised of at least two equilibria) and assume that their isotropy groups are isomorphic to $\mathbb{D}_k$ with $k \leq 4$. However, we assume that building block of a heteroclinic cycle involves only one equilibrium with a multiple expanding eigenvalue. In the case of several such equilibria, the bifurcation of a periodic orbit has codimension two or higher, which is beyond the scope of this paper. By contrast, no such periodic orbits bifurcate in a codimension one bifurcation if a building block involves an equilibrium with the isotropy group $\mathbb{D}_k$ with $k \geq 5$.

5.1 The case $\mathbb{D}_3$ and $\mathbb{D}_4$

Consider the Γ-equivariant system

$$\dot{\mathbf{x}} = f(\mathbf{x},\mu),\ \text{ where } f(\gamma\mathbf{x},\mu) = \gamma f(\mathbf{x},\mu) \text{ for all } \gamma \in \Gamma \subset SO(4), \tag{27}$$

and $f : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$ is a smooth map. We assume that the system possesses a pseudo-simple heteroclinic cycle with a building block $\{\xi_1 \to \ldots \xi_m;\ \gamma\}$. By $-c_j$, e_j and t_j we denote the non-radial eigenvalues of $df(\xi_j)$, $1 \leq j \leq m$. Let ξ_2 be an equilibrium with a two-dimensional expanding eigenspace (hence, $e_2 = t_2$) and a symmetry group $\Delta_2 = \mathbb{D}_k$, $k = 3$ or 4, acting naturally on the expanding eigenspace, and all other equilibria have one-dimensional expanding eigenspaces. A general $\mathbb{D}_k$-equivariant dynamical system in $\mathbb{C}$ in the leading order is $\dot{z} = \alpha z + \beta\bar{z}^2$ (for $k = 3$) and $\dot{z} = \alpha z + \beta_1 z^2\bar{z} + \beta_2\bar{z}^3$ ($k = 4$). A necessary condition for existence of a heteroclinic trajectory $\xi_2 \to \xi_3$ along the direction of real z is that $e_2 = \alpha > 0$ and $\beta > 0$ or $\beta_1 + \beta_2 > 0$ (for $k = 3$ or $k = 4$, respectively). Suppose that there exists $\mu_0 > 0$ such that

(i) $e_2 < 0$ for $-\mu_0 < \mu < 0$ and $e_2 > 0$ for $0 < \mu < \mu_0$;

(ii) for any $0 < \mu < \mu_0$ there exist heteroclinic connections $\kappa_j = (W_u(\xi_j) \cap P_j) \cap W_s(\xi_{j+1}) \neq \varnothing$, for all $1 \le j \le m$, where $\xi_{m+1} = \gamma \xi_1$.

Denote by X the group orbit of heteroclinic connections κ_j:

$$X = \cup_{\sigma \in \Gamma} \sigma \left(\bigcup_{1 \le j \le m} \kappa_j \right),$$

η is the product $\eta = \prod_{3 \le j \le m} \min(c_j/e_j, 1 - t_j/e_j)$, where we set $\eta = 1$ if $m = 2$, and $\zeta = 3$ (for $k = 3$) or $\zeta = 2\beta_2/(\beta_1 + \beta_2)$ (for $k = 4$).

Theorem 3

(a) If $\eta\zeta c_1 < e_1$ then there exist $\mu' > 0$ and $\delta > 0$, such that for any $0 < \mu < \mu'$ almost all trajectories escape from $B_\delta(X)$ as $t \to \infty$.

(b) If $\eta\zeta c_1 > e_1$ then generically there exists a periodic orbit bifurcating from X at $\mu = 0$. To be more precise, for any $\delta > 0$ we can find $\mu(\delta) > 0$ such that for all $0 < \mu < \mu(\delta)$ the system (27) possesses an asymptotically stable periodic orbit that belongs to $B_\delta(X)$.

We give the proof only for $k = 4$, for $k = 3$ it can be obtained by a simple modification combined with results of [16]. Since it follows closely the proof of theorem 2 *ibid*, some details are omitted and the reader is referred to that paper. We first formulate lemma 10 below, describing properties of trajectories of a generic $\mathbb{D}_4$-equivariant systems in $\mathbb{C}$, which in the leading order is

$$\dot{z} = \alpha z + \beta_1 z^2 \bar{z} + \beta_2 \bar{z}^3. \tag{28}$$

In polar coordinates, $z = r\mathrm{e}^{\mathrm{i}\theta}$, it takes the form

$$\begin{array}{rcl} \dot{r} & = & \alpha r + r^3(\beta_1 + \beta_2 \cos 4\theta), \\ \dot{\theta} & = & -\beta_2 r^2 \sin 4\theta. \end{array} \tag{29}$$

We assume that

$$\alpha > 0, \;\; \beta_2 > 0 \text{ and } \beta_1 + \beta_2 > 0. \tag{30}$$

The system has four invariant axes with $\theta = K\pi/4$, $K = 0, 1, 2, 3$. The two axes with even K are symmetric images of one another, as are the two axes with odd K. In case $\beta_1 - \beta_2 < 0$ there are four equilibria that are not at the origin with $r^2 = \alpha/(\beta_2 - \beta_1)$ and $\theta = (2k+1)\pi/4$, $k = 0, 1, 2, 3$. We consider the system in the sector $0 \le \theta < \pi/4$, the complement part of $\mathbb{C}$ is related to this sector by symmetries of the group $\mathbb{D}_4$.

Trajectories of the system satisfy

$$\frac{\mathrm{d}\tilde{r}}{\mathrm{d}\theta} = -\frac{2\alpha + 2\tilde{r}(\beta_1 + \beta_2 \cos 4\theta)}{\beta_2 \sin 4\theta}, \tag{31}$$

where we have denoted $\tilde{r} = r^2$. Re-writing this equation as

$$\frac{\mathrm{d}\tilde{r}}{\mathrm{d}\theta} + \tilde{r}\frac{2(\beta_1 + \beta_2 \cos 4\theta)}{\beta_2 \sin 4\theta} = -\frac{2\alpha}{\beta_2 \sin 4\theta},$$

multiplying it by $s(\theta)=(\sin 4\theta)^{(\beta_1+\beta_2)/2\beta_2}(1+\cos 4\theta)^{-\beta_1/2\beta_2}$ and integrating, we obtain that

$$r^2 s(\theta)=-\frac{2\alpha}{\beta_2}S(\theta)+C,\ \text{where}\ S(\theta)=\int_0^\theta \frac{s(\theta)}{\sin 4\theta}\mathrm{d}\theta, \tag{32}$$

which implies that

$$r^2 s(\theta)+\frac{2\alpha}{\beta_2}S(\theta)=r_0^2 s(\theta_0)+\frac{2\alpha}{\beta_2}S(\theta_0) \tag{33}$$

for the trajectory through the point (r_0,θ_0).

Lemma 10 *Let $\tau(r_0,\theta_0)$ denote the time it takes the trajectory of the system (29),(30) starting at (r_0,θ_0) to reach $r=1$ and $\vartheta(r_0,\theta_0)$ denote the value of θ at $r=1$. Then*

(i) $\tau(r_0,0)$ satisfies

$$\mathrm{e}^{\alpha\tau(r_0,0)}=\frac{r_0+\alpha/(\beta_1+\beta_2)}{r_0(1+\alpha/(\beta_1+\beta_2))}.$$

(ii) $\tau(r_0,\theta_0)$ satisfies

$$\tau(r_0,\theta_0)>\tau(r_0,0)\ \textit{for any}\ 0<\theta_0<\pi/4. \tag{34}$$

(iii) $\vartheta(r_0,\theta_0)$ satisfies

$$s(\vartheta(r_0,\theta_0))+\frac{2\alpha}{\beta_2}S(\vartheta(r_0,\theta_0))=r_0^2 s(\theta_0)+\frac{2\alpha}{\beta_2}S(\theta_0). \tag{35}$$

(iv) Given $C>0$, $\beta_1+\beta_2>0$ and $0<\theta_0<\pi/4$, for sufficiently small α and r_0

$$\mathrm{e}^{-C\tau(r_0,\theta_0)}\ll\vartheta(r_0,\theta_0).$$

The proof is similar to the proof of lemma 3(i-iv) in [16] and is omitted.

Proof of the theorem

As usual, we approximate trajectories in the vicinity of the cycle by superposition of local and global maps, $\phi_j:\ H_j^{(in)}\to H_j^{(out)}$ and $\psi_j:\ H_j^{(out)}\to H_{j+1}^{(in)}$, respectively, where $H_j^{(in)}$ and $H_j^{(out)}$ are cross sections transversal to the incoming and outgoing connections at an equilibrium ξ_j. We consider $g=\gamma\phi_1\psi_2\phi_2...\psi_m\phi_m\psi_1:\ H_1^{(out)}\to H_1^{(out)}$, where the γ is the symmetry in the definition of a building block. Since the expanding eigenspace of ξ_2 is two-dimensional, the contracting eigenspace of ξ_1 is two-dimensional as well. By the assumption of the theorem, other equilibria in the cycle have one-dimensional expanding and contracting eigenspaces. We employ the coordinates (w_j,q_j) in $H_j^{(in)}$ and (v_j,q_j) in $H_j^{(out)}$, similarly to [16]. We also employ the coordinates (ρ_1,θ_1) and (ρ_2,θ_2), in $H_1^{(out)}$ and $H_2^{(in)}$, respectively, such that $v_1=\rho_1\cos\theta_1$, $q_1=\rho_1\sin\theta_1$, $w_2=\rho_2\cos\theta_2$ and $q_2=\rho_2\sin\theta_2$. In the leading order the maps ϕ_1 is

$$(v_1^{(out)},q_1^{(out)})=\phi_1(w_1^{(in)},q_1^{(in)})=(v_{0,1}(w_1^{(in)})^{c_1/e_1},q_1^{(in)}(w_1^{(in)})^{c_1/e_1}),$$

which in polar coordinates takes the form

$$(\rho_1^{(out)}, \theta_1^{(out)}) = \phi_1(w_1^{(in)}, q_1^{(in)}) = (v_{0,1}(w_1^{(in)})^{c_1/e_1}, \arctan(q_1^{(in)}/v_{0,1})). \tag{36}$$

The maps ϕ_j, $j = 3, \dots, m$, are

$$(v_j^{(out)}, q_j^{(out)}) = \phi_j(w_j^{(in)}, q_j^{(in)}) = (v_{0,j}(w_j^{(in)})^{c_j/e_j}, q_j(w_j^{(in)})^{-t_j/e_j}). \tag{37}$$

(Here superscripts indicate coordinates in $H_1^{(in)}$ or $H_1^{(out)}$. Below, where it does not create ambiguity, we do not use superscripts.) In the leading order the map ψ_1 is

$$(\rho_2, \theta_2) = \psi_1(\rho_1, \theta_1) = (A\rho_1, \theta_1 + \Theta), \tag{38}$$

where generically $\Theta \neq N\pi/4$ for $N = 1, ..., 8$. The maps ψ_j, $j = 2, ..., m$, are

$$(w_1, q_1) = \psi_j(v_2, q_2) = (B_{j,11}v_2 + B_{j,12}q_2, B_{j,21}v_2 + B_{j,22}q_2). \tag{39}$$

Because of (i), for small μ the expanding eigenvalue of ξ_2 depends linearly on μ, therefore without restriction of generality we can assume that $e_2 = \mu$. Generically, all other eigenvalues and coefficients in the expressions for local and global maps do not vanish for sufficiently small μ and are of the order of one. We assume them to be constants independent of μ. From (ii), the eigenvalues satisfy $e_1 > 0$, $-c_1 < 0$ and $-c_2 < 0$.

For small enough $\tilde{\delta}$, in the scaled neighbourhoods $B_{\tilde{\delta}}(\xi_2)$ the restriction of the system to the unstable manifold of ξ_2 in the leading order is $\dot{z} = \mu z + \beta_1 z^3 + \beta_2 \bar{z}^3$, where we have denoted $z = w_2 + \mathrm{i}q_2$. We assume that the local bases near ξ_1 and ξ_2 are chosen in such a way that the heteroclinic connections $\gamma^{-1}\xi_m \to \xi_1$ and $\xi_2 \to \xi_3$ go along the directions $\arg(\theta_j) = 0$ for both $j = 1, 2$. In the complement subspace the system is approximated by the contractions $\dot{u} = -r_2 u$ and $\dot{v} = -c_2 v$. In terms of the functions $\tau(r, \theta)$ and $\vartheta(r, \theta)$ introduced in lemma 10, the map ϕ_2 is

$$(v_2, q_2) = \phi_2(\rho_2, \theta_2) = (v_{0,2}\mathrm{e}^{-c_2\tau(\rho_2,\theta_2)}, \sin\vartheta(\rho_2, \theta_2)).$$

According to lemma 10(iv), for small ρ_2 and μ

$$\mathrm{e}^{-c_2\tau(\rho_2,\theta_2)} \ll \sin\vartheta(\rho_2, \theta_2),$$

which implies that the superposition $\psi^* = \psi_3 ... \psi_m \phi_m \psi_1$ can be approximated as $\psi^*(v_2, q_2) \approx (B_{1,*}q_2^\eta, B_{2,*}q_2^\eta)$, where $\eta = \prod_{3 \le j \le m} \min(c_j/e_j, 1 - t_j/e_j)$ and the constants $B_{1,*}$ and $B_{2,*}$ depend on $B_{j,kl}$, $2 \le j \le m$, and eigenvalues of $\mathrm{d}f(\xi_j)$, $3 \le j \le m$. For small θ_1 we have $\sin\theta_1 \approx \tan\theta_1 \approx \theta_1$. Taking into account (36), (38) and lemma 10(iii), we obtain that

$$g(\rho_1, \theta_1) \approx \Big(C_1(\rho_1^2 A\beta_2 s(\Theta) + \mu S(\Theta))^{\eta\zeta c_1/2e_1}, C_2(\rho_1^2 A\beta_2 s(\Theta) + \mu S(\Theta))^{\eta\zeta/2}\Big), \tag{40}$$

where we have denoted $\zeta = 2\beta_2/(\beta_1 + \beta_2)$, $C_1 = v_{0,1}\beta_2^{-\eta\zeta c_1/2e_1}|B_{1,*}|^{c_1/e_1}$ and $C_2 = v_{0,1}^{-1}4^{-1}\beta_2^{-\eta\zeta/2}B_{2,*}$.

(a) From (40), the ρ-component of g satisfies

$$g_\rho(\rho_1,\theta_1) > C_3\rho_1^{\eta\zeta c_1/e_1},\ \text{ where } C_3 = C_1(A\beta_2 s(\Theta))^{\eta\zeta c_1/2e_1},$$

hence if $\eta\zeta c_1 < e_1$ then for any $0 < \delta < C_3^{e_1/(e_1-c_1\eta\zeta)}$ the iterates $g^n(\rho_1,\theta_1)$ with initial $0<\rho_1<\delta$ satisfy $g^n_\rho(\rho_1,\theta_1) > \delta$ for sufficiently large n.

(b) Assume that $\eta\zeta c_1 > e_1$. Existence and stability of a fixed point of the map g (40) for small μ can be proven by the same arguments as employed to prove theorem 2(b) in [16]. We omit the proof. The fixed point can be approximated by $(\rho_p,\theta_p) = (C_1(\mu S(\Theta))^{\eta\zeta c_1/2e_1}, C_2\mu S(\Theta)^{\eta\zeta/2})$. This fixed point is an intersection of a periodic orbit with $H_1^{(out)}$. The distance from (ρ_p,θ_p) to X depends on μ as $\mu^{c_1\eta\zeta/2e_1}$, therefore the trajectory approaches X as $\mu\to 0$. **QED**

5.2 The case $\mathbb{D}_k$, $k\geq 5$

In this subsection we prove that a bifurcation of a periodic orbit, that was discussed in the previous subsection, does not take place for $k\geq 5$:

Theorem 4 *Suppose that for $0<\mu<\mu_0$ the system (27) possesses a pseudo-simple heteroclinic cycle $X=\xi_1\to\ldots\to\xi_M$, where ξ_2 has a two-dimensional expanding eigenspace with the associated eigenvalue $e_2=\mu$ and the symmetry group $\Delta_2=\mathbb{D}_k$, $k\geq 5$, acting naturally on the expanding eigenspace. There exist $\varepsilon>0$ and $\mu'>0$, such that for any $0<\mu<\mu'$ almost all trajectories $\Phi(x,t)$ of the system (27), such that $d(\Phi(x_0,0),X)<\varepsilon$, satisfy $d(\Phi(x_0,t_0),X)>\varepsilon$ for some $t_0>0$. By $d(\cdot,\cdot)$ we denoted the distance between a point and a set.*

Proof: Similarly to the proof of theorem 1 in [16], we consider the map $\phi_2\psi_1\phi_1 : H_1^{(in)}\to H_2^{(out)}$ and prove existence of $\varepsilon>0$ such that

$$\phi_2\psi_1\phi_1(H_1^{(in)}(\varepsilon))\cap H_2^{(out)}(\varepsilon)=\varnothing, \tag{41}$$

where

$$H_1^{(in)}(\varepsilon)=\{(w,q)\in H_1^{(in)}\ :\ |(w,q)|<\varepsilon\} \text{ and } H_2^{(out)}(\varepsilon)=\{(v,q)\in H_2^{(out)}\ :\ |(v,q)|<\varepsilon\}.$$

Equation (41) shows that all points in $H_1^{(in)}(\varepsilon)$ are mapped outside $H_2^{(out)}(\varepsilon)$, which implies the statement of the theorem.

The maps ϕ_1 and ψ_1 are the same as for the $\mathbb{D}_4$ system, they are given by (36) and (38), respectively. In (38) generically $\Theta\neq N\pi/k$ for $N=1,2,...,2k$. Moreover, there exist $\Theta'>0$ and $\mu'>0$, such that $\min_{1\leq N\leq 2k}|\Theta-N\pi/k|>\Theta'$ for all sufficiently small δ and $0<\mu<\mu'$ (recall that δ is the distance from $H_2^{(in)}$ and $H_2^{(out)}$ to ξ_2).

For small enough δ, in a δ-neighbourhoods of ξ_2 the restriction of the system to the unstable manifold of ξ_2 in the leading order is $\dot z=\mu z+\beta_1 z^3+\beta_2\bar z^{k-1}$, where $z=w_2+\mathrm{i}q_2$.

In polar coordinates the system takes the form $\dot{r} = \alpha r + r^3\beta_1$, $\dot{\theta} = -\beta_2 r^{k-1}\sin k\theta$, which implies that the map $\phi_2(\rho_2^{in},\theta_2^{in}) = (v_2^{out}, q_2^{out})$ satisfies $q_2^{out} = \delta\tan\theta_2^{out}$ and

$$|\theta_2^{out} - \theta_2^{in}| < \int_{\rho_2^{in}}^{\delta} \frac{|\beta_2|}{|\beta_1|} r^{k-5}\mathrm{d}r = \frac{|\beta_2|}{|\beta_1|(k-4)}(\delta^{k-4} - (\rho_2^{in})^{k-4}). \tag{42}$$

We choose $0 < \delta < \Theta'/4$ and set

$$0 < \varepsilon < \min\left(\delta\tan\frac{\Theta'}{4},\ v_{0,1}\tan\frac{\Theta'}{4}\right). \tag{43}$$

Any $(w_1, q_1) \in H_1^{(in)}(\varepsilon)$ satisfies $q_1 < \varepsilon$, therefore (36) and (43) imply that $\theta_1 < \Theta'/4$. Hence, due to (38),(42) and (43), $|\theta_2 - N\pi/k| > \Theta'/4$ for any N. The steady state ξ_2 has k symmetric copies (under the action of symmetries $\sigma \in \Sigma_2$) of the heteroclinic connection $\kappa_2 : \xi_2 \to \xi_3$ which belong to the hyperplanes $\theta_2 = N\pi/k$ with some integer N's. Due to (42) and (43), the distance of (v_2, q_2) to any of these hyperplanes is larger than $\delta\tan(\Theta'/4)$, which implies (41). **QED**

6 Example: periodic orbit near a heteroclinic cycle in a $(\mathbb{D}_4 \,|\, \mathbb{D}_2; \mathbb{D}_4 \,|\, \mathbb{D}_2)$-equivariant system

We proved (see section 5) that an attracting periodic orbit can exist near a pseudo-simple heteroclinic cycle if the isotropy subgroup of one of its equilibria is $\mathbb{D}_3$ or $\mathbb{D}_4$. For the case when the isotropy subgroup is $\mathbb{D}_3$, examples of Γ-equivariant systems possessing periodic orbit in a neighbourhood of a heteroclinic cycle were given in [16] for $\Gamma = (\mathbb{D}_3 \,|\, \mathbb{Z}_1; \mathbb{D}_3 \,|\, \mathbb{Z}_1)$ and $\Gamma = (\mathbb{D}_3 \,|\, \mathbb{Z}_2; \mathbb{O} \,|\, \mathbb{V})$. The vector fields considered *ibid* were third order normal forms commuting with the considered actions of Γ. Here we present a numerical example of a heteroclinic cycle with a nearby attracting periodic orbit, where the isotropy subgroup of an equilibrium is $\mathbb{D}_4$ and a Γ-equivariant vector field is constructed using ideas employed in the proofs of lemma 8 and theorem 3.

We consider a Γ-equivariant dynamical system where $\Gamma = (\mathbb{D}_4 \,|\, \mathbb{D}_2; \mathbb{D}_4 \,|\, \mathbb{D}_2)$ (recall that the quaternionic group $\mathbb{D}_2$ is usually denoted by $\mathbb{V}$). The elements of Γ are:

$$\begin{array}{rcl}
(\mathbb{V} & ; & \mathbb{V}),\\
((1,0,0,\pm 1)/\sqrt{2} & ; & (\pm 1,0,0,\pm 1)/\sqrt{2}),\\
((1,0,0,\pm 1)/\sqrt{2} & ; & (0,\pm 1,\pm 1,0)/\sqrt{2}),\\
((0,1,\pm 1,0)/\sqrt{2} & ; & (\pm 1,0,0,\pm 1)/\sqrt{2}),\\
((0,1,\pm 1,0)/\sqrt{2} & ; & (0,\pm 1,\pm 1,0)/\sqrt{2}).
\end{array}$$

The group has five isotropy types of subgroups Σ satisfying $\dim\mathrm{Fix}\,\Sigma = 2$ (see appendix A). In agreement with appendix C, we take $\Sigma_1 =< \kappa_1 >\cong \mathbb{Z}_4$ and $\Sigma_2 =< \kappa_5 >\cong \mathbb{Z}_2$. For convenience, we use different notation for generating elements. Namely, we write $\Sigma_1 =< \gamma_1 >$ and $\Sigma_2 =< \gamma_2 >$, where

$$\gamma_1(s) = ((1,0,0,1)/\sqrt{2}; (1,0,0,(-1)^s)/\sqrt{2}) \text{ and}$$

$$\gamma_2(q,r,t) = ((0,1,(-1)^q,0)/\sqrt{2}; (0,(-1)^r,(-1)^t,0)/\sqrt{2}).$$

The action $(\mathbf{l};\mathbf{r}) : \mathbf{x} \to \mathbf{lxr^{-1}}$ on $\mathbb{R}^4$ of (some) elements of Γ is

$(\mathbf{l};\mathbf{r})$	$\mathbf{x} \to \mathbf{lxr^{-1}}$
$((0,0,0,1);(0,0,0,1))$	$\mathbf{x} \to (x_1,-x_2,-x_3,x_4)$
$((0,0,1,0);(0,0,1,0))$	$\mathbf{x} \to (x_3,x_4,-x_1,-x_2)$
$((1,0,0,1)/\sqrt{2};(1,0,0,1)/\sqrt{2})$	$\mathbf{x} \to (x_1,x_3,-x_2,x_4)$
$((0,1,1,0)/\sqrt{2};(0,1,1,0)/\sqrt{2})$	$\mathbf{x} \to (x_1,x_3,x_2,-x_4).$

(44)

The isotropy planes can be labelled as follows:

$$P_1(s) = \mathrm{Fix}\,\Sigma_1(s), \qquad \text{where } \Sigma_1(s) =< \gamma_1(s) >,$$

$$P_2(q,r,t) = \mathrm{Fix}\,\Sigma_2(q,r,t), \quad \text{where } \Sigma_2(q,r,t) =< \gamma_2(q,r,t) >,$$

hence there exist two different planes P_1 with $s = 0,1$ and eight different planes P_2 corresponding to $q,r,t = 0,1$. A plane P_1 contain four symmetry axes of two isotropy types with isotropy groups of the axes isomorphic to $\mathbb{D}_4$. An axis is an intersection of P_1 with two planes P_2 (and also with two other planes fixed by κ_4, that is irrelevant), namely $P_1(s)$ intersects with $P_2(0,r,t)$ and $P_2(1,r+s,t+s+1)$. The axes split into two isotropy classes, with odd or even $s+r+t$. A plane P_2 contains two isotropy axes which are intersections with $P_1(0)$ and $P_1(1)$.

We choose $P_1(0) = (x_1,0,0,x_4)$, $P_2(0,0,0) = (x_1,x_2,x_2,0)$ and, in agreement with (4), set

$$\mathbf{h}_1(r_1,\theta_1) = (r_1(1-r_1), \sin(4\theta_1), \ \mathbf{h}_2(r_2,\theta_2) = (r_2(1-r_2), -\sin(2\theta_1), \qquad (45)$$

where $x_1 = r_1\cos\theta_1$ and $x_4 = r_1\sin\theta_1$ in P_1 and $x_1 = r_2\cos\theta_2$ and $x_2 = (r_2\sin\theta_2)/\sqrt{2}$ in P_2. Hence, $\xi_1 \approx (1/\sqrt{2},0,0,-1/\sqrt{2}) \in P_1(0)\cap P_2(0,1,0)$ is unstable in P_1 and stable in P_2; $\xi_2 \approx (1,0,0,0) \in P_1(0)\cap P_2(0,0,0)$ is stable in P_1 and unstable in P_2. Following the proof of lemma 8, we construct the system (5)-(7) that possesses a heteroclinic cycle with a building block $\xi_1 \to \xi_2 \to \gamma\xi_1$, where $\gamma = ((1,0,0,0);(0,0,1,0))$. In agreement with theorem 1 in [16], the cycle is not asymptotically stable, hence trajectories starting near the cycle escape from it (see fig. 2(a)).

Theorem 3 states that a periodic orbit exists near a heteroclinic cycle with $\Delta_2 \cong \mathbb{D}_4$ if the multiple expanding eigenvalue e_2 is sufficiently small and $2c_1\beta_2/(\beta_1+\beta_2) > e_1$ (recall that α, β_1 and β_2 are the coefficients of the system (28)). To be more precise, in the proof we use the fact that the ratio α/β_2 is small. Therefore, we introduce a modified system

$$\dot{\mathbf{x}} = \mathbf{f}^*(\mathbf{x}), \text{ where } \mathbf{f}^*(\mathbf{x}) = \mathbf{f}(\mathbf{x}) + \sum_{\gamma^*\in G_2} \gamma^*\mathbf{g}^*((\gamma^*)^{-1}\mathbf{x}), \qquad (46)$$

$$\mathbf{g}^*(\mathbf{y}) = (0, 0, \frac{cy_3^3}{1+B|\pi^\perp \mathbf{y}|^2}, \frac{-by_4y_3^2}{1+B|\pi^\perp \mathbf{y}|^2}), \tag{47}$$

$G_2 = \Gamma/N_\Gamma(\Sigma_2(0,0,0,0))$ and $\mathbf{y} = (x_1, x_2, (x_3+x_4)/\sqrt(2), (x_3-x_4)/\sqrt(2))$.

In a small neighbourhood of ξ_2 the projection of the local field (46) into the plane $x_1 = x_2 = 0$ is

$$\dot{y}_3 = ay_3 + cy_3^3 - by_3y_4^2, \quad \dot{y}_4 = ay_4 + cy_4^3 - by_4y_3^2.$$

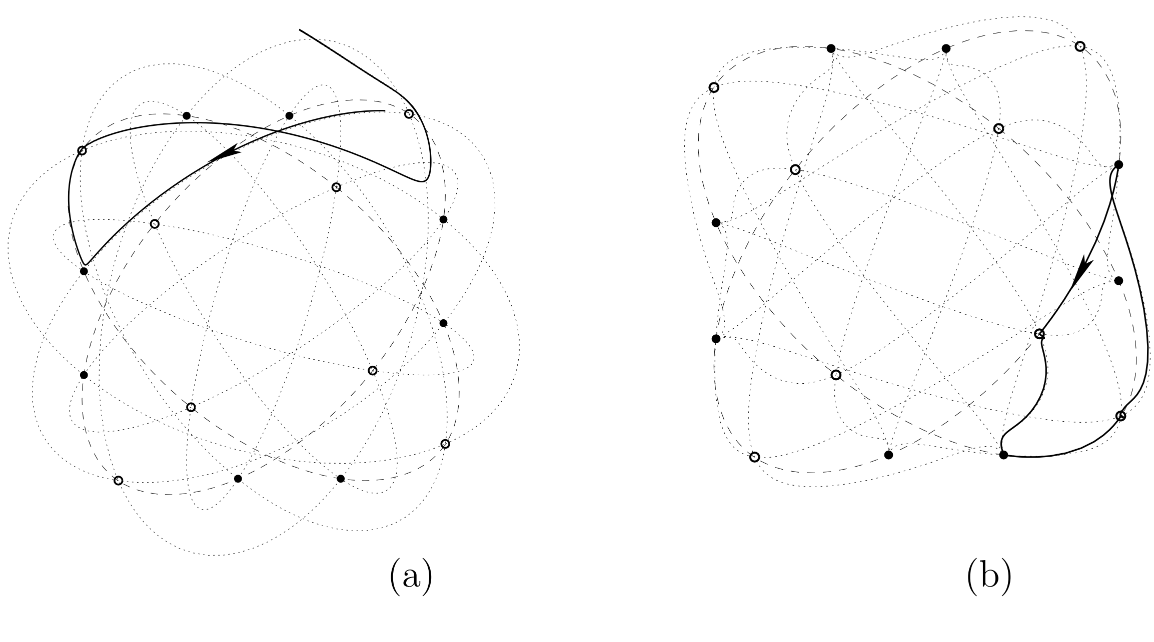
Comparing the above expression with (28), we obtain that

$$\beta_1 = (c-3b)/2 \text{ and } \beta_2 = (c+b)/4.$$

If the coefficient B in (47) is sufficiently large, then by the same arguments as applied in the proof in lemma 8, the system (46) possesses the heteroclinic cycle $\xi_1 \to \xi_2 \to \gamma\xi_1$. Theorem 3 indicates that for sufficiently large $b \gg c > 0$ there exists a stable periodic orbit close to the cycle. Therefore, we set

$$B = 100, \ b = 1000, \ c = 0.1 \ . \tag{48}$$

In agreement with our arguments, the system (46)-(48) has an attracting periodic orbit near the heteroclinic cycle, as shown on fig. 2(b).



(a) (b)

Figure 2: Projection of the heteroclinic connections $\xi_2 \to \xi_1$ (dashed lines), $\xi_1 \to \xi_2$ (dotted lines), a trajectory of the system $\dot{\mathbf{x}} = f(\mathbf{x})$) (a) and a periodic orbit of the system $\dot{\mathbf{x}} = f^*(\mathbf{x})$) (b) (solid lines) into the plane $< \mathbf{v}_1, \mathbf{v}_2 >$, where $\mathbf{v}_1 = (4, 2, 4, 1.5)$ and $\mathbf{v}_2 = (2, 4, -1.5, 4)$, (a). The steady state ξ_1 is denoted by a hollow circle and ξ_2 by filled one.

7 An example of stability when $\Gamma \not\subset SO(4)$

In this section we show that a family of subgroups $\Gamma \subset O(4)$, $\Gamma \not\subset SO(4)$, admits heteroclinic cycles involving multidimensional heteroclinic orbits. Following [3], we call such heteroclinic cycles *generalized*. We derive conditions for asymptotic stability of such generalized cycle and show that it involves as a subset a pseudo-simple heteroclinic cycle, that can be fragmentarily asymptotically stable. Numerical studies indicate that addition of small perturbation that breaks an $O(4)$ symmetry can result on emergence of asymptotically stable periodic orbit or on chaotic dynamics in the vicinity of a pseudo-simple heteroclinic cycle.

We shall in fact consider a class of subgroups of $O(4)$ defined as follows.
Let $(x_1, y_1, x_2, y_2) \in \mathbb{R}^4$ and $z_j = x_j + iy_j$. Fix an integer $n \geq 3$ and let Γ be the group generated by the transformations

$$\rho : \ (z_1, z_2) \mapsto (z_1, e^{\frac{2\pi i}{n}} z_2), \quad \kappa(z_1, z_2) \mapsto (\bar{z}_1, z_2), \quad \sigma : \ (z_1, z_2) \mapsto (z_1, \bar{z}_2) \tag{49}$$

(Choosing coordinates $z_1 = q_1 + iq_4$ and $z_2 = q_2 + iq_3$, we obtain that in quaternionic presentention the $SO(4)$ subgroup of Γ is $(\mathbb{D}_n \,|\, \mathbb{Z}_1; \mathbb{D}_n \,|\, \mathbb{Z}_1)$, in agreement with theorem 2.) This group action decomposes $\mathbb{R}^4$ into the direct sum of three irreducible representations of the dihedral group $\Gamma = \mathbb{D}_n \times \mathbb{Z}_2$:
(i) the trivial representation acting on the component x_1,
(ii) the one-dimensional representation acting on y_1 by $\kappa y_1 = -y_1$,
(iii) the two-dimensional natural representation of $\mathbb{D}_n$ acting on $z_2 = (x_2, y_2)$.
There are four types of fixed-point subspaces for this action:

- $L = P_1 \cap P_2 = Fix(\Gamma)$,
- $P_1 = \{(x_1, y_1, 0, 0)\} = Fix(\rho, \sigma)$,
- $P_2 = \{(x_1, 0, x_2, 0)\} = Fix(\kappa, \sigma)$,
- $V = \{(x_1, y_1, x_2, 0)\} = Fix(\sigma)$,
- $W = \{(x_1, 0, x_2, y_2)\} = Fix(\kappa)$.

When n is even there are two more types of invariant subspaces:

- $P_2' = \{(x_1, 0, x_2 \cos(\pi/n), x_2 \sin(\pi/n))\} = Fix(\kappa, \rho\sigma)$,
- $V' = \{(x_1, y_1, x_2 \cos(\pi/n), x_2 \sin(\pi/n)))\} = Fix(\rho\sigma)$.

Note that P_1 is fixed by Γ. When n is odd P_2 and V have $n-1$ symmetric copies $\rho^j P_2$, $\rho^j V$, $j = 1, \ldots, n-1$. When n is even each of P_2, P_2', V, V' has $n/2 - 1$ symmetric copies.
It can be shown that for an open set of Γ-equivariant vector fields, there exists an equilibrium ξ_1 on the negative semi-axis in L, an equilibrium ξ_2 on the positive semi-axis, and heteroclinic orbits lying in the planes P_1 and P_2 and realizing a cycle between ξ_1 and ξ_2. Moreover this cycle is pseudo-simple due to the action of the rotation ρ on the plane P_2, which forces the

eigenvalues along the x_2 direction in P_2 to be double. To fix ideas we assume the double eigenvalue is stable at ξ_1 and unstable at ξ_2. In order to study the stability of this pseudo-simple cycle we shall exploit a property that was observed in the case $n = 3$ in [16] and appears to also occur when $n > 3$. First, the two dimensional unstable manifold at ξ_2 lies entirely in the invariant subspace W, which also contains the axis L. Second, for an open set of vector fields any orbit on this unstable manifold lies in the stable manifold of ξ_1, hence realizing a two dimensional manifold of saddle-sink connections in W. Therefore the pseudo-simple heteroclinic cycle is part of a cycle involving multidimensional heteroclinic orbits, which was called a generalized heteroclinic cycle in [3]. Let us prove this claim.

Proposition 1 *There exists an open set $\mathcal{V}$ of Γ-equivariant smooth vector fields which possess a generalized heteroclinic cycle. This cycle, which we denote by $\mathcal{X}$, connects two equilibria ξ_1 and ξ_2 which lie on the negative, resp. positive semi axis in L. It is composed of a single heteroclinic orbit in P_1 and a two dimensional manifold of heteroclinic orbits in the space W. This manifold in W contains heteroclinic orbits in P_2 and in P_2' (when n is even), which realize two isotropy types of pseudo-simple heteroclinic cycles.*

Proof: Let us consider the group Γ_∞ defined by relations (49) where we replace the transformation ρ by $\rho_\varphi\ (z_1, z_2) \mapsto (z_1, e^{i\varphi} z_2)$, $\varphi \in S^1$. This group has the same invariant subspaces as Γ, but in addition any copy of the plane P_2 by ρ_φ is also invariant, and moreover W is spanned by letting ρ_φ rotate P_2 with any $\varphi \in S^1$. Therefore if a saddle-sink connection between equilibria ξ_1, ξ_2 lying on L exists in P_2, then a two dimensional manifold of connections exists in W. The fact that such equilibria and connections exist for an open set of smooth vector fields follows from a slight adaptation of lemma 8, which shows that the group Γ_∞ *admits* robust heteroclinic cycles with connections in P_1 and P_2. Since any Γ equivariant perturbation of this vector field leaves W, P_1 and P_2 invariant, we conclude by structural stability that generalized heteroclinic cycles persist for an open set of Γ equivariant smooth vector fields. The same argument applies if replacing P_2 by P_2' when n is even. **QED**

We denote by $e_j > 0$ and $-c_j < 0$ the non-radial eigenvalues at ξ_j, $j = 1, 2$, and further assume that $-c_1$ and e_2 are the double eigenvalues. Hence ξ_2 is a source while ξ_1 is a sink in W, along the eigendirections (x_2, y_2).

Theorem 5 *The generalized heteroclinic cycle $\mathcal{X}$ defined in Proposition 1 is asymptotically stable if $c_1c_2 > e_1e_2$ and is completely unstable if $c_1c_2 < e_1e_2$. Moreover there exists an open subset of $\mathcal{V}$ such that for any vector field in this subset, a pseudo-simple heteroclinic subcycle of $\mathcal{X}$ is fragmentarily asymptotically stable.*

Proof: As usual we want to define a first return map in the vicinity of the heteroclinic cycle, and to do so we decompose the dynamics close to $\mathcal{X}$ into local maps and global transition maps between suitably chosen cross-sections to the heteroclinic orbits near the equilibria. Possibly after a smooth Γ-equivariant change of coordinates we can always assume that in a neighborhood of the equilibria their stable and unstable manifolds are linear. Let v_j, resp.

$r_j e^{i\theta_j}$ denote the local coordinates near ξ_j along y_1, resp. z_2. The "radial" direction (along the axis L, coordinate x_1) can be neglected. We define the cross-sections along the (single) heteroclinic orbit from ξ_1 to ξ_2 (Fig. 3) by

$$\begin{aligned} H_1^{out} &= \{(v_1 = \varepsilon, r_1 > 0, \theta_1 < \pi/n)\} \\ H_2^{in} &= \{(v_2 = \varepsilon, r_2 > 0, \theta_2 < \pi/n)\} \end{aligned} \tag{50}$$

where $\varepsilon > 0$ is a small constant value.

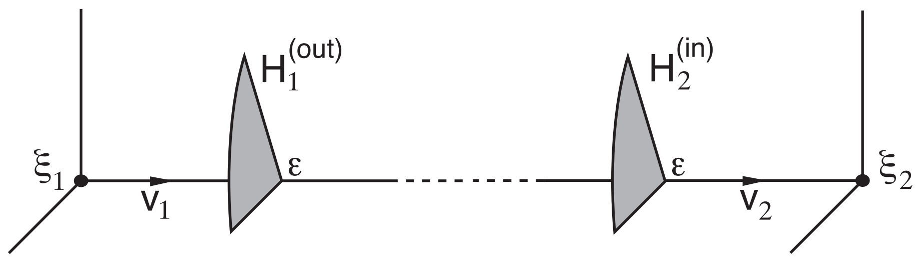


Figure 3: Cross-sections to the heteroclinic orbit $\xi_1 \to \xi_2$.

Similarly we define the cross-sections along the two-dimensional manifold of connections from ξ_2 to ξ_1 by (see Fig. 4):

$$\begin{aligned} H_1^{in} &= \{(v_1 > 0, r_1 = \varepsilon, \theta_1 < \pi/n)\} \\ H_2^{out} &= \{(v_2 > 0, r_2 = \varepsilon, \theta_2 < \pi/n)\} \end{aligned} \tag{51}$$

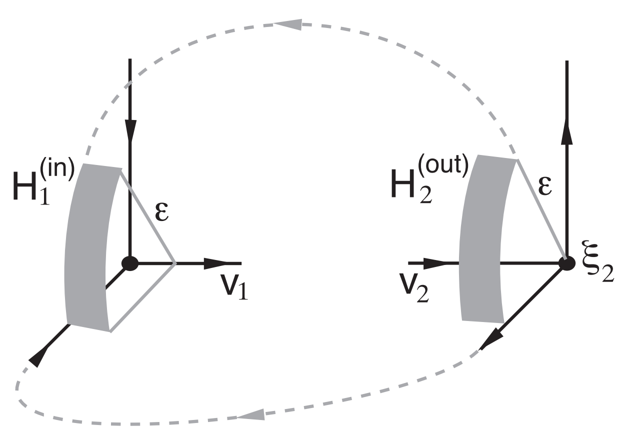


Figure 4: Cross-sections to the heteroclinic manifold $\xi_2 \to \xi_1$.

The boundaries of the cross-sections at the limit values $\theta_j = 0$ ($j = 1, 2$) lie in the space V while at $\theta_j = \pi/n$ they lie in the space ρV (when n is odd) or V' (when n is even). Since these spaces are flow-invariant, the sections defined above are mapped to each other by the flow in the order $H_1^{in} \to H_1^{out} \to H_2^{in} \to H_2^{out} \to H_1^{in}$. We can therefore define the local first hit maps $\Phi_j \;:\; H_j^{(in)} \to H_j^{(out)}$ and global maps $\Psi_j \;:\; H_j^{(out)} \to H_j^{(in)}$, $j = 1, 2$.

By choosing ε small enough and if non resonance conditions are satisfied between the eigenvalues at each equilibrium, we can approximate the local vector fields by their linear parts. Therefore near ξ_1 the flow is defined by the equations

$$\frac{d(r_1 e^{i\theta_1})}{dt} = -c_1 r_1 e^{i\theta_1} \quad \text{and} \quad \frac{dv_1}{dt} = e_1 v_1 \ ,$$

which gives

$$(r_1', \theta_1') = \Phi_1(v_1, \theta_1) = (v_1^{c_1/e_1}, \theta_1) \tag{52}$$

and near ξ_2 the flow is defined by

$$\frac{dv_2}{dt} = -c_2 v_2 \quad \text{and} \quad \frac{d(r_2 e^{i\theta_2})}{dt} = e_2 r_2 e^{i\theta_2} \ ,$$

which gives

$$(v_2', \theta_2') = \Phi_2(r_2, \theta_2) = (r_2^{c_2/e_2}, \theta_2) \tag{53}$$

The far map Ψ_1 is a Γ-equivariant near identity diffeomorphism which can be linearized under generic conditions. We therefore have

$$\Psi_1(r_1, \theta_1) = (ar_1, \theta_1) \tag{54}$$

where a is a positive constant.
The far map Ψ_2 is also Γ-equivariant, however it is not near identity and it can't be expressed as simply as Ψ_1. Let us set

$$\Psi_2(v_2, \theta_2) = (v(v_2, \theta_2), \theta(v_2, \theta_2)) \tag{55}$$

The component $v(v_2, \theta_2)$ vanishes when $v_2 = 0$, hence there exists a smooth function h such that $v_1'(v_2, \theta_2) = v_2 h(v_2, \theta_2)$. Moreover because Ψ_2 is a diffeomorphism $h(0, \theta_2) \neq 0$, which allows by a smooth change of variables to set $v_1'(v_2, \theta_2) = v_2 b(\theta_2)$ where b is a bounded function. The map $k(\theta_2) := \theta(0, \theta_2)$ is differentiably defined in the interval $[0, \pi/n]$ and has fixed-points at 0 and π/n.

Now we can define the first return map in H_1^{in} by $g = \Psi_2 \circ \Phi_2 \circ \Psi_1 \circ \Phi_1$ and we write $(v_1', \theta_1') = g(v_1, \theta_1)$.
Applying the above expressions for Φ_j and Ψ_j one obtains

$$v_1' = b(\theta_1) a^{\frac{c_2}{e_2}} v_1^{\frac{c_1 c_2}{e_1 e_2}} \tag{56}$$

Since b is a bounded function the iterates of the first component of g tend to 0 if and only if $c_1 c_2 > e_1 e_2$. This proves the first part of the theorem.
The second component of g has the form

$$\theta_1' = \theta(a^{\frac{c_2}{e_2}} v_1^{\frac{c_1 c_2}{e_1 e_2}}, \theta_1) \tag{57}$$

Assume $c_1c_2 > e_1e_2$, then by iteration the first argument of the function θ tends to 0. Therefore the dynamics of θ converges to the dynamics of the map k. By an argument similar to Prop. 4.9 of [9], k has generically hyperbolic fixed points at 0 and π/n. Moreover there exists an open subset of $\mathcal{V}$ such that for vector fields in this subset, k has no fixed point inside $(0, \pi/n)$. In this case we can conclude that the iterates of g converge to a pseudo-simple heteroclinic cycle. **QED**

In order to illustrate this result we built a $\mathbb{D}_n$ equivariant polynomial system with $n > 2$ satisfying the hypotheses of the theorem and performed numerical simulations. We use bifurcation method to find the equilibria and corresponding heteroclinic orbits. Applying classical methods in computing equivariant bifurcation systems [7] we construct

$$\begin{array}{l} \dot{z}_1 = a_1z_1 + a_2\bar{z}_1 + a_3z_1^2 + a_4\bar{z}_1^2 + a_5z_1\bar{z}_1 + a_6z_2\bar{z}_2 + a_7z_1^2\bar{z}_1 + a_8z_1z_2\bar{z}_2 + a_9(z_2^n + \bar{z}_2^n) \\ \dot{z}_2 = z_2[b_1 + b_2(z_1 + \bar{z}_1) + b_3z_1\bar{z}_1 + b_4z_2\bar{z}_2] + b_5\bar{z}_2^{n-1}, \end{array} \tag{58}$$

where a_1, a_2 and b_1 are small parameters. Suitable coefficient values for the system to possess generalized heteroclinic cycles can be found as was done in [16] in the $n = 3$ case.

We additionally assume that $a_3 + a_4 + a_5$ is close to 0 in order to ensure supercritical bifurcation of two equilibria on the x_1 axis. There is no loss of generality to take this sum equal to 0, so that the bifurcation is a pitchfork. Moreover in this bifurcation context it is suitable to take negative cubic coefficients in both equations, in order to keep the dynamics bounded. We normalize these coefficients to -1. Then the bifurcated equilibria are $\xi_1 = -\sqrt{a_1 + a_2}$ and $\xi_2 = +\sqrt{a_1 + a_2}$. The non radial eigenvalues at ξ_1 and ξ_2 are

$$\begin{array}{ll} e_1 = 2(a_1 - (a_3 - a_4)\sqrt{a_1 + a_2}), & -c_1 = b_1 - 2b_2\sqrt{a_1 + a_2} - a_1 - a_2 \\ e_2 = b_1 + 2b_2\sqrt{a_1 + a_2} - a_1 - a_2, & -c_2 = 2(a_1 + (a_3 - a_4)\sqrt{a_1 + a_2}) \end{array} \tag{59}$$

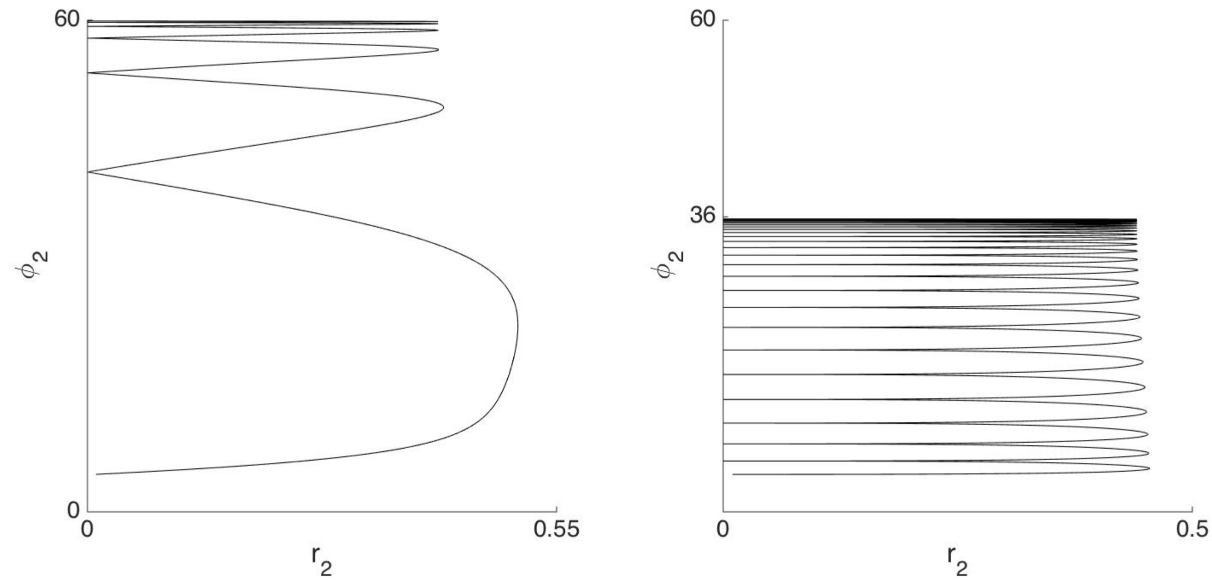
The heteroclinic cycles exist for a range of coefficient values which includes the following:

$$\begin{array}{l} a_1 = 0.2,\ a_2 = 0,\ a_3 = -0.3,\ a_4 = 0.05,\ a_5 = 0.25,\ a_6 = -0.6,\ a_7 = a_8 = -1 \\ b_1 = 0.05,\ b_2 = 0.4,\ b_3 = b_4 = -1,\ b_5 = -0.1 \end{array} \tag{60}$$

The eigenvalues are

$$\begin{array}{l} e_1 = 0.283406,\ c_1 = -0.428634 \\ e_2 = 0.228634,\ c_2 = -0.483406 \end{array}$$

so that the generalized heteroclinic cycle is an attractor. The numerical simulations (with Matlab) were done with $n = 3$ and $n = 5$. The two pictures in Figure 5 show the dynamics of the z_2 variable in polar coordinates: $z_2 = r_2e^{i\phi_2}$. The horizontal axis is the radial variable r_2 while the vertical axis is the angle ϕ_2 (in degrees). Observe that in both cases, taking an initial condition close to ξ_2 even with a small angle ϕ_2 (hence close to the plane P_2) the trajectory comes back to the vertical axis sequentially (as expected since it corresponds to going close to ξ_2), but with an increasing value of the angle. In the $n = 3$ case the angle converges to 60° while in the case $n = 5$ it converges to 36°. In both cases this corresponds to convergence to a pseudo-simple cycle with a connection in ρP_2. It is clear from this figure

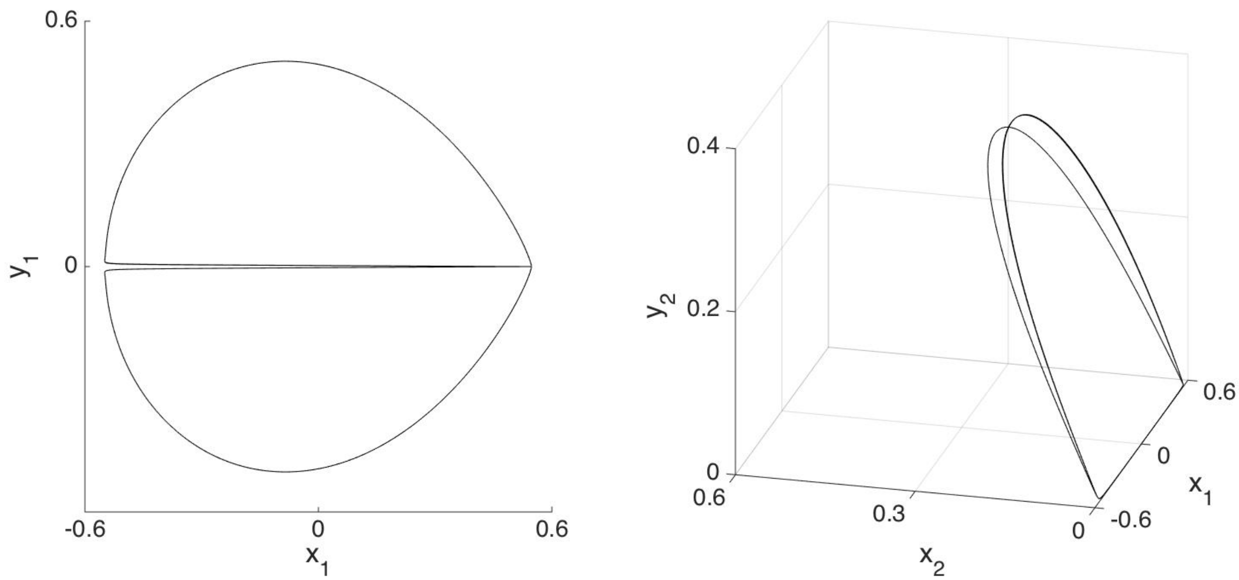


(a) Case $n = 3$ (b) Case $n = 5$

Figure 5: Dynamics of the $z_2 = r_2 e^{i\phi_2}$ variable in polar coordinates. Horizontal axis: r_2, vertical axis: ϕ_2 (in degrees).

that when $n = 3$ the convergence to the pseudo-simple cycle is faster and in particular the trajectory near the equilibria (near the vertical coordinate axis in the figure) is oblique while it is nearly horizontal in the $n = 5$ case. This is consistent with the results of [16] where the case $n = 3$ was studied using a different approach in which the double unstable eigenvalue e_2 is small enough for nonlinear effects to be felt by the flow near ξ_2. This argument doesn't work however when $n > 4$ because one essential property of the case $n = 3$ is that on the center manifold which exists at ξ_2 when e_2 is small enough, an unstable equilibrium point always exists near ξ_2 in P_2, which obliges the flow to "bend" back to P_2 or to ρP_2 in the vicinity of ξ_2. A similar idea holds when $n = 4$. The advantage of the method of [16] is that it does not require the existence of a generalized heteroclinic cycle, however only fragmentarily asymptotic stability can be proved in such case.

Let us assume now that a perturbation is added to the vector field, which breaks the symmetry κ. The symmetry group is therefore reduced to the action of $\mathbb{D}_n$ generated by the transformations ρ and $\kappa\sigma$. The invariant planes P_1, P_2 (and its copies by ρ^k) and V are preserved, but not the invariant space W. If the perturbation is not too large the equilibria in $L = P_1 \cap P_2$ and their heteroclinic connections in the invariant planes persist, hence a pseudosimple heteroclinic cycle exists, however we know it is completely unstable. The question is what happens to the dynamics when this perturbation is switched on. Some preliminary numerical experiments have been performed on the system (58), where $n = 5$ and the perturbation consists in replacing the terms $a_9(z_2^5 + \bar{z}_2^5)$ by $a_9 z_2^5 + a_9' \bar{z}_2^5$ and $b_2(z_1 + \bar{z}_1)z_2$ by $(b_2 z_1 + b_2' \bar{z}_1)z_2$, where $|a_9 - a_9'|$ and $|b_2 - b_2'|$ are small but non zero. Other coefficients are the same as in (60) except $a_1 = 0.25, a_2 = 0.05, b_1 = 0.2$. It has been observed that the dynamics remains in a neighborhood of the cycle and converges in certain cases to a periodic orbit (Fig. 6) while in other cases it exhibits a clear a aperiodic, possibly chaotic behavior (Fig. 7). The mathematical analysis of this behavior will be a subject for future study.



(a) Coordinates $x_1,\ y_1$

(b) Coordinates $x_1,\ x_2,\ y_2$

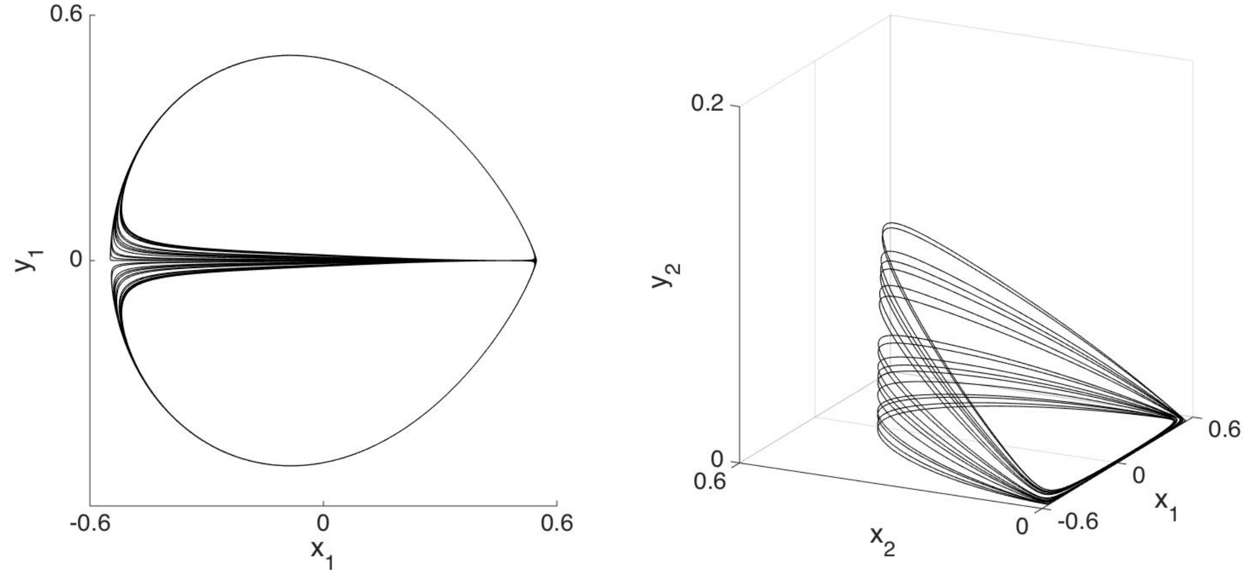
Figure 6: Asymptotic dynamics with $a_9 = 0.9$, $a'_9 = 1.05$, $b_2 = 0.292$, $b'_2 = 0.31$.

8 Conclusion

In this paper we completed the study of pseudo-simple heteroclinic cycles in $\mathbb{R}^4$, which have been discovered and distinguished from simple cycles only recently [15, 16]. Our primary contribution is a complete list of finite subgroups of $O(4)$ admitting pseudo-simple heteroclinic cycles. Similar to the completion of the classification of simple cycles in [15], and as projected ibid, this was achieved using the quaternionic presentation of such groups.

Up to now stability of pseudo-simple cycles had only been addressed in [16], where generic complete instability for the case $\Gamma \subset SO(4)$ was shown, and an example of a *fragmentarily asymptotically stable* cycle, an intermediate weak form of stability, with $\Gamma \not\subset SO(4)$ was given. We extended the stability analysis for pseudo-simple cycles in subsection 4.2 by identifying all subgroups of $O(4)$ admitting f.a.s. pseudo-simple heteroclinic cycles. A more comprehensive study, e.g. derivation of conditions for fragmentary asymptotic stability or calculation of stability indices along the heteroclinic connections as defined in [14], is beyond the scope of this work.

We have also studied the behaviour of trajectories close to pseudo-simple cycles. Namely,



(a) Coordinates $x_1,\ y_1$

(b) Coordinates $x_1,\ x_2,\ y_2$

Figure 7: Asymptotic dynamics with $a_9 = 1.1$, $a_9' = 0.85$, $b_2 = 0.28$, $b_2' = 0.32$.

we proved that asymptotically stable periodic orbits can bifurcate from the cycle in a codimension one bifurcation at a point where a multiple expanding eigenvalue vanishes. Necessary and sufficient conditions for such a bifurcation are given in theorems 3 and 4. In section 6 we illustrated this through a numerical example of a heteroclinic cycle with a nearby attracting periodic orbit with symmetry group $\Gamma = (\mathbb{D}_4 \,|\, \mathbb{D}_2; \mathbb{D}_4 \,|\, \mathbb{D}_2)$.

In contrast with [15], the proof of lemma 8 to characterize conditions for a group to be admissible relies upon an explicit construction of corresponding equivariant systems. This allows us to build examples of pseudo-simple heteroclinic cycles for any admissible group. As we noted (see remark 4), lemma 9 can be generalized to $\mathbb{R}^n$ with $n > 4$ to provide sufficient conditions for a subgroup of $O(n)$ to admit heteroclinic cycles. Moreover, the explicit construction of an equivariant system in $\mathbb{R}^n$ is applicable for this subgroup.

In addition to simple and pseudo-simple heteroclinic cycles other types of structurally stable heteroclinic cycles can exist in $\mathbb{R}^4$. One example is the generalized heteroclinic cycle that we studied in section 7. Another example is the cycle considered in [11]. To describe all robust heteroclinic cycles existing in $\mathbb{R}^4$ is an open problem which is beyond the scope of this paper.

Other possible continuations of our work include the full classification of pseudo-simple cycles in $\mathbb{R}^5$, similar to the full classification of homoclinic cycles in [13], as well as the study of networks, which are connected unions of more than one cycle. In principle we think this can be achieved by the same means as we used here, even though a complete classification of networks has not even been done for simple cycles yet, partial results to this end can be found in [2].

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A Elements of for the groups listed in Table 1, satisfying $\dim \mathrm{Fix}\, \gamma = 2$

We denote the quaternions $\mathbf{u} = (0,0,0,1)$, $\mathbf{v}_{\pm} = (0,1,\pm 1,0)/\sqrt{2}$, $\mathbf{v}'_{\pm} = (1,0,0,\pm 1)/\sqrt{2}$, $\mathbf{h}_{\pm,\pm,\pm} = (1,\pm 1,\pm 1,\pm 1)/2$, $\mathbf{w}_{\pm\pm} = (0,1,\pm\tau \pm \tau^{-1})/2$, $\mathbf{w}'_{\pm\pm} = (1,\pm\tau^{-1},\pm\tau,0)/2$, $\mathbf{w}''_{\pm\pm} = (\tau,\pm 1,\pm\tau^{-1},0)/2$, where $\tau = 2\cos(\pi/5) = (\sqrt{5}+1)/2$, $\tau^* = 2\sin(\pi/5) = \sqrt{5}(\tau)^{-1}$ and the permutation $\rho :\ (a,b,c,d) \mapsto\ (a,c,d,b)$.

group Γ	elements γ
$(\mathbb{D}_{K_1}\,\vert\,\mathbb{D}_{K_1};\mathbb{D}_{K_2}\,\vert\,\mathbb{D}_{K_2})$ K_1, K_2 odd, $m=\gcd(K_1,K_2)$ $\theta=\pi/m$, $\theta_1=\pi/K_1$, $\theta_2=\pi/K_2$	$\kappa_1(\pm,n) = ((\cos(n\theta),0,0,\pm\sin(n\theta));(\cos(n\theta),0,0,\sin(n\theta)))$ $\kappa_2(n_1,n_2) = ((0,\cos(n_1\theta_1),\sin(n_1\theta_1),0);(0,\cos(n_2\theta_2),\sin(n_2\theta_2),0))$
$(\mathbb{D}_{K_1}\,\vert\,\mathbb{D}_{K_1};\mathbb{D}_{K_2}\,\vert\,\mathbb{D}_{K_2})$ K_1 odd, K_2 even	$\kappa_1(\pm,n) = ((\cos(n\theta),0,0,\pm\sin(n\theta));(\cos(n\theta),0,0,\sin(n\theta)))$ $\kappa_2(n_1,n_2) = ((0,\cos(n_1\theta_1),\sin(n_1\theta_1),0);(0,\cos(n_2\theta_2),\sin(n_2\theta_2),0))$ $\kappa_3(n_1) = ((0,\cos(n_1\theta_1),\sin(n_1\theta_1),0);(0,0,0,1))$
$(\mathbb{D}_{2K_1}\,\vert\,\mathbb{D}_{2K_1};\mathbb{D}_{2K_2}\,\vert\,\mathbb{D}_{2K_2})$ $\theta=\pi/(2m)$, $\theta_1=\pi/(2K_1)$ $\theta_2=\pi/(2K_2)$	$\kappa_1(\pm,n) = ((\cos(n\theta),0,0,\pm\sin(n\theta));(\cos(n\theta),0,0,\sin(n\theta)))$ $\kappa_2(n_1) = ((0,\cos(n_1\theta_1),\sin(n_1\theta_1),0);(0,0,0,1))$ $\kappa_3(n_2) = ((0,0,0,1);(0,\cos(n_2\theta_2),\sin(n_2\theta_2),0))$ $\kappa_4(n_1,n_2) = ((0,\cos(n_1\theta_1),\sin(n_1\theta_1),0);(0,\cos(n_2\theta_2),\sin(n_2\theta_2),0))$
$(\mathbb{D}_{2K_1r}\,\vert\,\mathbb{Z}_{4K_1};\mathbb{D}_{2K_2r}\,\vert\,\mathbb{Z}_{4K_2})_s$ $m=\gcd(K_1,K_2)\gcd(r,K_1-sK_2)$ $\theta_1^*=\theta_1/r$, $\theta_2^*=\theta_2/r$	$\kappa_1(\pm,n) = ((\cos(n\theta),0,0,\sin(n\theta));(\cos(n\theta),0,0,\pm\sin(n\theta)))$ $\kappa_2(n_1,n_2,n_3) = ((0,\cos(n_1\theta_1+n_3\theta_1^*),\sin(n_1\theta_1+n_3\theta_1^*),0);$ $(0,\cos(n_2\theta_2+n_3s\theta_2^*),\cos(n_2\theta_2+n_3s\theta_2^*),0))$
$(\mathbb{D}_{2K_1r}\,\vert\,\mathbb{Z}_{2K_1};\mathbb{D}_{2K_2r}\,\vert\,\mathbb{Z}_{2K_2})_s$	$\kappa_1(\pm,n) = ((\cos(n\theta),0,0,\pm\sin(n\theta));(\cos(n\theta),0,0,\sin(n\theta)))$ $\kappa_2(n_1,n_2,n_3) = ((0,\cos(2n_1\theta_1+n_3\theta_1^*),\sin(2n_1\theta_1+n_3\theta_1^*),0);$ $(0,\cos(2n_2\theta_2+n_3s\theta_2^*),\sin(2n_2\theta_2+n_3s\theta_2^*),0)),$ $\kappa_3(n_1,n_2,n_3) = ((0,\cos((2n_1+1)\theta_1+n_3\theta_1^*),\sin((2n_1+1)\theta_1+n_3\theta_1^*),0);$ $(0,\cos((2n_2+1)\theta_2+n_3s\theta_2^*),\sin((2n_2+1)\theta_2+n_3s\theta_2^*),0))$
$(\mathbb{D}_{2K_1}\,\vert\,\mathbb{D}_{K_1};\mathbb{D}_{2K_2}\,\vert\,\mathbb{D}_{K_2})$ K_1, K_2 even $m=\gcd(K_1,K_2)$	$\kappa_1(\pm,n) = ((\cos(n\theta),0,0,\pm\sin(n\theta));(\cos(n\theta),0,0,\sin(n\theta)))$ $\kappa_2(n_1) = ((0,\cos(2n_1\theta_1),\sin(2n_1\theta_1),0);(0,0,0,1))$ $\kappa_3(n_2) = ((0,0,0,1);(0,\cos(2n_2\theta_2),\sin(2n_2\theta_2,0)))$ $\kappa_4(n_1,n_2) = ((0,\cos(2n_1\theta_1),\sin(2n_1)\theta_1),0);(0,\cos(2n_2\theta_2),\sin(2n_2\theta_2),0))$ $\kappa_5(n_1,n_2) = ((0,\cos((2n_1+1)\theta_1),\sin((2n_1+1)\theta_1),0);$ $(0,\cos((2n_2+1)\theta_2),\sin((2n_2+1)\theta_2),0))$
$(\mathbb{D}_{2K_1}\,\vert\,\mathbb{D}_{K_1};\mathbb{D}_{2K_2}\,\vert\,\mathbb{D}_{K_2})$ K_1, K_2, m odd	$\kappa_1(\pm,n) = ((\cos(n\theta),0,0,\pm\sin(n\theta));(\cos(n\theta),0,0,\sin(n\theta)))$ $\kappa_2(n_1) = ((0,\cos((2n_1+1)\theta_1),\sin((2n_1+1)\theta_1),0);(0,0,0,1))$ $\kappa_3(n_2) = ((0,0,0,1);(0,\cos((2n_2+1)\theta_2),\sin((2n_2+1)\theta_2),0))$ $\kappa_4(n_1,n_2) = ((0,\cos(2n_1\theta_1),\sin(2n_1\theta_1),0);(0,\cos(2n_2\theta_2),\sin(2n_2\theta_2),0))$ $\kappa_5(n_1,n_2) = ((0,\cos((2n_1+1)\theta_1),\sin((2n_1+1)\theta_1),0);$ $(0,\cos((2n_2+1)\theta_2),\sin((2n_2+1)\theta_2),0))$
$(\mathbb{D}_{2K_1}\,\vert\,\mathbb{D}_{K_1};\mathbb{D}_{2K_2}\,\vert\,\mathbb{D}_{K_2})$ K_1 even, K_2 odd	$\kappa_1(\pm,n) = ((\cos(n\theta),0,0,\pm\sin(n\theta));(\cos(n\theta),0,0,\sin(n\theta)))$ $\kappa_2(n_1) = ((0,\cos((2n_1+1)\theta_1),\sin((2n_1+1)\theta_1),0);(0,0,0,1))$ $\kappa_3(n_2) = ((0,0,0,1);(0,\cos(2n_2\theta_2),\sin(2n_2\theta_2),0))$ $\kappa_4(n_1,n_2) = ((0,\cos(2n_1\theta_1),\sin(2n_1\theta_1),0);(0,\cos(2n_2\theta_2),\sin(2n_2\theta_2),0))$ $\kappa_5(n_1,n_2) = ((0,\cos((2n_1+1)\theta_1),\sin((2n_1+1)\theta_1),0);$ $(0,\cos((2n_2+1)\theta_2),\sin((2n_2+1)\theta_2),0))$

Annex A continued.

group Γ	elements γ
$(\mathbb{D}_{2K_1}\,\vert\,\mathbb{D}_{K_1};\mathbb{D}_{2K_2}\,\vert\,\mathbb{Z}_{4K_2})$ K_1 even	$\kappa_1(\pm,n)=((\cos(n\theta),0,0,\pm\sin(n\theta));(\cos(n\theta),0,0,\sin(n\theta)))$ $\kappa_2(n_1)=((0,\cos(2n_1\theta_1),\sin(2n_1\theta_1),0);(0,0,0,1))$ $\kappa_3(n_1,n_2)=((0,\cos((2n_1+1)\theta_1),\sin((2n_1+1)\theta_1),0);$ $(0,\cos(n_2\theta_2),\sin(n_2\theta_2),0))$
$(\mathbb{D}_{2K_1}\,\vert\,\mathbb{D}_{K_1};\mathbb{D}_{2K_2}\,\vert\,\mathbb{Z}_{4K_2})$ K_1 odd	$\kappa_1(\pm,n)=((\cos(n\theta),0,0,\pm\sin(n\theta));(\cos(n\theta),0,0,\sin(n\theta)))$ $\kappa_2(n_1)=((0,\cos(2n_1\theta_1),\sin(2n_1\theta_1),0);(0,0,0,1))$ $\kappa_3(n_2)=((0,0,0,1);(0,\cos(n_2\theta_2),\sin(n_2\theta_2),0))$ $\kappa_4(n_1,n_2)=((0,\cos((2n_1+1)\theta_1),\sin((2n_1+1)\theta_1),0);$ $(0,\cos(n_2\theta_2),\sin(n_2\theta_2),0))$
$(\mathbb{D}_{2K_1}\,\vert\,\mathbb{D}_{K_1};\mathbb{D}_{K_2}\,\vert\,\mathbb{Z}_{2K_2})$ K_1,K_2 odd $\theta=\pi/m$	$\kappa_1(\pm,n)=((\cos(n\theta),0,0,\sin(n\theta));(\cos(n\theta),0,0,\pm\sin(n\theta)))$ $\kappa_2(n_1,n_2)=((0,\cos((2n_1+1)\theta_1),\sin((2n_1+1)\theta_1),0);$ $(0,\cos(n_2\theta_2),\sin(n_2\theta_2),0))$ $\kappa_3(n_2)=((0,0,0,1);(0,\cos(n_2\theta_2),\sin(n_2\theta_2),0))$
$(\mathbb{D}_{2K_1}\,\vert\,\mathbb{D}_{K_1};\mathbb{D}_{K_2}\,\vert\,\mathbb{Z}_{2K_2})$ K_1 even, K_2 odd	$\kappa_1(\pm,n)=((\cos(n\theta),0,0,\pm\sin(n\theta));(\cos(n\theta),0,0,\sin(n\theta)))$ $\kappa_2(n_1,n_2)=((0,\cos((2n_1+1)\theta_1),\sin((2n_1+1)\theta_1),0);$ $(0,\cos(n_2\theta_2),\sin(n_2\theta_2),0))$
$(\mathbb{D}_{3K}\,\vert\,\mathbb{D}_{3K};\mathbb{O}\,\vert\,\mathbb{O})$ K odd $\theta=\pi/(3K)$	$\kappa_1(\pm,\pm,\pm,\pm)=(1,0,0,\pm\sqrt{3})/2;h_{\pm\pm\pm})$ $\kappa_2(n,r)=((0,\cos(n\theta),\sin(n\theta),0);\rho^r\mathbf{u})$ $\kappa_3(n,r,\pm)=((0,\cos(n\theta),\sin(n\theta),0);\rho^r\mathbf{v}_\pm)$
$(\mathbb{D}_{6K}\,\vert\,\mathbb{D}_{6K};\mathbb{O}\,\vert\,\mathbb{O})$ K odd $\theta=\pi/(6K)$	$\kappa_1(\pm,\pm,\pm,\pm)=(1,0,0,\pm\sqrt{3})/2;h_{\pm\pm\pm}),\ \kappa_2(\pm,r)=((0,0,0,\pm1);\rho^r\mathbf{u})$ $\kappa_3(\pm,r,\pm)=((0,0,0,\pm1);\rho^r\mathbf{v}_\pm),\ \kappa_4(n,r)=((0,\cos(n\theta),\sin(n\theta),0);\rho^r\mathbf{u})$ $\kappa_5(n,r,\pm)=((0,\cos(n\theta),\sin(n\theta),0);\rho^r\mathbf{v}_\pm)$
$(\mathbb{D}_{4K}\,\vert\,\mathbb{D}_{4K};\mathbb{O}\,\vert\,\mathbb{O})$ $K\neq 3m$ $\theta=\pi/(4K)$	$\kappa_1(\pm,r,\pm)=(1,0,0,\pm1)/\sqrt{2};\rho^r\mathbf{v}'_\pm)$ $\kappa_2(\pm,r,\pm)=((0,0,0,\pm1);\rho^r\mathbf{v}_\pm),\ \kappa_3(n,r)=((0,\cos(n\theta),\sin(n\theta),0);\rho^r\mathbf{u})$ $\kappa_4(n,r,\pm)=((0,\cos(n\theta),\sin(n\theta),0);\rho^r\mathbf{v}_\pm),\ \kappa_5(\pm,r)=((0,0,0,\pm1);\rho^r\mathbf{u})$
$(\mathbb{D}_{12K}\,\vert\,\mathbb{D}_{12K};\mathbb{O}\,\vert\,\mathbb{O})$ $\theta=\pi/(12K)$	$\kappa_1(\pm,\pm,\pm,\pm)=(1,0,0,\pm\sqrt{3})/2;h_{\pm\pm\pm}),\ \kappa_2(\pm,r,\pm)=(1/\sqrt{2},0,0,\pm1/\sqrt{2});\rho^r\mathbf{v}'_\pm))$ $\kappa_3(\pm,r,\pm)=((0,0,0,\pm1);\rho^r\mathbf{v}_\pm),\ \kappa_4(n,r)=((0,\cos(n\theta),\sin(n\theta),0);\rho^r\mathbf{u})$ $\kappa_5(n,r,\pm)=((0,\cos(n\theta),\sin(n\theta),0);\rho^r\mathbf{v}_\pm),\ \kappa_6(\pm,r)=((0,0,0,\pm1);\rho^r\mathbf{u})$
$(\mathbb{D}_{3K}\,\vert\,\mathbb{Z}_{6K};\mathbb{O}\,\vert\,\mathbb{T})$ K odd, $\theta=\pi/(3K)$	$\kappa_1(\pm,\pm,\pm,\pm)=(1,0,0,\pm\sqrt{3})/2;h_{\pm\pm\pm})$ $\kappa_2(n,r,\pm)=((0,\cos(n\theta),\sin(n\theta),0);\rho^r\mathbf{v}_\pm)$
$(\mathbb{D}_{6K}\,\vert\,\mathbb{Z}_{12K};\mathbb{O}\,\vert\,\mathbb{T})$ $\theta=\pi/(6K)$	$\kappa_1(\pm,\pm,\pm,\pm)=(1/2,0,0,\pm\sqrt{3}/2);h_{\pm\pm\pm}),\ \kappa_2(\pm,r)=((0,0,0,\pm1);\rho^r\mathbf{u})$ $\kappa_3(n,r,\pm)=((0,\cos(n\theta),\sin(n\theta),0);\rho^r\mathbf{v}_\pm)$
$(\mathbb{D}_{4K}\,\vert\,\mathbb{D}_{2K};\mathbb{O}\,\vert\,\mathbb{T})$ K odd, $K\neq 3m$, $\theta=\pi/(4K)$	$\kappa_1(\pm,r,\pm)=(1,0,0,\pm1)/\sqrt{2};\rho^r\mathbf{v}'_\pm)),\ \kappa_2(n,r)=((0,\cos(2n\theta),\sin(2n\theta),0);\rho^r\mathbf{u})$ $\kappa_3(n,r,\pm)=((0,\cos((2n+1)\theta),\sin((2n+1)\theta),0);\rho^r\mathbf{v}_\pm)$ $\kappa_4(\pm,r)=((0,0,0,\pm1);\rho^r\mathbf{u})$

Annex A continued.

group Γ	elements γ
$(\mathbb{D}_{6K} \mid \mathbb{D}_{3K}; \mathbb{O} \mid \mathbb{T})$ K odd $\theta = \pi/(6K)$	$\kappa_1(\pm,\pm,\pm,\pm) = (1,0,0,\pm\sqrt{3})/2; h_{\pm\pm\pm}),$ $\kappa_2(\pm,r,\pm) = ((0,0,0,\pm 1); \rho^r \mathbf{v}_\pm),\ \kappa_3(n,r) = ((0,\cos(2n\theta),\sin(2n\theta),0); \rho^r \mathbf{u})$ $\kappa_4(n,r,\pm) = ((0,\cos((2n+1)\theta),\sin((2n+1)\theta),0); \rho^r \mathbf{v}_\pm)$
$(\mathbb{D}_{12K} \mid \mathbb{D}_{6K}; \mathbb{O} \mid \mathbb{T})$ K even $\theta = \pi/(12K)$	$\kappa_1(\pm,\pm,\pm,\pm) = (1,0,0,\pm\sqrt{3})/2; h_{\pm\pm\pm}),\ \kappa_2(\pm,r) = (0,0,0,\pm 1); \rho^r \mathbf{u}))$ $\kappa_3(n,r) = ((0,\cos(2n\theta),\sin(2n\theta),0); \rho^r \mathbf{u})$ $\kappa_4(n,r,\pm) = ((0,\cos((2n+1)\theta),\sin((2n+1)\theta),0); \rho^r \mathbf{v}_\pm)$
$(\mathbb{D}_{12K} \mid \mathbb{D}_{6K}; \mathbb{O} \mid \mathbb{T})$ K odd	$\kappa_1(\pm,\pm,\pm,\pm) = (1,0,0,\pm\sqrt{3})/2; h_{\pm\pm\pm}),\ \kappa_2(\pm,r,\pm) = (1,0,0,\pm 1)/\sqrt{2}; \rho^r \mathbf{v}'_\pm))$ $\kappa_3(n,r) = ((0,\cos(2n\theta),\sin(2n\theta),0); \rho^r \mathbf{u})$ $\kappa_4(n,r,\pm) = ((0,\cos((2n+1)\theta),\sin((2n+1)\theta),0); \rho^r \mathbf{v}_\pm)$ $\kappa_5(\pm,r) = ((0,0,0,\pm 1); \rho^r \mathbf{u})$
$(\mathbb{D}_{3K} \mid \mathbb{Z}_{2K}; \mathbb{O} \mid \mathbb{V})$ $\theta = \pi/(3K)$ K odd	$\kappa_1(\pm,\pm,\pm,\pm) = (1,0,0,\pm\sqrt{3})/2; h_{\pm\pm\pm})$ $\kappa_2(n,\pm) = ((0,\cos(3n\theta),\sin(3n\theta),0); \mathbf{v}_\pm)$ $\kappa'_2(n,\pm) = ((0,\cos(3n+1)\theta),\sin((3n+1)\theta),0); \rho\mathbf{v}_\pm)$ $\kappa''_2(n,\pm) = ((0,\cos(3n+2)\theta),\sin((3n+2)\theta),0); \rho^2\mathbf{v}_\pm)$
$(\mathbb{D}_{6K} \mid \mathbb{Z}_{4K}; \mathbb{O} \mid \mathbb{V})$ $\theta = \pi/(6K)$	$\kappa_1(\pm,\pm,\pm,\pm) = (1,0,0,\pm\sqrt{3})/2; h_{\pm\pm\pm})$ $\kappa_2(\pm,r) = ((0,0,0,\pm 1); \rho^r \mathbf{u}))$ $\kappa_3(n,\pm) = ((0,\cos(3n\theta),\sin(3n\theta),0); \mathbf{v}_\pm)$ $\kappa'_3(n,\pm) = ((0,\cos(3n+1)\theta),\sin((3n+1)\theta),0); \rho\mathbf{v}_\pm)$ $\kappa''_3(n,\pm) = ((0,\cos(3n+2)\theta),\sin((3n+2)\theta),0); \rho^2\mathbf{v}_\pm)$
$(\mathbb{D}_{3K} \mid \mathbb{D}_{3K}; \mathbb{I} \mid \mathbb{I})$ K odd, $K \neq 5m$ $\theta = \pi/(3K)$	$\kappa_1(\pm,\pm,\pm,\pm) = (1,0,0,\pm\sqrt{3})/2; h_{\pm\pm\pm})$ $\kappa'_1(\pm,r,\pm,\pm) = (1,0,0,\pm\sqrt{3})/2; \rho^r \mathbf{w}'_{\pm\pm})$ $\kappa_2(n,r) = ((0,\cos(n\theta),\sin(n\theta),0); \rho^r \mathbf{u})$ $\kappa'_2(n,r,\pm,\pm) = ((0,\cos(n\theta),\sin(n\theta),0); \rho^r \mathbf{w}_{\pm\pm})$
$(\mathbb{D}_{6K} \mid \mathbb{D}_{6K}; \mathbb{I} \mid \mathbb{I})$ K odd, $K \neq 5m$ $\theta = \pi/(6K)$	$\kappa_1(\pm,\pm,\pm,\pm) = (1,0,0,\pm\sqrt{3})/2; h_{\pm\pm\pm})$ $\kappa'_1(\pm,r,\pm,\pm) = (1,0,0,\pm\sqrt{3})/2; \rho^r \mathbf{w}'_{\pm\pm})$ $\kappa_2(n,r) = ((0,\cos(n\theta),\sin(n\theta),0); \rho^r \mathbf{u})$ $\kappa'_2(n,r,\pm,\pm) = ((0,\cos(n\theta),\sin(n\theta),0); \rho^r \mathbf{w}_{\pm\pm})$ $\kappa_3(\pm,r) = ((0,0,0,\pm 1); \rho^r \mathbf{u}),\ \kappa'_3(\pm,r,\pm,\pm) = ((0,0,0,\pm 1); \rho^r \mathbf{w}_{\pm\pm})$
$(\mathbb{D}_{5K} \mid \mathbb{D}_{5K}; \mathbb{I} \mid \mathbb{I})$ K odd, $K \neq 3m$ $\theta = \pi/(5K)$	$\kappa_1(\pm,r,\pm,\pm) = (\tau,0,0,\pm\tau^*)/2; \rho^r \mathbf{w}''_{\pm\pm})$ $\kappa_2(n,r) = ((0,\cos(n\theta),\sin(n\theta),0); \rho^r \mathbf{u})$ $\kappa'_2(n,r,\pm,\pm) = ((0,\cos(n\theta),\sin(n\theta),0); \rho^r \mathbf{w}_{\pm\pm})$
$(\mathbb{D}_{10K} \mid \mathbb{D}_{10K}; \mathbb{I} \mid \mathbb{I})$ $K \neq 3m$ $\theta = \pi/(10K)$	$\kappa_1(\pm,r,\pm,\pm) = (\tau,0,0,\pm\tau^*)/2; \rho^r \mathbf{w}''_{\pm\pm}),\ \kappa_2(n,r) = ((0,\cos(n\theta),\sin(n\theta),0); \rho^r \mathbf{u})$ $\kappa'_2(n,r,\pm,\pm) = ((0,\cos(n\theta),\sin(n\theta),0); \rho^r \mathbf{w}_{\pm\pm})$ $\kappa_3(\pm,r) = ((0,0,0,\pm 1); \rho^r \mathbf{u}),$ $\kappa'_3(\pm,r,\pm,\pm) = ((0,0,0,\pm 1); \rho^r \mathbf{w}_{\pm\pm})$

Annex A continued.

group Γ	elements γ
$(\mathbb{D}_{15K}\,\|\,\mathbb{D}_{15K};\mathbb{I}\,\|\,\mathbb{I})$ K odd $\theta=\pi/(15K)$	$\kappa_1(\pm,\pm,\pm,\pm)=(1,0,0,\pm\sqrt{3})/2;h_{\pm\pm\pm})$ $\kappa_1'(\pm,r,\pm,\pm)=(1,0,0,\pm\sqrt{3})/2;\rho^r\mathbf{w}'_{\pm\pm})$ $\kappa_2(\pm,r,\pm,\pm)=(\tau,0,0,\pm\tau^*)/2;\rho^r\mathbf{w}''_{\pm\pm})$ $\kappa_3(n,r)=((0,\cos(n\theta),\sin(n\theta),0);\rho^r\mathbf{u})$ $\kappa_3'(n,r,\pm,\pm)=((0,\cos(n\theta),\sin(n\theta),0);\rho^r\mathbf{w}_{\pm\pm})$
$(\mathbb{D}_{30K}\,\|\,\mathbb{D}_{30K};\mathbb{I}\,\|\,\mathbb{I})$ $\theta=\pi/(6K)$	$\kappa_1(\pm,\pm,\pm,\pm)=(1,0,0,\pm\sqrt{3})/2;h_{\pm\pm\pm})$ $\kappa_1'(\pm,r,\pm,\pm)=(1,0,0,\pm\sqrt{3})/2;\rho^r\mathbf{w}'_{\pm\pm})$ $\kappa_2(\pm,r,\pm,\pm)=(\tau,0,0,\pm\tau^*)/2;\rho^r\mathbf{w}''_{\pm\pm})$ $\kappa_3(n,r)=((0,\cos(n\theta),\sin(n\theta),0);\rho^r\mathbf{u})$ $\kappa_3'(n,r,\pm,\pm)=((0,\cos(n\theta),\sin(n\theta),0);\rho^r\mathbf{w}_{\pm\pm})$ $\kappa_4(\pm,r)=((0,0,0,\pm1);\rho^r\mathbf{u}),\ \kappa_4'(\pm,r,\pm,\pm)=((0,0,0,\pm1);\rho^r\mathbf{w}_{\pm\pm})$
$(\mathbb{D}_{2rK_1}\,\|\,\mathbb{Z}_{K_1};\mathbb{D}_{2rK_2}\,\|\,\mathbb{Z}_{K_2})_s$ K_1,K_2 odd , $m=\gcd(K_1,K_2)(K_2-sK_1)$ $\theta_1=\pi/K_1,\theta_2=\pi/K_2$ $\theta_1^*=\theta_1/(2r),\theta_2^*=\theta_2/(2r)$ $\theta=\pi/m$	$\kappa_1(n)=((\cos(n\theta),0,0,\sin(n\theta));(\cos(n\theta),0,0,\sin(n\theta)))$ $\kappa_2(n_1,n_2,n_3)=((0,\cos(2n_1\theta_1+n_3\theta_1^*),\sin(2n_1\theta_1+n_3\theta_1^*),0)$ $(0,\cos(2n_2\theta_2+sn_3\theta_2^*),\cos(2n_2\theta_2+sn_3\theta_2^*),0))$ $\kappa_3(n_1,n_2,n_3)=((0,\cos((2n_1+1)\theta_1+n_3\theta_1^*),\sin((2n_1+1)\theta_1+n_3\theta_1^*),0);$ $(0,\cos((2n_2+1)\theta_2+sn_3\theta_2^*),\sin((2n_2+1)\theta_2+sn_3\theta_2^*),0))$

B Conjugacy classes of isotropy subgroups of finite groups Γ satisfying $\dim \mathrm{Fix}\ (\Sigma)=2$ and $\dim \mathrm{Fix}\ (\Delta)=1$

We list subgroups Σ and Δ of Γ that satisfy $\dim \mathrm{Fix}\ (\Sigma)=2$ and $\dim \mathrm{Fix}\ (\Delta)=1$.

Γ	Σ	Δ
$(\mathbb{D}_{K_1}\,\|\,\mathbb{D}_{K_1};\mathbb{D}_{K_2}\,\|\,\mathbb{D}_{K_2})$ K_1, K_2 odd	$<\kappa_1(\pm,1)>$; $<\kappa_2(n_1,n_2)>$: n_1+n_2 even or odd	$<\kappa_1(\pm,1),\kappa_2(n_1,n_2)>$: n_1+n_2 even or odd
$(\mathbb{D}_{K_1}\,\|\,\mathbb{D}_{K_1};\mathbb{D}_{K_2}\,\|\,\mathbb{D}_{K_2})$ K_1 odd, K_2 even	$<\kappa_1(\pm,1)>$; $<\kappa_2(n_1,n_2)>$: n_2 even or odd, $<\kappa_3(n_1)>$	$<\kappa_1(\pm,1),\kappa_2(n_1,n_2)>$: n_2 even or odd
$(\mathbb{D}_{2K_1}\,\|\,\mathbb{D}_{2K_1};\mathbb{D}_{2K_2}\,\|\,\mathbb{D}_{2K_2})$ K_1+K_2 even	$<\kappa_1(\pm,1)>$; $<\kappa_2(n_1)>$: n_1 even or odd; $<\kappa_3(n_2)>$: n_2 even or odd; $<\kappa_4(n_1,n_2)>$: $n_1,\ n_2$ even or odd	$<\kappa_2(n_1),\kappa_3(n_2)>$: $n_1,\ n_2$ even or odd; $<\kappa_1(\pm,1),\kappa_4(n_1,n_2)>$: n_1+n_2 even or odd
$(\mathbb{D}_{2K_1}\,\|\,\mathbb{D}_{2K_1};\mathbb{D}_{2K_2}\,\|\,\mathbb{D}_{2K_2})$ K_1 odd, K_2 even	$<\kappa_1(\pm,1)>$, $<\kappa_2(n_1)>$: n_1 even or odd; $<\kappa_3(n_2)>$: n_2 even or odd; $<\kappa_4(n_1,n_2)>$: $n_1,\ n_2$ even or odd	$<\kappa_2(n_1),\kappa_3(n_2)>$: $n_1,\ n_2$ even or odd; $<\kappa_1(\pm,1),\kappa_4(n_1,n_2)>$: n_2 even or odd
$(\mathbb{D}_{2K_1r}\,\|\,\mathbb{Z}_{4K_1};\mathbb{D}_{2K_2r}\,\|\,\mathbb{Z}_{4K_2})_s$ K_1, K_2, r odd,	$<\kappa_1(+,1)>$; $<\kappa_1(-,1)>$; $<\kappa_2(n_1,n_2,n_3)>$: n_1+n_3, n_2+n_3 even or odd	$<\kappa_1(+,1),\kappa_2(n_1,n_2,n_3)>$: n_1+n_2 even or odd; $<\kappa_1(-,1),\kappa_2(n_1,n_2,n_3)>$: n_1+n_2 even or odd
$(\mathbb{D}_{2K_1r}\,\|\,\mathbb{Z}_{4K_1};\mathbb{D}_{2K_2r}\,\|\,\mathbb{Z}_{4K_2})_s$ K_1, K_2 odd, r even	$<\kappa_1(+,1)>$; $<\kappa_1(-,1)>$; $<\kappa_2(n_1,n_2,n_3)>$: $n_1+n_2,\ n_3$ even or odd	$<\kappa_1(+,1),\kappa_2(n_1,n_2,n_3)>$: n_3 even or odd; $<\kappa_1(-,1),\kappa_2(n_1,n_2,n_3)>$: n_3 even or odd
$(\mathbb{D}_{2K_1r}\,\|\,\mathbb{Z}_{4K_1};\mathbb{D}_{2K_2r}\,\|\,\mathbb{Z}_{4K_2})_s$ K_1 even, K_2, r odd	$<\kappa_1(+,1)>$; $<\kappa_1(-,1)>$; $<\kappa_2(n_1,n_2,n_3)>$: n_1, n_2+n_3 even or odd	$<\kappa_1(+,1),\kappa_2(n_1,n_2,n_3)>$: n_1 even or odd; $<\kappa_1(-,1),\kappa_2(n_1,n_2,n_3)>$: n_1 even or odd
$(\mathbb{D}_{2K_1r}\,\|\,\mathbb{Z}_{2K_1};\mathbb{D}_{2K_2r}\,\|\,\mathbb{Z}_{2K_2})_s$ K_1 even, K_2 odd	$<\kappa_1(+,1)>$; $<\kappa_1(-,1)>$; $<\kappa_2(n_1,n_2,n_3)>$: n_1 even or odd	$<\kappa_1(+,1),\kappa_2(n_1,n_2,n_3)>$: n_1 even or odd; $<\kappa_1(-,1),\kappa_2(n_1,n_2,n_3)>$: n_1 even or odd
$(\mathbb{D}_{2K_1}\,\|\,\mathbb{D}_{K_1};\mathbb{D}_{2K_2}\,\|\,\mathbb{D}_{K_2})$	$<\kappa_1(\pm,1)>$; $<\kappa_2(n_1)>$; $<\kappa_3(n_2)>$; $<\kappa_4(n_1,n_2)>$: n_1+n_2 even or odd; $<\kappa_5(n_1,n_2)>$	$<\kappa_2(n_1),\kappa_3(n_2)>$: n_1+n_2 even or odd; $<\kappa_1((-1)^s,1),\kappa_5(n_1,n_2)>$: $s+n_1+n_2$ even or odd
$(\mathbb{D}_{2K_1}\,\|\,\mathbb{D}_{K_1};\mathbb{D}_{2K_2}\,\|\,\mathbb{Z}_{4K_2})$ K_1 even	$<\kappa_1(\pm,1)>$; $<\kappa_2(n_1)>$; $<\kappa_3(n_1,n_2)>$: n_2 even or odd	$<\kappa_1((-1)^s,1),\kappa_3(n_1,n_2)>$: $s+n_2$ even or odd
$(\mathbb{D}_{2K_1}\,\|\,\mathbb{D}_{K_1};\mathbb{D}_{2K_2}\,\|\,\mathbb{Z}_{4K_2})$ K_1 odd	$<\kappa_1(\pm,1)>$; $<\kappa_2(n_1)>$: n_1 even or odd; $<\kappa_3(n_2)>$: n_2 even or odd; $<\kappa_4(n_1,n_2)>$: n_2 even or odd	$<\kappa_1(\pm,1),\kappa_4(n_1,n_2)>$: n_2 even or odd; $<\kappa_2(n_1),\kappa_4(n_1-K_1,n_2)>$: n_1,n_2 even or odd
$(\mathbb{D}_{2K_1}\,\|\,\mathbb{D}_{K_1};\mathbb{D}_{K_2}\,\|\,\mathbb{Z}_{2K_2})$ K_1, K_2 odd	$<\kappa_1(\pm,1)>$; $<\kappa_2(n_1,n_2)>$; $<\kappa_3(n_2)>$	$<\kappa_1((-1)^s,1),\kappa_2(n_1,n_2)>$: $s+n_1$ even or odd
$(\mathbb{D}_{2K_1}\,\|\,\mathbb{D}_{K_1};\mathbb{D}_{K_2}\,\|\,\mathbb{Z}_{2K_2})$ K_1 even, K_2 odd	$<\kappa_1(\pm,1>$; $<\kappa_2(n_1,n_2)>$	$<\kappa_1((-1)^s,1),\kappa_2(n_1,n_2)>$: $s+n_1$ even or odd
$(\mathbb{D}_{3K}\,\|\,\mathbb{D}_{3K};\mathbb{O}\,\|\,\mathbb{O})$ K odd	$<\kappa_1(+,\pm,\pm,\pm)>$; $<\kappa_2(n,r)>$; $<\kappa_3(n,r,\pm)>$	$<\kappa_1(+,(-1)^{q_1},(-1)^{q_2},(-1)^{q_3}),\kappa_3(n,0,(-1)^{q_1+q_2+1})>$: q_3+n even or odd

Γ	Σ	Δ
$(\mathbb{D}_{6K} \mid \mathbb{D}_{6K}; \mathbb{O} \mid \mathbb{O})$ K odd	$< \kappa_1(+,\pm,\pm,\pm) >$; $< \kappa_2(\pm,r) >$; $< \kappa_3(\pm,r,\pm) >$; $< \kappa_4(n,r) >$: n even or odd; $< \kappa_5(n,r,\pm) >$: n even or odd	$< \kappa_1(+,(-1)^{q_1},(-1)^{q_2},\pm), \kappa_5(n,0,(-1)^{q_1+q_2+1}) >$: n even or odd; $< \kappa_2(\pm,r), \kappa_4(n,r+1) >$; $< \kappa_2(\pm,r), \kappa_5(n,r,\pm) >$; $< \kappa_3(\pm,r,\pm), \kappa_4(n,r) >$: n even or odd
$(\mathbb{D}_{4K} \mid \mathbb{D}_{4K}; \mathbb{O} \mid \mathbb{O})$ K odd, $K \neq 3m$	$< \kappa_1(+,r,\pm) >$; $< \kappa_2(\pm,r,\pm) >$; $< \kappa_3(n,r) >$: n even or odd; $< \kappa_4(n,r,\pm) >$: n even or odd	$< \kappa_1(+,r,\pm), \kappa_3(n,r+1) >$: n even or odd; $< \kappa_2(\pm,r,\pm), \kappa_3(n,r) >$: n even or odd
$(\mathbb{D}_{12K} \mid \mathbb{D}_{12K}; \mathbb{O} \mid \mathbb{O})$	$< \kappa_1(+,\pm,\pm,\pm) >$; $< \kappa_2(+,r,\pm) >$; $< \kappa_3(\pm,r,\pm) >$; $< \kappa_4(n,r) >$: n even or odd; $< \kappa_5(n,r,\pm) >$: n even or odd	$< \kappa_1(+,(-1)^{q_1},(-1)^{q_2},\pm), \kappa_5(n,0,(-1)^{q_1+q_2+1}) >$: n even or odd; $< \kappa_2(+,r,\pm), \kappa_4(n,r+1) >$: n even or odd; $< \kappa_3(\pm,r,\pm), \kappa_4(n,r) >$: n even or odd
$(\mathbb{D}_{3K} \mid \mathbb{Z}_{6K}; \mathbb{O} \mid \mathbb{T})$ K odd	$< \kappa_1(+,(-1)^{q_1},(-1)^{q_2},(-1)^{q_3}) >$: $q_1+q_2+q_3$ even or odd; $< \kappa_2(n,r,\pm) >$	$< \kappa_1(+,(-1)^{q_1},(-1)^{q_2},(-1)^{q_3}), \kappa_2(n,0,(-1)^{q_1+q_2+1}) >$: q_1+n, $q_1+q_2+q_3$ even or odd
$(\mathbb{D}_{6K} \mid \mathbb{Z}_{12K}; \mathbb{O} \mid \mathbb{T})$ K odd	$< \kappa_1(+,(-1)^{q_1},(-1)^{q_2},(-1)^{q_3}) >$: $q_1+q_2+q_3$ even or odd; $< \kappa_3(n,r,\pm) >$: n even or odd	$< \kappa_1(+,(-1)^{q_1},(-1)^{q_2},(-1)^{q_3}), \kappa_3(n,0,(-1)^{q_1+q_2+1}) >$: $q_1+q_2+q_3$, n even or odd; $< \kappa_2(\pm,r), \kappa_3(n,r,\pm), >$ n even or odd
$(\mathbb{D}_{6K} \mid \mathbb{Z}_{12K}; \mathbb{O} \mid \mathbb{T})$ K even	$< \kappa_1(+,(-1)^{q_1},(-1)^{q_2},(-1)^{q_3}) >$: $q_1+q_2+q_3$ even or odd;$< \kappa_2(\pm,r) >$; $< \kappa_3(n,r,\pm) >$: n even or odd	$< \kappa_1(+,(-1)^{q_1},(-1)^{q_2},(-1)^{q_3}), \kappa_3(n,0,(-1)^{q_1+q_2+1}) >$: $q_1+q_2+q_3$, n even or odd; $< \kappa_2(\pm,r), \kappa_3(n,r,\pm), >$ n even or odd
$(\mathbb{D}_{4K} \mid \mathbb{D}_{2K}; \mathbb{O} \mid \mathbb{T})$ K odd, $K \neq 3m$	$< \kappa_1(+,r,\pm) >$; $< \kappa_2(n,r) >$; $< \kappa_3(n,r,\pm) >$	$< \kappa_1(+,r,(-1)^s), \kappa_3(n,r,\pm) >$: $s+n$ even or odd;
$(\mathbb{D}_{6K} \mid \mathbb{D}_{3K}; \mathbb{O} \mid \mathbb{T})$ K odd	$< \kappa_1(+,\pm,\pm,\pm) >$; $< \kappa_2(\pm,r,\pm) >$; $< \kappa_3(n,r) >$; $< \kappa_4(n,r,\pm) >$	$< \kappa_1(+,(-1)^{q_1},(-1)^{q_2},(-1)^{q_3}), \kappa_4(n,0,(-1)^{q_1+q_2+1}) >$: q_3+n even or odd; $< \kappa_2(\pm,r,(-1)^s), \kappa_3(n,r) >$: $s+n$ even or odd
$(\mathbb{D}_{6K} \mid \mathbb{D}_{3K}; \mathbb{O} \mid \mathbb{T})$ K even, $K \neq 2(2m+1)$	$< \kappa_1(+,\pm,\pm,\pm) >$; $< \kappa_2(\pm,r,\pm) >$; $< \kappa_3(n,r) >$; $< \kappa_4(n,r,\pm) >$	$< \kappa_1(+,(-1)^{q_1},(-1)^{q_2},(-1)^{q_3}), \kappa_4(n,0,(-1)^{q_1+q_2+1}) >$: q_3+n even or odd; $< \kappa_2(\pm,r,\pm), \kappa_3(n,r+1) >$; $< \kappa_2(\pm,r,(-1)^s), \kappa_4(n,r,(-1)^{s+1}) >$
$(\mathbb{D}_{12K} \mid \mathbb{D}_{6K}; \mathbb{O} \mid \mathbb{T})$ K odd	$< \kappa_1(+,\pm,\pm,\pm) >$; $< \kappa_2(+,r,\pm) >$; $< \kappa_3(n,r) >$; $< \kappa_4(n,r,\pm) >$	$< \kappa_1(+,(-1)^{q_1},(-1)^{q_2},(-1)^{q_3}), \kappa_4(n,0,(-1)^{q_1+q_2+1}) >$: q_3+n even or odd; $< \kappa_2(\pm,r,(-1)^s), \kappa_4(n,r,\pm) >$: $s+n$ even or odd
$(\mathbb{D}_{3K} \mid \mathbb{Z}_{2K}; \mathbb{O} \mid \mathbb{V})$ K odd	$< \kappa_1(+,\pm,\pm,\pm) >$; $< \kappa_2(n,\pm) >$	$< \kappa_1(+,(-1)^{q_1},(-1)^{q_2},(-1)^{q_3}), \kappa_2(n,(-1)^{q_1+q_2+1}) >$: q_2+q_3+n even or odd

Continuation of Annex B.

Γ	Σ	Δ
$(\mathbb{D}_{6K}\,\vert\,\mathbb{Z}_{4K};\mathbb{O}\,\vert\,\mathbb{V})$ K odd	$<\kappa_1(+,\pm,\pm,\pm)>;\ <\kappa_2(\pm,r)>;$ $<\kappa_3(n,\pm)>$: n even or odd	$<\kappa_1(+,(-1)^{q_1},(-1)^{q_2},(-1)^{q_3}),\kappa_2(n,r,(-1)^{q_1+q_2+1})>$: n even or odd; $<\kappa_2((-1)^s,0),\kappa_3(n,\pm)>$: $s+n$ even or odd
$(\mathbb{D}_{6K}\,\vert\,\mathbb{Z}_{4K};\mathbb{O}\,\vert\,\mathbb{V})$ K odd	$<\kappa_1(+,\pm,\pm,\pm)>;\ <\kappa_2(\pm,r)>;$ $<\kappa_3(n,\pm)>$: n even or odd	$<\kappa_1(+,(-1)^{q_1},(-1)^{q_2},(-1)^{q_3}),\kappa_3(n,r,(-1)^{q_1+q_2+1})>$: n even or odd; $<\kappa_2((-1)^s,0),\kappa_3(n,\pm)>$: n even or odd
$(\mathbb{D}_{3K}\,\vert\,\mathbb{D}_{3K};\mathbb{I}\,\vert\,\mathbb{I})$ K odd, $K\neq 5m$	$<\kappa_1>;\ <\kappa_2>$	$<\kappa_1(+,(-1)^{q_1},(-1)^{q_2},(-1)^{q_3}),\kappa_2'(n,0,(-1)^{q_1+q_2+1},(-1)^{q_1+q_3})>$: q_1+n even or odd; $<\kappa_1'(\pm,r,(-1)^{s_1},(-1)^{s_2}),\kappa_2(n,r)>$: $n+s_1+s_2$ even or odd
$(\mathbb{D}_{6K}\,\vert\,\mathbb{D}_{6K};\mathbb{I}\,\vert\,\mathbb{I})$ K odd, $K\neq 5m$	$<\kappa_1>;\ <\kappa_2>$: n even or odd;	$<\kappa_1(+,(-1)^{q_1},(-1)^{q_2},(-1)^{q_3}),\kappa_2'(n,0,(-1)^{q_1+q_2+1},(-1)^{q_1+q_3+1})>$: n even or odd; $<\kappa_1'(+,r,\pm,\pm),\kappa_2(n,r)>$: n even or odd; $<\kappa_2(n,r),\kappa_3((-1)^s,r+1)>$: $n+s$ even or odd; $<\kappa_2'(n,(-1)^{q_1},(-1)^{q_2}),\kappa_3'((-1)^s,r+1,(-1)^{q_1+1},(-1)^{q_1+q_2+1})>$: $n+s$ even or odd
$(\mathbb{D}_{6K}\,\vert\,\mathbb{D}_{6K};\mathbb{I}\,\vert\,\mathbb{I})$ K even, $K\neq 5m$	$<\kappa_1>;\ <\kappa_2>$: n even or odd;	$<\kappa_1(+,(-1)^{q_1},(-1)^{q_2},(-1)^{q_3}),\kappa_2'(n,0,(-1)^{q_1+q_2+1},(-1)^{q_1+q_3})>$: n even or odd; $<\kappa_1'(+,r,\pm),\kappa_2(n,r)>$: n even or odd; $<\kappa_2(n,r),\kappa_3(\pm,r+1)>$: n even or odd; $<\kappa_2'(n,(-1)^{q_1},(-1)^{q_2}),\kappa_3'(\pm,r+1,(-1)^{q_1+1},(-1)^{q_1+q_2})>$: n even or odd
$(\mathbb{D}_{5K}\,\vert\,\mathbb{D}_{5K};\mathbb{I}\,\vert\,\mathbb{I})$ K odd, $K\neq 3m$	$<\kappa_1(+,r,\pm,\pm)>;\ <\kappa_2>$	$<\kappa_1(+,r,(-1)^{q_1},(-1)^{q_2}),\kappa_2(n,r)>$: $n+q_1+q_2$ even or odd; $<\kappa_1(+,r,\pm,\pm),\kappa_2'(n,r+s,\pm,\pm)>$, $s=1,2$: $n+s$ even or odd
$(\mathbb{D}_{10K}\,\vert\,\mathbb{D}_{10K};\mathbb{I}\,\vert\,\mathbb{I})$ K odd, $K\neq 3m$	$<\kappa_1(+,r,\pm,\pm)>;\ <\kappa_2>;$ $<\kappa_3>$	$<\kappa_1(+,r,\pm,\pm),\kappa_2(n,r)>$: n even or odd; $<\kappa_1(+,r,\pm,\pm),\kappa_2'(n,r+s,\pm,\pm)>$, $s=1,2$: n even or odd; $<\kappa_2(n,r),\kappa_3((-1)^s,r+1)>$: $n+s$ even or odd; $<\kappa_2'(n,(-1)^{q_1},(-1)^{q_2}),\kappa_3'((-1)^s,r+1,(-1)^{q_1+1},(-1)^{q_1+q_2+1})>$: $n+s$ even or odd
$(\mathbb{D}_{10K}\,\vert\,\mathbb{D}_{10K};\mathbb{I}\,\vert\,\mathbb{I})$ K even, $K\neq 3m$	$<\kappa_1(+,r,\pm,\pm)>;\ <\kappa_2>;$ $<\kappa_3>$	$<\kappa_1(+,r,\pm,\pm),\kappa_2(n,r)>$: n even or odd; $<\kappa_1(+,r,\pm,\pm),\kappa_2'(n,r+s,\pm,\pm)>$, $s=1,2$: n even or odd; $<\kappa_2(n,r),\kappa_3((-1)^s,r+1)>$: n even or odd; $<\kappa_2'(n,(-1)^{q_1},(-1)^{q_2}),\kappa_3'((-1)^s,r+1,(-1)^{q_1+1},(-1)^{q_1+q_2+1})>$: n even or odd

Continuation of Annex B.

Γ	Σ	Δ
$(\mathbb{D}_{15K} \mid \mathbb{D}_{15K}; \mathbb{I} \mid \mathbb{I})$ K odd	$< \kappa_1 >$; $< \kappa_2(\pm, r, \pm) >$; $< \kappa_3 >$	$< \kappa_1(+, (-1)^{q_1}, (-1)^{q_2}, (-1)^{q_3}), \kappa_3'(n, 0, (-1)^{q_1+q_2+1})) >$: $q_q + n$ even or odd; $< \kappa_1'(+, (-1)^{s_1}, (-1)^{s_2}), \kappa_3(n, 0) >$: $s_1 + s_2 + n$ even or odd; $< \kappa_2(+, r, (-1)^{s_1}, (-1)^{s_2}), \kappa_3(n, r)) >$: $s_1 + s_2 + n$ even or odd; $< \kappa_2(+, (-1)^{s_1}, (-1)^{s_2}), \kappa_3'(n, 0, (-1)^{s_1+s_2+1})) >$: $s_1 + n$ even or odd
$(\mathbb{D}_{30K} \mid \mathbb{D}_{30K}; \mathbb{I} \mid \mathbb{I})$ K odd	$< \kappa_1 >$; $< \kappa_2(\pm, r, \pm) >$; $< \kappa_3 >$: n even or odd; $< \kappa_4 >$	$< \kappa_1(+, (-1)^{s_1}, (-1)^{s_2}, (-1)^{s_3}), \kappa_3'(n, 0, (-1)^{s_1+s_2+1})) >$: n even or odd; $< \kappa_1'(+, (-1)^{s_1}, (-1)^{s_2}), \kappa_3(n, 0) >$: n even or odd; $< \kappa_2((-1)^s, r, \pm), \kappa_3(n, r) >$: n even or odd $< \kappa_2((-1)^s, r, \pm), \kappa_3'(n, r+1, \mp, \pm) >$: n even or odd $< \kappa_3(n, r), \kappa_4((-1)^s, r+1) >$: $n + s$ even or odd $< \kappa_3'(n, r, \mp, \pm), \kappa_4'((-1)^s, r+1, \pm, \pm) >$: $n + s$ even or odd
$(\mathbb{D}_{30K} \mid \mathbb{D}_{30K}; \mathbb{I} \mid \mathbb{I})$ K even	$< \kappa_1 >$; $< \kappa_2(\pm, r, \pm) >$; $< \kappa_3 >$: n even or odd; $< \kappa_4 >$	$< \kappa_1(+, (-1)^{s_1}, (-1)^{s_2}, (-1)^{s_3}), \kappa_3'(n, 0, (-1)^{s_1+s_2+1})) >$: n even or odd; $< \kappa_1'(+, (-1)^{s_1}, (-1)^{s_2}), \kappa_3(n, 0) >$: n even or odd; $< \kappa_2((-1)^s, r, \pm), \kappa_3(n, r) >$: n even or odd $< \kappa_2((-1)^s, r, \pm), \kappa_3'(n, r+1, \mp, \pm) >$: n even or odd $< \kappa_3(n, r), \kappa_4((-1)^s, r+1) >$: n even or odd $< \kappa_3'(n, r, \mp, \pm), \kappa_4'((-1)^s, r+1, \pm, \pm) >$: n even or odd
$(\mathbb{D}_{2rK_1}/\mathbb{Z}_{K_1}; \mathbb{D}_{2rK_2}/\mathbb{Z}_{K_2})$	$< \kappa_1(1) >$, $< \kappa_2(n_1, n_2, n_3) >$, $< \kappa_3(n_1, n_2, n_3) >$	$< \kappa_1(1), \kappa_2(n_1, n_2, n_3) >$

Continuation of Annex B.

C Isotropy subgroups Σ_j and Δ_j and the element γ satisfying conditions C1-C6 of lemma 8, where $\Sigma_1 \cong \mathbb{Z}_k$ with $k \geq 3$. For the groups $(\mathbb{D}_{3K} \,|\, \mathbb{Z}_{6K}; \mathbb{O} \,|\, \mathbb{T})$ and $(\mathbb{D}_{15K} \,|\, \mathbb{D}_{15K}; \mathbb{I} \,|\, \mathbb{I})$ with odd K we list subgroups satisfying conditions of lemma 9.

Γ	Σ_j, Δ_j and γ
$(\mathbb{D}_{2K_1} \mid \mathbb{D}_{2K_1}; \mathbb{D}_{2K_2} \mid \mathbb{D}_{2K_2})$	$\Sigma_1 =< \kappa_1(+,1) >,\ \Sigma_2 =< \kappa_4(0,0) >,\ \Sigma_3 =< \kappa_2(K_1) >,\ \Sigma_4 =< \kappa_4(0,1) >$ $\Delta_1 =< \kappa_4(0,1), \kappa_1(+,1) >,\ \Delta_2 =< \kappa_1(+,1), \kappa_4(0,0) >,$ $\Delta_3 =< \kappa_4(0,0), \kappa_2(K_1) >,\ \Delta_4 =< \kappa_2(K_1), \kappa_4(0,1) >,\ \gamma = e$
$(\mathbb{D}_{2K_1 r} \mid \mathbb{Z}_{2K_1}; \mathbb{D}_{2K_2 r} \mid \mathbb{Z}_{2K_2})_s$	$\Sigma_1 =< \kappa_1(+,1) >,\ \Sigma_2 =< \kappa_2(0,0,0) >,\ \Sigma_3 =< \kappa_1(-,1) >,\ \Sigma_4 =< \kappa_2(1,0,0) >$ $\Delta_1 =< \kappa_2(1,0,0), \kappa_1(+,1) >,\ \Delta_2 =< \kappa_1(+,1), \kappa_2(0,0,0) >,$ $\Delta_3 =< \kappa_2(0,0,0), \kappa_1(-,1) >,\ \Delta_4 =< \kappa_2(-,1), \kappa_2(1,0,0) >,\ \gamma = e$
$(\mathbb{D}_{2K_1} \mid \mathbb{D}_{K_1}; \mathbb{D}_{2K_2} \mid \mathbb{D}_{K_2})$	$\Sigma_1 =< \kappa_1(+,1) >,\ \Sigma_2 =< \kappa_5(0,0) >$ $\Delta_1 =< \kappa_5(1,0), \kappa_1(+,1) >,\ \Delta_2 =< \kappa_1(+,1), \kappa_5(0,0) >,$ $\gamma = ((\cos\theta_1, 0, 0, -\sin\theta_1); (0, \cos\theta_2, \sin\theta_2, 0))$
$(\mathbb{D}_{2K_1} \mid \mathbb{D}_{K_1}; \mathbb{D}_{2K_2} \mid \mathbb{Z}_{4K_2})$ K_1 even	$\Sigma_1 =< \kappa_1(+,1) >,\ \Sigma_2 =< \kappa_3(0,0) >$ $\Delta_1 =< \kappa_3(0,1), \kappa_1(+,1) >,\ \Delta_2 =< \kappa_1(+,1), \kappa_3(0,0) >,$ $\gamma = ((0, \cos\theta_1, \sin\theta_1, 0); (\cos\theta_2, 0, 0, -\sin\theta_2))$
$(\mathbb{D}_{2K_1} \mid \mathbb{D}_{K_1}; \mathbb{D}_{2K_2} \mid \mathbb{Z}_{4K_2})$ K_1 odd	$\Sigma_1 =< \kappa_1(+,1) >,\ \Sigma_2 =< \kappa_4(0,0) >,\ \Sigma_3 =< \kappa_2(K_1) >,\ \Sigma_4 =< \kappa_4(0,1) >$ $\Delta_1 =< \kappa_4(0,1), \kappa_1(+,1) >,\ \Delta_2 =< \kappa_1(+,1), \kappa_4(0,0) >,$ $\Delta_3 =< \kappa_4(0,0), \kappa_2(K_1) >,\ \Delta_4 =< \kappa_2(K_1), \kappa_4(0,1) >,\ \gamma = e$
$(\mathbb{D}_{2K_1} \mid \mathbb{D}_{K_1}; \mathbb{D}_{K_2} \mid \mathbb{Z}_{2K_2})$ K_2 odd	$\Sigma_1 =< \kappa_1(+,1) >,\ \Sigma_2 =< \kappa_2(0,0) >$ $\Delta_1 =< \kappa_2(1,0), \kappa_1(+,1) >,\ \Delta_2 =< \kappa_1(+,1), \kappa_2(0,0) >,$ $\gamma = ((\cos\theta_1, 0, 0, -\sin\theta_1); (0,1,0,0))$
$(\mathbb{D}_{3K} \mid \mathbb{D}_{3K}; \mathbb{O} \mid \mathbb{O})$ K odd	$\Sigma_1 =< \kappa_1(+,+,+,+) >,\ \Sigma_2 =< \kappa_3(0,0,-) >$ $\Delta_1 =< \kappa_3(3K,0,-), \kappa_1(+,+,+,+) >,\ \Delta_2 =< \kappa_1(+,+,+,+), \kappa_3(0,0,-) >,$ $\gamma = ((1,0,0,0); (0,0,0,1))$
$(\mathbb{D}_{6K} \mid \mathbb{D}_{6K}; \mathbb{O} \mid \mathbb{O})$ K odd	$\Sigma_1 =< \kappa_1(+,+,+,+) >,\ \Sigma_2 =< \kappa_5(0,0,-) >,\ \Sigma_3 =< \kappa_5(3K,0,+) >$ $\Delta_1 =< \kappa_5(3K,0,-), \kappa_1(+,+,+,+) >,\ \Delta_2 =< \kappa_1(+,+,+,+), \kappa_5(0,0,-) >,$ $\Delta_3 =< \kappa_5(0,0,-), \kappa_5(3K,0,+) >,\ \gamma = ((1,0,0,0); (0,1,0,0))$
$(\mathbb{D}_{4K} \mid \mathbb{D}_{4K}; \mathbb{O} \mid \mathbb{O})$	$\Sigma_1 =< \kappa_1(+,0,+) >,\ \Sigma_2 =< \kappa_3(0,1) >,\ \Sigma_3 =< \kappa_2(+,1,+) >,\ \Sigma_4 =< \kappa_3(1,1) >$ $\Delta_1 =< \kappa_3(1,1), \kappa_1(+,0,+) >,\ \Delta_2 =< \kappa_1(+,0,+), \kappa_3(0,1) >,$ $\Delta_3 =< \kappa_3(0,1), \kappa_2(+,1,+) >,\ \Delta_4 =< \kappa_2(+,1,+), \kappa_1(+,0,+) >,\ \gamma = e$
$(\mathbb{D}_{12K} \mid \mathbb{D}_{12K}; \mathbb{O} \mid \mathbb{O})$	$\Sigma_1 =< \kappa_2(+,0,+) >,\ \Sigma_2 =< \kappa_4(0,1) >,\ \Sigma_3 =< \kappa_3(+,1,+) >,\ \Sigma_4 =< \kappa_4(1,1) >$ $\Delta_1 =< \kappa_4(1,1), \kappa_2(+,0,+) >,\ \Delta_2 =< \kappa_2(+,0,+), \kappa_4(0,1) >,$ $\Delta_3 =< \kappa_4(0,1), \kappa_3(+,1,+) >,\ \Delta_4 =< \kappa_3(+,1,+), \kappa_2(+,0,+) >,\ \gamma = e$
$(\mathbb{D}_{3K} \mid \mathbb{Z}_{6K}; \mathbb{O} \mid \mathbb{T})$	$\Sigma_1 =< \kappa_1(+,+,+,+) >,\ \Sigma_2 =< \kappa_2(0,0,-) >,\ \Sigma_3 =< \kappa_1(+,+,+,-) >,\ \Sigma_4 =< \kappa_2(1,0,-) >,$ $\Delta_1 =< \kappa_2(1,0,-), \kappa_1(+,+,+,+) >,\ \Delta_2 =< \kappa_1(+,+,+,+), \kappa_2(0,0,-) >,$ $\Delta_3 =< \kappa_2(0,0,-), \kappa_1(+,+,+,-) >,\ \Delta_4 =< \kappa_1(+,+,+,-), \kappa_2(-,0,-) >,\ \gamma = e$

Γ	Σ_j, Δ_j and γ
$(\mathbb{D}_{6K} \,\vert\, \mathbb{Z}_{12K}; \mathbb{O} \,\vert\, \mathbb{T})$	$\Sigma_1 =< \kappa_1(+,+,+,+) >,\ \Sigma_2 =< \kappa_3(0,0,-) >,\ \Sigma_3 =< \kappa_1(+,+,+,-) >,\ \Sigma_4 =< \kappa_3(1,0,-) >$ $\Delta_1 =< \kappa_3(1,0,-), \kappa_1(+,+,+,+) >,\ \Delta_2 =< \kappa_1(+,+,+,+), \kappa_3(0,0,-) >,$ $\Delta_3 =< \kappa_3(0,0,-), \kappa_1(+,+,+,-) >,\ \Delta_4 =< \kappa_2(+,+,+,-), \kappa_3(1,0,-) >,\ \gamma = e$
$(\mathbb{D}_{4K} \,\vert\, \mathbb{D}_{2K}; \mathbb{O} \,\vert\, \mathbb{T})$ $K \neq 3m$ odd	$\Sigma_1 =< \kappa_1(+,0,+) >,\ \Sigma_2 =< \kappa_3(0,0,+) >,$ $\Delta_1 =< \kappa_3(1,0,+), \kappa_1(+,0,+) >,\ \Delta_2 =< \kappa_1(+,0,+), \kappa_3(0,0,+) >,$ $\gamma = ((\cos\theta_1, 0, 0, -\sin\theta_1); (0,1,1,0)/\sqrt{2})$
$(\mathbb{D}_{6K} \,\vert\, \mathbb{D}_{3K}; \mathbb{O} \,\vert\, \mathbb{T})$	$\Sigma_1 =< \kappa_1(+,+,+,+) >,\ \Sigma_2 =< \kappa_4(0,0,-) >,$ $\Delta_1 =< \kappa_4(1,0,-), \kappa_1(+,+,+,+) >,\ \Delta_2 =< \kappa_1(+,+,+,+), \kappa_4(0,0,-) >,$ $\gamma = ((\cos\theta_1, 0, 0, -\sin\theta_1); (0,1,-1,0)/\sqrt{2})$
$(\mathbb{D}_{3K} \,\vert\, \mathbb{Z}_{2K}; \mathbb{O} \,\vert\, \mathbb{V})$ K odd	$\Sigma_1 =< \kappa_1(+,+,+,+) >,\ \Sigma_2 =< \kappa_2(0,-) >,$ $\Delta_1 =< \kappa_2(3K,-), \kappa_1(+,+,+,+) >,\ \Delta_2 =< \kappa_1(+,+,+,+), \kappa_2(0,-) >,$ $\gamma = ((1,0,0,0); (0,0,0,1))$
$(\mathbb{D}_{6K} \,\vert\, \mathbb{Z}_{4K}; \mathbb{O} \,\vert\, \mathbb{V})$	$\Sigma_1 =< \kappa_1(+,+,+,+) >,\ \Sigma_2 =< \kappa_3(0,-) >,\ \Sigma_3 =< \kappa_2(+,0) >,\ \Sigma_4 =< \kappa_3(1,-) >$ $\Delta_1 =< \kappa_3(1,-), \kappa_1(+,+,+,+) >,\ \Delta_2 =< \kappa_1(+,+,+,+), \kappa_3(0,-) >,$ $\Delta_3 =< \kappa_3(0,-), \kappa_2(+,0) >,\ \Delta_4 =< \kappa_2(+,0), \kappa_3(1,-) >,\ \gamma = e$
$(\mathbb{D}_{3K} \,\vert\, \mathbb{D}_{3K}; \mathbb{I} \,\vert\, \mathbb{I})$ K odd	$\Sigma_1 =< \kappa_1'(+,0,+,+) >,\ \Sigma_2 =< \kappa_2(0,0) >,$ $\Delta_1 =< \kappa_2(3K,0), \kappa_1'(+,0,+,+) >,\ \Delta_2 =< \kappa_1'(+,0,+,+), \kappa_2(0,0) >,\ \gamma = ((1,0,0,0); (0,0,0,1))$
$(\mathbb{D}_{6K} \,\vert\, \mathbb{D}_{6K}; \mathbb{I} \,\vert\, \mathbb{I})$	$\Sigma_1 =< \kappa_1'(+,0,+,+) >,\ \Sigma_2 =< \kappa_2(0,0) >,\ \Sigma_3 =< \kappa_3(+,1) >,\ \Sigma_4 =< \kappa_2(1,0) >,$ $\Delta_1 =< \kappa_2(1,0), \kappa_1'(+,0,+,+) >,\ \Delta_2 =< \kappa_1'(+,0,+,+), \kappa_2(0,0) >,$ $\Delta_3 =< \kappa_2(0,0), \kappa_3(+,1) >,\ \Delta_4 =< \kappa_3(+,1), \kappa_2(1,0) >,\ \gamma = e$
$(\mathbb{D}_{5K} \,\vert\, \mathbb{D}_{5K}; \mathbb{I} \,\vert\, \mathbb{I})$ K odd	$\Sigma_1 =< \kappa_1(+,0,+,+) >,\ \Sigma_2 =< \kappa_2(0,0) >,$ $\Delta_1 =< \kappa_2(5K,0), \kappa_1(+,0,+,+) >,\ \Delta_2 =< \kappa_1(+,0,+,+), \kappa_2(0,0) >,\ \gamma = ((1,0,0,0); (0,0,0,1))$
$(\mathbb{D}_{10K} \,\vert\, \mathbb{D}_{10K}; \mathbb{I} \,\vert\, \mathbb{I})$	$\Sigma_1 =< \kappa_1(+,0,+,+) >,\ \Sigma_2 =< \kappa_2(0,0) >,\ \Sigma_3 =< \kappa_3(+,1) >,\ \Sigma_4 =< \kappa_2(1,0) >,$ $\Delta_1 =< \kappa_2(1,0), \kappa_1(+,0,+,+) >,\ \Delta_2 =< \kappa_1(+,0,+,+), \kappa_2(0,0) >,$ $\Delta_3 =< \kappa_2(0,0), \kappa_3(+,1) >,\ \Delta_4 =< \kappa_3(+,1), \kappa_2(1,0) >,\ \gamma = e$
$(\mathbb{D}_{15K} \,\vert\, \mathbb{D}_{15K}; \mathbb{I} \,\vert\, \mathbb{I})$	$\Sigma_1 =< \kappa_1'(+,0,+,+) >,\ \Sigma_2 =< \kappa_3(0,0) >,\ \Sigma_3 =< \kappa_2(+,0,+,+) >,\ \Sigma_4 =< \kappa_3(1,0) >,$ $\Delta_1 =< \kappa_3(1,0), \kappa_1'(+,0,+,+) >,\ \Delta_2 =< \kappa_1'(+,0,+,+), \kappa_3(0,0) >,$ $\Delta_3 =< \kappa_3(0,0), \kappa_2(+,0,+,+) >,\ \Delta_4 =< \kappa_3(1,0), \kappa_2(+,0,+,+) >,\ \gamma = e$
$(\mathbb{D}_{30K} \,\vert\, \mathbb{D}_{30K}; \mathbb{I} \,\vert\, \mathbb{I})$	$\Sigma_1 =< \kappa_1'(+,0,+,+) >,\ \Sigma_2 =< \kappa_3(0,0) >,\ \Sigma_3 =< \kappa_2(+,0,+,+) >,\ \Sigma_4 =< \kappa_3(1,0) >,$ $\Delta_1 =< \kappa_3(1,0), \kappa_1'(+,0,+,+) >,\ \Delta_2 =< \kappa_1'(+,0,+,+), \kappa_3(0,0) >,$ $\Delta_3 =< \kappa_3(0,0), \kappa_2(+,0,+,+) >,\ \Delta_4 =< \kappa_3(1,0), \kappa_2(+,0,+,+) >,\ \gamma = e$
$(\mathbb{D}_{2rK_1} \,\vert\, \mathbb{Z}_{K_1}; \mathbb{D}_{2rK_2} \,\vert\, \mathbb{Z}_{K_2})$	$\Sigma_1 =< \kappa_1(1) >,\ \Sigma_2 =< \kappa_2(0,0,0) >,$ $\Delta_1 = \Delta_2 =< \kappa_1(1), \kappa_2(0,0,0) >,\ \gamma = e$

Continuation of Annex C