Error analysis for global minima of semilinear optimal control problems

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Dedicated to Eduardo Casas on the occasion of his 60th birthday.

Abstract: In [1] we consider an optimal control problem subject to a semilinear elliptic PDE together with its variational discretization, where we provide a condition which allows to decide whether a solution of the necessary first order conditions is a global minimum. This condition can be explicitly evaluated at the discrete level. Furthermore, we prove that if the above condition holds uniformly with respect to the discretization parameter the sequence of discrete solutions converges to a global solution of the corresponding limit problem. With the present work we complement our investigations of [1] in that we prove an error estimate for those discrete global solutions. Numerical experiments confirm our analytical findings.

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1 Introduction

In this work we are concerned with the error analysis of a variational discretization of the control problem

$$(\mathbb{P}) \qquad \min_{u \in U_{ad}} J(u) := \frac{1}{2}\|y - y_0\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)}$$

subject to

$$-\Delta y + \phi(y) = u \quad \text{in } \Omega, \tag{1}$$

$$y = 0 \quad \text{on } \partial\Omega, \tag{2}$$

and the pointwise constraints

$$u_a \le u(x) \le u_b \qquad \text{for a.e. } x \in \Omega,$$

$$y_a(x) \le y(x) \le y_b(x) \qquad \forall\, x \in K \subset \Omega,$$

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where the precise assumptions on the data of the problem will be given in Section 2.1. In [1] the authors considered the same class of problems and established a sufficient condition for the global minima of ($\mathbb{P}$) assuming particular types of growth conditions for the nonlinearity ϕ. The same result was established for the variational discrete counterpart of ($\mathbb{P}$), and it was shown that a sequence of the computed discrete global minima converges to a global minimum of the continuous control problem but without discussing the corresponding rate of convergence. Hence, our aim in this study is to investigate this convergence rate.

The organization of the paper is as follows: in § 2.1 we formulate the control problem and give the exact assumptions on the data. In § 2.2 and in § 2.3 we review the results concerning the state equation and the control problem ($\mathbb{P}$), respectively. The variational discretization of ($\mathbb{P}$) is considered in § 2.4 while § 3 is devoted to the error analysis. Finally, in § 4 we verify our theoretical findings by a numerical example.

Before starting, we give a short list of literature considering the problem ($\mathbb{P}$). For a broad overview, we refer the reader to the references of the respective citations. In [4] the problem ($\mathbb{P}$) is studied when the controls are of boundary type, and the necessary first order conditions are established. Compare [3] where the function ϕ is linear, and [8] where the pointwise constraints are imposed on the gradient of the state.

The regularity of the optimal controls of ($\mathbb{P}$) and their associated multipliers are investigated in [12] and [11], where also the sufficient second order conditions are discussed. Compare [9, 6, 7] for second order conditions when the set K contains finitely/infinitely many points, and [13] for the role of those conditions in PDE constrained control problems.

Finite element discretization of problem ($\mathbb{P}$) under more general setting is studied in [10], and in [19] where a wider class of perturbations are considered. The convergence of the discrete solutions to the continuous solutions is verified there but without rates. However, when the set K contains finitely many points, convergence rates are established in [23] for finite dimensional controls, and in [5] for control functions. Only in [25] error analysis is studied for general pointwise state constraints in K. There, Pfefferer at al. prove an error estimate for discrete solutions in the vicinity of a local solution which satisfies a quadratic growth condition. Error analysis for linear-quadratic control problems can be found in e.g. [11], [14] and [24]. A detailed discussion of discretization concepts and error analysis in PDE-constrained control problems can be found in [20, 21] and[17, Chapter 3].

2 Problem Setting and discretization

2.1 Assumptions

- $\Omega \subset \mathbb{R}^2$ is a bounded, convex and polygonal domain.
- K is a (possibly empty) compact subset of Ω.
- $u_a \in \mathbb{R} \cup \{-\infty\}$ and $u_b \in \mathbb{R} \cup \{\infty\}$ with $u_a \leq u_b$.
- $y_a, y_b \in C_0(\Omega) \cap W^{2,\infty}(\Omega)$ are given functions that satisfy $y_a(x) < y_b(x)$, $x \in K$.

- $y_0 \in L^2(\Omega)$ and $\alpha > 0$ are given.
- $\phi : \mathbb{R} \to \mathbb{R}$ is of class C^2 and monotonically increasing.
- There exist $r > 1$ and $M \geq 0$ such that

$$|\phi''(s)| \leq M\phi'(s)^{\frac{1}{r}} \quad \text{for all } s \in \mathbb{R}, \tag{3}$$

  where ϕ' and ϕ'' denote the first and second derivative of ϕ, respectively.

2.2 The State Equation

Recall that a function $y \in H_0^1(\Omega)$ is called a weak solution of (1), (2) if

$$\int_\Omega \nabla y \cdot \nabla v + \phi(y)v \, dx = \int_\Omega uv \, dx \quad \forall\, v \in H_0^1(\Omega). \tag{4}$$

Theorem 2.1 *For every $u \in L^2(\Omega)$ the boundary value problem* (1), (2) *admits a unique weak solution $y \in H_0^1(\Omega) \cap H^2(\Omega)$. Moreover, there exists $c > 0$ such that*

$$\|y\|_{H^2(\Omega)} \leq c\big(1 + \|u\|_{L^2(\Omega)}\big). \tag{5}$$

Proof. The existence and uniqueness of the solution y in $H_0^1(\Omega)$ follows from the monotone operator theorem. Using the method of Stampacchia one can show, in addition, that $y \in L^\infty(\Omega)$. Utilizing the boundedness of y and the properties of the nonlinearity ϕ, one can show $y \in H^2(\Omega)$ and the estimate (5) using the regularity results from [16, Chapter 4]. For a detailed proof compare for instance [4]. □

In the light of Theorem 2.1, we introduce the control–to–state operator

$$\mathcal{G} : L^2(\Omega) \to H_0^1(\Omega) \cap H^2(\Omega) \tag{6}$$

such that $y := \mathcal{G}(u)$ is the solution to (4) for a given $u \in L^2(\Omega)$.

Lemma 2.2 *Let $\mathcal{G}$ be the mapping introduced in* (6)*. Then there exists $c > 0$ depending only on Ω such that*

$$\|\mathcal{G}(u) - \mathcal{G}(v)\|_{L^2(\Omega)} \leq c\|u - v\|_{L^2(\Omega)} \qquad \forall\, u, v \in L^2(\Omega).$$

Proof. Given $u, v \in L^2(\Omega)$ let $y_u := \mathcal{G}(u)$ and $y_v := \mathcal{G}(v)$. Using Poincaré's inequality, the monotonicity of ϕ and (4) we have

$$\begin{aligned}
\|y_u - y_v\|^2_{L^2(\Omega)} &\leq c \int_\Omega |\nabla(y_u - y_v)|^2 \, dx \\
&\leq \int_\Omega |\nabla(y_u - y_v)|^2 + [\phi(y_u) - \phi(y_v)](y_u - y_v) \, dx \\
&= \int_\Omega (u - v)(y_u - y_v) \, dx \leq \|u - v\|_{L^2(\Omega)} \|y_u - y_v\|_{L^2(\Omega)},
\end{aligned}$$

which implies the result. □

Lemma 2.3 *Let $\mathcal{G}$ be the mapping introduced in* (6)*. Then for any $m > 0$ there exists $L(m) > 0$ such that*

$$\|\mathcal{G}(u) - \mathcal{G}(v)\|_{H^2(\Omega)} \le L(m)\|u - v\|_{L^2(\Omega)}$$

for all u, $v \in L^2(\Omega)$ with $\|u\|_{L^2(\Omega)}$, $\|v\|_{L^2(\Omega)} \le m$.

Proof. Defining again $y_u := \mathcal{G}(u)$, $y_v := \mathcal{G}(v)$ we infer from Theorem 2.1 and the continuous embedding $H^2(\Omega) \hookrightarrow C(\bar{\Omega})$ that $\|y_u\|_{L^\infty(\Omega)}$, $\|y_v\|_{L^\infty(\Omega)} \le c_m$ for some $c_m > 0$ depending on m. Clearly, $y_u - y_v$ belongs to $H_0^1(\Omega) \cap H^2(\Omega)$ and satisfies

$$-\Delta(y_u - y_v) = (u_1 - u_2) - [\phi(y_u) - \phi(y_v)] \quad \text{in } \Omega.$$

Using a standard a–priori estimate, the Lipschitz continuity of ϕ on bounded sets and Lemma 2.2 we infer that

$$\begin{aligned}\|y_1 - y_2\|_{H^2(\Omega)} &\le c\big(\|u - v\|_{L^2(\Omega)} + \|\phi(y_u) - \phi(y_v)\|_{L^2(\Omega)}\big)\\ &\le L(m)\big(\|u - v\|_{L^2(\Omega)} + \|y_u - y_v\|_{L^2(\Omega)}\big)\\ &\le L(m)\|u - v\|_{L^2(\Omega)},\end{aligned}$$

where $L(m)$ is a constant depending on m. This completes the proof. □

2.3 The Optimal Control Problem $(\mathbb{P})$

Using the control-to-state operator $\mathcal{G}$ defined in (6), the reduced form of our optimal control problem reads

$$(\mathbb{P}) \qquad \begin{array}{l}\min_{u\in U_{ad}} J(u) := \frac{1}{2}\|y - y_0\|^2_{L^2(\Omega)} + \dfrac{\alpha}{2}\|u\|^2_{L^2(\Omega)}\\ \text{subject to } y = \mathcal{G}(u) \text{ and } y|_K \in Y_{ad},\end{array}$$

where

$$\begin{aligned}U_{ad} &:= \{v \in L^2(\Omega) : u_a \le v(x) \le u_b \text{ a.e. in } \Omega\},\\ Y_{ad} &:= \{z \in C(K) : y_a(x) \le z(x) \le y_b(x) \text{ for all } x \in K\}.\end{aligned}$$

It is well–known that $(\mathbb{P})$ admits at least one solution provided that a feasible point exists (compare [4]). Moreover, if a solution of $(\mathbb{P})$ satisfies some constraint qualification, then one can guarantee the existence of a multiplier associated with the pointwise state constraints and the necessary first order conditions can be established. A typical constraint qualification for a local solution $\bar{u}$ of problem $(\mathbb{P})$ is the linearized Slater condition which reads: there exist $u_0 \in U_{ad}$ and $\delta > 0$ such that

$$y_a(x) + \delta \le \mathcal{G}(\bar{u})(x) + \mathcal{G}'(\bar{u})(u_0 - \bar{u})(x) \le y_b(x) - \delta \qquad \forall\, x \in K. \tag{7}$$

The next result is a consequence of [4, Theorem 5.2].

Theorem 2.4 *Let $\bar{u} \in U_{ad}$ be a local solution of problem $(\mathbb{P})$ satisfying* (7)*. Then there exist $\bar{p} \in W_0^{1,s}(\Omega)$ for $1 < s < 2$ and a regular Borel measure*

$\bar{\mu} \in \mathcal{M}(K)$ *such that with* $\bar{y} \in H_0^1(\Omega) \cap H^2(\Omega)$ *there holds*

$$\int_\Omega \nabla \bar{y} \cdot \nabla v + \phi(\bar{y}) v \, dx = \int_\Omega \bar{u} v \, dx \quad \forall\, v \in H_0^1(\Omega), \qquad \bar{y}_{|K} \in Y_{ad}, \tag{8}$$

$$\begin{aligned} &\int_\Omega \bar{p}(-\Delta v) + \phi'(\bar{y}) \bar{p} v \, dx \\ &\quad = \int_\Omega (\bar{y} - y_0) v \, dx + \int_K v \, d\bar{\mu} \quad \forall\, v \in H_0^1(\Omega) \cap H^2(\Omega), \end{aligned} \tag{9}$$

$$\int_\Omega (\bar{p} + \alpha \bar{u})(u - \bar{u}) \, dx \geq 0 \qquad \forall\, u \in U_{ad}, \tag{10}$$

$$\int_K (z - \bar{y}) \, d\bar{\mu} \leq 0 \qquad \forall\, z \in Y_{ad}. \tag{11}$$

Note that in view of (10) $\bar{u}$ is the L^2–projection of $-\frac{1}{\alpha}\bar{p}$ onto U_{ad} so that

$$\bar{u}(x) = \min\big(\max\big(u_a, -\tfrac{1}{\alpha}\bar{p}(x)\big), u_b\big) \quad \forall\, x \in \Omega,$$

Since $\bar{p} \in W^{1,s}(\Omega)$ for $1 < s < 2$ it follows from [22, Corollary A.6] that $\bar{u} \in W^{1,s}(\Omega)$ for $1 < s < 2$ as well. Furthermore, it is well known that the multiplier $\bar{\mu}$ associated with the pointwise state constraints is concentrated at the points in K where the state constraints are active. We state this more precisely in the next proposition whose proof can be found in [11]. Compare also the proof in [3] when the bounds y_a, y_b are constant functions.

Proposition 2.5 *Let* $\bar{\mu} \in \mathcal{M}(K)$ *and* $\bar{y} \in C_0(\Omega)$ *satisfy* (11). *Then there holds*

$$\begin{aligned} supp(\bar{\mu}_b) &\subset \{x \in K : \bar{y}(x) = y_b(x)\}, \\ supp(\bar{\mu}_a) &\subset \{x \in K : \bar{y}(x) = y_a(x)\}, \end{aligned}$$

where $\bar{\mu} = \bar{\mu}_b - \bar{\mu}_a$ *with* $\bar{\mu}_b, \bar{\mu}_a \geq 0$ *is the Jordan decomposition of* $\bar{\mu}$.

We note that the problem ($\mathbb{P}$) is in general nonconvex since the state equation is not linear. In other words, the problem ($\mathbb{P}$) can have several solutions. A decision of which of these solution is a global minimum proves difficult in general. However, it is shown in [1] that if the nonlinearity ϕ of the state equation enjoys certain growth conditions, namely (3), then one can establish a condition that helps to decide if a given point satisfying the first order conditions is a global minimum. We state this condition of global optimality in the next result, but before that we first need to introduce the following constant:

$$\eta(\alpha, r) := \alpha^{\frac{\rho}{2}} C_q^{\frac{2-2r}{r}} M^{-1} \left(\frac{r-1}{2r-1}\right)^{\frac{1-r}{r}} q^{1/q} r^{1/r} \rho^{\rho/2} (2-\rho)^{\frac{\rho}{2}-1}. \tag{12}$$

Here, $q := \frac{3r-2}{r-1}$, $\rho := \frac{r+q}{rq}$, while M and r appear in (3). Furthermore, C_q is an upper bound on the optimal constant in the Gagliardo-Nirenberg inequality

$$\|f\|_{L^q} \leq C \|f\|_{L^2}^{\frac{2}{q}} \|\nabla f\|_{L^2}^{\frac{q-2}{q}} \qquad \forall f \in H^1(\mathbb{R}^2).$$

For sharp upper bounds for the constant C, see for instance [1, Theorem 7.3].

Theorem 2.6 *Suppose that* $\bar{u} \in U_{ad}$*,* $\bar{y} \in H^2(\Omega) \cap H_0^1(\Omega)$*,* $\bar{p} \in W_0^{1,s}(\Omega)$ *(*$1 < s < 2$*),* $\bar{\mu} \in \mathcal{M}(K)$ *is a solution of* (8)–(11)*. If*

$$\|\bar{p}\|_{L^q(\Omega)} \leq \eta(\alpha, r), \tag{13}$$

then $\bar{u}$ *is a global minimum for Problem* ($\mathbb{P}$)*. If the inequality (13) is strict, then* $\bar{u}$ *is the unique global minimum.*

2.4 Variational Discretization

Let $\mathcal{T}_h$ be an admissible triangulation of the polygonal domain $\Omega \subset \mathbb{R}^2$ with

$$\overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} \overline{T}.$$

Here $h := \max_{T \in \mathcal{T}_h} \operatorname{diam}(T)$ is the maximum mesh size, while $\operatorname{diam}(T)$ stands for the diameter of the triangle T. We introduce the following spaces of linear finite elements:

$$\begin{aligned} X_h &:= \{v_h \in C(\bar{\Omega}) : v_h|_T \text{ is a linear polynomial on each } T \in \mathcal{T}_h\}, \\ X_{h0} &:= \{v_h \in X_h : v_{h|\partial\Omega} = 0\}. \end{aligned}$$

The Lagrange interpolation operator I_h is defined by

$$I_h : C(\bar{\Omega}) \to X_h, \qquad I_h y := \sum_{i=1}^{n} y(x_i)\phi_i,$$

where $\{x_1, \ldots, x_n\}$ denote the nodes in the triangulation $\mathcal{T}_h$ and $\{\phi_1, \ldots, \phi_n\}$ are the basis functions of the space X_h which satisfy $\phi_i(x_j) = \delta_{ij}$.

The finite element discretization of (4) reads: for a given $u \in L^2(\Omega)$, find $y_h \in X_{h0}$ such that

$$\int_\Omega \nabla y_h \cdot \nabla v_h + \phi(y_h) v_h \, dx = \int_\Omega u v_h \, dx \quad \forall\, v_h \in X_{h0}. \tag{14}$$

Using the monotonicity of ϕ and the Brouwer fixed-point theorem one can show that (14) admits a unique solution $y_h \in X_{h0}$. Hence, analogously to (6), we introduce the discrete control–to–state operator

$$\mathcal{G}_h : L^2(\Omega) \to X_{h0} \tag{15}$$

such that $y_h := \mathcal{G}_h(u)$ is the solution of (14).

The variational discretization (see [18]) of Problem ($\mathbb{P}$) reads:

$$(\mathbb{P}_h) \qquad \begin{array}{l} \min_{u \in U_{ad}} J_h(u) := \frac{1}{2}\|y_h - y_0\|^2_{L^2(\Omega)} + \dfrac{\alpha}{2}\|u\|^2_{L^2(\Omega)} \\ \text{subject to } y_h = \mathcal{G}_h(u),\ (y_h(x_j))_{x_j \in \mathcal{N}_h} \in Y^h_{ad}, \end{array}$$

where we define

$$Y^h_{ad} := \{(z_j)_{x_j \in \mathcal{N}_h} \,|\, y_a(x_j) \leq z_j \leq y_b(x_j), x_j \in \mathcal{N}_h\},$$

with the set of nodes

$$\mathcal{N}_h := \{x_j \,|\, x_j \text{ is a vertex of } T \in \mathcal{T}_h, \text{ where } T \cap K \neq \emptyset\}.$$

We remark that $y_a(x_j) < y_b(x_j), x_j \in \mathcal{N}_h$ provided that h is small enough. This follows from the fact that $\text{dist}(x_j, K) \leq h, x_j \in \mathcal{N}_h$ and y_a, y_b are continuous functions with $y_a(x) < y_b(x), x \in K$.

In an analogous way to that of problem $(\mathbb{P})$, one can show that $(\mathbb{P}_h)$ admits at least one solution, denoted by $\bar{u}_h$, provided that a feasible point exists. In practice one calculates candidates for solutions of $(\mathbb{P}_h)$ by solving the system of necessary first order conditions which reads: find $\bar{u}_h \in U_{ad}, \bar{y}_h \in X_{h0}, \bar{p}_h \in X_{h0}$ and $\bar{\mu}_j \in \mathbb{R}, x_j \in \mathcal{N}_h$ such that

$$\int_\Omega \nabla \bar{y}_h \cdot \nabla v_h + \phi(\bar{y}_h) v_h \, dx = \int_\Omega \bar{u}_h v_h \, dx \quad \forall\, v_h \in X_{h0}, \qquad (\bar{y}_h(x_j))_{x_j \in \mathcal{N}_h} \in Y_{ad}^h \tag{16}$$

$$\int_\Omega \nabla \bar{p}_h \cdot \nabla v_h + \phi'(\bar{y}_h) \bar{p}_h v_h \, dx = \int_\Omega (\bar{y}_h - y_0) v_h \, dx + \sum_{x_j \in \mathcal{N}_h} \bar{\mu}_j v_h(x_j) \qquad \forall\, v_h \in X_{h0}, \tag{17}$$

$$\int_\Omega (\bar{p}_h + \alpha \bar{u}_h)(u - \bar{u}_h) \, dx \geq 0 \qquad \forall\, u \in U_{ad}, \tag{18}$$

$$\sum_{x_j \in \mathcal{N}_h} \bar{\mu}_j (z_j - \bar{y}_h(x_j)) \leq 0 \qquad \forall\, (z_j)_{x_j \in \mathcal{N}_h} \in Y_{ad}^h. \tag{19}$$

As in the continuous case, there exist multipliers $\bar{p}_h$ and $\bar{\mu}_j \in \mathbb{R}, x_j \in \mathcal{N}_h$ solving (16)–(19) provided that the local solution $\bar{u}_h$ satisfies the linearized Slater condition, that is, there exist $u_0 \in U_{ad}$ and $\delta > 0$ such that

$$y_a(x_j) + \delta \leq \mathcal{G}_h(\bar{u}_h)(x_j) + \mathcal{G}_h'(\bar{u}_h)(u_0 - \bar{u}_h)(x_j) \leq y_b(x_j) - \delta,\ x_j \in \mathcal{N}_h. \tag{20}$$

It will be convenient in the upcoming analysis to associate with the multipliers $(\bar{\mu}_j)_{x_j \in \mathcal{N}_h}$ from the system (16)–(19) the measure $\bar{\mu}_h \in \mathcal{M}(\Omega)$ defined by

$$\bar{\mu}_h := \sum_{x_j \in \mathcal{N}_h} \bar{\mu}_j \delta_{x_j}, \tag{21}$$

where δ_{x_j} is the Dirac measure at x_j. We can easily deduce from (19) the following result about the support of the measure $\bar{\mu}_h$.

Proposition 2.7 *Let $\bar{\mu}_h \in \mathcal{M}(\Omega)$ be the measure introduced in* (21) *satisfying* (19)*. Then there holds*

$$\begin{aligned} \mathit{supp}(\bar{\mu}_h^b) &\subset \{x_j \in \mathcal{N}_h : \bar{y}_h(x_j) = y_b(x_j)\}, \\ \mathit{supp}(\bar{\mu}_h^a) &\subset \{x_j \in \mathcal{N}_h : \bar{y}_h(x_j) = y_a(x_j)\}. \end{aligned}$$

where $\bar{\mu}_h = \bar{\mu}_h^b - \bar{\mu}_h^a$ with $\bar{\mu}_h^b, \bar{\mu}_h^a \geq 0$ is the Jordan decomposition of $\bar{\mu}_h$.

Analogously to Theorem 2.6, we have the next theorem about global solutions of problem $(\mathbb{P}_h)$. The proof can be found in [1].

Theorem 2.8 *Suppose that $\bar{u}_h \in U_{ad}$, $\bar{y}_h \in X_{h0}$, $\bar{p}_h \in X_{h0}$, $(\bar{\mu}_j)_{x_j \in \mathcal{N}_h}$ is a solution of* (16)–(19)*. If*

$$\|\bar{p}_h\|_{L^q(\Omega)} \leq \eta(\alpha, r), \tag{22}$$

then $\bar{u}_h$ is a global minimum for Problem $(\mathbb{P}_h)$. If the inequality (22) is strict, then $\bar{u}_h$ is the unique global minimum.

3 Error Analysis

Let $\{\mathcal{T}_h\}_{0<h\le h_0}$ be a sequence of admissible triangulations of Ω. We assume that the sequence $\{\mathcal{T}_h\}_{0<h\le h_0}$ is quasi-uniform in the sense that each $T \in \mathcal{T}_h$ is contained in a ball of radius $\gamma^{-1}h$ and contains a ball of radius γh for some $\gamma > 0$ independent of h. In addition we make the following assumption concerning the set K:

Assumption 1 For every $h > 0$ there exists a set of triangles $\mathbb{T}_h \subset \mathcal{T}_h$ such that

$$K = \bigcup_{T\in\mathbb{T}_h} \bar{T}.$$

In what follows we consider a sequence $(\bar{u}_h, \bar{y}_h, \bar{p}_h, \bar{\mu}_h)_{0<h\le h_1}$ of solutions of (16)–(19) satisfying

$$\|\bar{p}_h\|_{L^q(\Omega)} \le \kappa\eta(\alpha, r) \qquad \text{for } 0 < h \le h_1 \tag{23}$$

for some $\kappa \in (0, 1)$ that is independent of h. We immediately infer from Theorem 2.8 that $\bar{u}_h$ is the unique global minimum of $(\mathbb{P}_h)$ and we are interested in the convergence properties of these solutions as $h \to 0$. It is shown in [1] (see Theorem 4.2 and its proof) that there exist $\bar{u} \in U_{ad}$, $\bar{p} \in L^q(\Omega)$ and $\bar{\mu} \in \mathcal{M}(K)$ such that

$$\bar{u}_h \to \bar{u} \text{ in } L^2(\Omega), \quad \bar{p}_h \rightharpoonup \bar{p} \text{ in } L^q(\Omega), \quad \bar{\mu}_h \rightharpoonup \bar{\mu} \text{ in } \mathcal{M}(K)$$

and $(\bar{u}, \bar{y} = \mathcal{G}(\bar{u}), \bar{p}, \bar{\mu})$ is a solution of (8)–(11). Since

$$\|\bar{p}\|_{L^q(\Omega)} \le \liminf_{h\to 0} \|\bar{p}_h\|_{L^q(\Omega)} \le \kappa\eta(\alpha, r), \tag{24}$$

Theorem 2.6 implies that $\bar{u}$ is the unique global optimum of $(\mathbb{P})$. The aim in the remaining part of this paper is to prove error estimates for $\bar{u}_h - \bar{u}$ and the corresponding optimal states $\bar{y}_h - \bar{y}$. Our main results read:

Theorem 3.1 *Suppose that (23) holds and let $\bar{u}_h$, $\bar{u}$ be the unique global minima of $(\mathbb{P}_h)$ and $(\mathbb{P})$ respectively. Then we have for any $1 < s < 2$ that*

$$\|\bar{u}_h - \bar{u}\|_{L^2(\Omega)} \le c_s\sqrt{|\ln h|}\, h^{\frac{3}{2}-\frac{1}{s}} \tag{25}$$

$$\|\bar{y}_h - \bar{y}\|_{H^1(\Omega)} + \|\bar{y}_h - \bar{y}\|_{L^\infty(\Omega)} \le c_s\sqrt{|\ln h|}\, h^{\frac{3}{2}-\frac{1}{s}}. \tag{26}$$

Remark 1 In [25] Pfefferer at al. for problems in two and three dimensions present a similar error estimate for discrete (local) solutions in the vicinity of a local solution which satisfies a quadratic growth condition. Assuming (23) we here use different techniques to prove an error estimate for the unique global discrete solutions which converge to the unique global solution of our optimization problem.

Before we start presenting the proof of this result we collect some results concerning the uniform boundedness of the discrete optimal control $\bar u_h$, its state $\bar y_h$ and the associated multipliers $\bar p_h$ and $\bar\mu_h$.

Lemma 3.2 *Let $\bar u_h \in U_{ad}$, $\bar y_h$, $\bar p_h \in X_{h0}$ and $(\bar\mu_j)_{x_j\in\mathcal{N}_h}$ be a solution of (16)–(19) satisfying*

$$\|\bar p_h\|_{L^q(\Omega)} \le \eta(\alpha, r), \quad 0 < h \le h_0.$$

Then there exists a constant $C > 0$, which is independent of h, such that

$$\|\bar u_h\|_{L^2(\Omega)},\ \|\bar y_h\|_{H^1(\Omega)},\ \|\bar y_h\|_{L^\infty(\Omega)}\ ,\ \|\bar\mu_h\|_{\mathcal{M}(K)} \le C. \tag{27}$$

Proof. The uniform boundedness of $\|\bar u_h\|_{L^2(\Omega)}$, $\|\bar y_h\|_{H^1(\Omega)}$, $\|\bar\mu_h\|_{\mathcal{M}(\tilde K)}$ is shown in [1, Lemma 4.1] while the one of $\|\bar y_h\|_{L^\infty(\Omega)}$ is a consequence of the uniform convergence [1, (4.16)]. □

Next, let us introduce the auxiliary functions $\tilde y^h \in H^2(\Omega)\cap H^1_0(\Omega)$, $\tilde y_h \in X_{h0}$, $\tilde p_h \in X_{h0}$ as the solutions of

$$\int_\Omega \nabla\tilde y^h\cdot\nabla v + \phi(\tilde y^h)v\,dx = \int_\Omega \bar u_h v\,dx \qquad \forall\, v\in H^1_0(\Omega), \tag{28}$$

$$\int_\Omega \nabla\tilde y_h\cdot\nabla v_h + \phi(\tilde y_h)v_h\,dx = \int_\Omega \bar u v_h\,dx \qquad \forall\, v_h\in X_{h0}, \tag{29}$$

$$\begin{aligned}&\int_\Omega \nabla\tilde p_h\cdot\nabla v_h + \phi'(\bar y)\tilde p_h v_h\,dx\\ &\quad= \int_\Omega (\bar y - y_0)v_h\,dx + \int_K v_h\,d\bar\mu \qquad \forall\, v_h\in X_{h0}.\end{aligned} \tag{30}$$

Lemma 3.3 *Let $\tilde y^h$, $\tilde y_h$ and $\tilde p_h$ be as above and Ω_0 an open set such that $\overline{\Omega}_0\subset\Omega$ and $K\subset\Omega_0$. Then we have*

$$\|\bar y_h - \tilde y^h\|_{L^2(\Omega)} + h\,\|\bar y_h - \tilde y^h\|_{L^\infty(\Omega)} \le ch^2\big(\|\bar u_h\|_{L^2(\Omega)} + 1\big), \tag{31}$$

$$\|\tilde y_h - \bar y\|_{L^2(\Omega)} + h\,\|\tilde y_h - \bar y\|_{L^\infty(\Omega)} \le ch^2\big(\|\bar u\|_{L^2(\Omega)} + 1\big), \tag{32}$$

$$\|\tilde y_h - \bar y\|_{L^\infty(\Omega_0)} \le c|\ln h| h^{3-\frac{2}{s}}\big(\|\bar u\|_{W^{1,s}(\Omega)} + 1\big), \tag{33}$$

$$\|\tilde p_h - \bar p\|_{L^2(\Omega)} \le ch\big(\|\bar y - y_0\|_{L^2(\Omega)} + \|\bar\mu\|_{\mathcal{M}(K)}\big). \tag{34}$$

Proof. The estimates (31) and (32) can be found as Theorems 1 and 2 in [10]. On the other hand, (33) follows from [25, Theorem 3.5]. Finally, the estimate (34) is a consequence of [2, Theorem 3]. □

Proof of Theorem 3.1: Testing (10) with $\bar u_h$ and (18) with $\bar u$ and adding the resulting inequalities gives

$$\int_\Omega (\bar p_h - \bar p)(\bar u - \bar u_h) - \alpha(\bar u_h - \bar u)^2\,dx \ge 0$$

from which we obtain

$$\begin{aligned}\alpha\|\bar u - \bar u_h\|^2_{L^2(\Omega)} &\le \int_\Omega (\bar u - \bar u_h)(\bar p_h - \bar p)\,dx\\ &= \underbrace{\int_\Omega (\tilde p_h - \bar p)(\bar u - \bar u_h)\,dx}_{S_1} + \int_\Omega (\bar p_h - \tilde p_h)(\bar u - \bar u_h)\,dx.\end{aligned} \tag{35}$$

We see that from (16) and (29) with the choice $v_h = \bar{p}_h - \tilde{p}_h$ that

$$
\begin{aligned}
&\int_\Omega (\bar{p}_h - \tilde{p}_h)(\bar{u} - \bar{u}_h)\,dx = \int_\Omega [\phi(\tilde{y}_h) - \phi(\bar{y}_h)](\bar{p}_h - \tilde{p}_h)\,dx \\
&\quad + \int_\Omega \nabla(\tilde{y}_h - \bar{y}_h)\cdot\nabla(\bar{p}_h - \tilde{p}_h)\,dx \\
&\quad = \underbrace{\int_\Omega (\bar{y}_h - \bar{y})(\tilde{y}_h - \bar{y}_h)\,dx}_{S_2} + \underbrace{\int_K (\tilde{y}_h - \bar{y}_h)\,d\bar{\mu}_h - \int_K (\tilde{y}_h - \bar{y}_h)\,d\bar{\mu}}_{S_3} \\
&\quad + \underbrace{\int_\Omega [\phi(\tilde{y}_h) - \phi(\bar{y}_h)](\bar{p}_h - \tilde{p}_h)\,dx - \int_\Omega [\phi'(\bar{y}_h)\bar{p}_h - \phi'(\bar{y})\tilde{p}_h](\tilde{y}_h - \bar{y}_h)\,dx}_{S_4},
\end{aligned}
$$

where we utilized (17) and (30) with the test function $v_h = \tilde{y}_h - \bar{y}_h$ to rewrite the term containing the gradients in the first equality. Consequently, adding the terms S_2, S_3, S_4 to S_1 in (35) gives

$$
\alpha\|\bar{u} - \bar{u}_h\|^2_{L^2(\Omega)} \le \sum_{i=1}^4 S_i. \tag{36}
$$

Young's inequality together with (34) implies that

$$
\begin{aligned}
S_1 &\le \|\tilde{p}_h - \bar{p}\|_{L^2(\Omega)}\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \le \frac{1}{2\alpha\epsilon}\|\tilde{p}_h - \bar{p}\|^2_{L^2(\Omega)} + \frac{\alpha\epsilon}{2}\|\bar{u} - \bar{u}_h\|^2_{L^2(\Omega)} \\
&\le \frac{c}{\alpha\epsilon}h^2\big(\|\bar{y} - y_0\|^2_{L^2(\Omega)} + \|\bar{\mu}\|^2_{\mathcal{M}(K)}\big) + \frac{\alpha\epsilon}{2}\|\bar{u} - \bar{u}_h\|^2_{L^2(\Omega)} \\
&= \frac{\alpha\epsilon}{2}\|\bar{u} - \bar{u}_h\|^2_{L^2(\Omega)} + c_\epsilon h^2.
\end{aligned}
$$

In a similar way we deduce with the help of (32)

$$
\begin{aligned}
S_2 &= -\|\bar{y}_h - \bar{y}\|^2_{L^2(\Omega)} + \int_\Omega (\bar{y}_h - \bar{y})(\tilde{y}_h - \bar{y})\,dx \\
&\le -\|\bar{y}_h - \bar{y}\|^2_{L^2(\Omega)} + \|\bar{y}_h - \bar{y}\|_{L^2(\Omega)}\|\tilde{y}_h - \bar{y}\|_{L^2(\Omega)} \\
&\le -\|\bar{y}_h - \bar{y}\|^2_{L^2(\Omega)} + \frac{\epsilon}{2}\|\bar{y}_h - \bar{y}\|^2_{L^2(\Omega)} + \frac{1}{2\epsilon}\|\tilde{y}_h - \bar{y}\|^2_{L^2(\Omega)} \\
&\le (\frac{\epsilon}{2} - 1)\|\bar{y}_h - \bar{y}\|^2_{L^2(\Omega)} + \frac{c}{\epsilon}h^4\big(\|\bar{u}\|^2_{L^2(\Omega)} + 1\big) \\
&\le (\frac{\epsilon}{2} - 1)\|\bar{y}_h - \bar{y}\|^2_{L^2(\Omega)} + c_\epsilon h^4.
\end{aligned}
$$

Let us next consider the first integral in S_3. Using $\bar{\mu}_h = \bar{\mu}_h^b - \bar{\mu}_h^a$, Proposition 2.7, the fact that $y_a \le \bar{y} \le y_b$ on K, Lemma 3.2 and (33) we have

$$
\begin{aligned}
&\int_K (\tilde{y}_h - \bar{y}_h)\,d\bar{\mu}_h = \int_K (\tilde{y}_h - y_b)\,d\bar{\mu}_h^b - \int_K (\tilde{y}_h - y_a)\,d\bar{\mu}_h^a \\
&\le \int_K (\tilde{y}_h - \bar{y})\,d\bar{\mu}_h^b + \int_K (\bar{y} - \tilde{y}_h)\,d\bar{\mu}_h^a \le \|\tilde{y}_h - \bar{y}\|_{L^\infty(\Omega_0)}\|\bar{\mu}_h\|_{\mathcal{M}(K)} \\
&\le c|\ln h|h^{3-\frac{2}{s}}\big(\|\bar{u}\|_{W^{1,s}(\Omega)} + 1\big).
\end{aligned} \tag{37}
$$

To estimate the second integral in S_3 we use Proposition 2.5, the fact that $I_h y_a \le \bar{y}_h \le I_h y_b$ in K, a well–known interpolation estimate and (33) to obtain

$$\begin{aligned}
&\int_K (\bar{y}_h - \tilde{y}_h)\, d\bar{\mu} = \int_K (\bar{y}_h - \tilde{y}_h)\, d\bar{\mu}_b - \int_K (\bar{y}_h - \tilde{y}_h)\, d\bar{\mu}_a \\
&\le \int_K (I_h y_b - y_b)\, d\bar{\mu}_b + \int_K (\bar{y} - \tilde{y}_h)\, d\bar{\mu}_b + \int_K (\tilde{y}_h - \bar{y})\, d\bar{\mu}_a + \int_K (y_a - I_h y_a)\, d\bar{\mu}_a \\
&\le \|\bar{\mu}\|_{\mathcal{M}(K)} \Big(\|\tilde{y}_h - \bar{y}\|_{L^\infty(K)} + \|y_a - I_h y_a\|_{L^\infty(K)} + \|y_b - I_h y_b\|_{L^\infty(K)} \Big) \\
&\le c\|\bar{\mu}\|_{\mathcal{M}(K)} \Big(|\ln h| h^{3-\frac{2}{s}} (\|\bar{u}\|_{W^{1,s}(\Omega)} + 1) + ch^2 \big(\|y_a\|_{W^{2,\infty}(\Omega)} + \|y_b\|_{W^{2,\infty}(\Omega)} \big) \Big) \\
&\le c|\ln h| h^{3-\frac{2}{s}} \|\bar{\mu}\|_{\mathcal{M}(K)} \big(\|\bar{u}\|_{W^{1,s}(\Omega)} + 1 \big). \qquad (38)
\end{aligned}$$

Combining (37) and (38) yields

$$S_3 \le c|\ln h| h^{3-\frac{2}{s}} \big(\|\bar{u}\|_{W^{1,s}(\Omega)} + 1 \big).$$

Let us next turn S_4, which we rewrite as

$$\begin{aligned}
S_4 = &\underbrace{\int_\Omega [\phi(\tilde{y}_h) - \phi(\bar{y}_h) + \phi'(\bar{y}_h)(\bar{y}_h - \tilde{y}_h)] \bar{p}_h \, dx}_{S_{4.1}} + \underbrace{\int_\Omega [\phi(\tilde{y}^h) - \phi(\bar{y}) + \phi'(\bar{y})(\bar{y} - \tilde{y}^h)] \bar{p} \, dx}_{S_{4.2}} \\
&+ \underbrace{\int_\Omega [\phi(\tilde{y}^h) - \phi(\bar{y}) + \phi'(\bar{y})(\bar{y} - \tilde{y}^h)](\tilde{p}_h - \bar{p}) \, dx}_{S_{4.3}} + \underbrace{\int_\Omega [\phi(\bar{y}) - \phi(\tilde{y}_h) + \phi'(\bar{y})(\tilde{y}_h - \bar{y})] \tilde{p}_h \, dx}_{S_{4.4}} \\
&+ \underbrace{\int_\Omega [\phi(\bar{y}_h) - \phi(\tilde{y}^h) + \phi'(\bar{y})(\tilde{y}^h - \bar{y}_h)] \tilde{p}_h \, dx}_{S_{4.5}}.
\end{aligned}$$

In order to estimate $S_{4.1}$ we first observe that $S_{4.1} = R_h(u_h)$ for the choice $y_h = \tilde{y}_h$, where $R_h(u_h)$ is defined at the bottom of page 266 in [1]. Retracing the steps in [1] leading to (3.11) we infer that

$$|S_{4.1}| \le 2\alpha^{-\frac{\rho}{2}} L_r C_q^{\frac{2r-2}{r}} d_r e_r \|\bar{p}_h\|_{L^q(\Omega)} \Big(\frac{1}{2} \|\tilde{y}_h - \bar{y}_h\|^2_{L^2(\Omega)} + \frac{\alpha}{2} \|u_h - \bar{u}_h\|^2_{L^2(\Omega)} \Big), \qquad (39)$$

where q and ρ are defined immediately after (12), while

$$L_r = M\big(\frac{r-1}{2r-1}\big)^{(r-1)/r}, \quad d_r = q^{-1/q} r^{-1/r} \rho^{-\rho}, e_r = \big(1 - \frac{\rho}{2}\big)^{1-\frac{\rho}{2}} \big(\frac{\rho}{2}\big)^{\frac{\rho}{2}}.$$

In view of the definition of $\eta(\alpha, r)$ and (23) this implies

$$\begin{aligned}
|S_{4.1}| &\le \eta(\alpha, r)^{-1} \|\bar{p}_h\|_{L^q(\Omega)} \Big(\frac{1}{2} \|\tilde{y}_h - \bar{y}_h\|^2_{L^2(\Omega)} + \frac{\alpha}{2} \|\bar{u} - \bar{u}_h\|^2_{L^2(\Omega)} \Big) \\
&\le \kappa \Big(\frac{1}{2} \|\tilde{y}_h - \bar{y}_h\|^2_{L^2(\Omega)} + \frac{\alpha}{2} \|\bar{u} - \bar{u}_h\|^2_{L^2(\Omega)} \Big).
\end{aligned}$$

Since

$$\begin{aligned}
\|\tilde{y}_h - \bar{y}_h\|^2_{L^2(\Omega)} &\le (1+\epsilon) \|\bar{y}_h - \bar{y}\|^2_{L^2(\Omega)} + c_\epsilon \|\tilde{y}_h - \bar{y}\|^2_{L^2(\Omega)} \\
&\le (1+\epsilon) \|\bar{y}_h - \bar{y}\|^2_{L^2(\Omega)} + c_\epsilon h^4
\end{aligned}$$

by (32), we finally obtain

$$|S_{4.1}| \le \kappa\Big(\frac{1+\epsilon}{2}\|\bar{y}_h - \bar{y}\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|\bar{u} - \bar{u}_h\|^2_{L^2(\Omega)}\Big) + c_\epsilon h^4.$$

Using (24), (31) and (27), we derive in a similar way

$$\begin{aligned}|S_{4.2}| &\le \eta(\alpha, r)^{-1}\|\bar{p}\|_{L^q(\Omega)}\Big(\frac{1}{2}\|\tilde{y}^h - \bar{y}\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|\bar{u}_h - \bar{u}\|^2_{L^2(\Omega)}\Big)\\ &\le \kappa\Big(\frac{1+\epsilon}{2}\|\bar{y}_h - \bar{y}\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|\bar{u}_h - \bar{u}\|^2_{L^2(\Omega)}\Big) + c_\epsilon\|\tilde{y}^h - \bar{y}_h\|^2_{L^2(\Omega)}\\ &\le \kappa\Big(\frac{1+\epsilon}{2}\|\bar{y}_h - \bar{y}\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|\bar{u}_h - \bar{u}\|^2_{L^2(\Omega)}\Big) + c_\epsilon h^4.\end{aligned}$$

Since $\phi \in C^2$ and $\|\tilde{y}^h\|_{L^\infty(\Omega)}$ is uniformly bounded in h (in view of (31) and (27)) we infer with the help of Lemma 2.2 and (34)

$$\begin{aligned}|S_{4.3}| &\le c\|\tilde{y}^h - \bar{y}\|_{L^\infty(\Omega)}\|\tilde{y}^h - \bar{y}\|_{L^2(\Omega)}\|\tilde{p}_h - \bar{p}\|_{L^2(\Omega)}\\ &\le ch\|\bar{u}_h - \bar{u}\|_{L^2(\Omega)}\big(\|\bar{y} - y_0\|_{L^2(\Omega)} + \|\bar{\mu}\|_{\mathcal{M}(K)}\big)\\ &\le \frac{\alpha\epsilon}{2}\|\bar{u}_h - \bar{u}\|^2_{L^2(\Omega)} + c_\epsilon h^2.\end{aligned}$$

In a similar way we obtain using (32) and (31)

$$\begin{aligned}|S_{4.4}| + |S_{4.5}| &\le c\big(\|\tilde{y}_h - \bar{y}\|_{L^2(\Omega)} + \|\tilde{y}^h - \bar{y}_h\|_{L^2(\Omega)}\big)\|\tilde{p}_h\|_{L^2(\Omega)}\\ &\le ch^2\big(\|\bar{u}\|_{L^2(\Omega)} + \|\bar{u}_h\|_{L^2(\Omega)} + 1\big)\|\tilde{p}_h\|_{L^2(\Omega)}\\ &\le ch^2,\end{aligned}$$

where we note that $\|\tilde{p}_h\|_{L^2(\Omega)}$ is uniformly bounded for sufficiently small h in view of (34). Collecting the estimates for $S_{4.1}, \dots, S_{4.5}$, we conclude that S_4 can be bounded by

$$S_4 \le \kappa(1+\epsilon)\|\bar{y}_h - \bar{y}\|^2_{L^2(\Omega)} + (\kappa\alpha + \frac{\alpha\epsilon}{2})\|\bar{u}_h - \bar{u}\|^2_{L^2(\Omega)} + c_\epsilon h^2.$$

Inserting the estimates of the terms $S_1, \dots, S_4$ into (36) yields

$$\begin{aligned}\alpha\|\bar{u}_h - \bar{u}\|^2_{L^2(\Omega)} \le \big(\kappa(1+\epsilon) + (\frac{\epsilon}{2} - 1)\big)\|\bar{y}_h - \bar{y}\|^2_{L^2(\Omega)} + \alpha(\kappa + \epsilon)\|\bar{u}_h - \bar{u}\|^2_{L^2(\Omega)}\\ + c_\epsilon|\ln h|h^{3-\frac{2}{s}}\big(\|\bar{u}\|_{W^{1,s}(\Omega)} + 1\big).\end{aligned}$$

Since $\kappa < 1$, choosing $\epsilon > 0$ to be small enough in the above expression yields the existence of $c > 0$ independent of h such that

$$\|\bar{u}_h - \bar{u}\|^2_{L^2(\Omega)} + \|\bar{y}_h - \bar{y}\|^2_{L^2(\Omega)} \le c|\ln h|h^{3-\frac{2}{s}}\big(\|\bar{u}\|_{W^{1,s}(\Omega)} + 1\big). \qquad (40)$$

Let us next establish an upper bound for $\|\nabla(\bar{y}_h - \bar{y})\|_{L^2(\Omega)}$. To this end we introduce $R_h\bar{y}$ as the Ritz projection of $\bar{y}$, i.e

$$\int_\Omega \nabla R_h\bar{y} \cdot \nabla w_h\, dx = \int_\Omega \nabla\bar{y} \cdot \nabla w_h\, dx \qquad \forall\, w_h \in X_{h0}.$$

Let us first derive an upper bound on $\|\nabla(\bar{y}_h - R_h\bar{y})\|_{L^2(\Omega)}$. To begin, from the definition of $R_h\bar{y}$ and the weak formulation of $\bar{y}$ we have

$$\int_\Omega \nabla R_h\bar{y} \cdot \nabla w_h\, dx = \int_\Omega \nabla\bar{y} \cdot \nabla w_h\, dx = \int_\Omega \bar{u}w_h\, dx - \int_\Omega \phi(\bar{y})w_h\, dx \quad \forall\, w_h \in X_{h0}.$$

If we combine this relation with (16) we obtain for all $w_h \in X_{h0}$ that

$$\int_\Omega \nabla(R_h\bar{y} - \bar{y}_h) \cdot \nabla w_h \, dx = \int_\Omega (\bar{u} - \bar{u}_h) w_h \, dx + \int_\Omega [\phi(\bar{y}_h) - \phi(\bar{y})] w_h \, dx.$$

Using $w_h = R_h\bar{y} - \bar{y}_h$ in the previous variational equation and observing that $\|\bar{y}_h\|_{L^\infty(\Omega)}$ is uniformly bounded in h we deduce that

$$\begin{aligned} &\int_\Omega |\nabla(R_h\bar{y} - \bar{y}_h)|^2 \, dx \\ &\le \big(\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} + \|\phi(\bar{y}_h) - \phi(\bar{y})\|_{L^2(\Omega)}\big) \|R_h\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \\ &\le c\big(\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} + \|\bar{y}_h - \bar{y}\|_{L^2(\Omega)}\big) \|\nabla(R_h\bar{y} - \bar{y}_h)\|_{L^2(\Omega)} \end{aligned}$$

by Poincaré's inequality. Thus,

$$\|\nabla(\bar{y}_h - R_h\bar{y})\|_{L^2(\Omega)} \le c\big(\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} + \|\bar{y}_h - \bar{y}\|_{L^2(\Omega)}\big), \tag{41}$$

which together with a standard error bound for the Ritz projection and (40) implies

$$\begin{aligned} \|\nabla(\bar{y}_h - \bar{y})\|_{L^2(\Omega)} &\le \|\nabla(\bar{y}_h - R_h\bar{y})\|_{L^2(\Omega)} + \|\nabla(R_h\bar{y} - \bar{y})\|_{L^2(\Omega)} \\ &\le c\big(\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} + \|\bar{y}_h - \bar{y}\|_{L^2(\Omega)}\big) + ch\|\bar{y}\|_{H^2(\Omega)} \\ &\le c|\ln h| h^{3-\frac{2}{s}} \big(\|\bar{u}\|_{W^{1,s}(\Omega)} + 1\big). \end{aligned}$$

It remains to prove the uniform estimate for $\bar{y}_h - \bar{y}$. We obtain from (31), the continuous embedding $H^2(\Omega) \hookrightarrow C(\bar{\Omega})$, Lemma 2.3 and (40) that

$$\begin{aligned} &\|\bar{y}_h - \bar{y}\|_{L^\infty(\Omega)} \le \|\bar{y}_h - \tilde{y}^h\|_{L^\infty(\Omega)} + \|\,\tilde{y}^h - \bar{y}\|_{L^\infty(\Omega)} \\ &\le ch\big(\|\bar{u}_h\|_{L^2(\Omega)} + 1\big) + c\|\tilde{y}^h - \bar{y}\|_{H^2(\Omega)} \le ch + c\|\bar{u}_h - \bar{u}\|_{L^2(\Omega)} \\ &\le c|\ln h| h^{3-\frac{2}{s}} \big(\|\bar{u}\|_{W^{1,s}(\Omega)} + 1\big). \end{aligned}$$

This completes the proof of Theorem 3.1.

Remark 2 The choice $K = \bar{\Omega}$ is in fact allowed for Problem ($\mathbb{P}$) provided that the bounds y_a, $y_b \in C(\bar{\Omega})$ satisfy in addition to $y_a < y_b$ in $\bar{\Omega}$ the compatibility condition $y_a < 0 < y_b$ on $\partial\Omega$. In this case the set $\mathcal{N}_h$, which appears in the discrete optimal control problem, should be defined as

$$\mathcal{N}_h := \{x_j | x_j \text{ is a vertex of } T \in \mathcal{T}_h, x_j \notin \partial\Omega\}.$$

We claim that the assertion of Theorem 3.1 remains valid in this setting. To see this, we note that the only change in the proof concerns the term S_3 which now reads

$$\int_{\bar{\Omega}} (\tilde{y}_h - \bar{y}) \, d\bar{\mu}_h - \int_{\bar{\Omega}} (\tilde{y}_h - \bar{y}) \, d\bar{\mu}.$$

However, using the fact that y_a, $y_b \in C(\bar{\Omega})$, $y_a < y_b$ in $\bar{\Omega}$ and $y_a < 0 < y_b$ on $\partial\Omega$, it can be shown that there exists $\Omega_0 \subset\subset \Omega$ such that $\mathrm{supp}(\bar{\mu}) \subset \Omega_0$ and $\mathrm{supp}(\bar{\mu}_h) \subset \Omega_0$ for h small enough, see [11, Corollary 5.4]. We may then use again (33) and argue in the same way as before.

4 Numerical Example

We now examine numerically the error bounds established in Theorem 3.1. For this purpose, we consider the following example taken from [25, Section 7], in which Problem ($\mathbb{P}$) is considered with the following choice for the data: $\Omega := (0,1) \times (0,1)$, $\phi(s) = s^3$, $\alpha = 10^{-2}$, $y_0 := -1$, $U_{ad} = L^2(\Omega)$, $y_b = \infty$ and

$$y_a(x) := -\frac{1}{2} + \frac{1}{2}\min(x_1 + x_2, 1 + x_1 - x_2, 1 - x_1 + x_2, 2 - x_1 - x_2).$$

It was shown in [1] that this example admits a unique global solution. In fact, it is easy to see that (3) holds for $\phi(s) = s^3$ with $r = 2$ and $M = 2\sqrt{3}$, hence $q = 4$ in (12). After applying the variational discretization, the numerical solution of the resulting discrete optimality system (16)–(19) is obtained by the semismooth Newton method proposed in [15] whose extension to semilinear elliptic control problems is straightforward. Consequently, the condition (22) from Theorem 2.8 now reads

$$\|\bar{p}_h\|_{L^4(\Omega)} \le 5^{-\frac{5}{8}} 3^{\frac{3}{8}} \sqrt{2} C_4^{-1} \alpha^{\frac{3}{8}} =: \eta(\alpha),$$

where $C_4 \approx 0.648027075$ is an upper bound for the constant in Gagliardo-Nirenberg inequality, precisely it is the bound $C_4^{(3)}$ from [1, Theorem 7.3]. Figure 1 compares the quantities $\|\bar{p}_h\|_{L^4(\Omega)}$ and $\eta(\alpha)$ for several choices of α, including $\alpha = 10^{-2}$. It can be seen from this figure that the previous condition is satisfied strictly which in turn implies that the considered example admits a unique global solution. The global minimum of the considered example together with its state and the associated multipliers are presented graphically in Figure 2. We see that the state constraints are active at one point, namely $\tilde{x} := (\frac{1}{2}, \frac{1}{2})$, and the corresponding multiplier is approximately given by

$$\bar{\mu}_h^a = 0.3386\,\delta_{\tilde{x}},$$

where $\delta_{\tilde{x}}$ is a Dirac measure at $\tilde{x}$. We can easily find a polygonal subdomain $K \subset\subset \Omega$ that contains the active point $\tilde{x}$ so that Assumption 1 holds. Consequently, we are expecting the bound $\sqrt{|\ln h|} h^{\frac{3}{2}-\frac{1}{s}}$, or equivalently $h^{1-\varepsilon}$ for arbitrarily small $\varepsilon > 0$, for the computed errors according to Theorem 3.1.

To deduce the convergence rates numerically, we compute the experimental order of convergence (EOC) which is defined as

$$\mathrm{EOC} := \frac{\log E(h_i) - \log E(h_{i-1})}{\log h_i - \log h_{i-1}},$$

where E is a given positive error functional and h_{i-1}, h_i are two consecutive mesh sizes. For our experiment, we consider the error functionals

$$\begin{aligned} E_{u_{L2}}(h_i) &:= \|\bar{u}_{ref} - \bar{u}_{h_i}\|_{L^2(\Omega)}, \\ E_{y_{H1}}(h_i) &:= \|\bar{y}_{ref} - \bar{y}_{h_i}\|_{H^1(\Omega)}, \\ E_{y_{L2}}(h_i) &:= \|\bar{y}_{ref} - \bar{y}_{h_i}\|_{L^2(\Omega)}, \\ E_{y_{L\infty}}(h_i) &:= \|\bar{y}_{ref} - \bar{y}_{h_i}\|_{L^\infty(\Omega)}, \end{aligned}$$

and denote the corresponding experimental orders of convergence by $\mathrm{EOC}_{u_{L2}}$, $\mathrm{EOC}_{y_{H1}}$, $\mathrm{EOC}_{y_{L2}}$ and $\mathrm{EOC}_{y_{L\infty}}$, respectively. Furthermore, we consider the sequence of mesh sizes $h_i = 2^{-i}\sqrt{2}$, for $i = 1, \dots, 9$. Since we don't have the exact solution at hand, we consider the numerical solution computed at mesh size $h_{10} = 2^{-10}\sqrt{2}$ to be the reference solution, that is, we define $\bar{u}_{ref} := \bar{u}_{h_{10}}$ and $\bar{y}_{ref} := \bar{y}_{h_{10}}$.

Figure 3 shows the values of our error functionals in dependence of h, and also illustrates the order of convergence. The computed values of the associated EOC are presented in Table 1.

From the numerical findings we see that as the mesh size h decreases the errors $E_{u_{L2}}(h)$ and $E_{y_{H1}}(h)$ behave like $O(h)$ which indicates that the convergence rate, namely $O(h^{1-\varepsilon})$ for arbitrarily small $\varepsilon > 0$, predicted in Theorem 3.1 is optimal. On the other hand, for $E_{y_{L2}}(h)$ and $E_{y_{L\infty}}(h)$ we see the behaviour $O(h^2)$ and $O(h^{1.6})$, respectively, from which we conclude that the error bounds for the discrete optimal state in the spaces $L^2(\Omega)$ and $L^\infty(\Omega)$ which are deduced from the error bound of the discrete optimal control via the Lipschitz continuity of the control-to-state map are not sharp.

In fact, the $O(h^2)$ behaviour of $E_{y_{L2}}(h)$ could be explained in the light of the work [26] where it was shown that for an elliptic control problem with finitely many pointwise inequality state constraints the error of the discrete optimal state in $L^2(\Omega)$ is of order h^{4-d} up to logarithmic factor in $d = 2$ or $d = 3$ space dimensions when the control problem is discretized by continuous, piecewise linear finite elements.

Table 1 EOC for the optimal control and its state.

Levels	$\mathrm{EOC}_{u_{L2}}$	$\mathrm{EOC}_{y_{H1}}$	$\mathrm{EOC}_{y_{L2}}$	$\mathrm{EOC}_{y_{L\infty}}$
1-2	1.186801	0.776001	1.124945	0.895581
2-3	1.187645	0.833842	1.464788	1.058334
3-4	1.078183	0.948273	1.708822	1.758387
4-5	1.027290	0.985352	1.794456	1.657899
5-6	1.016702	0.997996	1.831198	1.514376
6-7	1.033565	1.007509	1.864317	1.631527
7-8	1.101321	1.034964	1.936853	1.702538
8-9	1.338363	1.160921	2.210162	1.747642

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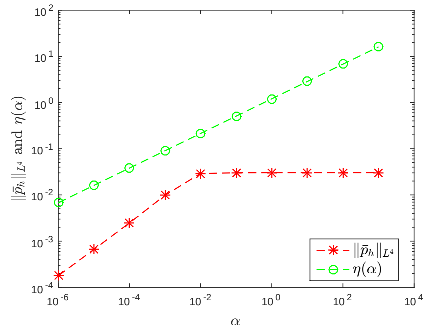
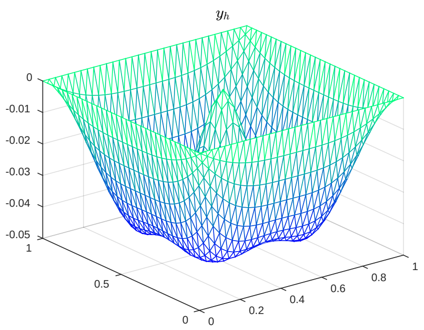
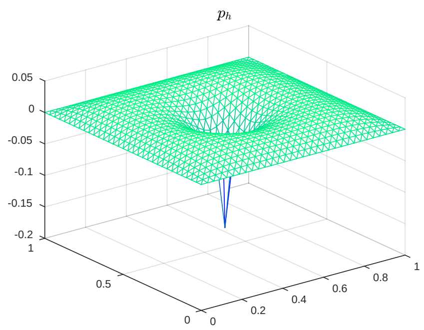


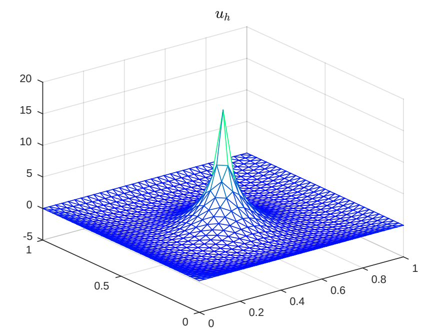
Figure 1 $\|\bar{p}_h\|_{L^4}$ and $\eta(\alpha)$ vs. α.



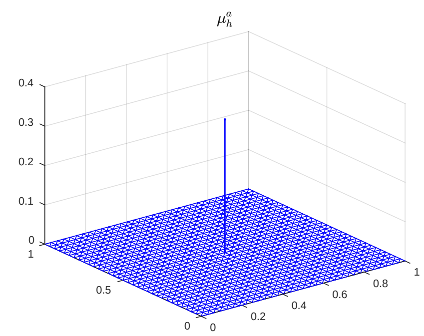
(a) The optimal state $\bar{y}_h$.



(b) The adjoint state $\bar{p}_h$.

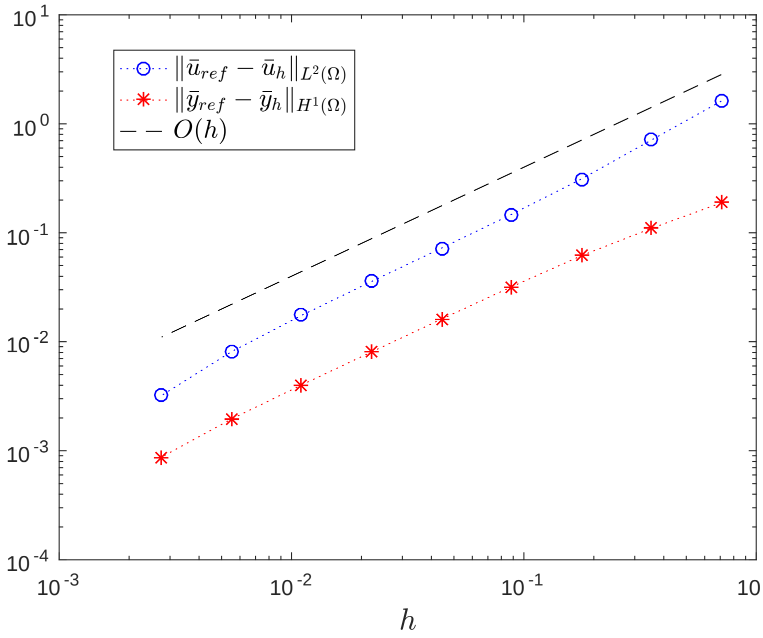


(c) The optimal control $\bar{u}_h$.

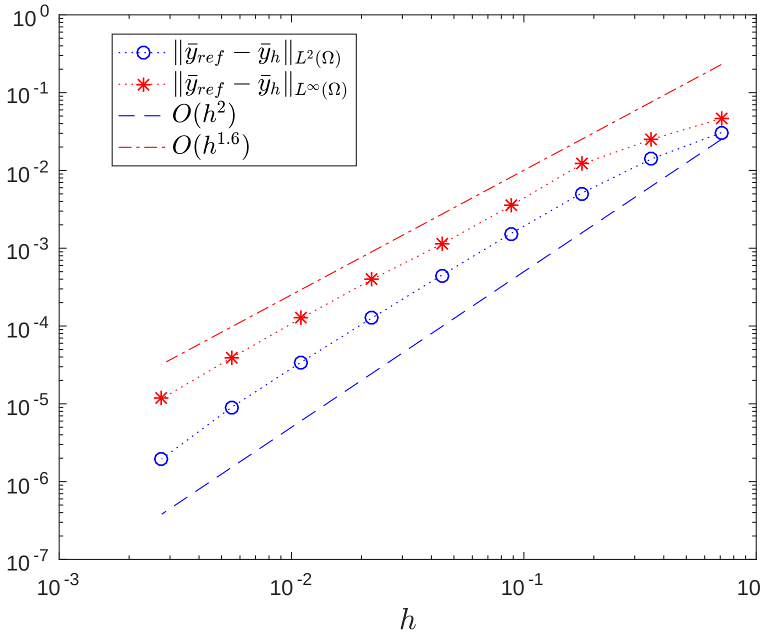


(d) The multiplier $\bar{\mu}_h^a = 0.3386\,\delta_{\tilde{x}}$, $\tilde{x} := (\frac{1}{2}, \frac{1}{2})$.

Figure 2 The unique global minimum together with its state and the associated multipliers.



(a) $E_{u_{L2}}(h)$ and $E_{y_{H1}}(h)$ v.s. h.



(b) $E_{y_{L2}}(h)$ and $E_{y_{L\infty}}(h)$ v.s. h.

Figure 3 Errors for the optimal control and its state versus the mesh size.

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