Variational method for reconstructing the source in elliptic systems from boundary observations

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Abstract: In this paper we investigate the problem of identifying the source term f in the elliptic system

$$-\nabla\cdot\big(Q\nabla\Phi\big)=f \text{ in } \Omega\subset R^d, d\in\{2,3\},\ Q\nabla\Phi\cdot\vec{n}=j \text{ on } \partial\Omega \text{ and } \Phi=g \text{ on } \partial\Omega$$

from a single noisy measurement couple (j_δ, g_δ) of the Neumann and Dirichlet data (j,g) with noise level $\delta>0$. In this context, the diffusion matrix Q is given. A variational method of Tikhonov-type regularization with specific misfit term and quadratic stabilizing penalty term is suggested to tackle this inverse problem. The method proves to be a modified variant of the Lavrentiev regularization with implicit forward operator. Using the variational discretization concept, where the PDE is discretized with piecewise linear, continuous finite elements, we show the convergence of regularized finite element approximations. Moreover, we derive an error bound and corresponding convergence rates for discrete regularized solutions under a suitable source condition, which typically occurs in the theory of Lavrentiev regularization. For the numerical solution we propose a conjugate gradient method. To illustrate the theoretical results, a numerical case study is presented which supports our analytical findings.

Key words and phrases: Inverse source problem, Tikhonov and Lavrentiev regularization, finite element method, source condition, convergence rates, ill-posedness, conjugate gradient method, Neumann problem, Dirichlet problem.

AMS Subject Classifications: 35R25; 47A52; 35R30; 65J20; 65J22.

1 Introduction

Let Ω be an open, bounded and connected domain of R^d, $d\in\{2,3\}$, with Lipschitz boundary $\partial\Omega$. We consider the elliptic system

$$-\nabla\cdot\big(Q\nabla\Phi\big)=f \text{ in } \Omega, \tag{1.1}$$

$$Q\nabla\Phi\cdot\vec{n}=j^\dagger \text{ on } \partial\Omega \text{ and} \tag{1.2}$$

$$\Phi=g^\dagger \text{ on } \partial\Omega, \tag{1.3}$$

where $\vec{n}$ is the unit outward normal on $\partial\Omega$ and the diffusion matrix Q is given. Furthermore, we assume that $Q:=(q_{rs})_{1\le r,s\le d}\in L^\infty(\Omega)^{d\times d}$ is symmetric and satisfies the uniformly ellipticity condition

$$Q(x)\xi\cdot\xi=\sum_{1\le r,s\le d} q_{rs}(x)\xi_r\xi_s\ge \underline{q}|\xi|^2 \text{ a.e. in } \Omega \tag{1.4}$$

for all $\xi=(\xi_r)_{1\le r\le d}\in R^d$ with some constant $\underline{q}>0$.

The system (1.1)–(1.3) is overdetermined, i.e. if the Neumann and Dirichlet boundary conditions $j^\dagger\in H^{-1/2}(\partial\Omega):=H^{1/2}(\partial\Omega)^*$, $g^\dagger\in H^{1/2}(\partial\Omega)$, and the source term $f\in L^2(\Omega)$ are given, then there may be no Φ satisfying this system. In this paper we assume that the system is consistent and our aim is to reconstruct a function $f\in L^2(\Omega)$ and a function $\Phi\in H^1(\Omega)$ in the system (1.1)–(1.3) from a noisy measurement couple $(j_\delta, g_\delta)\in H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)$ of the exact Neumann and Dirichlet data $\big(j^\dagger, g^\dagger\big)$, where $\delta>0$ stands for the measurement error, i.e. we assume the noise model

$$\big\|j_\delta-j^\dagger\big\|_{H^{-1/2}(\partial\Omega)}+\big\|g_\delta-g^\dagger\big\|_{H^{1/2}(\partial\Omega)}\le\delta. \tag{1.5}$$

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M. Hinze gratefully acknowledges support of the Lothar Collatz Center for Computing in Science, University of Hamburg
B. Hofmann gratefully acknowledges support by the German Research Foundation (DFG) under grant HO 1454/10-1
T.N.T. Quyen gratefully acknowledges support of the Alexander von Humboldt-Foundation

To formulate precisely the problem, we first give some notations. Let us denote by $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ the continuous Dirichlet trace operator with $\gamma^{-1} : H^{1/2}(\partial\Omega) \to H^1(\Omega)$ its continuous right inverse operator, i.e. $(\gamma \circ \gamma^{-1})g = g$ for all $g \in H^{1/2}(\partial\Omega)$. We set

$$H^1_\diamond(\Omega) := \left\{ u \in H^1(\Omega) \;\Big|\; \int_{\partial\Omega} \gamma u dx = 0 \right\} \text{ and } H^{1/2}_\diamond(\partial\Omega) := \left\{ g \in H^{1/2}(\partial\Omega) \;\Big|\; \int_{\partial\Omega} g(x) dx = 0 \right\}$$

and denote by C_Ω the positive constant appearing in the Poincaré-Friedrichs inequality (cf. [36])

$$C_\Omega \int_\Omega \varphi^2 \le \int_\Omega |\nabla\varphi|^2 \text{ for all } \varphi \in H^1_\diamond(\Omega). \tag{1.6}$$

Since $H^1_0(\Omega) := \{ u \in H^1(\Omega) \mid \gamma u = 0 \} \subset H^1_\diamond(\Omega)$, the inequality (1.6) is in particular valid for all $\varphi \in H^1_0(\Omega)$. Furthermore, by (1.4), the coercivity condition

$$\|\varphi\|^2_{H^1(\Omega)} \le \frac{1 + C_\Omega}{C_\Omega} \int_\Omega |\nabla\varphi|^2 \le \frac{1 + C_\Omega}{C_\Omega \underline{q}} \int_\Omega Q\nabla\varphi \cdot \nabla\varphi \tag{1.7}$$

holds for all $\varphi \in H^1_\diamond(\Omega)$.

Now, for any fixed $(j, g) \in H^{-1/2}(\partial\Omega) \times H^{1/2}_\diamond(\partial\Omega)$ we can simultaneously consider the Neumann problem

$$-\nabla \cdot (Q\nabla u) = f \text{ in } \Omega \text{ and } Q\nabla u \cdot \vec{n} = j \text{ on } \partial\Omega \tag{1.8}$$

as well as the Dirichlet problem

$$-\nabla \cdot (Q\nabla v) = f \text{ in } \Omega \text{ and } v = g \text{ on } \partial\Omega. \tag{1.9}$$

By the aid of (1.7) and the Riesz representation theorem, we conclude that for each $f \in L^2(\Omega)$ there exists a unique weak solution u of the problem (1.8) in the sense that $u \in H^1_\diamond(\Omega)$ and satisfies the identity

$$\int_\Omega Q\nabla u \cdot \nabla\varphi = \langle j, \gamma\varphi \rangle + (f, \varphi) \tag{1.10}$$

for all $\varphi \in H^1_\diamond(\Omega)$, where notation $\langle j, g \rangle$ stands for the value of the function $j \in H^{-1/2}(\partial\Omega)$ at $g \in H^{1/2}(\partial\Omega)$ and the notation (f, φ) is the scalar inner product of f and φ in the space $L^2(\Omega)$. Then we can define the *Neumann operator*

$$\mathcal{N} : L^2(\Omega) \to H^1_\diamond(\Omega) \text{ with } f \mapsto \mathcal{N}_f j,$$

which maps each $f \in L^2(\Omega)$ to the unique weak solution $\mathcal{N}_f j := u$ of the problem (1.8). Similarly, the problem (1.9) also attains a unique weak solution v in the sense that $v \in H^1(\Omega)$, $\gamma v = g$ and the identity

$$\int_\Omega Q\nabla v \cdot \nabla\psi = (f, \psi) \tag{1.11}$$

holds for all $\psi \in H^1_0(\Omega)$. The *Dirichlet operator* is defined as

$$\mathcal{D} : L^2(\Omega) \to H^1_\diamond(\Omega) \text{ with } f \mapsto \mathcal{D}_f g,$$

which maps each $f \in L^2(\Omega)$ to the unique weak solution $\mathcal{D}_f g := v$ of the problem (1.9). Therefore, for any fixed $f \in L^2(\Omega)$ we can define the so-called *Neumann-to-Dirichlet map*

$$\Lambda_f : H^{-1/2}(\partial\Omega) \to H^{1/2}_\diamond(\partial\Omega), \quad j \mapsto \Lambda_f j := \gamma\mathcal{N}_f j.$$

We mention that since $H^1_0(\Omega) \subset H^1_\diamond(\Omega)$, we from (1.10) have that $\int_\Omega Q\nabla\mathcal{N}_f j \cdot \nabla\psi = (f, \psi)$ for all $\psi \in H^1_0(\Omega)$. In view of (1.11) we therefore conclude

$$\Lambda_f j = g \text{ if and only if } \mathcal{N}_f j = \mathcal{D}_f g,$$

where the identities

$$\mathcal{N}_f j = \mathcal{N}_f 0 + \mathcal{N}_0 j \quad \text{and} \quad \mathcal{D}_f g = \mathcal{D}_f 0 + \mathcal{D}_0 g \tag{1.12}$$

are satisfied, and the operators $f \mapsto \mathcal{N}_f 0$ and $f \mapsto \mathcal{D}_f 0$ are linear and bounded from $L^2(\Omega)$ into itself. Furthermore,

$$\Lambda_f j = \gamma \mathcal{N}_f j = \gamma \mathcal{N}_0 j + \gamma \mathcal{N}_f 0 = \Lambda_0 j + \Lambda_f 0,$$

where $\Lambda_0 j$ is linear, self-adjoint, bounded and invertible, as the diffusion Q is smooth enough (cf. [34]).

As in Electrical Impedance Tomography or Calderón's problem [4, 15, 34] one can pose the question whether the source distribution f inside a physical domain Ω can be determined from an *infinite* number of observations on the boundary $\partial\Omega$, i.e. from the Neumann-to-Dirichlet map Λ_f:

$$f_1, f_2 \in L^2(\Omega) \quad \text{with} \quad \Lambda_{f_1} = \Lambda_{f_2} \quad \Rightarrow \quad f_1 = f_2 \ ?$$

To our best knowledge, the above question is still open so far. In case an observation Λ_δ of Λ_f being available one can use a certain regularization method to approximate the sought source. For example, one can consider for operator norms $\|\cdot\|_*$ a minimizer of the problem

$$\min_{f \in L^2(\Omega)} \|\Lambda_f - \Lambda_\delta\|_*^2 + \rho \|f - f^*\|_{L^2(\Omega)}^2$$

as a reconstruction along the lines of Tikhonov's regularization method, where $\rho > 0$ is the regularization parameter and f^* is an a-priori estimate of the sought source.

However, in practice we have only a *finite* number of observations and the task is to reconstruct the identified source, at least by numerical approximations. Furthermore, for simplicity of exposition we below restrict ourselves to the case of just one observation pair (j_δ, g_δ) being available, while the approach described here can be easily extended to multiple measurements $\left(j_\delta^i, g_\delta^i\right)_{i=1,\ldots,I}$, see Section 7, Ex. 7.2. The inverse problem is thus stated as follows.

$$\textit{Given } \left(j^\dagger, g^\dagger\right) \in H^{-1/2}(\partial\Omega) \times H_\diamond^{1/2}(\partial\Omega) \textit{ with } \Lambda_f j^\dagger = g^\dagger, \textit{ find } f \in L^2(\Omega). \tag{$\mathcal{IP}$}$$

In other word, the interested problem is, for given $\left(j^\dagger, g^\dagger\right) \in H^{-1/2}(\partial\Omega) \times H_\diamond^{1/2}(\partial\Omega)$, to find some $f \in L^2(\Omega)$ and consequently $\Phi \in H_\diamond^1(\Omega)$ such that the system (1.1)–(1.3) is satisfied in the weak sense. Precisely, we define the general solution set

$$\mathcal{I}\left(j^\dagger, g^\dagger\right) := \left\{ f \in L^2(\Omega) \mid \Lambda_f j^\dagger = g^\dagger \right\} = \left\{ f \in L^2(\Omega) \mid \mathcal{N}_f j^\dagger = \mathcal{D}_f g^\dagger \right\} \tag{1.13}$$

of the inverse problem ($\mathcal{IP}$). The source identification problem as described here is well known to be not uniquely determined from boundary observations (see a counterexample in [3]), i.e. the set $\mathcal{I}\left(j^\dagger, g^\dagger\right)$ fails to be a singleton. Since not the Neumann-to-Dirichlet map is given, but only one pair $\left(j^\dagger, g^\dagger\right)$, the problem is even highly underdetermined. Thus instead we will search for uniquely determined f^*-minimum-norm solutions $f^\dagger$ (cf. Section 4) which is the minimizer of the problem

$$\min_{f \in \mathcal{I}(j^\dagger, g^\dagger)} \|f - f^*\|_{L^2(\Omega)}^2. \tag{$\mathcal{IP}-MN$}$$

Due to Lemma 4.4 below, the set $\mathcal{I}\left(j^\dagger, g^\dagger\right)$ is non-empty, closed and convex, hence $f^\dagger$ is uniquely determined. On the other hand, for all $f \in \mathcal{I}\left(j^\dagger, g^\dagger\right)$ the equation $\mathcal{N}_f j^\dagger = \mathcal{D}_f g^\dagger$ is fulfilled. However, we have to solve this equation with noise data $(j_\delta, g_\delta) \in H^{-1/2}(\partial\Omega) \times H_\diamond^{1/2}(\partial\Omega)$ of $\left(j^\dagger, g^\dagger\right)$ satisfying (1.5). The simplest variety of regularization may be to consider a minimizer of the Tikhonov functional

$$\|\mathcal{N}_f j_\delta - \mathcal{D}_f g_\delta\|_{\mathcal{X}}^2 + \rho \|f - f^*\|_{L^2(\Omega)}^2$$

over $f \in L^2(\Omega)$ as an approximate solution to $f^\dagger$, where $\mathcal{X} = L^2(\Omega)$ or $\mathcal{X} = H^1(\Omega)$.

In present work we adopt the variational approach of Kohn and Vogelius [28, 29, 30] in using cost functions containing the gradient of forward operators to the above mentioned inverse source problem. More precisely, we use the convex function

$$\mathcal{J}_\delta(f) := \int_\Omega Q \nabla \left(\mathcal{N}_f j_\delta - \mathcal{D}_f g_\delta\right) \cdot \nabla \left(\mathcal{N}_f j_\delta - \mathcal{D}_f g_\delta\right) dx, \tag{1.14}$$

(cf. Lemma 2.3) instead of the mapping $f \mapsto \|\mathcal{N}_f j_\delta - \mathcal{D}_f g_\delta\|^2_{\mathcal{X}}$, together with Tikhonov regularization and consider the *unique* solution $f_{\rho,\delta}$ of the strictly convex minimization problem

$$\min_{f\in L^2(\Omega)} \mathcal{J}_\delta(f) + \rho\|f - f^*\|^2_{L^2(\Omega)} \tag{$\mathcal{P}_{\rho,\delta}$}$$

as the regularized solution. The motivation in using this cost functional $\mathcal{J}_\delta$ as misfit functional is that for all $\xi \in L^2(\Omega)$ the inequality

$$\mathcal{J}_0(\xi) := \int_\Omega Q\nabla\left(\mathcal{N}_\xi j^\dagger - \mathcal{D}_\xi g^\dagger\right)\cdot\nabla\left(\mathcal{N}_\xi j^\dagger - \mathcal{D}_\xi g^\dagger\right)dx \ge \frac{C_\Omega \underline{q}}{1+C_\Omega}\left\|\mathcal{N}_\xi j^\dagger - \mathcal{D}_\xi g^\dagger\right\|^2_{H^1(\Omega)} \ge 0$$

holds true and $\mathcal{J}_0(f) = 0$ at any $f \in \mathcal{I}\left(j^\dagger, g^\dagger\right)$. The advantage is evident, because the minimizer $f_{\rho,\delta}$ satisfies the equation

$$f - f^* = -\frac{1}{\rho}\left(\mathcal{N}_f j_\delta - \mathcal{D}_f g_\delta\right) \tag{1.15}$$

(see Theorem 3.2 below). Due to formula (1.15), the approach proves to be a modified variant of the Lavrentiev regularization (see, e.g., [2, 13, 25, 43]) with implicit forward operator. Furthermore, for convenience in numerical analysis with the finite element methods we consider here the Tikhonov regularization. The use of different convex penalty terms, e.g. total variation, may be a work for us in future.

Let $\mathcal{N}^h_f j_\delta$ and $\mathcal{D}^h_f g_\delta$ be corresponding approximations of the solution maps $\mathcal{N}_f j_\delta$ and $\mathcal{D}_f g_\delta$ in the finite dimensional space $\mathcal{V}^h_1$ of piecewise linear, continuous finite elements. We then consider the discrete regularized problem corresponding to $(\mathcal{P}_{\rho,\delta})$, i.e., the following strictly convex minimization problem

$$\min_{f\in L^2(\Omega)} \int_\Omega Q\nabla\left(\mathcal{N}^h_f j_\delta - \mathcal{D}^h_f g_\delta\right)\cdot\nabla\left(\mathcal{N}^h_f j_\delta - \mathcal{D}^h_f g_\delta\right)dx + \rho\|f - f^*\|^2_{L^2(\Omega)}. \tag{$\mathcal{P}^h_{\rho,\delta}$}$$

Using the variational discretization concept introduced in [24], we show in Section 3 that the unique solution $f^h_{\rho,\delta}$ of the problem $\left(\mathcal{P}^h_{\rho,\delta}\right)$ automatically belongs to the finite dimensional space $\mathcal{V}^h_1$. Thus, a discretization of the admissible set $L^2(\Omega)$ can be avoided.

As $h, \delta \to 0$ and with an appropriate a-priori regularization parameter choice $\rho = \rho(h,\delta)$, we in Section 4 prove that the sequence $\left(f^h_{\rho,\delta}\right)$ converges to $f^\dagger$ in the $L^2(\Omega)$-norm. Furthermore, the corresponding state sequences $\left(\mathcal{N}^h_{f^h_{\rho,\delta}} j_\delta\right)$ and $\left(\mathcal{D}^h_{f^h_{\rho,\delta}} g_\delta\right)$ converge in the $H^1(\Omega)$-norm to $\Phi^\dagger = \Phi^\dagger(f^\dagger, j^\dagger, g^\dagger)$ solving (1.1)–(1.3).

Section 5 is devoted to convergence rates. In this section we also show that if $f \in \mathcal{I}\left(j^\dagger, g^\dagger\right)$ and there is a function $w \in L^2(\Omega)$ such that

$$f - f^* = \mathcal{N}_w j^\dagger - \mathcal{D}_w g^\dagger \tag{1.16}$$

then $f = f^\dagger$, i.e. f is the *unique* f^*-minimum-norm solution of the identification problem. Condition (1.16) appears to be a *source condition* which is typical for Lavrentiev regularization and allows for convergence rates. Precisely, for the known matrix $Q \in C^{0,1}(\Omega)^{d\times d}$ and the exact data $\left(j^\dagger, g^\dagger\right) \in H^{1/2}(\partial\Omega) \times H^{3/2}(\partial\Omega)$ we derive the convergence rates

$$\left\|\mathcal{N}^h_{f^h_{\rho,\delta}} j_\delta - \mathcal{D}^h_{f^h_{\rho,\delta}} g_\delta\right\|^2_{H^1(\Omega)} + \rho\left\|f^h_{\rho,\delta} - f^\dagger\right\|^2_{L^2(\Omega)} = \mathcal{O}\left(\delta^2 + h^2 + h\rho + \delta\rho + \rho^2\right)$$

and

$$\left\|\mathcal{N}^h_{f^h_{\rho,\delta}} j_\delta - \Phi^\dagger\right\|^2_{H^1(\Omega)} + \left\|\mathcal{D}^h_{f^h_{\rho,\delta}} g_\delta - \Phi^\dagger\right\|^2_{H^1(\Omega)} = \mathcal{O}\left(\delta^2\rho^{-1} + h^2\rho^{-1} + h + \delta + \rho\right)$$

will be established. Finally, for the numerical solution of the discrete regularized problem $\left(\mathcal{P}^h_{\rho,\delta}\right)$ we employ a conjugate gradient algorithm. Numerical results show an efficiency of our theoretical findings.

The source identification problem in PDEs arises in many branches of applied science such as electroencephalography, geophysical prospecting and pollutant detection, and attracted great attention from many scientists in the last 30 years or so. For surveys on this subject we may consult in [8, 16, 19, 23, 26, 42] and the references therein. Up to now, only a limited number of works was investigated the general source identification problem and obtained results concentrated on numerical analysis for the identification problem. In [20, 32, 33] authors have used the dual reciprocity boundary element methods to simulate numerically for the above mentioned identification problem. In case some priori knowledge of the identified source is available, such as a point source, a characteristic function or a harmonic function, numerical methods treating the problem have been obtained in [5, 6, 31, 39]. A survey of the problem of simultaneously identifying the source term and coefficients in elliptic systems from *distributed* observations can be found in [38], where further references can be found.

In the present paper, the general source identification problem in elliptic partial differential equations from a single noisy measurement couple of Neumann and Dirichlet data is studied. So far, we have not yet found investigations on the discretization analysis for this source recovery problem, a fact which also motivated the research presented in the paper. By using a suitable version of the Tikhonov-type regularization with some non-standard misfit term we could outline that the source distribution inside the physical domain Ω can be reconstructed from a *finite* number of observations on the boundary $\partial\Omega$, at least by numerical approximations. The specific regularization approach proves to be a version of Lavrentiev regularization with implicit forward operator. One of the main results of the paper is to show convergence of the finite element discretized Tikhonov-regularized solutions to a sought source function. Another main result is the interpretation of an occurring condition of solution smoothness as a range-type source condition of Lavrentiev's regularization method. This allows us to establish error bounds and corresponding convergence rates for the regularized solutions.

We conclude this introduction with a remark that since the main interest is to clearly state our ideas, we only treat the model elliptic problem (1.1) while the approach described here can be easily extended to more general models, e.g., for the source identification problem in diffusion-reaction equations

$$-\nabla \cdot \big(Q\nabla\Phi\big) + \kappa^2\Phi = f \text{ in } \Omega,\ Q\nabla\Phi\cdot\vec{n} + \sigma\Phi = j^\dagger \text{ on } \partial\Omega \text{ and } \Phi = g^\dagger \text{ on } \partial\Omega \tag{1.17}$$

from a measurement (j_δ, g_δ) of $\big(j^\dagger, g^\dagger\big)$, where Q satisfying the condition (1.4), $0 \neq \kappa = \kappa(x) \in L^\infty(\Omega)$, i.e the set $\{x \in \Omega | \kappa(x) \neq 0\}$ has positive Lebesgue measure, and $\sigma = \sigma(x) \in L^\infty(\partial\Omega)$ with $\sigma \geq 0$ are given. The variational approach is now formulated as the minimizing problem with the misfit

$$\int_\Omega Q\nabla\left(R_f j_\delta - D_f g_\delta\right)\cdot\nabla\left(R_f j_\delta - D_f g_\delta\right)dx + \int_\Omega \kappa^2\left(R_f j_\delta - D_f g_\delta\right)^2 dx + \int_{\partial\Omega}\sigma\left(R_f j_\delta - D_f g_\delta\right)^2 dx$$

over $f \in L^2(\Omega)$, where R and D are the Robin operator and the Dirichlet operator relating with the equation (1.17), respectively. Recently obtained results concerning the inverse source problem for (1.17) on the identification together with numerical algorithms can be found in, e.g., [1, 3, 9, 10].

Throughout the paper we use the standard notion of Sobolev spaces $H^1(\Omega)$, $H^1_0(\Omega)$, $W^{k,p}(\Omega)$, etc from, for example, [44]. If not stated otherwise we write $\int_\Omega \cdots$ instead of $\int_\Omega \cdots dx$.

2 Preliminaries

We first mention that the solution $\mathcal{N}_f j$ satisfies the following estimate

$$\begin{aligned}\|\mathcal{N}_f j\|_{H^1(\Omega)} &\leq \frac{1+C_\Omega}{C_\Omega \underline{q}}\left(\|j\|_{H^{-1/2}(\partial\Omega)}\|\gamma\|_{\mathcal{L}\left(H^1(\Omega),H^{1/2}(\partial\Omega)\right)} + \|f\|_{L^2(\Omega)}\right)\\ &\leq C_{\mathcal{N}}\left(\|j\|_{H^{-1/2}(\partial\Omega)} + \|f\|_{L^2(\Omega)}\right),\end{aligned} \tag{2.1}$$

where $C_{\mathcal{N}} := \frac{1+C_\Omega}{C_\Omega \underline{q}}\max\left(1, \|\gamma\|_{\mathcal{L}\left(H^1(\Omega),H^{1/2}(\partial\Omega)\right)}\right)$. Next, we can rewrite $\mathcal{D}_f g = v_0 + G$, where $G = \gamma^{-1}g$ and $v_0 \in H^1_0(\Omega)$ is the unique solution to the variational problem $\int_\Omega Q\nabla v_0\cdot\nabla\psi = (f,\psi) - \int_\Omega Q\nabla G\cdot\nabla\psi$ for all $\psi \in H_0^1(\Omega)$. Since $\|G\|_{H^1(\Omega)} \le \left\|\gamma^{-1}\right\|_{\mathcal{L}\left(H^{1/2}(\partial\Omega),H^1(\Omega)\right)} \|g\|_{H^{1/2}(\partial\Omega)}$, we thus obtain the priori estimate

$$\begin{aligned}\|\mathcal{D}_f g\|_{H^1(\Omega)} &\le \|v_0\|_{H^1(\Omega)} + \|G\|_{H^1(\Omega)} \le \frac{1+C_\Omega}{C_\Omega \underline{q}} \left(\|f\|_{L^2(\Omega)} + d\|Q\|_{L^\infty(\Omega)^{d\times d}} \|G\|_{H^1(\Omega)}\right) + \|G\|_{H^1(\Omega)} \\ &\le \frac{1+C_\Omega}{C_\Omega \underline{q}} \|f\|_{L^2(\Omega)} + \left(d\frac{1+C_\Omega}{C_\Omega \underline{q}}\|Q\|_{L^\infty(\Omega)^{d\times d}} + 1\right) \left\|\gamma^{-1}\right\|_{\mathcal{L}\left(H^{1/2}(\partial\Omega),H^1(\Omega)\right)} \|g\|_{H^{1/2}(\partial\Omega)} \\ &\le C_{\mathcal{D}} \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}\right), \end{aligned} \tag{2.2}$$

where $C_{\mathcal{D}} := \max\left(\frac{1+C_\Omega}{C_\Omega \underline{q}}, \left(d\frac{1+C_\Omega}{C_\Omega \underline{q}}\|Q\|_{L^\infty(\Omega)^{d\times d}} + 1\right) \left\|\gamma^{-1}\right\|_{\mathcal{L}\left(H^{1/2}(\partial\Omega),H^1(\Omega)\right)}\right)$ and $Q := (q_{rs})_{1\le r,s\le d} \in L^\infty(\Omega)^{d\times d}$ with $\|Q\|_{L^\infty(\Omega)^{d\times d}} := \max_{1\le r,s\le d} \|q_{rs}\|_{L^\infty(\Omega)}$.

In view of the identities (1.12), where $\mathcal{N}_f 0$ and $\mathcal{D}_f 0$ are linear operators of f, we deduce for each $f \in L^2(\Omega)$ that the action of Fréchet derivatives of $\mathcal{N}_f j$ and $\mathcal{D}_f g$ in the direction $\xi \in L^2(\Omega)$ are given by

$$\mathcal{N}'_f j(\xi) := \mathcal{N}'(f)\xi = \mathcal{N}_\xi 0 \quad \text{and} \quad \mathcal{D}'_f g(\xi) := \mathcal{D}'(f)\xi = \mathcal{D}_\xi 0. \tag{2.3}$$

Lemma 2.1. *(i) The map* $\mathbf{T} : L^2(\Omega) \to L^2(\Omega)$ *defined by*

$$\mathbf{T}(f) := \mathcal{N}_f 0 - \mathcal{D}_f 0$$

is linear, bounded and self-adjoint, i.e. $(\mathbf{T}(f), w) = (f, \mathbf{T}(w))$ *for all* $f, w \in L^2(\Omega)$.

(ii) For any fixed $(j, g) \in H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$ *the map* $\mathbf{L} : L^2(\Omega) \to L^2(\Omega)$ *defined by*

$$\mathbf{L}(f) := \mathcal{N}_f j - \mathcal{D}_f g = \mathbf{T}(f) + \mathcal{N}_0 j - \mathcal{D}_0 g$$

is affine linear, continuous and monotone, i.e. $(\mathbf{L}(f) - \mathbf{L}(w), f - w) \ge 0$ *for all* $f, w \in L^2(\Omega)$.

Proof. (i) It follows from (1.10) that

$$(\mathbf{T}(f), w) = \int_\Omega Q\nabla\mathcal{N}_w 0 \cdot \nabla(\mathcal{N}_f 0 - \mathcal{D}_f 0) = \int_\Omega Q\nabla\mathcal{N}_w 0 \cdot \nabla\mathcal{N}_f 0 - \int_\Omega Q\nabla\mathcal{N}_w 0 \cdot \nabla\mathcal{D}_f 0, \tag{2.4}$$

and similarly,

$$(f, \mathbf{T}(w)) = \int_\Omega Q\nabla\mathcal{N}_f 0 \cdot \nabla\mathcal{N}_w 0 - \int_\Omega Q\nabla\mathcal{N}_f 0 \cdot \nabla\mathcal{D}_w 0. \tag{2.5}$$

Using (1.10)–(1.11) again, we get

$$\begin{aligned} \int_\Omega Q\nabla\mathcal{N}_w 0 \cdot \nabla\mathcal{D}_f 0 = (w, \mathcal{D}_f 0) = \int_\Omega Q\nabla\mathcal{D}_w 0 \cdot \nabla\mathcal{D}_f 0 \\ \int_\Omega Q\nabla\mathcal{N}_f 0 \cdot \nabla\mathcal{D}_w 0 = (f, \mathcal{D}_w 0) = \int_\Omega Q\nabla\mathcal{D}_f 0 \cdot \nabla\mathcal{D}_w 0 \end{aligned} \tag{2.6}$$

and the self-adjoint property of $\mathbf{T}$ now follows directly from (2.4)-(2.6).

(ii) Denoting by $\xi := f - w$, we get from (1.10) that

$$(\mathbf{L}(f) - \mathbf{L}(w), f - w) = (\mathbf{T}(f) - \mathbf{T}(w), f - w) = (\mathbf{T}(\xi), \xi) = \int_\Omega Q\nabla\mathcal{N}_\xi 0 \cdot \nabla(\mathcal{N}_\xi 0 - \mathcal{D}_\xi 0). \tag{2.7}$$

We further have from (1.10)–(1.11) that

$$\int_\Omega Q\nabla\mathcal{D}_\xi 0 \cdot \nabla(\mathcal{N}_\xi 0 - \mathcal{D}_\xi 0) = 0. \tag{2.8}$$

Combining (2.7) with (2.8), we arrive at $(\mathbf{L}(f) - \mathbf{L}(w), f - w) = \int_\Omega Q\nabla(\mathcal{N}_\xi 0 - \mathcal{D}_\xi 0) \cdot \nabla(\mathcal{N}_\xi 0 - \mathcal{D}_\xi 0) \ge 0$, which finishes the proof. □

We now briefly discuss the character of the null-space $\ker(\mathbf{T})$ of the operator $\mathbf{T}$ occurring in the above lemma. For simplicity we assume that the known matrix $Q = I_d$, the $d \times d$-unit matrix. Then the counterexample of [3] shows that $\ker(\mathbf{T}) \neq \{0\}$. Furthermore, one can easily see that for any $\Psi \in C_c^2(\Omega)$, the space of all functions having second-order derivatives with compact support in Ω, the negative Laplace $-\Delta\Psi$ of Ψ satisfies $-\Delta\Psi \in \ker(\mathbf{T})$.

We remark that due to the inequality

$$\frac{C_\Omega q}{1 + C_\Omega}\|\varphi\|_{H^1(\Omega)}^2 \leq \int_\Omega Q\nabla\varphi \cdot \nabla\varphi \leq d\|Q\|_{L^\infty(\Omega)^{d\times d}}\|\varphi\|_{H^1(\Omega)}^2, \quad \forall\varphi \in H_\diamond^1(\Omega), \tag{2.9}$$

the expression

$$[u, v] := \int_\Omega Q\nabla u \cdot \nabla v \tag{2.10}$$

generates a scalar inner product on the space $H_\diamond^1(\Omega)$ which is equivalent to the usual one.

Let $\tilde{\mathbf{T}}^* : H_\diamond^1(\Omega) \to L^2(\Omega)$ be the adjoint operator of $\tilde{\mathbf{T}} : L^2(\Omega) \to H_\diamond^1(\Omega)$, where $H_\diamond^1(\Omega)$ is equipped with the scalar inner product (2.10) above and $\tilde{\mathbf{T}}(f) := \mathbf{T}(f) \in H_\diamond^1(\Omega)$. For all $f \in L^2(\Omega)$ and $\phi \in H_\diamond^1(\Omega)$ we thus have

$$\left[\tilde{\mathbf{T}}f, \phi\right] = \int_\Omega Q\nabla\mathcal{N}_f 0 \cdot \nabla\phi - \int_\Omega Q\nabla\mathcal{D}_f 0 \cdot \nabla\phi = (f, \phi) - \int_\Omega Q\nabla\mathcal{D}_f 0 \cdot \nabla\phi = (f, \tilde{\mathbf{T}}^*\phi), \tag{2.11}$$

by (1.10). We now decompose $H_\diamond^1(\Omega)$ into the orthogonal direct sum $H_\diamond^1(\Omega) = H_0^1(\Omega) \oplus H_0^1(\Omega)^\perp$ with respect to the inner product (2.10). We note for all $g \in H_\diamond^{1/2}(\partial\Omega)$ that $\mathcal{D}_0 g \in H_0^1(\Omega)^\perp$. Furthermore,

$$\forall\ g_1,\ g_2 \in H_\diamond^{1/2}(\partial\Omega),\ g_1 \neq g_2 \quad \Rightarrow \quad \mathcal{D}_0 g_1 \neq \mathcal{D}_0 g_2$$

which implies $\dim H_0^1(\Omega)^\perp \geq \dim H_\diamond^{1/2}(\partial\Omega) = \infty$. For all $f \in L^2(\Omega)$ we deduce from (1.11) and (2.11) that

$$\phi \in H_0^1(\Omega) \Leftrightarrow \int_\Omega Q\nabla\mathcal{D}_f 0 \cdot \nabla\phi = (f, \phi) \Leftrightarrow (f, \tilde{\mathbf{T}}^*\phi) = 0 \Leftrightarrow \tilde{\mathbf{T}}^*\phi = 0 \Leftrightarrow \phi \in \ker\tilde{\mathbf{T}}^*,$$

or in other words $\ker\tilde{\mathbf{T}}^* = H_0^1(\Omega)$. Furthermore, for all $\widehat{\phi} \in H_0^1(\Omega)^\perp$ we get $\int_\Omega Q\nabla\mathcal{D}_f 0 \cdot \nabla\widehat{\phi} = 0$, since $\mathcal{D}_f 0 \in H_0^1(\Omega)$. Again, the equation (2.11) follows that

$$(f, \widehat{\phi}) = (f, \tilde{\mathbf{T}}^*\widehat{\phi}) \quad \text{for all} \quad f \in L^2(\Omega),\ \widehat{\phi} \in H_0^1(\Omega)^\perp.$$

Therefore, $\tilde{\mathbf{T}}^*_{|H_0^1(\Omega)^\perp}$ is the compact embedding $H_0^1(\Omega)^\perp \hookrightarrow L^2(\Omega)$. The operator $\tilde{\mathbf{T}} : L^2(\Omega) \to H_\diamond^1(\Omega)$ and hence $\mathbf{T} : L^2(\Omega) \to L^2(\Omega)$ as a composition with a continuous embedding operator are compact operators. Next, we show that $\dim(\tilde{\mathbf{T}}) = \infty$. For deriving a contradiction we assume that $\dim\operatorname{range}(\tilde{\mathbf{T}}) < \infty$. Then we can write $H_\diamond^1(\Omega) = \operatorname{range}(\tilde{\mathbf{T}}) \oplus \operatorname{range}(\tilde{\mathbf{T}})^\perp$ with respect to the inner product (2.10). By (2.11), for all $\varphi \in \operatorname{range}(\tilde{\mathbf{T}})^\perp$ we get $\int_\Omega Q\nabla\mathcal{D}_f 0 \cdot \nabla\varphi = (f, \varphi)$ holding for all $f \in L^2(\Omega)$ which implies that $\varphi \in H_0^1(\Omega)$. Therefore, $H_0^1(\Omega)^\perp \subset \left(\operatorname{range}(\tilde{\mathbf{T}})^\perp\right)^\perp = \overline{\operatorname{range}(\tilde{\mathbf{T}})} = \operatorname{range}(\tilde{\mathbf{T}})$ and this yields the contradiction $\infty = \dim H_0^1(\Omega)^\perp \leq \dim\operatorname{range}(\tilde{\mathbf{T}}) < \infty$. Consequently, $\mathbf{T} : L^2(\Omega) \to L^2(\Omega)$ is compact operator and possesses an infinite dimensional range. Therefore, it should be noted at this point that all linear operator equations with forward operator $\mathbf{T}$ mapping in $L^2(\Omega)$ are *ill-posed of type II* in the sense of Nashed ([35]).

Finding an element $f \in \mathcal{I}\left(j^\dagger, g^\dagger\right)$, i.e. a solution to the identification problem $(\mathcal{IP})$, is equivalent to solving an operator equation (see for notations and details Lemma 2.1 above)

$$F(f, j, g) = 0, \quad \text{where} \quad F(f, j, g) := \mathbf{L}(f), \quad \text{and} \quad j := j^\dagger,\ g := g^\dagger. \tag{2.12}$$

Such an implicit inverse problem model was generally introduced in the monograph [7, Section 1.2]. Due to Lemma 2.1, the forward operator F in the operator equation (2.12) is affine linear with respect to f and we can rewrite (2.12) as $\mathbf{T}(f) + B(j^\dagger, g^\dagger) = 0$ with the remaining term $B(j, g) = \mathcal{N}_0 j - \mathcal{D}_0 g$, where for fixed $(j^\dagger, g^\dagger)$ the magnitude $B(j^\dagger, g^\dagger)$ is a constant function in $L^2(\Omega)$ and $\mathbf{T}$ is a linear, bounded and self-adjoint operator monotone mapping in $L^2(\Omega)$ such that $\mathbf{L}$ is a continuous, affine linear and monotone operator. The monotonicity of $\mathbf{L}$ allows us to apply Lavrentiev regularization for stabilizing the inverse problem for given noisy data (j_δ, g_δ) by using the solution of the singularly perturbed version

$$F(f, j_\delta, g_\delta) + \rho\,(f - f^*) = \mathcal{N}_f j_\delta - \mathcal{D}_f g_\delta + \rho\,(f - f^*) = 0 \tag{2.13}$$

of the original operator equation as regularized solution. One simply sees that the uniquely determined Lavrentiev-regularized solution satisfying (2.13) and the Tikhonov-regularized solution $f_{\rho,\delta}$ satisfying (1.15) coincide. Hence, computing the regularized solutions is rather simple and requires, given noisy data j_δ and g_δ, only to solve one Neumann problem for finding $\mathcal{N}_f j_\delta$ as well as one Dirichlet problem for finding $\mathcal{D}_f g_\delta$. No extremal problems have to be solved.

Remark 2.2. For analytic reasons, not for computational use, one can make the Lavrentiev regularization explicit as

$$\mathbf{T}(f) + \rho\,(f - f^*) = \mathcal{D}_0 g_\delta - \mathcal{N}_0 j_\delta, \tag{2.14}$$

where the solutions of Dirichlet and Neumann problems for the homogeneous elliptic partial differential equation with noisy data in the form $-B(j^\dagger, g^\dagger)$ occur on the right-hand side of equation (2.14). This is a singularly perturbed version of the equation $\mathbf{T}(f) = \mathcal{D}_0 g_\delta - \mathcal{N}_0 j_\delta$, which is ill-posed in the sense of Nashed as discussed above and moreover locally ill-posed everywhere (cf. [13, Definition 1.1]). Taking into account the saturation result from [37], our Lavrentiev-regularized solutions cannot converge to the f^*-minimum norm solution with a convergence rate better than $\|f_{\rho,\delta} - f^*\|_{L^2(\Omega)} = \mathcal{O}(\sqrt{\delta})$ as $\delta \to 0$. As Corollary 5.5 will show, we achieve this optimal rate with our method.

Lemma 2.3. *The function $\mathcal{J}_\delta$ defined by* (1.14) *is convex and weakly sequentially lower semi-continuous.*

Proof. The above operator $\tilde{\mathbf{T}} : L^2(\Omega) \to H^1_\diamond(\Omega)$ is compact, so it is continuous. Using the equivalent inner product (2.10), we therefore conclude that

$$\mathcal{J}_\delta(f) = [\mathcal{N}_f j_\delta - \mathcal{D}_f g_\delta, \mathcal{N}_f j_\delta - \mathcal{D}_f g_\delta] = \Big[\tilde{\mathbf{T}}(f) + \mathcal{N}_0 j_\delta - \mathcal{D}_0 g_\delta, \tilde{\mathbf{T}}(f) + \mathcal{N}_0 j_\delta - \mathcal{D}_0 g_\delta\Big]$$

is convex and weakly sequentially lower semi-continuous, which finishes the proof. □

Proposition 2.4. *The problem $(\mathcal{P}_{\rho,\delta})$ attains a unique solution $f_{\rho,\delta}$, which as regularized solution represents an approximation of the f^*-minimum-norm solution $f^\dagger$ to the identification problem $(\mathcal{IP})$.*

Proof. The proof of existence of solutions is based on Lemma 2.3 in combining with arguments of [40, Proposition 4.1], therefore omitted here. Furthermore, since the cost function of $(\mathcal{P}_{\rho,\delta})$ is strictly convex, the minimizer is unique. □

3 Finite element discretization

Let $(\mathcal{T}^h)_{0<h<1}$ be a family of regular and quasi-uniform triangulations of the domain $\overline{\Omega}$ with the mesh size h. For the definition of the discretization space of the state functions let us denote

$$\mathcal{V}_1^h := \left\{ v^h \in C(\overline{\Omega}) \mid v^h{}_{|T} \in \mathcal{P}_1(T),\ \forall T \in \mathcal{T}^h \right\} \quad \text{and} \quad \mathcal{V}_{1,\diamond}^h := \mathcal{V}_1^h \cap H^1_\diamond(\Omega) \text{ and } \mathcal{V}_{1,0}^h := \mathcal{V}_1^h \cap H^1_0(\Omega) \subset \mathcal{V}_{1,\diamond}^h,$$

where $\mathcal{P}_1$ consists all polynomial functions of degree less than or equal to 1.

Proposition 3.1. *(i) Let f be in $L^2(\Omega)$ and j be in $H^{-1/2}(\partial\Omega)$. Then the variational equation*

$$\int_\Omega Q\nabla u^h \cdot \nabla \varphi^h = \left(f, \varphi^h\right) + \left\langle j, \gamma\varphi^h \right\rangle \text{ for all } \varphi^h \in \mathcal{V}_{1,\diamond}^h \tag{3.1}$$

admits a unique solution $u^h \in \mathcal{V}_{1,\diamond}^h$. Furthermore, the estimate

$$\left\|u^h\right\|_{H^1(\Omega)} \le C_{\mathcal{N}} \left(\|f\|_{L^2(\Omega)} + \|j\|_{H^{-1/2}(\partial\Omega)} \right) \tag{3.2}$$

is satisfied. The map $\mathcal{N}^h : L^2(\Omega) \to \mathcal{V}^h_{1,\diamond}$ from each $f \in L^2(\Omega)$ to the unique solution $u^h =: \mathcal{N}^h_f j$ of (3.1) is then called the discrete Neumann operator.

(ii) Let f be in $L^2(\Omega)$ and g be in $H^{1/2}_\diamond(\partial\Omega)$. The equation

$$\int_\Omega Q\nabla v^h \cdot \nabla \psi^h = \left(f, \psi^h\right) \text{ for all } \psi^h \in \mathcal{V}^h_{1,0} \tag{3.3}$$

has a unique solution $v^h \in \mathcal{V}^h_{1,\diamond}$ with $\gamma v^h = g$. Furthermore, the inequality

$$\left\|v^h\right\|_{H^1(\Omega)} \le C_{\mathcal{D}} \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}\right) \tag{3.4}$$

is satisfied. The map $\mathcal{D}^h : L^2(\Omega) \to \mathcal{V}^h_{1,\diamond}$ from each $f \in L^2(\Omega)$ to the unique solution $v^h =: \mathcal{D}^h_f g$ of (3.3) is called the discrete Dirichlet operator.

Similar to (2.3), one sees that the discrete operators $\mathcal{N}^h$, $\mathcal{D}^h$ are Fréchet differentiable on $L^2(\Omega)$. For each $f \in L^2(\Omega)$ the Fréchet derivatives $\mathcal{N}^{h\prime}(f)\xi =: \mathcal{N}^{h\prime}_f j(\xi) \in \mathcal{V}^h_{1,\diamond}$ and $\mathcal{D}^{h\prime}(f)\xi =: \mathcal{D}^{h\prime}_f g(\xi) \in \mathcal{V}^h_{1,0}$ in the direction $\xi \in L^2(\Omega)$ satisfy the equations

$$\int_\Omega Q\nabla \mathcal{N}^{h\prime}_f j(\xi) \cdot \nabla \varphi^h = \left(\xi, \varphi^h\right) \quad \text{and} \quad \int_\Omega Q\nabla \mathcal{D}^{h\prime}_f g(\xi) \cdot \nabla \psi^h = \left(\xi, \psi^h\right) \tag{3.5}$$

for all $\varphi^h \in \mathcal{V}^h_{1,\diamond}$ and $\psi^h \in \mathcal{V}^h_{1,0}$. We now can introduce the strictly convex, discrete cost function

$$\Upsilon^h_{\rho,\delta}(f) := \mathcal{J}^h_\delta(f) + \rho\,\|f - f^*\|^2_{L^2(\Omega)} \text{ with } \mathcal{J}^h_\delta(f) := \int_\Omega Q\nabla\left(\mathcal{N}^h_f j_\delta - \mathcal{D}^h_f g_\delta\right) \cdot \nabla\left(\mathcal{N}^h_f j_\delta - \mathcal{D}^h_f g_\delta\right).$$

Theorem 3.2. *The problem*

$$\min_{f \in L^2(\Omega)} \Upsilon^h_{\rho,\delta}(f) \tag{$\mathcal{P}^h_{\rho,\delta}$}$$

attains a unique minimizer f which satisfies the equation

$$f - f^* = -\frac{1}{\rho}\left(\mathcal{N}^h_f j_\delta - \mathcal{D}^h_f g_\delta\right). \tag{3.6}$$

Remark 3.3. Since $\mathcal{N}^h_f j_\delta$ and $\mathcal{D}^h_f g_\delta$ are both in $\mathcal{V}^1_h$, so is f, provided that $f^* \in \mathcal{V}^1_h$. Thus, taking this into account, a discretization of the set $L^2(\Omega)$ can be avoided.

Proof of Theorem 3.2. The existence and uniqueness of a minimizer to the problem $\left(\mathcal{P}^h_{\rho,\delta}\right)$ are exactly obtained as in the continuous case, therefore omitted here. It remains to show (3.6).

Let $f \in L^2(\Omega)$ be the minimizer to $\left(\mathcal{P}^h_{\rho,\delta}\right)$. The first-order optimality condition yields that $\Upsilon^{h\,\prime}_{\rho,\delta}(f)\xi = \mathcal{J}^{h\prime}_\delta(f)\xi + 2\rho(\xi, f - f^*) = 0$ for all $\xi \in L^2(\Omega)$, where

$$\begin{aligned}
\mathcal{J}^{h\prime}_\delta(f)\xi &= 2\int_\Omega Q\nabla\left(\mathcal{N}^{h\prime}_f j_\delta(\xi) - \mathcal{D}^{h\prime}_f g_\delta(\xi)\right) \cdot \nabla\left(\mathcal{N}^h_f j_\delta - \mathcal{D}^h_f g_\delta\right) \\
&= 2\int_\Omega Q\nabla \mathcal{N}^{h\prime}_f j_\delta(\xi) \cdot \nabla\left(\mathcal{N}^h_f j_\delta - \mathcal{D}^h_f g_\delta\right) + 2\int_\Omega Q\nabla \mathcal{D}^h_f g_\delta \cdot \nabla \mathcal{D}^{h\prime}_f g_\delta(\xi) - 2\int_\Omega Q\nabla \mathcal{N}^h_f j_\delta \cdot \nabla \mathcal{D}^{h\prime}_f g_\delta(\xi).
\end{aligned}$$

By (3.5), it follows that $\int_\Omega Q\nabla \mathcal{N}^{h\prime}_f j_\delta(\xi) \cdot \nabla\left(\mathcal{N}^h_f j_\delta - \mathcal{D}^h_f g_\delta\right) = \left(\xi, \mathcal{N}^h_f j_\delta - \mathcal{D}^h_f g_\delta\right)$ while (3.3) and (3.1) yield

$$\int_\Omega Q\nabla \mathcal{D}^h_f g_\delta \cdot \nabla \mathcal{D}^{h\prime}_f g_\delta(\xi) = \left(f, \mathcal{D}^{h\prime}_f g_\delta(\xi)\right)$$

and

$$\int_\Omega Q\nabla \mathcal{N}^h_f j_\delta \cdot \nabla \mathcal{D}^{h\prime}_f g_\delta(\xi) = \left(f, \mathcal{D}^{h\prime}_f g_\delta(\xi)\right) + \left\langle j_\delta, \gamma \mathcal{D}^{h\prime}_f g_\delta(\xi)\right\rangle = \left(f, \mathcal{D}^{h\prime}_f g_\delta(\xi)\right).$$

We thus infer that $\mathcal{J}^{h\prime}_\delta(f)(\xi) = 2\left(\xi, \mathcal{N}^h_f j_\delta - \mathcal{D}^h_f g_\delta\right)$ and so obtain $\left(\xi, \frac{1}{\rho}\left(\mathcal{N}^h_f j_\delta - \mathcal{D}^h_f g_\delta\right) + f - f^*\right) = 0$ for all $\xi \in L^2(\Omega)$, which finishes the proof. ∎

4 Convergence

From now on C is a generic positive constant which is independent of the mesh size h of $\mathcal{T}^h$, the noise level δ and the regularization parameter ρ. Before presenting the convergence of finite element approximations we here state some auxiliary results.

Lemma 4.1. *A projection operator $\Pi^h_\diamond : L^1(\Omega) \to \mathcal{V}^h_{1,\diamond}$ exists such that*

$$\Pi^h_\diamond \varphi^h = \varphi^h \text{ for all } \varphi^h \in \mathcal{V}^h_{1,\diamond} \text{ and } \Pi^h_\diamond\big(H^1_0(\Omega)\big) \subset \mathcal{V}^h_{1,0} \subset \mathcal{V}^h_{1,\diamond}.$$

Furthermore, it satisfies the properties

$$\lim_{h\to 0} \left\|\vartheta - \Pi^h_\diamond \vartheta\right\|_{H^1(\Omega)} = 0 \quad \text{for all } \vartheta \in H^1_\diamond(\Omega) \tag{4.1}$$

and

$$\left\|\vartheta - \Pi^h_\diamond \vartheta\right\|_{H^1(\Omega)} \le Ch\|\vartheta\|_{H^2(\Omega)} \text{ for all } \vartheta \in H^1_\diamond(\Omega) \cap H^2(\Omega). \tag{4.2}$$

Proof. Let $\Pi^h : L^1(\Omega) \to \mathcal{V}^h_1$ be the Clement's mollification interpolation operator, see [18] and some generalizations [11, 12, 41]. We then define the operator

$$\Pi^h_\diamond \vartheta := \Pi^h \vartheta - \frac{1}{|\partial\Omega|}\int_{\partial\Omega} \gamma \Pi^h \vartheta \in \mathcal{V}^h_{1,\diamond}, \quad \forall \vartheta \in L^1(\Omega)$$

which has the properties (4.1) and (4.2). The proof is completed. □

On the basis of (4.1) and (4.2) we introduce for each $\Phi \in H^1_\diamond(\Omega)$

$$\varrho^h_\Phi := \left\|\Phi - \Pi^h_\diamond \Phi\right\|_{H^1(\Omega)}. \tag{4.3}$$

We note that $\lim_{h\to 0} \varrho^h_\Phi = 0$ and

$$0 \le \varrho^h_\Phi \le Ch \tag{4.4}$$

in case $\Phi \in H^2(\Omega)$. Furthermore, let $(f, j, g) \in L^2(\Omega) \times H^{-1/2}(\partial\Omega) \times H^{1/2}_\diamond(\partial\Omega)$ be fixed, we denote by

$$\alpha^h_{f,j} = \left\|\mathcal{N}^h_f j - \mathcal{N}_f j\right\|_{H^1(\Omega)} \text{ and } \beta^h_{f,g} = \left\|\mathcal{D}^h_f g - \mathcal{D}_f g\right\|_{H^1(\Omega)}. \tag{4.5}$$

Then $\lim_{h\to 0} \alpha^h_{f,j} = \lim_{h\to 0} \beta^h_{f,g} = 0$. In particular, if $\mathcal{N}_f j \in H^2(\Omega)$ and $\mathcal{D}_f g \in H^2(\Omega)$, the error estimates

$$\alpha^h_{f,j} \le Ch \text{ and } \beta^h_{f,g} \le Ch \tag{4.6}$$

are satisfied (cf. [14, 17]).

Lemma 4.2. *Let (f_1, j_1, g_1) and (f_2, j_2, g_2) be arbitrary in $L^2(\Omega) \times H^{-1/2}(\partial\Omega) \times H^{1/2}_\diamond(\partial\Omega)$. Then the estimates*

$$\left\|\mathcal{N}^h_{f_1} j_1 - \mathcal{N}^h_{f_2} j_2\right\|_{H^1(\Omega)} \le C_{\mathcal{N}}\left(\|f_1 - f_2\|_{L^2(\Omega)} + \|j_1 - j_2\|_{H^{-1/2}(\partial\Omega)}\right) \tag{4.7}$$

and

$$\left\|\mathcal{D}^h_{f_1} g_1 - \mathcal{D}^h_{f_2} g_2\right\|_{H^1(\Omega)} \le C_{\mathcal{D}}\left(\|f_1 - f_2\|_{L^2(\Omega)} + \|g_1 - g_2\|_{H^{1/2}(\partial\Omega)}\right) \tag{4.8}$$

hold for all $h \ge 0$.

Proof. According to the definition of the discrete Neumann operator, we have for all $\varphi^h \in \mathcal{V}^h_{1,\diamond}$ that

$$\int_\Omega Q\nabla \mathcal{N}^h_{f_i} j_i \cdot \nabla \varphi^h = \langle j_i, \gamma\varphi^h\rangle + \left(f_i, \varphi^h\right) \text{ with } i = 1, 2.$$

Thus, $\Phi^h_{\mathcal{N}} := \mathcal{N}^h_{f_1} j_1 - \mathcal{N}^h_{f_2} j_2$ is the unique solution to the variational problem

$$\int_\Omega Q\nabla \Phi^h_{\mathcal{N}} \cdot \nabla \varphi^h = \langle j_1 - j_2, \gamma\varphi^h\rangle + \left(f_1 - f_2, \varphi^h\right)$$

for all $\varphi^h \in \mathcal{V}^h_{1,\diamond}$ and so that (4.7) is satisfied. Likewise, we also obtain (4.8). The proof is completed. □

Lemma 4.3. *Let $(\mathcal{T}^{h_n})$ be a sequence of triangulations with $\lim_{n\to\infty} h_n = 0$. Assume that $(j_{\delta_n}, g_{\delta_n})$ is a sequence in $H^{-1/2}(\partial\Omega) \times H^{1/2}_\diamond(\partial\Omega)$ convergent to (j_δ, g_δ) in the $H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$-norm and (f_n) is a sequence in $L^2(\Omega)$ weakly convergent in $L^2(\Omega)$ to f, then there holds the inequality*

$$\liminf_{n\to\infty} \mathcal{J}^{h_n}_{\delta_n}(f_n) \ge \mathcal{J}_\delta(f). \tag{4.9}$$

Proof. In view of (3.2), the sequence $(\mathcal{N}^{h_n}_{f_n} j_{\delta_n})_n \subset H^1_\diamond(\Omega)$ is bounded in the $H^1(\Omega)$-norm, a subsequence not relabelled and an element $\Phi_\mathcal{N} \in H^1_\diamond(\Omega)$ exist such that $(\mathcal{N}^{h_n}_{f_n} j_{\delta_n})$ converges weakly in $H^1(\Omega)$ to $\Phi_\mathcal{N}$. Using the operator $\Pi^{h_n}_\diamond$ in Lemma 4.1, for all $\varphi \in H^1_\diamond(\Omega)$ we have that

$$\langle j_{\delta_n}, \gamma\varphi\rangle + (f_n, \varphi) = \left\langle j_{\delta_n}, \gamma\Pi^{h_n}_\diamond\varphi\right\rangle + \left(f_n, \Pi^{h_n}_\diamond\varphi\right) + \left\langle j_{\delta_n}, \gamma\left(\varphi - \Pi^{h_n}_\diamond\varphi\right)\right\rangle + \left(f_n, \varphi - \Pi^{h_n}_\diamond\varphi\right),$$

where $\Pi^{h_n}_\diamond\varphi \in \mathcal{V}^{h_n}_{1,\diamond}$. We note that

$$\begin{aligned}\left|\left\langle j_{\delta_n}, \gamma\left(\varphi - \Pi^{h_n}_\diamond\varphi\right)\right\rangle\right| &\le \|j_{\delta_n}\|_{H^{-1/2}(\partial\Omega)} \left\|\gamma\left(\varphi - \Pi^{h_n}_\diamond\varphi\right)\right\|_{H^{1/2}(\partial\Omega)} \\ &\le \|j_{\delta_n}\|_{H^{-1/2}(\partial\Omega)} \|\gamma\|_{\mathcal{L}\left(H^1(\Omega), H^{1/2}(\partial\Omega)\right)} \left\|\varphi - \Pi^{h_n}_\diamond\varphi\right\|_{H^1(\Omega)} \le C\left\|\varphi - \Pi^{h_n}_\diamond\varphi\right\|_{H^1(\Omega)} \to 0\end{aligned}$$

as $n \to \infty$. Likewise, $\lim_{n\to\infty}\left|\left(f_n, \varphi - \Pi^{h_n}_\diamond\varphi\right)\right| = 0$. We thus get for all $\varphi \in H^1_\diamond(\Omega)$ that

$$\begin{aligned}&\langle j_\delta, \gamma\varphi\rangle + (f, \varphi) = \lim_{n\to\infty}\left(\langle j_{\delta_n}, \gamma\varphi\rangle + (f_n, \varphi)\right) = \lim_{n\to\infty}\left(\left\langle j_{\delta_n}, \gamma\Pi^{h_n}_\diamond\varphi\right\rangle + \left(f_n, \Pi^{h_n}_\diamond\varphi\right)\right) \\ &= \lim_{n\to\infty}\int_\Omega Q\nabla\mathcal{N}^{h_n}_{f_n} j_{\delta_n}\cdot\nabla\Pi^{h_n}_\diamond\varphi = \lim_{n\to\infty}\int_\Omega Q\nabla\mathcal{N}^{h_n}_{f_n} j_{\delta_n}\cdot\nabla\varphi + \lim_{n\to\infty}\int_\Omega Q\nabla\mathcal{N}^{h_n}_{f_n} j_{\delta_n}\cdot\nabla\left(\Pi^{h_n}_\diamond\varphi - \varphi\right) \\ &= \lim_{n\to\infty}\int_\Omega Q\nabla\mathcal{N}^{h_n}_{f_n} j_{\delta_n}\cdot\nabla\varphi = \int_\Omega Q\nabla\Phi_\mathcal{N}\cdot\nabla\varphi\end{aligned}$$

and so that $\Phi_\mathcal{N} = \mathcal{N}_f j_\delta$. Similarly, we can show that the sequence $(\mathcal{D}^{h_n}_{f_n} j_{g_n})_n$ has a subsequence not relabelled which converges weakly in $H^1(\Omega)$ to $\mathcal{D}_f g_\delta$. The inequality (4.9) now follows from the weakly sequentially lower semi-continuous property of the norm defined by (2.10). The proof is completed. □

Lemma 4.4. *The problem*

$$\min_{f\in\mathcal{I}(j^\dagger, g^\dagger)} \|f - f^*\|^2_{L^2(\Omega)} \tag{$\mathcal{IP}-MN$}$$

attains a unique solution, which is called the f^*-minimum-norm solution of the identification problem.

Proof. Since $\left(j^\dagger, g^\dagger\right)$ is the exact data, the set $\mathcal{I}\left(j^\dagger, g^\dagger\right)$ is non-empty. Furthermore, in view of the notations $\mathbf{T}$ and $\mathbf{L}$ of Lemma 2.1 we can rewrite

$$\mathcal{I}\left(j^\dagger, g^\dagger\right) = \left\{f \in L^2(\Omega) \;\middle|\; \mathbf{T}(f) = \mathcal{D}_0 g^\dagger - \mathcal{N}_0 j^\dagger\right\}.$$

Since the operator $\mathbf{T}$ is linear, we deduce that $\mathcal{I}\left(j^\dagger, g^\dagger\right)$ is closed and convex, the problem $(\mathcal{IP}-MN)$ then has a unique solution, which finishes the proof. □

We now show the convergence of finite element approximations to the identification problem.

Theorem 4.5. *Let $f^\dagger$ be the unique f^*-minimum-norm solution to the identification problem $(\mathcal{IP})$, which solves the minimization problem $(\mathcal{IP}-MN)$. Assume that $\lim_{n\to\infty} h_n = 0$ and (δ_n) and (ρ_n) any positive sequences such that*

$$\rho_n \to 0,\; \frac{\delta_n}{\sqrt{\rho_n}} \to 0,\; \frac{\alpha^{h_n}_{f^\dagger, j^\dagger}}{\sqrt{\rho_n}} \to 0 \text{ and } \frac{\beta^{h_n}_{f^\dagger, g^\dagger}}{\sqrt{\rho_n}} \to 0 \text{ as } n \to \infty, \tag{4.10}$$

where $\alpha^{h_n}_{f^\dagger, j^\dagger}$ and $\beta^{h_n}_{f^\dagger, g^\dagger}$ are defined by (4.5). Furthermore, assume that $(j_{\delta_n}, g_{\delta_n})$ is a sequence in $H^{-1/2}(\partial\Omega) \times H^{1/2}_\diamond(\partial\Omega)$ satisfying

$$\left\|j_{\delta_n} - j^\dagger\right\|_{H^{-1/2}(\partial\Omega)} + \left\|g_{\delta_n} - g^\dagger\right\|_{H^{1/2}(\partial\Omega)} \le \delta_n$$

and $f_n := f^{h_n}_{\rho_n,\delta_n}$ *is the unique minimizer of* $\left(\mathcal{P}^{h_n}_{\rho_n,\delta_n}\right)$ *for each* $n \in N$*. Then:*

(i) The sequence (f_n) *converges in the* $L^2(\Omega)$*-norm to* $f^\dagger$*.*

(ii) The corresponding state sequences $\left(\mathcal{N}^{h_n}_{f_n} j_{\delta_n}\right)$ *and* $\left(\mathcal{D}^{h_n}_{f_n} g_{\delta_n}\right)$ *converge in the* $H^1(\Omega)$*-norm to the unique weak solution* $\Phi^\dagger = \Phi^\dagger\left(f^\dagger, j^\dagger, g^\dagger\right)$ *of the boundary value problem* (1.1)–(1.3)*.*

Before going to prove the theorem, we make the following short remark.

Remark 4.6. In case the weak solution $\Phi^\dagger = \Phi^\dagger\left(f^\dagger, j^\dagger, g^\dagger\right)$ of (1.1)–(1.3) belonging to $H^2(\Omega)$, the estimate (4.6) shows that $0 \le \alpha^{h_n}_{f^\dagger,j^\dagger},\ \beta^{h_n}_{f^\dagger,g^\dagger} \le Ch_n$. Therefore, in view of (4.10), the above convergences (i) and (ii) are obtained if the sequence (ρ_n) is chosen such that

$$\rho_n \to 0,\ \frac{\delta_n}{\sqrt{\rho_n}} \to 0 \text{ and } \frac{h_n}{\sqrt{\rho_n}} \to 0 \text{ as } n \to \infty.$$

By regularity theory for elliptic boundary value problems, the regularity assumption $\Phi^\dagger \in H^2(\Omega)$ is satisfied if the diffusion matrix $Q \in C^{0,1}(\Omega)^{d\times d}$, $j^\dagger \in H^{1/2}(\partial\Omega)$, $g^\dagger \in H^{3/2}(\partial\Omega)$ and either $\partial\Omega$ is smooth of the class $C^{0,1}$ or the domain Ω is convex (see, for example, [21, 44]).

Proof of Theorem 4.5. We have from the optimality of f_n that

$$\mathcal{J}^{h_n}_{\delta_n}(f_n) + \rho_n\|f_n - f^*\|^2_{L^2(\Omega)} \le \mathcal{J}^{h_n}_{\delta_n}(f^\dagger) + \rho_n\|f^\dagger - f^*\|^2_{L^2(\Omega)}. \tag{4.11}$$

Since at $f^\dagger$ there holds the equation $\mathcal{N}_{f^\dagger} j^\dagger = \mathcal{D}_{f^\dagger} g^\dagger$, we infer from Lemma 4.2 that

$$\begin{aligned}
&\mathcal{J}^{h_n}_{\delta_n}(f^\dagger) \le C\left\|\mathcal{N}^{h_n}_{f^\dagger} j_{\delta_n} - \mathcal{D}^{h_n}_{f^\dagger} g_{\delta_n}\right\|^2_{H^1(\Omega)} \\
&= C\left\|\mathcal{N}^{h_n}_{f^\dagger} j_{\delta_n} - \mathcal{N}^{h_n}_{f^\dagger} j^\dagger + \mathcal{D}^{h_n}_{f^\dagger} g^\dagger - \mathcal{D}^{h_n}_{f^\dagger} g_{\delta_n} + \mathcal{N}^{h_n}_{f^\dagger} j^\dagger - \mathcal{N}_{f^\dagger} j^\dagger + \mathcal{D}_{f^\dagger} g^\dagger - \mathcal{D}^{h_n}_{f^\dagger} g^\dagger\right\|^2_{H^1(\Omega)} \\
&\le C\Bigg(\left\|\mathcal{N}^{h_n}_{f^\dagger} j_{\delta_n} - \mathcal{N}^{h_n}_{f^\dagger} j^\dagger\right\|_{H^1(\Omega)} + \left\|\mathcal{D}^{h_n}_{f^\dagger} g_{\delta_n} - \mathcal{D}^{h_n}_{f^\dagger} g^\dagger\right\|_{H^1(\Omega)} \\
&\quad + \left\|\mathcal{N}^{h_n}_{f^\dagger} j^\dagger - \mathcal{N}_{f^\dagger} j^\dagger\right\|_{H^1(\Omega)} + \left\|\mathcal{D}_{f^\dagger} g^\dagger - \mathcal{D}^{h_n}_{f^\dagger} g^\dagger\right\|_{H^1(\Omega)}\Bigg)^2 \qquad (4.12) \\
&\le C\left(\|j_{\delta_n} - j^\dagger\|_{H^{-1/2}(\partial\Omega)} + \|g_{\delta_n} - g^\dagger\|_{H^{1/2}(\partial\Omega)} + \alpha^{h_n}_{f^\dagger,j^\dagger} + \beta^{h_n}_{f^\dagger,g^\dagger}\right)^2 \le C\left(\delta_n^2 + \left(\alpha^{h_n}_{f^\dagger,j^\dagger}\right)^2 + \left(\beta^{h_n}_{f^\dagger,g^\dagger}\right)^2\right)
\end{aligned}$$

which implies from (4.11) that

$$\lim_{n\to\infty} \mathcal{J}^{h_n}_{\delta_n}(f_n) = 0 \tag{4.13}$$

and, by the assumption (4.10),

$$\limsup_{n\to\infty} \|f_n - f^*\|^2_{L^2(\Omega)} \le \left\|f^\dagger - f^*\right\|^2_{L^2(\Omega)}. \tag{4.14}$$

So that the sequence (f_n) is bounded in the $L^2(\Omega)$-norm. A subsequence not relabelled and an element $\widehat{f} \in L^2(\Omega)$ exist such that (f_n) converges weakly in $L^2(\Omega)$ to $\widehat{f}$ and

$$\left\|\widehat{f} - f^*\right\|^2_{L^2(\Omega)} \le \liminf_{n\to\infty} \|f_n - f^*\|^2_{L^2(\Omega)}. \tag{4.15}$$

For any $f \in L^2(\Omega)$ we denote by $\mathcal{J}_0(f) := \int_\Omega Q\nabla\left(\mathcal{N}_f j^\dagger - \mathcal{D}_f g^\dagger\right) \cdot \nabla\left(\mathcal{N}_f j^\dagger - \mathcal{D}_f g^\dagger\right)$. By (1.7), we have

$$\left\|\mathcal{N}_{\widehat{f}} j^\dagger - \mathcal{D}_{\widehat{f}} g^\dagger\right\|^2_{H^1(\Omega)} \le \frac{1 + C_\Omega}{C_\Omega \underline{q}} \mathcal{J}_0\big(\widehat{f}\big). \tag{4.16}$$

Furthermore, applying Lemma 4.3, we have that $\mathcal{J}_0\big(\widehat{f}\big) \le \liminf_{n\to\infty} \mathcal{J}_{\delta_n}^{h_n}(f_n) = 0$, here we used (4.13). Combining this with (4.16), we get $\mathcal{N}_{\widehat{f}} j^\dagger = \mathcal{D}_{\widehat{f}} g^\dagger$ which infers $\widehat{f} \in \mathcal{I}\left(j^\dagger, g^\dagger\right)$. Now we show $\widehat{f} = f^\dagger$ and the sequence (f_n) converges to $\widehat{f}$ in the $L^2(\Omega)$-norm. By the definition of the f^*-minimum-norm solution and (4.14)–(4.15), we get that

$$\left\|f^\dagger - f^*\right\|^2_{L^2(\Omega)} \le \left\|\widehat{f} - f^*\right\|^2_{L^2(\Omega)} \le \liminf_{n\to\infty} \|f_n - f^*\|^2_{L^2(\Omega)} \le \limsup_{n\to\infty} \|f_n - f^*\|^2_{L^2(\Omega)} \le \left\|f^\dagger - f^*\right\|^2_{L^2(\Omega)}$$

and so that $\left\|f^\dagger - f^*\right\|^2_{L^2(\Omega)} = \left\|\widehat{f} - f^*\right\|^2_{L^2(\Omega)} = \lim_{n\to\infty} \|f_n - f^*\|^2_{L^2(\Omega)}$. By the uniqueness of the minimum-norm solution and the sequence (f_n) weakly converging in $L^2(\Omega)$ to $\widehat{f}$, we conclude that $\widehat{f} = f^\dagger$ and the sequence (f_n) in fact converges in the $L^2(\Omega)$-norm to $\widehat{f}$.

Finally, we show the sequences $(\mathcal{N}^{h_n}_{f_n} j_{\delta_n})$ and $(\mathcal{D}^{h_n}_{f_n} g_{\delta_n})$ converge to $\Phi^\dagger = \mathcal{N}_{f^\dagger} j^\dagger = \mathcal{D}_{f^\dagger} g^\dagger$ in the $H^1(\Omega)$-norm. Indeed, by Lemma 4.2, we obtain that

$$\begin{aligned}
\left\|\mathcal{N}^{h_n}_{f_n} j_{\delta_n} - \mathcal{N}_{f^\dagger} j^\dagger\right\|_{H^1(\Omega)} &\le \left\|\mathcal{N}^{h_n}_{f_n} j_{\delta_n} - \mathcal{N}^{h_n}_{f^\dagger} j^\dagger\right\|_{H^1(\Omega)} + \left\|\mathcal{N}^{h_n}_{f^\dagger} j^\dagger - \mathcal{N}_{f^\dagger} j^\dagger\right\|_{H^1(\Omega)} \\
&\le C\left(\left\|j_{\delta_n} - j^\dagger\right\|_{H^{-1/2}(\partial\Omega)} + \left\|f_n - f^\dagger\right\|_{L^2(\Omega)} + \alpha^{h_n}_{f^\dagger, j^\dagger}\right) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Similarly, we also get $\left\|\mathcal{D}^{h_n}_{f_n} g_{\delta_n} - \mathcal{D}_{f^\dagger} g^\dagger\right\|_{H^1(\Omega)} \le C\left(\left\|g_{\delta_n} - g^\dagger\right\|_{H^{1/2}(\partial\Omega)} + \left\|f_n - f^\dagger\right\|_{L^2(\Omega)} + \beta^{h_n}_{f^\dagger, g^\dagger}\right) \to 0$ as n tends to ∞, which finishes the proof. □

5 Convergence rates

In this section we investigate convergence rates for the regularized solutions with respect to the f^*-minimum-norm solution of the source identification problem. We recall that solutions of Tikhonov's and Lavrentiev's method coincide here. The condition (1.16) (repeated as equation (5.1) below) will play a prominent role in this context.

Remark 5.1. Assume that $f \in \mathcal{I}\left(j^\dagger, g^\dagger\right)$ and a function $w \in L^2(\Omega)$ exists such that $f - f^* = \mathcal{N}_w j^\dagger - \mathcal{D}_w g^\dagger$. Then f is the *unique* f^*-minimum-norm solution of the problem $(\mathcal{IP} - MN)$.

Indeed, we have with $\xi \in \left\{\xi \in L^2(\Omega) \mid \mathcal{N}_\xi j^\dagger = \mathcal{D}_\xi g^\dagger\right\}$ that

$$\begin{aligned}
&\frac{1}{2}\|\xi - f^*\|^2_{L^2(\Omega)} - \frac{1}{2}\|f - f^*\|^2_{L^2(\Omega)} \\
&= \frac{1}{2}\|\xi - f\|^2_{L^2(\Omega)} + (f - f^*, \xi - f) \ge (f - f^*, \xi - f) = \left(\mathcal{N}_w j^\dagger - \mathcal{D}_w g^\dagger, \xi - f\right) \\
&= \int_\Omega Q\nabla\mathcal{N}_\xi j^\dagger \cdot \nabla\left(\mathcal{N}_w j^\dagger - \mathcal{D}_w g^\dagger\right) - \left\langle j^\dagger, \gamma\left(\mathcal{N}_w j^\dagger - \mathcal{D}_w g^\dagger\right)\right\rangle \\
&\quad - \int_\Omega Q\nabla\mathcal{N}_f j^\dagger \cdot \nabla\left(\mathcal{N}_w j^\dagger - \mathcal{D}_w g^\dagger\right) + \left\langle j^\dagger, \gamma\left(\mathcal{N}_w j^\dagger - \mathcal{D}_w g^\dagger\right)\right\rangle \\
&= \int_\Omega Q\nabla\left(\mathcal{N}_\xi j^\dagger - \mathcal{N}_f j^\dagger\right) \cdot \nabla\left(\mathcal{N}_w j^\dagger - \mathcal{D}_w g^\dagger\right).
\end{aligned}$$

Since $\gamma\mathcal{N}_\xi j^\dagger = \gamma\mathcal{N}_f j^\dagger = g^\dagger$, it follows that $\mathcal{N}_\xi j^\dagger - \mathcal{N}_f j^\dagger \in H^1_0(\Omega)$. We thus obtain from the last inequality

$$\begin{aligned}
\frac{1}{2}\|\xi - f^*\|^2_{L^2(\Omega)} - \frac{1}{2}\|f - f^*\|^2_{L^2(\Omega)} &\ge \int_\Omega Q\nabla\mathcal{N}_w j^\dagger \cdot \nabla\left(\mathcal{N}_\xi j^\dagger - \mathcal{N}_f j^\dagger\right) - \int_\Omega Q\nabla\mathcal{D}_w g^\dagger \cdot \nabla\left(\mathcal{N}_\xi j^\dagger - \mathcal{N}_f j^\dagger\right) \\
&= \left(w, \mathcal{N}_\xi j^\dagger - \mathcal{N}_f j^\dagger\right) + \left\langle j^\dagger, \gamma\left(\mathcal{N}_\xi j^\dagger - \mathcal{N}_f j^\dagger\right)\right\rangle - \left(w, \mathcal{N}_\xi j^\dagger - \mathcal{N}_f j^\dagger\right) = 0,
\end{aligned}$$

which finishes the proof.

We are now in a position to state the main theorem on convergence rates for the general case of finite element discretized regularized solutions with noise level ($\delta > 0$ and $h > 0$).

Theorem 5.2. *Assume that for $f \in \mathcal{I}\left(j^\dagger, g^\dagger\right)$ there exists a function $w \in L^2(\Omega)$ such that the source condition*

$$f - f^* = \mathcal{N}_w j^\dagger - \mathcal{D}_w g^\dagger \tag{5.1}$$

holds true. Then, according to Remark 5.1, f is the uniquely determined f^*-minimum-norm solution $f^\dagger$ and minimizer of the problem $(\mathcal{IP}-MN)$. Moreover, we have the error estimate and convergence rate

$$\begin{aligned}\left\|\mathcal{N}^h_{f^h} j_\delta - \mathcal{D}^h_{f^h} g_\delta\right\|^2_{H^1(\Omega)} &+ \rho\left\|f^h - f^\dagger\right\|^2_{L^2(\Omega)} \\ &= \mathcal{O}\left(\delta^2 + \left(\alpha^h_{f^\dagger,j^\dagger}\right)^2 + \left(\beta^h_{f^\dagger,g^\dagger}\right)^2 + \rho\varrho^h_{\mathcal{N}_w j^\dagger} + \rho\varrho^h_{\mathcal{N}_{f^\dagger} j^\dagger} + \rho\varrho^h_{\mathcal{D}_0\gamma\mathcal{N}_w j^\dagger - g^\dagger} + \delta\rho + \rho^2\right),\end{aligned} \tag{5.2}$$

where $f^h := f^h_{\rho,\delta}$ is the unique minimizer of $\left(\mathcal{P}^h_{\rho,\delta}\right)$ and $\mathcal{D}_0\gamma\mathcal{N}_w j^\dagger - g^\dagger$ is the unique weak solution to the Dirichlet problem

$$-\nabla\cdot(Q\nabla v) = 0 \text{ in } \Omega \text{ and } v = \gamma\mathcal{N}_w j^\dagger - g^\dagger \text{ on } \partial\Omega$$

and $\alpha^h_{f^\dagger,j^\dagger}$, $\beta^h_{f^\dagger,g^\dagger}$, $\varrho^h_{\mathcal{N}_w j^\dagger}$, $\varrho^h_{\mathcal{N}_{f^\dagger} j^\dagger}$ and $\varrho^h_{\mathcal{D}_0\gamma\mathcal{N}_w j^\dagger - g^\dagger}$ come from (4.3) and (4.5).

Remark 5.3. In case (cf. Remark 4.6) $\mathcal{N}_{f^\dagger} j^\dagger, \mathcal{N}_w j^\dagger, \mathcal{D}_0\gamma\mathcal{N}_w j^\dagger - g^\dagger \in H^2(\Omega)$, by (4.4) and (4.6), we have

$$0 \le \alpha^h_{f^\dagger,j^\dagger}, \beta^h_{f^\dagger,g^\dagger}, \varrho^h_{\mathcal{N}_w j^\dagger}, \varrho^h_{\mathcal{N}_{f^\dagger} j^\dagger}, \varrho^h_{\mathcal{D}_0\gamma\mathcal{N}_w j^\dagger - g^\dagger} \le Ch$$

and so that the convergence rate

$$\left\|\mathcal{N}^h_{f^h} j_\delta - \mathcal{D}^h_{f^h} g_\delta\right\|^2_{H^1(\Omega)} + \rho\left\|f^h - f^\dagger\right\|^2_{L^2(\Omega)} = \mathcal{O}\left(\delta^2 + h^2 + h\rho + \delta\rho + \rho^2\right)$$

is obtained.

Remark 5.4. Let $\Phi^\dagger = \Phi^\dagger\left(f^\dagger, j^\dagger, g^\dagger\right)$ be the weak solution of (1.1)–(1.3). Then the convergence rate

$$\begin{aligned}&\left\|\mathcal{N}^h_{f^h} j_\delta - \Phi^\dagger\right\|^2_{H^1(\Omega)} + \left\|\mathcal{D}^h_{f^h} g_\delta - \Phi^\dagger\right\|^2_{H^1(\Omega)} \\ &= \mathcal{O}\left(\delta^2\rho^{-1} + \left(\alpha^h_{f^\dagger,j^\dagger}\right)^2\rho^{-1} + \left(\beta^h_{f^\dagger,g^\dagger}\right)^2\rho^{-1} + \varrho^h_{\mathcal{N}_w j^\dagger} + \varrho^h_{\mathcal{N}_{f^\dagger} j^\dagger} + \varrho^h_{\mathcal{D}_0\gamma\mathcal{N}_w j^\dagger - g^\dagger} + \delta + \rho + \alpha^h_{f^\dagger,j^\dagger} + \beta^h_{f^\dagger,g^\dagger}\right)\end{aligned}$$

is also established. Indeed, the desired equation directly follows from (5.2) and the following inequalities

$$\begin{aligned}\left\|\mathcal{N}^h_{f^h} j_\delta - \mathcal{N}_{f^\dagger} j^\dagger\right\|_{H^1(\Omega)} &\le C\left(\left\|j_\delta - j^\dagger\right\|_{H^{-1/2}(\partial\Omega)} + \left\|f^h - f^\dagger\right\|_{L^2(\Omega)} + \alpha^h_{f^\dagger,j^\dagger}\right) \\ &\le C\left(\delta + \left\|f^h - f^\dagger\right\|_{L^2(\Omega)} + \alpha^h_{f^\dagger,j^\dagger}\right)\end{aligned}$$

and $\left\|\mathcal{D}^h_{f^h} g_\delta - \mathcal{D}_{f^\dagger} g^\dagger\right\|_{H^1(\Omega)} \le C\left(\delta + \left\|f^h - f^\dagger\right\|_{L^2(\Omega)} + \beta^h_{f^\dagger,g^\dagger}\right)$, here we used Lemma 4.2.

Proof of Theorem 5.2. In view of (4.12) we first have that $\mathcal{J}^h_\delta\left(f^\dagger\right) \le C\left(\delta^2 + \left(\alpha^h_{f^\dagger,j^\dagger}\right)^2 + \left(\beta^h_{f^\dagger,g^\dagger}\right)^2\right)$. The optimality of f^h yields $\mathcal{J}^h_\delta\left(f^h\right) + \rho\left\|f^h - f^*\right\|^2_{L^2(\Omega)} \le \mathcal{J}^h_\delta\left(f^\dagger\right) + \rho\left\|f^\dagger - f^*\right\|^2_{L^2(\Omega)}$. This gives

$$\begin{aligned}\rho\left\|f^h - f^*\right\|_{L^2(\Omega)} &\le C\rho^{1/2}\left(\delta^2 + \left(\alpha^h_{f^\dagger,j^\dagger}\right)^2 + \left(\beta^h_{f^\dagger,g^\dagger}\right)^2\right)^{1/2} + \rho\left\|f^\dagger - f^*\right\|_{L^2(\Omega)} \\ &\le C\left(\delta^2 + \left(\alpha^h_{f^\dagger,j^\dagger}\right)^2 + \left(\beta^h_{f^\dagger,g^\dagger}\right)^2 + \rho\right),\end{aligned} \tag{5.3}$$

and

$$\mathcal{J}_\delta^h\left(f^h\right)+\rho\|f^h-f^\dagger\|^2_{L^2(\Omega)} \le \mathcal{J}_\delta^h\left(f^\dagger\right)+\rho\left(\|f^\dagger-f^*\|^2_{L^2(\Omega)}-\|f^h-f^*\|^2_{L^2(\Omega)}+\|f^h-f^\dagger\|^2_{L^2(\Omega)}\right)$$
$$\le C\left(\delta^2+\left(\alpha^h_{f^\dagger,j^\dagger}\right)^2+\left(\beta^h_{f^\dagger,g^\dagger}\right)^2\right)+2\rho\left(f^\dagger-f^*,f^\dagger-f^h\right). \tag{5.4}$$

Since $\mathcal{N}_{f^\dagger}j^\dagger=\mathcal{D}_{f^\dagger}g^\dagger$, it follows from (5.1) that

$$\left(f^\dagger-f^*,f^\dagger-f^h\right)=\left(f^\dagger-f^h,\mathcal{N}_wj^\dagger-\mathcal{N}_{f^\dagger}j^\dagger\right)+\left(f^\dagger-f^h,\mathcal{D}_{f^\dagger}g^\dagger-\mathcal{D}_wg^\dagger\right). \tag{5.5}$$

From (1.10), we infer

$$\left(f^\dagger,\mathcal{N}_wj^\dagger-\mathcal{N}_{f^\dagger}j^\dagger\right)=\int_\Omega Q\nabla\mathcal{N}_{f^\dagger}j^\dagger\cdot\nabla\left(\mathcal{N}_wj^\dagger-\mathcal{N}_{f^\dagger}j^\dagger\right)-\left\langle j^\dagger,\gamma\left(\mathcal{N}_wj^\dagger-\mathcal{N}_{f^\dagger}j^\dagger\right)\right\rangle,$$

and

$$\left(f^h,\mathcal{N}_wj^\dagger-\mathcal{N}_{f^\dagger}j^\dagger\right)=\int_\Omega Q\nabla\mathcal{N}_{f^h}j^\dagger\cdot\nabla\left(\mathcal{N}_wj^\dagger-\mathcal{N}_{f^\dagger}j^\dagger\right)-\left\langle j^\dagger,\gamma\left(\mathcal{N}_wj^\dagger-\mathcal{N}_{f^\dagger}j^\dagger\right)\right\rangle.$$

This in turn implies

$$\left(f^\dagger-f^h,\mathcal{N}_wj^\dagger-\mathcal{N}_{f^\dagger}j^\dagger\right)=\int_\Omega Q\nabla\left(\mathcal{N}_{f^\dagger}j^\dagger-\mathcal{N}_{f^h}j^\dagger\right)\cdot\nabla\left(\mathcal{N}_wj^\dagger-\mathcal{N}_{f^\dagger}j^\dagger\right)$$
$$=\int_\Omega Q\nabla\left(\mathcal{N}_{f^\dagger}j^\dagger-\mathcal{D}_{f^h}g^\dagger\right)\cdot\nabla\left(\mathcal{N}_wj^\dagger-\mathcal{N}_{f^\dagger}j^\dagger\right)+\int_\Omega Q\nabla\left(\mathcal{D}_{f^h}g^\dagger-\mathcal{N}_{f^h}j^\dagger\right)\cdot\nabla\left(\mathcal{N}_wj^\dagger-\mathcal{N}_{f^\dagger}j^\dagger\right). \tag{5.6}$$

Since $\gamma\left(\mathcal{D}_{f^\dagger}g^\dagger-\mathcal{D}_wg^\dagger\right)=0$, it follows from (1.11) that

$$\left(f^\dagger-f^h,\mathcal{D}_{f^\dagger}g^\dagger-\mathcal{D}_wg^\dagger\right)=\int_\Omega Q\nabla\mathcal{D}_{f^\dagger}g^\dagger\cdot\nabla\left(\mathcal{D}_{f^\dagger}g^\dagger-\mathcal{D}_wg^\dagger\right)-\int_\Omega Q\nabla\mathcal{D}_{f^h}g^\dagger\cdot\nabla\left(\mathcal{D}_{f^\dagger}g^\dagger-\mathcal{D}_wg^\dagger\right)$$
$$=\int_\Omega Q\nabla\left(\mathcal{D}_{f^\dagger}g^\dagger-\mathcal{D}_{f^h}g^\dagger\right)\cdot\nabla\left(\mathcal{D}_{f^\dagger}g^\dagger-\mathcal{D}_wg^\dagger\right) \tag{5.7}$$

holds. We thus infer from (5.5)–(5.7) the identity

$$\left(f^\dagger-f^*,f^\dagger-f^h\right)=\int_\Omega Q\nabla\left(\mathcal{D}_{f^h}g^\dagger-\mathcal{N}_{f^h}j^\dagger\right)\cdot\nabla\left(\mathcal{N}_wj^\dagger-\mathcal{N}_{f^\dagger}j^\dagger\right)$$
$$+\int_\Omega Q\nabla\left(\mathcal{N}_{f^\dagger}j^\dagger-\mathcal{D}_{f^h}g^\dagger\right)\cdot\nabla\left(\mathcal{N}_wj^\dagger-\mathcal{N}_{f^\dagger}j^\dagger\right)+\int_\Omega Q\nabla\left(\mathcal{D}_{f^\dagger}g^\dagger-\mathcal{D}_{f^h}g^\dagger\right)\cdot\nabla\left(\mathcal{D}_{f^\dagger}g^\dagger-\mathcal{D}_wg^\dagger\right). \tag{5.8}$$

We note again that $\mathcal{N}_{f^\dagger}j^\dagger=\mathcal{D}_{f^\dagger}g^\dagger$ and $\gamma\left(\mathcal{D}_{f^\dagger}g^\dagger-\mathcal{D}_{f^h}g^\dagger\right)=0$. Then, together with (1.10) and (1.11), the last two terms on the right hand side of (5.8) satisfy

$$\int_\Omega Q\nabla\left(\mathcal{N}_{f^\dagger}j^\dagger-\mathcal{D}_{f^h}g^\dagger\right)\cdot\nabla\left(\mathcal{N}_wj^\dagger-\mathcal{N}_{f^\dagger}j^\dagger\right)+\int_\Omega Q\nabla\left(\mathcal{D}_{f^\dagger}g^\dagger-\mathcal{D}_{f^h}g^\dagger\right)\cdot\nabla\left(\mathcal{D}_{f^\dagger}g^\dagger-\mathcal{D}_wg^\dagger\right)$$
$$=\int_\Omega Q\nabla\left(\mathcal{D}_{f^\dagger}g^\dagger-\mathcal{D}_{f^h}g^\dagger\right)\cdot\nabla\left(\mathcal{N}_wj^\dagger-\mathcal{D}_wg^\dagger\right)$$
$$=\left(w,\mathcal{D}_{f^\dagger}g^\dagger-\mathcal{D}_{f^h}g^\dagger\right)+\left\langle j^\dagger,\gamma\left(\mathcal{D}_{f^\dagger}g^\dagger-\mathcal{D}_{f^h}g^\dagger\right)\right\rangle-\left(w,\mathcal{D}_{f^\dagger}g^\dagger-\mathcal{D}_{f^h}g^\dagger\right)=0.$$

Thus, we obtain from (5.8)

$$\left(f^\dagger-f^*,f^\dagger-f^h\right)=\int_\Omega Q\nabla\left(\mathcal{D}_{f^h}g^\dagger-\mathcal{N}_{f^h}j^\dagger\right)\cdot\nabla\left(\mathcal{N}_wj^\dagger-\mathcal{N}_{f^\dagger}j^\dagger\right).$$

Next, we abbreviate $W=\mathcal{N}_wj^\dagger-\mathcal{N}_{f^\dagger}j^\dagger$ and note

$$\gamma W=\gamma\mathcal{N}_wj^\dagger-g^\dagger. \tag{5.9}$$

Then we get

$$
\begin{aligned}
\left(f^\dagger - f^*, f^\dagger - f^h\right) &= \int_\Omega Q\nabla\left(\mathcal{D}_{f^h} g^\dagger - \mathcal{N}_{f^h} j^\dagger\right)\cdot\nabla W \\
&= \int_\Omega Q\nabla\left(\mathcal{D}_{f^h} g^\dagger - \mathcal{D}^h_{f^h} g^\dagger\right)\cdot\nabla W - \int_\Omega Q\nabla\left(\mathcal{N}_{f^h} j^\dagger - \mathcal{N}^h_{f^h} j^\dagger\right)\cdot\nabla W \\
&\quad + \int_\Omega Q\nabla\left(\mathcal{D}^h_{f^h} g^\dagger - \mathcal{D}^h_{f^h} g_\delta\right)\cdot\nabla W - \int_\Omega Q\nabla\left(\mathcal{N}^h_{f^h} j^\dagger - \mathcal{N}^h_{f^h} j_\delta\right)\cdot\nabla W \\
&\quad + \int_\Omega Q\nabla\left(\mathcal{D}^h_{f^h} g_\delta - \mathcal{N}^h_{f^h} j_\delta\right)\cdot\nabla W := I_1 + I_2 + I_3.
\end{aligned} \tag{5.10}
$$

To prepare the estimation of those three addends we start with writing

$$
\begin{aligned}
&\int_\Omega Q\nabla\left(\mathcal{D}_{f^h} g^\dagger - \mathcal{D}^h_{f^h} g^\dagger\right)\cdot\nabla W \\
&= \int_\Omega Q\nabla\left(\mathcal{D}_{f^h} g^\dagger - \mathcal{D}^h_{f^h} g^\dagger\right)\cdot\nabla\mathcal{D}_0\gamma W + \int_\Omega Q\nabla\left(\mathcal{D}_{f^h} g^\dagger - \mathcal{D}^h_{f^h} g^\dagger\right)\cdot\nabla\Pi^h_\diamond\left(W - \mathcal{D}_0\gamma W\right) \\
&\quad + \int_\Omega Q\nabla\left(\mathcal{D}_{f^h} g^\dagger - \mathcal{D}^h_{f^h} g^\dagger\right)\cdot\nabla\left(W - \mathcal{D}_0\gamma W - \Pi^h_\diamond\left(W - \mathcal{D}_0\gamma W\right)\right).
\end{aligned}
$$

Since $\mathcal{D}_{f^h} g^\dagger - \mathcal{D}^h_{f^h} g^\dagger \in H^1_0(\Omega)$, we then get

$$
\int_\Omega Q\nabla\left(\mathcal{D}_{f^h} g^\dagger - \mathcal{D}^h_{f^h} g^\dagger\right)\cdot\nabla\mathcal{D}_0\gamma W = \int_\Omega Q\nabla\mathcal{D}_0\gamma W\cdot\nabla\left(\mathcal{D}_{f^h} g^\dagger - \mathcal{D}^h_{f^h} g^\dagger\right) = 0.
$$

Since

$$
\gamma\left(W - \mathcal{D}_0\gamma W\right) = \gamma W - \gamma\mathcal{D}_0\gamma W = \gamma W - \gamma W = 0,
$$

we infer $\Pi^h_\diamond\left(W - \mathcal{D}_0\gamma W\right) \in \mathcal{V}^h_{1,0} = \mathcal{V}^h_1 \cap H^1_0(\Omega)$ and then obtain from (1.11) and (3.3) that

$$
\int_\Omega Q\nabla\left(\mathcal{D}_{f^h} g^\dagger - \mathcal{D}^h_{f^h} g^\dagger\right)\cdot\nabla\Pi^h_\diamond\left(W - \mathcal{D}_0\gamma W\right) = 0
$$

holds. Hence we have

$$
\begin{aligned}
&\left|\int_\Omega Q\nabla\left(\mathcal{D}_{f^h} g^\dagger - \mathcal{D}^h_{f^h} g^\dagger\right)\cdot\nabla W\right| \\
&= \left|\int_\Omega Q\nabla\left(\mathcal{D}_{f^h} g^\dagger - \mathcal{D}^h_{f^h} g^\dagger\right)\cdot\nabla\left(W - \mathcal{D}_0\gamma W - \Pi^h_\diamond\left(W - \mathcal{D}_0\gamma W\right)\right)\right| \\
&\le C\left\|\mathcal{D}_{f^h} g^\dagger - \mathcal{D}^h_{f^h} g^\dagger\right\|_{H^1(\Omega)}\left\|W - \mathcal{D}_0\gamma W - \Pi^h_\diamond\left(W - \mathcal{D}_0\gamma W\right)\right\|_{H^1(\Omega)} \\
&\le C\left(\left\|f^h\right\|_{L^2(\Omega)} + \left\|g^\dagger\right\|_{H^{1/2}(\partial\Omega)}\right)\left(\left\|W - \Pi^h_\diamond W\right\|_{H^1(\Omega)} + \left\|\mathcal{D}_0\gamma W - \Pi^h_\diamond\mathcal{D}_0\gamma W\right\|_{H^1(\Omega)}\right) \\
&\le C\left(\left\|f^h\right\|_{L^2(\Omega)} + \left\|g^\dagger\right\|_{H^{1/2}(\partial\Omega)}\right) \\
&\quad\cdot\left(\left\|\mathcal{N}_w j^\dagger - \Pi^h_\diamond\mathcal{N}_w j^\dagger\right\|_{H^1(\Omega)} + \left\|\mathcal{N}_{f^\dagger} j^\dagger - \Pi^h_\diamond\mathcal{N}_{f^\dagger} j^\dagger\right\|_{H^1(\Omega)} + \left\|\mathcal{D}_0\gamma W - \Pi^h_\diamond\mathcal{D}_0\gamma W\right\|_{H^1(\Omega)}\right) \\
&= C\left(\left\|f^h\right\|_{L^2(\Omega)} + \left\|g^\dagger\right\|_{H^{1/2}(\partial\Omega)}\right)\left(\varrho^h_{\mathcal{N}_w j^\dagger} + \varrho^h_{\mathcal{N}_{f^\dagger} j^\dagger} + \varrho^h_{\mathcal{D}_0\gamma\mathcal{N}_w j^\dagger - g^\dagger}\right),
\end{aligned} \tag{5.11}
$$

where we use (5.9). Similarly, since $\Pi^h_\diamond W \in \mathcal{V}^h_\diamond$ and by (1.10) and (3.1), we get

$$
\left|\int_\Omega Q\nabla\left(\mathcal{N}_{f^h} j^\dagger - \mathcal{N}^h_{f^h} j^\dagger\right)\cdot\nabla W\right| \le C\left(\left\|f^h\right\|_{L^2(\Omega)} + \left\|j^\dagger\right\|_{H^{-1/2}(\partial\Omega)}\right)\left(\varrho^h_{\mathcal{N}_w j^\dagger} + \varrho^h_{\mathcal{N}_{f^\dagger} j^\dagger}\right). \tag{5.12}
$$

Now we are in the position to estimate $I_1 - I_3$. Combining (5.11) with (5.12), we obtain with the help of

(5.3)

$$\begin{aligned}
\rho|I_1| &= \rho\left|\int_\Omega Q\nabla\left(\mathcal{D}_{f^h}g^\dagger - \mathcal{D}^h_{f^h}g^\dagger\right)\cdot\nabla W - \int_\Omega Q\nabla\left(\mathcal{N}_{f^h}j^\dagger - \mathcal{N}^h_{f^h}j^\dagger\right)\cdot\nabla W\right| \\
&\le C\rho\left(\|f^h\|_{L^2(\Omega)} + \|g^\dagger\|_{H^{1/2}(\partial\Omega)} + \|j^\dagger\|_{H^{-1/2}(\partial\Omega)}\right)\left(\varrho^h_{\mathcal{N}_w j^\dagger} + \varrho^h_{\mathcal{N}_{f^\dagger}j^\dagger} + \varrho^h_{\mathcal{D}_0\gamma\mathcal{N}_w j^\dagger - g^\dagger}\right) \\
&\le C\left(\delta^2 + \left(\alpha^h_{f^\dagger,j^\dagger}\right)^2 + \left(\beta^h_{f^\dagger,g^\dagger}\right)^2 + \rho\right)\left(\varrho^h_{\mathcal{N}_w j^\dagger} + \varrho^h_{\mathcal{N}_{f^\dagger}j^\dagger} + \varrho^h_{\mathcal{D}_0\gamma\mathcal{N}_w j^\dagger - g^\dagger}\right) \\
&\quad + C\rho\left(\varrho^h_{\mathcal{N}_w j^\dagger} + \varrho^h_{\mathcal{N}_{f^\dagger}j^\dagger} + \varrho^h_{\mathcal{D}_0\gamma\mathcal{N}_w j^\dagger - g^\dagger}\right) \\
&\le C\left(\delta^2 + \left(\alpha^h_{f^\dagger,j^\dagger}\right)^2 + \left(\beta^h_{f^\dagger,g^\dagger}\right)^2 + \rho\varrho^h_{\mathcal{N}_w j^\dagger} + \rho\varrho^h_{\mathcal{N}_{f^\dagger}j^\dagger} + \rho\varrho^h_{\mathcal{D}_0\gamma\mathcal{N}_w j^\dagger - g^\dagger}\right). \qquad (5.13)
\end{aligned}$$

Now, using Lemma 4.2, we arrive at

$$\begin{aligned}
\rho|I_2| &= \rho\left|\int_\Omega Q\nabla\left(\mathcal{D}^h_{f^h}g^\dagger - \mathcal{D}^h_{f^h}g_\delta\right)\cdot\nabla W - \int_\Omega Q\nabla\left(\mathcal{N}^h_{f^h}j^\dagger - \mathcal{N}^h_{f^h}j_\delta\right)\cdot\nabla W\right| \\
&\le C\rho\left(\left\|\mathcal{D}^h_{f^h}g^\dagger - \mathcal{D}^h_{f^h}g_\delta\right\|_{H^1(\Omega)} + \left\|\mathcal{N}^h_{f^h}j^\dagger - \mathcal{N}^h_{f^h}j_\delta\right\|_{H^1(\Omega)}\right) \\
&\le C\rho\left(\|g_\delta - g^\dagger\|_{H^{1/2}(\partial\Omega)} + \|j_\delta - j^\dagger\|_{H^{-1/2}(\partial\Omega)}\right) \le C\delta\rho. \qquad (5.14)
\end{aligned}$$

Since for a.e. in Ω the matrix $Q(x)$ is positive definite, the root $Q(x)^{1/2}$ is then well defined. Thus, using the Cauchy-Schwarz inequality and Young's inequality, we estimate I_3 as

$$\begin{aligned}
\rho|I_3| &= \rho\left|\int_\Omega Q\nabla\left(\mathcal{D}^h_{f^h}g_\delta - \mathcal{N}^h_{f^h}j_\delta\right)\cdot\nabla W\right| = \rho\left|\int_\Omega Q^{1/2}\nabla\left(\mathcal{D}^h_{f^h}g_\delta - \mathcal{N}^h_{f^h}j_\delta\right)\cdot Q^{1/2}\nabla W\right| \\
&\le \rho\left(\int_\Omega Q^{1/2}\nabla\left(\mathcal{D}^h_{f^h}g_\delta - \mathcal{N}^h_{f^h}j_\delta\right)\cdot Q^{1/2}\nabla\left(\mathcal{D}^h_{f^h}g_\delta - \mathcal{N}^h_{f^h}j_\delta\right)\right)^{1/2}\left(\int_\Omega Q^{1/2}\nabla W\cdot Q^{1/2}\nabla W\right)^{1/2} \\
&\le C\rho\left(\mathcal{J}^h_\delta(f^h)\right)^{1/2} \le C^2\rho^2 + \frac{1}{4}\mathcal{J}^h_\delta(f^h) \le C\rho^2 + \frac{1}{4}\mathcal{J}^h_\delta(f^h). \qquad (5.15)
\end{aligned}$$

It follows from (5.10) and (5.13)–(5.15) that

$$\begin{aligned}
&2\rho\left(f^\dagger - f^*, f^\dagger - f^h\right) \\
&\le C\left(\delta^2 + \left(\alpha^h_{f^\dagger,j^\dagger}\right)^2 + \left(\beta^h_{f^\dagger,g^\dagger}\right)^2 + \rho\varrho^h_{\mathcal{N}_w j^\dagger} + \rho\varrho^h_{\mathcal{N}_{f^\dagger}j^\dagger} + \rho\varrho^h_{\mathcal{D}_0\gamma\mathcal{N}_w j^\dagger - g^\dagger} + \rho\delta + \rho^2\right) + \frac{1}{2}\mathcal{J}^h_\delta(f^h)
\end{aligned}$$

holds, which together with (5.4) implies

$$\begin{aligned}
&\frac{1}{2}\mathcal{J}^h_\delta(f^h) + \rho\|f^h - f^\dagger\|^2_{L^2(\Omega)} \\
&\le C\left(\delta^2 + \left(\alpha^h_{f^\dagger,j^\dagger}\right)^2 + \left(\beta^h_{f^\dagger,g^\dagger}\right)^2 + \rho\varrho^h_{\mathcal{N}_w j^\dagger} + \rho\varrho^h_{\mathcal{N}_{f^\dagger}j^\dagger} + \rho\varrho^h_{\mathcal{D}_0\gamma\mathcal{N}_w j^\dagger - g^\dagger} + \rho\delta + \rho^2\right). \qquad (5.16)
\end{aligned}$$

Since $\left\|\mathcal{D}^h_{f^h}g_\delta - \mathcal{N}^h_{f^h}j_\delta\right\|^2_{H^1(\Omega)} \le C\mathcal{J}^h_\delta(f^h)$, (5.2) now directly follows from (5.16), which finishes the proof. □

The proof of Theorem 5.2 is much simpler if one considers with focus on the Lavrentiev regularization setting the noisy data, but the non-discretized case ($\delta > 0$, but $h = 0$). Taking into account Remark 2.2 we will show this by formulating and proving the following corollary. This proof gives also essential insights into the character of (5.1) as a range-type source condition, which is the benchmark condition in the theory of Lavrentiev's regularization method.

Corollary 5.5. *Assume that, for the f^*-minimum-norm solution $f^\dagger$, there exists a function $w \in L^2(\Omega)$ such that the source condition $f^\dagger - f^* = \mathcal{N}_w j^\dagger - \mathcal{D}_w g^\dagger$ holds true. Then we have the convergence rate*

$$\|f_{\rho,\delta} - f^\dagger\|_{L^2(\Omega)} = \mathcal{O}(\sqrt{\delta}) \qquad \text{as} \qquad \delta \to 0$$

whenever the regularization parameter ρ is chosen a priori as $\underline{c}\sqrt{\delta} \le \rho(\delta) \le \overline{c}\sqrt{\delta}$ with some constants $0 < \underline{c} \le \overline{c} < \infty$.

Proof. The Lavrentiev-regularized solutions $f = f_{\rho,\delta}$ under consideration solve the equation (2.14) with the monotone bounded linear operator $\mathbf{T}$ and (5.1) represents the corresponding benchmark source condition $f^\dagger - f^* \in \mathrm{range}(\mathbf{T})$, since

$$\begin{aligned}\mathbf{T}(w - f^\dagger) &= \mathcal{N}_{w-f^\dagger}0 - \mathcal{D}_{w-f^\dagger}0 = \mathcal{N}_w 0 - \mathcal{N}_{f^\dagger}0 - \mathcal{D}_w 0 + \mathcal{D}_{f^\dagger}0\\ &= \mathcal{N}_w 0 + \mathcal{N}_0 j^\dagger - (\mathcal{N}_{f^\dagger}0 + \mathcal{N}_0 j^\dagger) - (\mathcal{D}_w 0 + \mathcal{D}_0 g^\dagger) + \mathcal{D}_{f^\dagger}0 + \mathcal{D}_0 g^\dagger\\ &= \mathcal{N}_w j^\dagger - \mathcal{D}_w g^\dagger - (\mathcal{N}_{f^\dagger} j^\dagger - \mathcal{D}_{f^\dagger} g^\dagger)\\ &= \mathcal{N}_w j^\dagger - \mathcal{D}_w g^\dagger,\end{aligned}$$

by the equation $\mathcal{N}_{f^\dagger} j^\dagger - \mathcal{D}_{f^\dagger} g^\dagger = 0$. Under this benchmark source condition, however, the rate assertion with associated choice of the regularization parameter ρ of the corollary is a well-known result of Lavrentiev regularization theory (cf., e.g., [43, Theorem 2.2] and [25, § 4.2]) whenever the noise on the right-hand side of (2.14) is of order δ. Precisely, we have to show that the norm difference between the noisy right-hand side $\mathcal{D}_0 g_\delta - \mathcal{N}_0 j_\delta$ of (2.14) and its the noise-free counterpart $\mathcal{D}_0 g^\dagger - \mathcal{N}_0 j^\dagger$ can be be estimated above as by $K\delta$ with some positive constant K for sufficiently small $\delta > 0$. Indeed, by the inequality

$$\|(\mathcal{D}_0 g_\delta - \mathcal{N}_0 j_\delta) - (\mathcal{D}_0 g^\dagger - \mathcal{N}_0 j^\dagger)\|_{L^2(\Omega)} \le C_{\mathcal{N}} \|j_\delta - j^\dagger\|_{H^{-1/2}(\partial\Omega)} + C_{\mathcal{D}} \|g_\delta - g^\dagger\|_{H^{1/2}(\partial\Omega)}$$

based on the formulas (4.7) and (4.8) for $h = 0$ of Lemma 4.2 we derive this with $K = \max\{C_{\mathcal{N}}, C_{\mathcal{D}}\}$ by taking into account the noise assumption (1.5). The proof is completed. ☐

6 Conjugate gradient method

In this section we will utilize the conjugate gradient (CG) method (see, for example, [22, 27]) to find the minimizes of the strictly convex, discrete regularized problem $\left(\mathcal{P}^h_{\rho,\delta}\right)$. Let $\nabla\Upsilon^h_{\rho,\delta}(f) = 2\left(\mathcal{N}^h_f j_\delta - \mathcal{D}^h_f g_\delta\right) + 2\rho(f - f^*)$ be the L^2-gradient of the cost function $\Upsilon^h_{\rho,\delta}$ at f (see Proof of Theorem 3.2), where $f^* \in \mathcal{V}^h_1$. Then the sequence of iterates via this algorithm is generated by $f^0 \in L^2(\Omega) \cap \mathcal{V}^h_1$ and $f^{k+1} := f^k + t^k d^k$ for $k \ge 0$, where

$$d^k := \begin{cases} -\nabla\Upsilon^h_{\rho,\delta}(f^k) & \text{if } k = 0,\\ -\nabla\Upsilon^h_{\rho,\delta}(f^k) + \beta^k d^{k-1} & \text{if } k > 0\end{cases} \quad \text{with } \beta^k := \frac{\|\nabla\Upsilon^h_{\rho,\delta}(f^k)\|^2}{\|\nabla\Upsilon^h_{\rho,\delta}(f^{k-1})\|^2} \text{ and } t^k := \arg\min_{t\ge 0} \Upsilon^h_{\rho,\delta}(f^k + t d^k).$$

A short computation shows that

$$t^k = -\frac{\int_\Omega Q\nabla\left(\mathcal{N}^h_{d^k}0 - \mathcal{D}^h_{d^k}0\right)\cdot\nabla\left(\mathcal{N}^h_{f^k} j_\delta - \mathcal{D}^h_{f^k} g_\delta\right) + \rho\left(d^k, f^k - f^*\right)}{\int_\Omega Q\nabla\left(\mathcal{N}^h_{d^k}0 - \mathcal{D}^h_{d^k}0\right)\cdot\nabla\left(\mathcal{N}^h_{d^k}0 - \mathcal{D}^h_{d^k}0\right) + \rho\left\|d^k\right\|^2_{L^2(\Omega)}} = -\frac{1}{2}\frac{\left(d^k, \nabla\Upsilon^h_{\rho,\delta}(f^k)\right)}{\left(d^k, \mathcal{N}^h_{d^k}0 - \mathcal{D}^h_{d^k}0\right) + \rho\left\|d^k\right\|^2_{L^2(\Omega)}}.$$

Consequently, the CG method then reads as follows: giving an initial approximation $f^0 \in \mathcal{V}^h_1$, number of iterations N and a positive constants τ_1, τ_2. Computing

$$\nabla\Upsilon^h_{\rho,\delta}(f^0) = 2\left(\mathcal{N}^h_{f^0} j_\delta - \mathcal{D}^h_{f^0} g_\delta\right) + 2\rho(f^0 - f^*),\ d^0 = -\nabla\Upsilon^h_{\rho,\delta}(f^0),\ t^0 = \frac{1}{2}\frac{\left\|d^0\right\|^2_{L^2(\Omega)}}{\left(d^0, \mathcal{N}^h_{d^0}0 - \mathcal{D}^h_{d^0}0\right) + \rho\left\|d^0\right\|^2_{L^2(\Omega)}}$$

and setting

$$f^1 = f^0 + t^0 d^0 \text{ and } k = 1, \quad \text{Tolerance} := \left\|\nabla\Upsilon^h_{\rho,\delta}(f^k)\right\|_{L^2(\Omega)} - \tau_1 - \tau_2\left\|\nabla\Upsilon^h_{\rho,\delta}(f^0)\right\|_{L^2(\Omega)}.$$

while $(Tolerance > 0)$ & $(k \le N)$ **do**

$$\overline{r} = \left\|\nabla \Upsilon^h_{\rho,\delta}(f^{k-1})\right\|^2_{L^2(\Omega)}, \quad r = \left\|\nabla \Upsilon^h_{\rho,\delta}(f^{k})\right\|^2_{L^2(\Omega)}, \quad \beta^k = \frac{r}{\overline{r}},$$

$$d^k = -\nabla \Upsilon^h_{\rho,\delta}(f^k) + \beta^k d^{k-1}, \quad t^k = -\frac{1}{2}\frac{\left(d^k, \nabla \Upsilon^h_{\rho,\delta}(f^k)\right)}{\left(d^k, \mathcal{N}^h_{d^k}0 - \mathcal{D}^h_{d^k}0\right) + \rho\left\|d^k\right\|^2_{L^2(\Omega)}},$$

$$f^{k+1} = f^k + t^k d^k,$$

$$k := k+1, \quad \text{Tolerance} := \left\|\nabla \Upsilon^h_{\rho,\delta}(f^k)\right\|_{L^2(\Omega)} - \tau_1 - \tau_2\left\|\nabla \Upsilon^h_{\rho,\delta}(f^0)\right\|_{L^2(\Omega)}.$$

end

Algorithm 1: CG iteration

7 Numerical test

In this section we illustrate the theoretical result with numerical examples. For this purpose we consider the the boundary value problem

$$-\nabla \cdot \left(Q\nabla\Phi\right) = f^\dagger \text{ in } \Omega, \tag{7.1}$$

$$Q\nabla\Phi \cdot \vec{n} = j^\dagger \text{ on } \partial\Omega \text{ and} \tag{7.2}$$

$$\Phi = g^\dagger \text{ on } \partial\Omega \tag{7.3}$$

with $\Omega = \{x = (x_1, x_2) \in R^2 \mid -1 < x_1, x_2 < 1\}$. We assume that entries of the known symmetric diffusion matrix Q are discontinuous which are defined as

$$q_{11} = 3\chi_{\Omega_{11}} + \chi_{\Omega\setminus\Omega_{11}}, \; q_{12} = \chi_{\Omega_{12}}, \; q_{22} = 4\chi_{\Omega_{22}} + 2\chi_{\Omega\setminus\Omega_{22}},$$

where χ_D is the characteristic function of the Lebesgue measurable set D and

$$\Omega_{11} := \left\{(x_1,x_2) \in \Omega \;\middle|\; |x_1| \le 1/2 \text{ and } |x_2| \le 1/2\right\}, \; \Omega_{12} := \left\{(x_1,x_2) \in \Omega \;\middle|\; |x_1| + |x_2| \le 1/2\right\} \text{ and}$$
$$\Omega_{22} := \left\{(x_1,x_2) \in \Omega \;\middle|\; x_1^2 + x_2^2 \le 1/4\right\}.$$

The identified source function $f^\dagger \in L^2(\Omega)$ in (7.1) is assumed to be discontinuous and defined as

$$f^\dagger = 2\chi_{\Omega_1} - \chi_{\Omega_2} + \frac{5\pi}{7\pi - 192}\chi_{\Omega\setminus(\Omega_1\cup\Omega_2)},$$

where

$$\Omega_1 := \left\{(x_1,x_2) \in \Omega \;\middle|\; 9(x_1+1/2)^2 + 16(x_2-1/2)^2 \le 1\right\} \text{ and}$$
$$\Omega_2 := \left\{(x_1,x_2) \in \Omega \;\middle|\; (x_1-1/2)^2 + (x_2+1/2)^2 \le 1/16\right\}.$$

For the discretization we divide the interval $(-1,1)$ into ℓ equal segments and so that the domain $\Omega = (-1,1)^2$ is divided into $2\ell^2$ triangles, where the diameter of each triangle is $h_\ell = \frac{\sqrt{8}}{\ell}$. In the minimization problem $\left(\mathcal{P}^h_{\rho,\delta}\right)$ we take $h = h_\ell$ and $\rho = \rho_\ell = 0.01h_\ell$. We use Algorithm 1 which is described in the last paragraph of Section 6 for computing the numerical solution of the problem $\left(\mathcal{P}^{h_\ell}_{\rho_\ell,\delta_\ell}\right)$. As an a-priori estimate and the initial approximation we choose $f^* := 0$ and $f^0(x) := \chi_{(0,1]\times[-1,1]} - \chi_{[-1,0]\times[-1,1]}$.

Example 7.1. In this first example $j^\dagger \in H^{-1/2}(\partial\Omega)$ in the equation (7.2) is chosen to be the piecewise constant function defined by

$$\begin{aligned} j^\dagger = \chi_{(0,1]\times\{-1\}} - \chi_{[-1,0]\times\{1\}} + 2\chi_{(0,1]\times\{1\}} - 2\chi_{[-1,0]\times\{-1\}} \\ + 3\chi_{\{-1\}\times(-1,0]} - 3\chi_{\{1\}\times(0,1)} + 4\chi_{\{1\}\times(-1,0]} - 4\chi_{\{-1\}\times(0,1)}. \end{aligned} \tag{7.4}$$

Then $g^\dagger \in H^{1/2}_\diamond(\partial\Omega)$ in the equation (7.3) is defined as $g^\dagger = \gamma\mathcal{N}_{f^\dagger}j^\dagger$, where $\mathcal{N}_{f^\dagger}j^\dagger$ is the unique weak solution of (7.1)–(7.2). We mention that, to avoid a so-called inverse crime, we generate the given data on a finer grid than those used in the computations. For this purpose we first solve the problem (7.1)–(7.2) on the very fine grid with $\ell = 128$, and then use this numerical approximation as substitute for $(j^\dagger, g^\dagger)$ in our computational considerations below.

For observations with noise we assume that

$$(j_{\delta_\ell}, g_{\delta_\ell}) = \left(j^\dagger + \theta_\ell \cdot R_{j^\dagger}, g^\dagger + \theta_\ell \cdot R_{g^\dagger}\right) \quad \text{for some} \quad \theta_\ell > 0 \quad \text{depending on} \quad \ell, \tag{7.5}$$

where $R_{j^\dagger}$ and $R_{g^\dagger}$ are $\partial M^{h_\ell} \times 1$-matrices of random numbers on the interval $(-1,1)$ which are generated by the MATLAB function "rand", and ∂M^{h_ℓ} is the number of boundary nodes of the triangulation $\mathcal{T}^{h_\ell}$. The measurement error is then computed as $\delta_\ell = \left\|j_{\delta_\ell} - j^\dagger\right\|_{L^2(\partial\Omega)} + \left\|g_{\delta_\ell} - g^\dagger\right\|_{L^2(\partial\Omega)}$. To satisfy the condition $\delta_\ell \cdot \rho_\ell^{-1/2} \to 0$ as $\ell \to \infty$ in Theorem 4.5 we below take $\theta_\ell = h_\ell\sqrt{\rho_\ell}$.

We start with the coarsest level $\ell = 4$. At each iteration k we compute

$$\text{Tolerance} := \left\|\nabla\Upsilon^{h_\ell}_{\rho_\ell,\delta_\ell}\left(f^k_\ell\right)\right\|_{L^2(\Omega)} - \tau_1 - \tau_2\left\|\nabla\Upsilon^{h_\ell}_{\rho_\ell,\delta_\ell}\left(f^0_\ell\right)\right\|_{L^2(\Omega)},$$

where $\tau_1 := 10^{-6}h_\ell^{1/2}$ and $\tau_2 := 10^{-4}h_\ell^{1/2}$. Then the iteration was stopped if Tolerance ≤ 0 or the number of iterations reached the maximum iteration count of 600. After obtaining the numerical solution of the first iteration process with respect to the coarsest level $\ell = 4$, we use its interpolation on the next finer mesh $\ell = 8$ as an initial approximation f^0 for the algorithm on this finer mesh, and proceed similarly on the preceding refinement levels.

Let f_ℓ be the function which is obtained at *the final iterate* of Algorithm 1 corresponding to the refinement level ℓ. Furthermore, let $\mathcal{N}^{h_\ell}_{f_\ell}j_{\delta_\ell}$ and $\mathcal{D}^{h_\ell}_{f_\ell}g_{\delta_\ell}$ denote *the computed numerical solution* to the Neumann and Dirichlet problem

$$-\nabla\cdot(Q\nabla u) = f_\ell \text{ in } \Omega \text{ and } Q\nabla u\cdot\vec{n} = j_{\delta_\ell} \text{ on } \partial\Omega \quad \text{and} \quad -\nabla\cdot(Q\nabla v) = f_\ell \text{ in } \Omega \text{ and } v = g_\ell \text{ on } \partial\Omega,$$

respectively. The notations $\mathcal{N}^{h_\ell}_{f^\dagger}j^\dagger$ and $\mathcal{D}^{h_\ell}_{f^\dagger}g^\dagger$ of *the exact numerical solutions* are to be understood similarly. We use the following abbreviations for the errors

$$L^2_f = \left\|f_\ell - f^\dagger\right\|_{L^2(\Omega)}, L^2_{\mathcal{N}} = \left\|\mathcal{N}^{h_\ell}_{f_\ell}j_{\delta_\ell} - \mathcal{N}^{h_\ell}_{f^\dagger}j^\dagger\right\|_{L^2(\Omega)},\ H^1_{\mathcal{N}} = \left\|\mathcal{N}^{h_\ell}_{f_\ell}j_{\delta_\ell} - \mathcal{N}^{h_\ell}_{f^\dagger}j^\dagger\right\|_{H^1(\Omega)} \text{ and}$$
$$L^2_{\mathcal{D}} = \left\|\mathcal{D}^{h_\ell}_{f_\ell}g_{\delta_\ell} - \mathcal{D}^{h_\ell}_{f^\dagger}g^\dagger\right\|_{L^2(\Omega)},\ H^1_{\mathcal{D}} = \left\|\mathcal{D}^{h_\ell}_{f_\ell}g_{\delta_\ell} - \mathcal{D}^{h_\ell}_{f^\dagger}g^\dagger\right\|_{H^1(\Omega)}.$$

The numerical results are summarized in Table 1 and Table 2, where we present the refinement level ℓ, mesh size h_ℓ of the triangulation, regularization parameter ρ_ℓ, measured noise δ_ℓ, number of iterations, value of tolerances and errors L^2_f, $L^2_{\mathcal{N}}$, $L^2_{\mathcal{D}}$, $H^1_{\mathcal{N}}$ and $H^1_{\mathcal{D}}$. Their experimental order of convergence (EOC) is presented in Table 3, where $\mathrm{EOC}_\Theta := \dfrac{\ln\Theta(h_1) - \ln\Theta(h_2)}{\ln h_1 - \ln h_2}$ and $\Theta(h)$ is an error function of the mesh size h.

All figures presented correspond to $\ell = 64$. Figure 1 from left to right shows the computed numerical solution f_ℓ of the algorithm at the final 579^{th}-iteration, and the differences $\mathcal{N}^{h_\ell}_{f^\dagger}j^\dagger - \mathcal{N}^{h_\ell}_{f_\ell}j_{\delta_\ell}$, $\mathcal{D}^{h_\ell}_{f^\dagger}g^\dagger - \mathcal{D}^{h_\ell}_{f_\ell}g_{\delta_\ell}$ and $\mathcal{D}^{h_\ell}_{f_\ell}g_{\delta_\ell} - \mathcal{N}^{h_\ell}_{f_\ell}j_{\delta_\ell}$.

Convergence history					
ℓ	h_ℓ	ρ_ℓ	δ_ℓ	**Iterate**	**Tolerance**
4	0.7071	0.7071e-2	0.1916	312	-3.0822e-5
8	0.3536	0.3536e-2	9.3172e-2	387	-1.2739e-6
16	0.1766	0.1766e-2	4.1174e-2	461	-1.4029e-6
32	8.8388e-2	0.8839e-3	2.0932e-2	505	-1.8559e-7
64	4.4194e-2	0.4419e-3	7.2765e-3	579	-7.3540e-9

Table 1: Refinement level ℓ, mesh size h_ℓ of the triangulation, regularization parameter ρ_ℓ, measurement noise δ_ℓ, number of iterates and value of Tolerance.

Convergence history					
ℓ	L^2_f	$L^2_{\mathcal{N}}$	$L^2_{\mathcal{D}}$	$H^1_{\mathcal{N}}$	$H^1_{\mathcal{D}}$
4	0.5215	2.0441e-2	2.0396e-2	6.9952e-2	6.9713e-2
8	0.3309	6.3175e-3	6.3083e-3	3.1374e-2	3.1311e-2
16	0.1915	2.0132e-3	2.0122e-3	1.7276e-2	1.7243e-2
32	0.1073	5.5434e-4	5.5426e-4	8.9136e-3	8.9130e-3
64	5.2568e-2	1.4669e-4	1.4666e-4	3.9352e-3	3.9347e-3

Table 2: Errors L^2_f, $L^2_{\mathcal{N}}$, $L^2_{\mathcal{D}}$, $H^1_{\mathcal{N}}$ and $H^1_{\mathcal{D}}$.

Experimental order of convergence					
ℓ	$\mathbf{EOC}_{L^2_f}$	$\mathbf{EOC}_{L^2_{\mathcal{N}}}$	$\mathbf{EOC}_{L^2_{\mathcal{D}}}$	$\mathbf{EOC}_{H^1_{\mathcal{N}}}$	$\mathbf{EOC}_{H^1_{\mathcal{D}}}$
4	–	–	–	–	–
8	0.6563	1.6940	1.6930	1.1568	1.1548
16	0.7891	1.6499	1.6485	0.8608	0.8607
32	0.8357	1.8606	1.8601	0.9547	0.9520
64	1.0294	1.9180	1.9181	1.1796	1.1797
Mean of EOC	0.8276	1.7806	1.7799	1.0380	1.0368

Table 3: Experimental order of convergence between finest and coarsest level for L^2_f, $L^2_{\mathcal{N}}$, $L^2_{\mathcal{D}}$, $H^1_{\mathcal{N}}$ and $H^1_{\mathcal{D}}$.

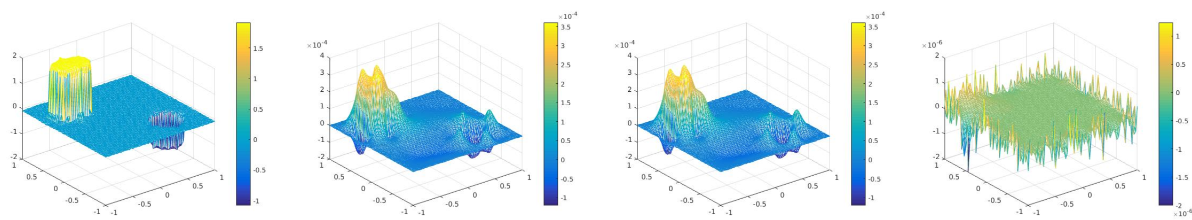

Figure 1: Computed numerical solution f_ℓ of the algorithm at the final iteration, and the differences $\mathcal{N}^{h_\ell}_{f^\dagger} j^\dagger - \mathcal{N}^{h_\ell}_{f_\ell} j_{\delta_\ell}$, $\mathcal{D}^{h_\ell}_{f^\dagger} g^\dagger - \mathcal{D}^{h_\ell}_{f_\ell} g_{\delta_\ell}$ and $\mathcal{D}^{h_\ell}_{f_\ell} g_{\delta_\ell} - \mathcal{N}^{h_\ell}_{f_\ell} j_{\delta_\ell}$.

Example 7.2. In present example we assume that multiple measurements are available, say $\left(j^i_\delta, g^i_\delta\right)_{i=1,\dots,I}$. Then, problem $\left(\mathcal{P}^h_{\rho,\delta}\right)$ in Section 3 is given by

$$\min_{f\in L^2(\Omega)} \bar{\Upsilon}^h_{\rho,\delta}(f) := \min_{f\in L^2(\Omega)} \left(\underbrace{\frac{1}{I}\sum_{i=1}^{I} \int_\Omega Q\nabla\left(\mathcal{N}^h_f j^i_\delta - \mathcal{D}^h_f g^i_\delta\right)\cdot\nabla\left(\mathcal{N}^h_f j^i_\delta - \mathcal{D}^h_f g^i_\delta\right)}_{:=\bar{\mathcal{J}}^h_\delta(q)} + \rho\left\|f - f^*\right\|^2_{L^2(\Omega)} \right) \qquad \left(\bar{\mathcal{P}}^h_{\rho,\delta}\right),$$

which also attains a solution $\bar{f}^h_{\rho,\delta}$. The Neumann boundary condition in the equation (7.2) is chosen in the same form as (7.4), i.e.

$$\begin{aligned} j^\dagger_{(A,B,C,D)} &= A\cdot\chi_{(0,1]\times\{-1\}} - A\cdot\chi_{[-1,0]\times\{1\}} + B\cdot\chi_{(0,1]\times\{1\}} - B\cdot\chi_{[-1,0]\times\{-1\}} \\ &\quad + C\cdot\chi_{\{-1\}\times(-1,0]} - C\cdot\chi_{\{1\}\times(0,1)} + D\cdot\chi_{\{1\}\times(-1,0]} - D\cdot\chi_{\{-1\}\times(0,1)}, \end{aligned} \tag{7.6}$$

and depends on the constants A, B, C and D. Let $g^\dagger_{(A,B,C,D)} := \gamma\mathcal{N}_{f^\dagger} j^\dagger_{(A,B,C,D)}$ and assume that noisy observations are given by

$$\left(j^{(A,B,C,D)}_{\delta_\ell}, g^{(A,B,C,D)}_{\delta_\ell}\right) = \left(j^\dagger_{(A,B,C,D)} + \theta\cdot R_{j^\dagger_{(A,B,C,D)}}, g^\dagger_{(A,B,C,D)} + \theta\cdot R_{g^\dagger_{(A,B,C,D)}}\right), \tag{7.7}$$

where $R_{j^\dagger_{(A,B,C,D)}}$ and $R_{g^\dagger_{(A,B,C,D)}}$ denote $\partial M^{h_\ell} \times 1$-matrices of random numbers on the interval $(-1,1)$. Different from (7.5), the constant θ appeared in the equation (7.7) is now independent of the grid level ℓ.

In the case $(A,B,C,D) = (1,2,3,4)$ we have a single noisy measurement couple, i.e. $I = 1$. We now fix $D = 4$, and let (A,B,C) take all permutations $\mathcal{S}_3$ of the set $\{1,2,3\}$. Then, the equations (7.6)–(7.7) generate $I = 6$ measurements. Similarly, if (A,B,C,D) takes all permutations $\mathcal{S}_4$ of $\{1,2,3,4\}$ we get $I = 16$ measurements. With $\theta = 0.1$ and $\ell = 64$ we compute the noise level

$$\bar{\delta}_\ell = \begin{cases} \|j^{(1,2,3,4)}_{\delta_\ell} - j^\dagger_{(1,2,3,4)}\|_{L^2(\partial\Omega)} + \|g^{(1,2,3,4)}_{\delta_\ell} - g^\dagger_{(1,2,3,4)}\|_{L^2(\partial\Omega)} & \text{if} \quad (A,B,C,D) = (1,2,3,4), \\ \dfrac{1}{6}\sum_{(A,B,C)\in\mathcal{S}_3} \|j^{(A,B,C,4)}_{\delta_\ell} - j^\dagger_{(A,B,C,4)}\|_{L^2(\partial\Omega)} + \|g^{(A,B,C,4)}_{\delta_\ell} - g^\dagger_{(A,B,C,4)}\|_{L^2(\partial\Omega)} & \text{if} \quad D = 4, \\ \dfrac{1}{16}\sum_{(A,B,C,D)\in\mathcal{S}_4} \|j^{(A,B,C,D)}_{\delta_\ell} - j^\dagger_{(A,B,C,D)}\|_{L^2(\partial\Omega)} + \|g^{(A,B,C,D)}_{\delta_\ell} - g^\dagger_{(A,B,C,D)}\|_{L^2(\partial\Omega)}. & \end{cases}$$

The corresponding numerical results for the multiple measurement case are presented in the Table 4.

Numerical results for $\ell = 64$, $\theta = 0.1$ with multiple observations								
I	**Iterate**	**Tolerance**	$\bar{\delta}_\ell$	L^2_q	$L^2_{\mathcal{N}}$	$L^2_{\mathcal{D}}$	$H^1_{\mathcal{N}}$	$H^1_{\mathcal{D}}$
1	531	-3.2313e-8	0.3292	0.3280	5.9096e-3	5.9090e-3	0.1225	0.1221
6	517	-7.1620e-9	0.3331	0.2583	4.3125e-3	4.3122e-3	7.9322e-2	7.9320e-2
16	536	-6.4706e-8	0.3289	0.1747	2.8465e-3	2.8461e-3	5.2318e-2	5.2314e-2

Table 4: Numerical results for $\ell = 64$, $\theta = 0.1$, and with multiple measurements $I = 1, 6, 16$.

Finally, in Figure 2 from left to right we show the interpolation $I_1^{h_\ell} f^\dagger$ of the exact source and the computed numerical solution q_ℓ of the algorithm at the final iteration for $\ell = 64$, $\theta = 0.1$, and $I = 16, 6, 1$, respectively.

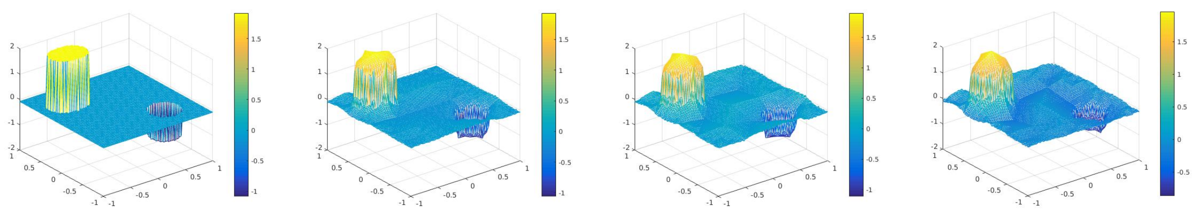

Figure 2: Interpolation $I_1^{h_\ell} f^\dagger$, computed numerical solution f_ℓ of the algorithm at the final iteration for $\ell = 64$, $\theta = 0.1$, and with multiple measurements $I = 16, 6, 1$, respectively.

Acknowledgments

The authors would like to thank the referees and the editor for their valuable comments and suggestions which helped to significantly improve the present paper.

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