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**Further dense properties of the space of circle
diffeomorphisms with a Liouville rotation number**

Philipp Kunde

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PHILIPP KUNDE

Department of Mathematics, University of Hamburg, Hamburg, Germany

Abstract

In continuation of Matsumoto's paper [Ma12] we show that various subspaces are C^∞ -dense in the space of orientation-preserving C^∞ -diffeomorphisms of the circle with rotation number α , where $\alpha \in \mathbb{S}^1$ is any prescribed Liouville number. In particular, for every odometer \mathcal{O} of product type we prove the denseness of the subspace of diffeomorphisms which are orbit-equivalent to \mathcal{O} .

Key words: circle diffeomorphisms, orbit equivalence, rotation number, approximation by conjugation-method, odometer

AMS subject classification: 37E10 (primary), 37A20, 37C05, 37E45 (secondary).

Introduction

Let F be the group of orientation-preserving C^∞ -diffeomorphisms of the circle. Furthermore, for $\alpha \in \mathbb{S}^1$ we consider the subspace F_α of F consisting of all the C^∞ -diffeomorphisms of the circle with rotation number α . If α is irrational, for any $f \in F_\alpha$ there is a unique orientation-preserving homeomorphism H_f of the circle such that $f = H_f \circ R_\alpha \circ H_f^{-1}$ and $H_f(0) = 0$, where R_α denotes the rotation by α on \mathbb{S}^1 . J.-C. Yoccoz proved that the subspace O_α of F_α of all the diffeomorphisms, for which H_f are C^∞ -diffeomorphisms, is C^∞ -dense in F_α ([Yo95]).

In the following, let α be a Liouville number. By [He79], chapter IV, section 6, the unique f -invariant probability measure μ_f is given by $\mu_f = (H_f)_*m$, where m is the Lebesgue measure on \mathbb{S}^1 , and μ_f is either equivalent to m (then H_f maps any Lebesgue null set to a null set and H_f is called absolutely continuous) or singular to m (then H_f maps some Lebesgue null set to a conull set and H_f is called singular). In [Ma12] S. Matsumoto considered several subspaces of F_α according to the regularity of H_f :

- $G_{0,\text{sing}}$: H_f is singular and is not d -Hölder for any $d \in (0, 1)$.
- $G_{0,\text{ac}}$: H_f is absolutely continuous and is not d -Hölder for any $d \in (0, 1)$.
- For $\beta \in (0, 1)$ G_β : H_f is bi- β -Hölder, but is not d -Hölder for any $d \in (\beta, 1)$.
- $G_{1,\text{sing}}$: H_f is singular and is bi- d -Hölder for any $d \in (0, 1)$.
- $G_{1,\text{ac}}$: H_f is absolutely continuous and is d -Hölder for any $d \in (0, 1)$, but is not bi-Lipschitz.
- For $k \in \mathbb{N}$ G_k : H_f is a C^k -diffeomorphism, but is not a C^{k+1} -diffeomorphism.

Then Matsumoto proved that $G_{0,\text{sing}}$ is C^∞ -dense in F_α in [Ma13]. In [Ma12], Theorem 1, the C^∞ -denseness of all the other spaces is shown. In this paper we examine the subsequent subspaces of G_β :

Theorem 1. *For any Liouville number α and for any $\beta \in (0, 1)$ the subspaces*

- $G_{\beta,\text{sing}}$: H_f is singular, bi- β -Hölder, but is not d -Hölder for any $d \in (\beta, 1)$
- $G_{\beta,\text{ac}}$: H_f is absolutely continuous, bi- β -Hölder, but is not d -Hölder for any $d \in (\beta, 1)$

are C^∞ -dense in F_α .

This statement was conjectured in [Ma12], Remark 1.6, but not pursued.

In the second part of this paper we study odometers of product type.

Definition 1. Let $(d_k)_{k \in \mathbb{N}}$ be a sequence of positive integers. We consider the compact metrisable space $X = \prod_{k \in \mathbb{N}} \{0, \dots, d_k - 1\}$ with Borel algebra \mathcal{B} on it. For $x = (x_k)_{k \in \mathbb{N}} \in X$ we set $r(x) := \inf \{k : x_k < d_k - 1\}$. Hereby, we define the transformation $T : X \rightarrow X$, $Tx = (y_k)_{k \in \mathbb{N}}$, where

$$y_k = \begin{cases} 0 & \text{for } k < r(x) \\ x_k + 1 & \text{for } k = r(x) \\ x_k & \text{for } k > r(x) \end{cases}$$

- Let μ be a continuous measure on (X, \mathcal{B}) which is ergodic and quasi-invariant with respect to T . Then (X, \mathcal{B}, μ, T) is called a measured odometer and we will denote it by $\mathcal{O}((d_k)_{k \in \mathbb{N}}, \mu)$.
- Let $\mathcal{O}((d_k)_{k \in \mathbb{N}})$ be a measured odometer and assume that for every $k \in \mathbb{N}$ ν_k is a probability measure on $\{0, 1, \dots, d_k - 1\}$ such that the probability of every digit is positive and the product measure $\nu = \prod_{k \in \mathbb{N}} \nu_k$ is non-atomic on $\mathcal{O}((d_k)_{k \in \mathbb{N}})$. Then ν is ergodic and quasi-invariant under T . We call $\mathcal{O}((d_k)_{k \in \mathbb{N}}, \nu)$ an odometer of product type and denote it also by $\mathcal{O}((d_k)_{k \in \mathbb{N}}, \{\nu_k\})$. We also use the notation $\nu_k^{(i)} = \nu_k(\{i - 1\})$.

Moreover, we recall the notion of orbit equivalence (also referred to as “weak-equivalence” or “Dye-equivalence”): The non-singular systems $(X_1, \mathcal{B}_1, \mu_1, T_1)$ and $(X_2, \mathcal{B}_2, \mu_2, T_2)$ are orbit equivalent if there is an isomorphism ψ of $(X_1, \mathcal{B}_1, \mu_1)$ onto $(X_2, \mathcal{B}_2, \mu_2)$ such that $\psi(\{T_1^i x\}_{i \in \mathbb{Z}}) = \{T_2^i \psi(x)\}_{i \in \mathbb{Z}}$ almost everywhere. Y. Katznelson proved that for every odometer of product type the set of C^∞ -diffeomorphisms of the circle, which are orbit equivalent to this odometer, is C^∞ -dense in the set of all C^∞ -diffeomorphisms with irrational rotation number ([Ka79], Theorem 2.7). We obtain such a statement in the restricted space F_α for any Liouville number α :

Theorem 2. *Let $\alpha \in \mathbb{S}^1$ be a Liouville number. For every odometer of product type $\mathcal{O} = \mathcal{O}((d_k)_{k \in \mathbb{N}}, \{\nu_k\})$ the set of C^∞ -diffeomorphisms of the circle which are orbit-equivalent to \mathcal{O} is C^∞ -dense in F_α .*

We point out that it is still an open problem to find a smooth realization of an odometer (cf. [FK04], Problem 7.10).

1 Proof of Theorem 1

The proof bases upon the “approximation by conjugation”-method developed by D. Anosov and A. Katok ([AK70]): We construct the desired diffeomorphisms as limits of conjugates $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$, where $\alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} \in \mathbb{Q}$, $(p_{n+1}, q_{n+1}) = 1$ and $H_n = H_{n-1} \circ h_n$ with an orientation-preserving circle diffeomorphism h_n satisfying $h_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ h_n$.

1.1 Denseness of $G_{\beta, \text{sing}}$

Fix $\beta \in (0, 1)$. We use a C^∞ -function $\psi : \mathbb{R} \rightarrow [0, 1]$ satisfying $\psi((-\infty, 0]) = 0$, $\psi([\frac{1}{4}, \infty)) = 1$ and ψ is strictly monotone increasing on $[0, \frac{1}{4}]$. For any $t \in (0, 1)$ we define the orientation-preserving diffeomorphism \hat{h}_t of the circle as follows

$$\hat{h}_t(x) = \begin{cases} (1 - \psi(t^{-1}x))tx + \psi(t^{-1}x)t^{-1}x & \text{if } x \in [0, \frac{t}{4}] \\ t^{-1}x & \text{if } x \in [\frac{t}{4}, \frac{t}{1+t}] \\ (1 - \psi(t^{-1}x - \frac{1}{t+1}))t^{-1}x + \psi(t^{-1}x - \frac{1}{t+1}) \cdot (t(x-1) + 1) & \text{if } x \in [\frac{t}{1+t}, \frac{t}{1+t} + \frac{t}{4}] \\ t \cdot (x-1) + 1 & \text{if } x \in [\frac{t}{1+t} + \frac{t}{4}, 1] \end{cases}$$

See figure 1 for a visualisation of such a map. In particular, \hat{h}_t is a smooth joining of the two

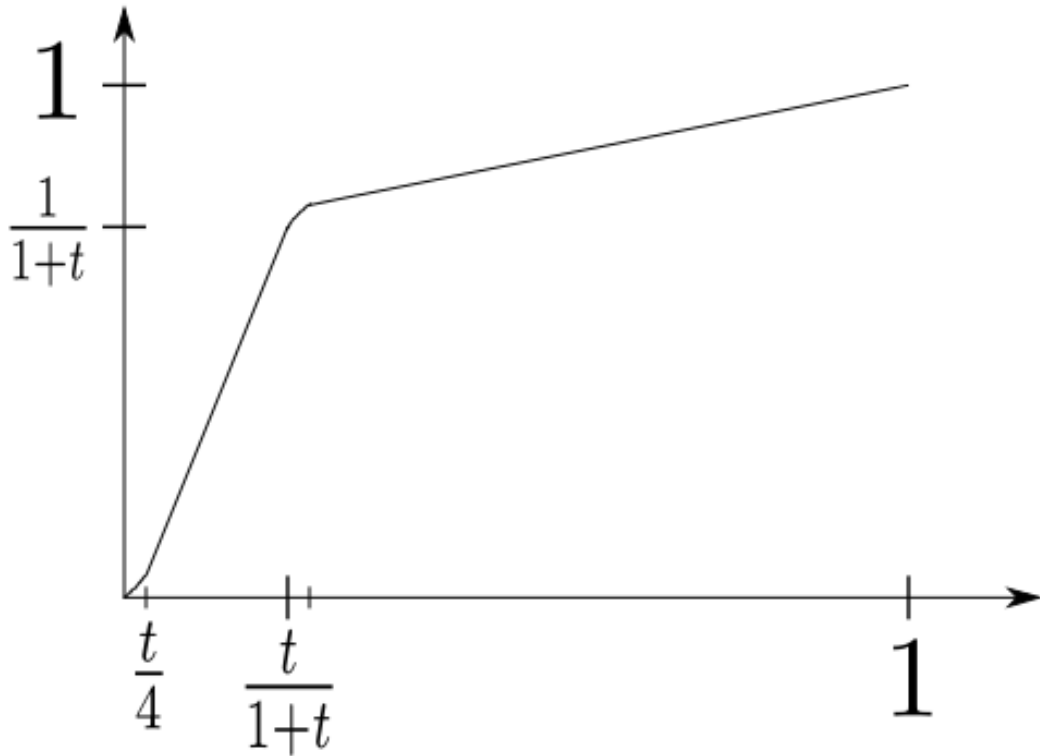


Figure 1: Qualitative shape of the function \hat{h}_t .

affine functions $x \mapsto t^{-1}x$ and $x \mapsto t \cdot (x-1) + 1$, which coincide at $x = \frac{t}{1+t}$. Moreover, we observe for any $r \in \mathbb{N}$

$$(1) \quad |||\hat{h}_t|||_r \leq C_r \cdot t^{-m(r)}$$

with some constant $C(r) > 0$ and an integer $m(r) \geq r - 1$ which are independent of t . The notation $|||\cdot|||_r$ is the same as in [Ma12].

We present step n of the inductive process of our construction. Hence, we have already defined the orientation-preserving diffeomorphism $H_{n-1} = h_1 \circ \dots \circ h_{n-1}$ as well as the numbers $\alpha_{n-1} = \frac{p_{n-1}}{q_{n-1}} \in \mathbb{Q}$, $t_{n-1} = \frac{c_{n-1}}{d_{n-1}} \in \mathbb{Q}$ and $Q_{n-1} \in \mathbb{N}$. We put

$$(2) \quad Q_n = 12 \cdot d_{n-1} \cdot (d_{n-1} + c_{n-1}) \cdot Q_{n-1} \cdot q_n$$

and

$$(3) \quad t_n = Q_n^{1-\beta_n^{-1}},$$

where the numbers q_n and $\beta_n \in [\beta, \beta_{n-1}]$ will be determined later (see Lemma 1.5). In particular, part 4 of Lemma 1.5 shows $t_n \in \mathbb{Q}$. Note that

$$(4) \quad Q_n^{-1} = (t_n Q_n^{-1})^{\beta_n}.$$

Let h_n be the lift of \hat{h}_{t_n} by the cyclic Q_n -fold covering map π_{Q_n} such that $\text{Fix}(h_n) \neq \emptyset$. In particular, we have $h_n \circ R_{\frac{1}{Q_n}} = R_{\frac{1}{Q_n}} \circ h_n$. Since Q_n is a multiple of q_n , this yields $h_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ h_n$ as well as

$$\|H_n^{-1} - H_{n-1}^{-1}\|_0 = \|(h_n^{-1} - \text{id}) \circ H_{n-1}^{-1}\|_0 = \|h_n^{-1} - \text{id}\|_0 \leq Q_n^{-1}.$$

Then we obtain

$$\|H_{n+k}^{-1} - H_{n-1}^{-1}\|_0 \leq \sum_{l=0}^k \|H_{n+l}^{-1} - H_{n+l-1}^{-1}\|_0 \leq \sum_{l=0}^k Q_{n+l}^{-1}.$$

Since $\sum_{n=1}^{\infty} Q_n^{-1} < \infty$ by Lemma 1.5, $(H_n^{-1})_{n \in \mathbb{N}}$ is a Cauchy sequence. This shows the uniform convergence of $(H_n^{-1})_{n \in \mathbb{N}}$ to a continuous map $H^{-1} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$. Additionally, H^{-1} is monotone as a uniform limit of homeomorphisms. Due to Lemma 1.6 the sequence of C^∞ -diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ converges to a diffeomorphism f in the Diff^∞ -topology. Hereby, we conclude $H^{-1} \circ f = R_\alpha \circ H^{-1}$. Hence, H^{-1} is a homeomorphism.

Moreover, we introduce the intervals

$$\hat{I}_{t_n} = \left[\frac{1}{1+t_n} + \frac{t_n^2}{4}, 1 \right], \quad \hat{J}_{t_n} = \left[\frac{t_n}{1+t_n} + \frac{t_n}{4}, 1 \right] \quad \text{and} \quad \hat{K}_{t_n} = \left[\frac{t_n}{4}, \frac{7t_n}{12} \right].$$

By construction $\hat{h}_{t_n}|_{\hat{J}_{t_n}}$ is an affine transformation with slope t_n . Due to $t_n < 1$ we have $\hat{I}_{t_n} \subset \hat{J}_{t_n}$. Additionally, we observe $\hat{h}_{t_n}(\hat{J}_{t_n}) = \hat{I}_{t_n}$ and $\hat{K}_{t_n} \subset \left[\frac{t_n}{4}, \frac{t_n}{1+t_n} \right]$.

Finally, we define $I_n = \pi_{Q_n}^{-1}(\hat{I}_{t_n})$, $J_n = \pi_{Q_n}^{-1}(\hat{J}_{t_n})$ as well as $K_n = \pi_{Q_n}^{-1}(\hat{K}_{t_n})$. By direct computation, $m(J_n) = \frac{1}{1+t_n} - \frac{t_n}{4} \geq 1 - 2t_n$, $m(I_n) = \frac{t_n}{1+t_n} - \frac{t_n^2}{4} \leq t_n$ (where m stands for the Lebesgue measure) and every component of K_n has length $\frac{t_n}{3Q_n}$.

Now we are able to prove the singularity of H by an approach similar to [Ma12], section 3:

Lemma 1.1. *H is singular.*

Proof. Let $C_n = \bigcap_{i=1}^n J_i$ and $C = \bigcap_{i=1}^{\infty} J_i$. By construction of the number Q_n any component of J_{n-1} is a $(Q_n^{-1}\mathbb{Z})/\mathbb{Z}$ -interval. Then

$$m(C_n) \geq \prod_{i=1}^n (1 - 2t_i).$$

Note that $\prod_{n=1}^{\infty} (1 - a_n)$, $a_n \geq 0$, converges to a positive number if and only if $\sum_{n=1}^{\infty} a_n$ converges ([Kn64], section 28, Theorem 4). Since

$$\sum_{n=1}^{\infty} t_n = \sum_{n=1}^{\infty} Q_n^{1-\beta_n^{-1}} \leq \sum_{n=1}^{\infty} Q_n^{1-\beta_n^{-1}} < \infty$$

by part 3 of Lemma 1.5, we obtain $m(C) > 0$. This yields

$$\mu_f(H(C)) = (H_*m)(H(C)) = m(C) > 0.$$

By construction of h_j with the aid of the Q_j -fold covering map $(Q_{n+1}^{-1}\mathbb{Z})/\mathbb{Z}$ is pointwise fixed under h_j , $j > n$. Since any component of J_n is a $(Q_{n+1}^{-1}\mathbb{Z})/\mathbb{Z}$ -interval, we get $h_j(J_n) = J_n$ for $j > n$. Then we obtain for any $j > n$:

$$H_j(J_n) = H_n(J_n) = H_{n-1}h_n(J_n) = H_{n-1}(I_n),$$

which yields $J_n = H_j^{-1}H_{n-1}(I_n)$, where $H_0 = \text{id}$. This shows that the uniform limit H^{-1} of H_j^{-1} satisfies $J_n = H^{-1}(H_{n-1}(I_n))$ for any $n \in \mathbb{N}$. Hereby, we observe

$$H(C_n) = H\left(\bigcap_{i=1}^n J_i\right) = \bigcap_{i=1}^n H(J_i) = \bigcap_{i=1}^n H_{i-1}(I_i).$$

In order to have $H_{i-1}(A) \subset H_{i-2}(I_{i-1})$ a set $A \subset \mathbb{S}^1$ has to satisfy $h_{i-1}(A) \subset I_{i-1}$ which implies the condition $A \subset J_{i-1}$. Since $m(I_i) \leq t_i$ and the slope of $h_{i-1}|_{J_{i-1}}$ is equal to t_{i-1} , this yields $m(H(C_n)) \leq \prod_{i=1}^{n-1} t_i$ which converges to 0 as $n \rightarrow \infty$. Therefore, $m(H(C)) = 0$.

Finally, we note that μ_f is not equivalent to m because $\mu_f(H(C)) > 0$ and $m(H(C)) = 0$. Hence, H is a singular map. \square

In the next steps, we examine the Hölder continuity of H :

Lemma 1.2. *H is not d -Hölder for any $d \in (\beta, 1)$.*

Proof. For any component \tilde{K}_i of K_i there is a component \tilde{K}_{i+1} of K_{i+1} such that $\tilde{K}_{i+1} \subset \tilde{K}_i$. This proves the existence of a component \tilde{K}_n which is contained in $\bigcap_{i=1}^n K_i$. By construction, $H_n|_{\tilde{K}_n}$ is an affine transformation of slope $t_1^{-1} \cdots t_n^{-1}$. In the following, we denote $\tilde{K}_n = [x', y']$. In particular, we have $|y' - x'| = 3^{-1}t_n Q_n^{-1}$. Moreover, we define $x, y \in \mathbb{S}^1$ by $H^{(n+1)}(x) = x'$ as well as $H^{(n+1)}(y) = y'$ using the notation $H^{(n+1)} = H_n^{-1}H$. Its inverse $(H^{(n+1)})^{-1}$ is the uniform limit of $h_{n+m}^{-1} \circ \cdots \circ h_{n+2}^{-1} \circ h_{n+1}^{-1}$ as $m \rightarrow \infty$. Since $\sum_{i=n+1}^{\infty} Q_i^{-1} \leq 3^{-1}t_n Q_n^{-1}$ for any $n \in \mathbb{N}$ by Lemma 1.5 and the maps h_i are Q_i^{-1} -cyclic, we have

$$\begin{aligned} |x - y| &= \left| (H^{(n+1)})^{-1}(x') - (H^{(n+1)})^{-1}(y') \right| \\ &\leq \left| (H^{(n+1)})^{-1}(x') - x' \right| + |x' - y'| + \left| y' - (H^{(n+1)})^{-1}(y') \right| \\ &\leq \sum_{i=n+1}^{\infty} Q_i^{-1} + |x' - y'| + \sum_{i=n+1}^{\infty} Q_i^{-1} \\ &\leq 3^{-1}t_n Q_n^{-1} + |x' - y'| + 3^{-1}t_n Q_n^{-1} \\ &= 3|x' - y'|. \end{aligned}$$

With the aid of $|y' - x'| = 3^{-1}t_n Q_n^{-1}$ we estimate

$$\begin{aligned} |H(x) - H(y)| &= |H_n(x') - H_n(y')| = t_1^{-1} \cdots t_n^{-1} \cdot |x' - y'| = t_1^{-1} \cdots t_n^{-1} \cdot 3^{-1}t_n Q_n^{-1} \\ &= 3^{-1}t_1^{-1} \cdots t_{n-1}^{-1} \cdot Q_n^{-1} \end{aligned}$$

as well as

$$|x - y|^d \leq 3^d |x' - y'|^d = 3^d (3^{-1}t_n Q_n^{-1})^d = t_n^d Q_n^{-d}.$$

Using equation 3 both estimates together yield in case of $\beta_n < d$ (which is fulfilled for sufficiently large $n \in \mathbb{N}$ due to $\beta_n \rightarrow \beta < d$):

$$\begin{aligned} \frac{|H(x) - H(y)|}{|x - y|^d} &\geq 3^{-1}t_1^{-1} \cdots t_{n-1}^{-1} \cdot Q_n^{d-1} \cdot t_n^{-d} \\ &= 3^{-1}t_1^{-1} \cdots t_{n-1}^{-1} \cdot Q_n^{d-1} \cdot Q_n^{\beta_n^{-1}d-d} \\ &= 3^{-1}t_1^{-1} \cdots t_{n-1}^{-1} \cdot Q_n^{\beta_n^{-1}d-1} \\ &\geq 3^{-1}t_1^{-1} \cdots t_{n-1}^{-1}. \end{aligned}$$

Since this expression can be arbitrarily large, we conclude that H cannot be d -Hölder for any $d \in (\beta, 1)$. \square

Lemma 1.3. H^{-1} is β -Hölder.

Proof. For any pair of $x, y \in \mathbb{S}^1$, $x \neq y$, there is $n \in \mathbb{N}$ such that $t_{n+1}Q_{n+1}^{-1} \leq |x - y| \leq t_n Q_n^{-1}$. Since the Lipschitz constant of h_i^{-1} is t_i^{-1} , we have

$$\begin{aligned} |H_n^{-1}(x) - H_n^{-1}(y)| &\leq t_1^{-1} \cdots t_n^{-1} \cdot |x - y| = t_1^{-1} \cdots t_n^{-1} \cdot |x - y|^{1-\beta} \cdot |x - y|^\beta \\ &\leq t_1^{-1} \cdots t_n^{-1} \cdot (t_n Q_n^{-1})^{1-\beta} \cdot |x - y|^\beta \\ &= t_1^{-1} \cdots t_{n-1}^{-1} \cdot (Q_n^{1-\beta_n^{-1}})^{-\beta} \cdot Q_n^{\beta-1} \cdot |x - y|^\beta \\ &= t_1^{-1} \cdots t_{n-1}^{-1} \cdot Q_n^{-1+\beta\beta_n^{-1}} \cdot |x - y|^\beta \end{aligned}$$

By Lemma 1.5 this shows

$$(5) \quad |H_n^{-1}(x) - H_n^{-1}(y)| \leq |x - y|^\beta.$$

There are two possible cases

- Case 1: $|H_n^{-1}(x) - H_n^{-1}(y)| \geq Q_{n+1}^{-1}$
Since h_{n+1}^{-1} is Q_{n+1}^{-1} -cyclic, we get in this case $|H_{n+1}^{-1}(x) - H_{n+1}^{-1}(y)| \leq 2 |H_n^{-1}(x) - H_n^{-1}(y)|$. With the aid of Lemma 1.5 we see that the numbers Q_i , $i > n + 1$, grow fast enough such that

$$|H^{-1}(x) - H^{-1}(y)| \leq 3 |H_n^{-1}(x) - H_n^{-1}(y)| \leq 3 |x - y|^\beta$$

using equation (5) in the last step.

- Case 2: $|H_n^{-1}(x) - H_n^{-1}(y)| < Q_{n+1}^{-1}$
Once again, we exploit the fact that h_{n+1}^{-1} is Q_{n+1}^{-1} -cyclic. In the case under consideration, this yields $|H_{n+1}^{-1}(x) - H_{n+1}^{-1}(y)| \leq Q_{n+1}^{-1}$. With the aid of equation 4 we get

$$|H^{-1}(x) - H^{-1}(y)| \leq 2Q_{n+1}^{-1} \leq 2(t_{n+1}Q_{n+1}^{-1})^\beta \leq 2|x - y|^\beta$$

Hence, H^{-1} is β -Hölder. \square

Lemma 1.4. H is β -Hölder.

Proof. As above, for any pair of $x, y \in \mathbb{S}^1$, $x \neq y$, there is $n \in \mathbb{N}$ such that $t_{n+1}Q_{n+1}^{-1} \leq |x - y| \leq t_n Q_n^{-1}$. Let $x' = H^{(n+2)}(x)$ and $y' = H^{(n+2)}(y)$. Recall $H^{(n+2)} = H_{n+1}^{-1}H$. Since H is the uniform limit of H_n and the maps h_i are Q_i^{-1} -cyclic and the numbers Q_i , $i > n + 1$, are sufficiently large due to Lemma 1.5, we have $|x' - y'| \leq 2|x - y|$. Once again, we have to examine two cases:

- Case 1: $|x' - y'| \geq Q_{n+1}^{-1}$
Since h_{n+1} is Q_{n+1}^{-1} -cyclic, we get in this case $|x'' - y''| \leq 2|x' - y'|$ for $x'' = h_{n+1}(x')$ and $y'' = h_{n+1}(y')$. Since the Lipschitz constant of h_i is t_i^{-1} , we obtain by the same calculations as in the first case of the previous Lemma

$$\begin{aligned} |H(x) - H(y)| &= |H_n(x'') - H_n(y'')| \leq t_1^{-1} \cdots t_n^{-1} \cdot |x'' - y''| \\ &\leq t_1^{-1} \cdots t_n^{-1} \cdot 2|x' - y'| \\ &\leq t_1^{-1} \cdots t_n^{-1} \cdot 4|x - y| \\ &\leq 4|x - y|^\beta. \end{aligned}$$

- Case 2: $|x' - y'| < Q_{n+1}^{-1}$
Since h_{n+1} is Q_{n+1}^{-1} -cyclic, we have

$$|x'' - y''| = |h_{n+1}(x') - h_{n+1}(y')| \leq Q_{n+1}^{-1} = (t_{n+1}Q_{n+1}^{-1})^{\beta_{n+1}} \leq |x - y|^{\beta_{n+1}}$$

Hereby, we conclude

$$\begin{aligned} |H(x) - H(y)| &= |H_n(x'') - H_n(y'')| \\ &\leq t_1^{-1} \cdots t_n^{-1} \cdot |x'' - y''|^{1-\beta_{n+1}} \cdot |x'' - y''|^{\beta_{n+1}} \\ &\leq t_1^{-1} \cdots t_n^{-1} \cdot (Q_{n+1}^{-1})^{1-\beta_{n+1}} \cdot |x'' - y''|^{\beta_{n+1}} \\ &\leq |x - y|^\beta \end{aligned}$$

using Lemma 1.5, part 4, in the last step.

Hence, H is β -Hölder. \square

Finally, we want to prove convergence of the sequence $(f_n)_{n \in \mathbb{N}}$ in $\text{Diff}^\infty(\mathbb{S}^1)$. For this purpose, we deduce the subsequent statement.

Lemma 1.5. Let $(l_n)_{n \in \mathbb{N}}$ be a strictly increasing sequence of natural numbers with $\sum_{n=1}^\infty \frac{1}{l_n} < \infty$ and C_{l_n} be the constants from [Ma12], Lemma 2.4. For any Liouvillean number α there are sequences $\alpha_n = \frac{p_n}{q_n}$ of rational numbers and $(\beta_n)_{n \in \mathbb{N}}$ of real numbers, such that $\beta_n \searrow \beta$ and the following conditions are satisfied:

1. For every $n \in \mathbb{N}$:

$$|\alpha - \alpha_n| < \frac{1}{2^{n+1} \cdot l_n \cdot C_{l_n} \cdot \|H_n\|_{l_n+1}^{l_n+1}}.$$

2. For every $n \in \mathbb{N}$:

$$\sum_{i=n+1}^{\infty} Q_i^{-1} \leq 3^{-1} t_n Q_n^{-1}$$

3. For every $n \in \mathbb{N}$:

$$t_1^{-1} \cdots t_{n-1}^{-1} \cdot Q_n^{-1+\beta\beta_n^{-1}} \leq 1 \quad \text{and} \quad Q_n^{1-\beta_n^{-1}} \leq \frac{1}{n^2}.$$

4. $t_n \in \mathbb{Q}$ for every $n \in \mathbb{N}$.

Proof. Since the numbers t_i , $i < n$, are independent of q_n and $1 > \beta_{n-1} > \beta$, we can demand the number q_n to be sufficiently large such that

$$(6) \quad t_1^{-1} \cdots t_{n-1}^{-1} \cdot q_n^{\frac{1}{2} \left(1 - \frac{\beta_{n-1}}{\beta}\right)} \leq 1$$

as well as

$$q_n^{1-\beta_n^{-1}} \leq \frac{1}{n^2}.$$

Additionally, we can satisfy the second property of the Lemma by choosing the numbers q_i sufficiently large in each step.

By equations (1) and (3) we have

$$\begin{aligned} \|h_n\|_r &\leq C_r \cdot Q_n^{r-1} \cdot \left(Q_n^{\beta_n^{-1}-1}\right)^{m(r)} \leq C_r \cdot Q_n^{\beta_n^{-1} \cdot m(r)} \\ &= C_r \cdot (12 \cdot d_{n-1} \cdot (d_{n-1} + c_{n-1}) \cdot Q_{n-1})^{\beta_n^{-1} m(r)} \cdot q_n^{\beta_n^{-1} m(r)} \\ &\leq C_{r,n-1} \cdot q_n^{\beta_n^{-1} m(r)} \end{aligned}$$

for any $r \in \mathbb{N}$ due to the condition $\beta_n \geq \beta$. Using [Mal2], Lemma 2.3, we obtain

$$\|H_n\|_r = \|H_{n-1} \circ h_n\|_r \leq \tilde{C}_r \cdot \|H_{n-1}\|_r^r \cdot \|h_n\|_r^r \leq \hat{C}_{r,n-1} \cdot q_n^{\beta_n^{-1} m(r) \cdot r},$$

where $\hat{C}_{r,n-1}$ is a constant independent of q_n . In particular, we can demand $q_n \geq \hat{C}_{l_n+1,n-1}$. Then we get

$$\|H_n\|_{l_n+1} \leq q_n^{\beta_n^{-1} m(l_n+1) \cdot (l_n+1)+1} \leq q_n^{A_n}$$

using the notation $A_n = \lceil \beta^{-1} m(l_n+1) \cdot (l_n+1) + 1 \rceil$.

Since α is a Liouvillean number, we find a rational number $\alpha_n = \frac{p_n}{q_n}$, p_n, q_n relatively prime, satisfying the above restrictions and

$$|\alpha - \alpha_n| = \left| \alpha - \frac{p_n}{q_n} \right| < \frac{|\alpha - \alpha_{n-1}|}{2^{n+1} \cdot l_n \cdot C_{l_n} \cdot q_n^{A_n \cdot (l_n+1)}} \leq \frac{1}{2^{n+1} \cdot l_n \cdot C_{l_n} \cdot \|H_n\|_{l_n+1}^{l_n+1}}.$$

After the number q_n is determined with respect to these restrictions we can choose a number $\beta_n \in \left[\beta + \frac{\beta_{n-1}-\beta}{2}, \beta_{n-1} \right)$ such that $t_n = Q_n^{1-\beta_n^{-1}} \in \mathbb{Q}$. Then we have

$$\begin{aligned} -1 + \beta\beta_n^{-1} &= \beta_n^{-1} \cdot (-\beta_n + \beta) \leq \beta^{-1} \cdot \left(- \left(\beta + \frac{\beta_{n-1}-\beta}{2} \right) + \beta \right) \\ &= \beta^{-1} \frac{\beta - \beta_{n-1}}{2} = \frac{1}{2} \left(1 - \frac{\beta_{n-1}}{\beta} \right) < 0. \end{aligned}$$

By condition 6 this yields

$$t_1^{-1} \cdots t_{n-1}^{-1} \cdot Q_n^{-1+\beta\beta_n^{-1}} < t_1^{-1} \cdots t_{n-1}^{-1} \cdot q_n^{-1+\beta\beta_n^{-1}} \leq t_1^{-1} \cdots t_{n-1}^{-1} \cdot q_n^{\frac{1}{2}\left(1-\frac{\beta_{n-1}}{\beta}\right)} \leq 1.$$

□

The previous Lemma shows that the requirements of the following convergence result deduced in [Kul16], Lemma 5.8, are fulfilled.

Lemma 1.6. *Let $\varepsilon > 0$ be arbitrary and $(l_n)_{n \in \mathbb{N}}$ be a strictly increasing sequence of natural numbers satisfying $\sum_{n=1}^{\infty} \frac{1}{l_n} < \varepsilon$. Furthermore, we assume that in our constructions the following conditions are fulfilled:*

$$|\alpha - \alpha_1| < \varepsilon \quad \text{and} \quad |\alpha - \alpha_n| \leq \frac{1}{2 \cdot l_n \cdot C_{l_n} \cdot \|H_n\|^{l_n+1}} \quad \text{for every } n \in \mathbb{N},$$

where C_{l_n} are the constants from [Ma12], Lemma 2.4. Then the sequence of diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ converges in the Diff^∞ -topology to a smooth diffeomorphism f , for which $d_\infty(f, R_\alpha) < 3 \cdot \varepsilon$ holds.

Hence, the rotation R_α is contained in the C^∞ -closure of $G_{\beta, \text{sing}}$. Since this closure is invariant under conjugation by any $h \in F$ and O_α is C^∞ -dense in F_α ([Yo95]), we obtain the C^∞ -denseness of $G_{\beta, \text{sing}}$ in F_α by the same reasoning as in [Ma12], section 2.1.

1.2 Denseness of $G_{\beta, \text{ac}}$

Fix $\beta \in (0, 1)$. We slightly modify the construction of the map \hat{h}_t from the previous chapter. Once again, we use a C^∞ -map $\psi : \mathbb{R} \rightarrow [0, 1]$ satisfying $\psi((-\infty, 0]) = 0$, $\psi([\frac{1}{4}, \infty)) = 1$ and ψ is strictly monotone increasing on $[0, \frac{1}{4}]$. Then, for any $t \in (0, 1)$ we define the orientation-preserving diffeomorphism $\check{h}_t : [0, 1] \rightarrow [0, 1]$ as follows

$$\check{h}_t(x) = \begin{cases} (1 - \psi(t^{-1}x))x + \psi(t^{-1}x)t^{-1}x & \text{if } x \in [0, \frac{t}{4}] \\ t^{-1}x & \text{if } x \in [\frac{t}{4}, \frac{t}{1+t}] \\ (1 - \psi(t^{-1}x - \frac{1}{t+1}))t^{-1}x + \psi(t^{-1}x - \frac{1}{t+1}) \cdot (t(x-1) + 1) & \text{if } x \in [\frac{t}{1+t}, \frac{t}{1+t} + \frac{t}{4}] \\ t \cdot (x-1) + 1 & \text{if } x \in [\frac{t}{1+t} + \frac{t}{4}, 1 - \frac{t}{4}] \\ (1 - \psi(t^{-1}(x-1 + \frac{t}{4}))) \cdot (t(x-1) + 1) + \psi(t^{-1}(x-1 + \frac{t}{4}))x & \text{if } x \in [1 - \frac{t}{4}, 1] \end{cases}$$

Note that \check{h}_t coincides with the identity in a neighbourhood of the boundary. Using the maps $C_n : [0, \frac{1}{2^{n+1}}] \rightarrow [0, 1]$, $C_n(x) = 2^{n+1} \cdot x$, we construct the orientation-preserving circle diffeomorphism h_{t_n} as follows:

$$h_{t_n}(x) = \begin{cases} C_n^{-1} \circ \check{h}_{t_n} \circ C_n(x) & \text{if } x \in [0, \frac{1}{2^{n+1}}] \\ x & \text{if } x \in [\frac{1}{2^{n+1}}, 1] \end{cases},$$

where we define the numbers $Q_n = 2^n \cdot 12 \cdot d_{n-1} \cdot (d_{n-1} + c_{n-1}) \cdot Q_{n-1} \cdot q_n$ and t_n as in the previous section. This time, we demand the additional requirement

$$(7) \quad t_n \leq 2^{-(n+2)}$$

which can be satisfied by choosing q_n sufficiently large in the proof of Lemma 1.5.

Let h_n be the lift of h_{t_n} by the cyclic Q_n -fold covering map π_{Q_n} such that $\text{Fix}(h_n) \neq \emptyset$. By the same reasoning as above, the sequence $(H_n^{-1})_{n \in \mathbb{N}}$ converges to a homeomorphism H^{-1} .

First of all, we prove the absolute continuity of H by the same method as in [Ma12], section 4:

Lemma 1.7. *H is absolutely continuous.*

Proof. We introduce the sets

$$\hat{L}_n = [2^{-(n+1)}, 1] \quad \text{and} \quad L_n = \pi_{Q_n}^{-1}(\hat{L}_n).$$

According to our construction h_n is the identity on L_n . Let $X = \bigcap_{n=1}^{\infty} L_n$. Then we have

$$m(X) \geq 1 - \sum_{n=1}^{\infty} m(\mathbb{S}^1 \setminus L_n) = 1 - \sum_{n=1}^{\infty} 2^{-(n+1)} = \frac{1}{2}.$$

Since H is the identity on the positive measure set X , we have for any Borel set B $\mu_f(B \cap X) = m(B \cap X)$ and $\mu_f(X) = m(X) > 0$.

Assume that μ_f is not equivalent to m . Then μ_f is singular to m and there is a Borel set $B \subset \mathbb{S}^1$ such that $m(B) = 1$ and $\mu_f(B) = 0$. But then we obtain the contradiction $m(B \cap X) = m(X) > 0$ but $\mu_f(B \cap X) \leq \mu_f(B) = 0$.

Hence, H is absolutely continuous. \square

We start to examine the Hölder-continuity of H .

Lemma 1.8. *H is not d-Hölder for any $d \in (\beta, 1)$.*

Proof. Let

$$K_{t_n} = \left[\frac{t_n}{4 \cdot 2^{n+1}}, \frac{7t_n}{12 \cdot 2^{n+1}} \right] \quad \text{and} \quad K_n = \pi_{Q_n}^{-1}(K_{t_n}).$$

For any component \tilde{K}_i of K_i there is a component \tilde{K}_{i+1} of K_{i+1} such that $\tilde{K}_{i+1} \subset \tilde{K}_i$. This proves the existence of a component \tilde{K}_n which is contained in $\bigcap_{i=1}^n K_i$. By construction, $H_n|_{\tilde{K}_n}$ is an affine transformation of slope $t_1^{-1} \cdots t_n^{-1}$. In the following, we denote $\tilde{K}_n = [x', y']$. In particular, we have $|y' - x'| = 3^{-1} t_n Q_n^{-1} \cdot 2^{-(n+1)}$.

According to this we modify the second requirement of Lemma 1.5 as follows $\sum_{i=n+1}^{\infty} Q_i^{-1} \leq 3^{-1} t_n Q_n^{-1} \cdot 2^{-(n+1)}$. Then we get for $x, y \in \mathbb{S}^1$ defined by $H^{(n+1)}(x) = x'$ as well as $H^{(n+1)}(y) = y'$: $|x - y| \leq 3|x' - y'|$. With the aid of $|y' - x'| = 3^{-1} t_n Q_n^{-1} \cdot 2^{-(n+1)}$ we estimate

$$|H(x) - H(y)| = |H_n(x') - H_n(y')| = t_1^{-1} \cdots t_n^{-1} \cdot |x' - y'| = t_1^{-1} \cdots t_{n-1}^{-1} \cdot 3^{-1} Q_n^{-1} \cdot 2^{-(n+1)}$$

as well as

$$|x - y|^d \leq 3^d |x' - y'|^d = 3^d \left(3^{-1} t_n Q_n^{-1} 2^{-(n+1)} \right)^d = t_n^d Q_n^{-d} \cdot 2^{-(n+1)d}.$$

Using equation 3 both estimates together yield in case of $\beta_n < d$ (which is fulfilled for sufficiently large $n \in \mathbb{N}$ due to $\beta_n \rightarrow \beta < d$):

$$\begin{aligned} \frac{|H(x) - H(y)|}{|x - y|^d} &\geq 3^{-1} t_1^{-1} \cdots t_{n-1}^{-1} \cdot Q_n^{d-1} \cdot t_n^{-d} \cdot 2^{(d-1) \cdot (n+1)} \\ &> 3^{-1} t_1^{-1} \cdots t_{n-1}^{-1} \cdot Q_n^{\beta_n^{-1} d - 1} \cdot 2^{-(n+1)} \\ &\geq 3^{-1} t_1^{-1} \cdots t_{n-2}^{-1}. \end{aligned}$$

(where we used the additional requirement $t_{n-1} \leq 2^{-(n+1)}$ from equation 7 in the last step). Since this expression can be arbitrarily large, we conclude that H cannot be d -Hölder for any $d \in (\beta, 1)$. \square

Lemma 1.9. H and H^{-1} are β -Hölder.

Proof. Since the Lipschitz constants of h_i and h_i^{-1} are equal to t_i^{-1} , we can copy the proofs of Lemma 1.3 and Lemma 1.4. \square

Then we conclude the C^∞ -denseness of $G_{\beta,ac}$ by the same reasoning as in the previous section.

2 Proof of Theorem 2

For any $n \in \mathbb{N}$ we use a C^∞ -map $\psi_n : \mathbb{R} \rightarrow [0, 1]$ satisfying $\psi_n((-\infty, 0]) = 0$, $\psi_n([\frac{1}{4 \cdot 2^n}, \infty)) = 1$ and ψ is strictly monotone increasing on $[0, \frac{1}{4 \cdot 2^n}]$. We define the orientation-preserving diffeomorphism \tilde{h}_n of the circle as follows

- If $x \in [0, \frac{1}{4 \cdot 2^n \cdot d_n}]$

$$\tilde{h}_n(x) = (1 - \psi_n(d_n x)) d_n \nu_n^{(d_n)} x + \psi_n(d_n x) d_n \nu_n^{(1)} x$$

- For $i = 0, \dots, d_n - 1$ and $x \in [\frac{i}{d_n} + \frac{1}{4 \cdot 2^n \cdot d_n}, \frac{i+1}{d_n}]$:

$$\tilde{h}_n(x) = d_n \nu_n^{(i+1)} \cdot \left(x - \frac{i}{d_n} \right) + \sum_{k=0}^i \nu_n^{(k)}$$

- For $i = 1, \dots, d_n - 1$ and $x \in [\frac{i}{d_n}, \frac{i}{d_n} + \frac{1}{4 \cdot 2^n \cdot d_n}]$:

$$\begin{aligned} \tilde{h}_n(x) &= \left(1 - \psi_n \left(d_n \cdot \left(x - \frac{i}{d_n} \right) \right) \right) \cdot \left(d_n \nu_n^{(i)} \left(x - \frac{i}{d_n} \right) + \sum_{k=0}^{i-1} \nu_n^{(k)} \right) \\ &\quad + \psi_n \left(d_n \cdot \left(x - \frac{i}{d_n} \right) \right) \cdot \left(d_n \nu_n^{(i+1)} \left(x - \frac{i}{d_n} \right) + \sum_{k=0}^i \nu_n^{(k)} \right) \end{aligned}$$

using the notation $\nu_n^{(0)} = 0$. See figure 2 for a visualisation of such a map.

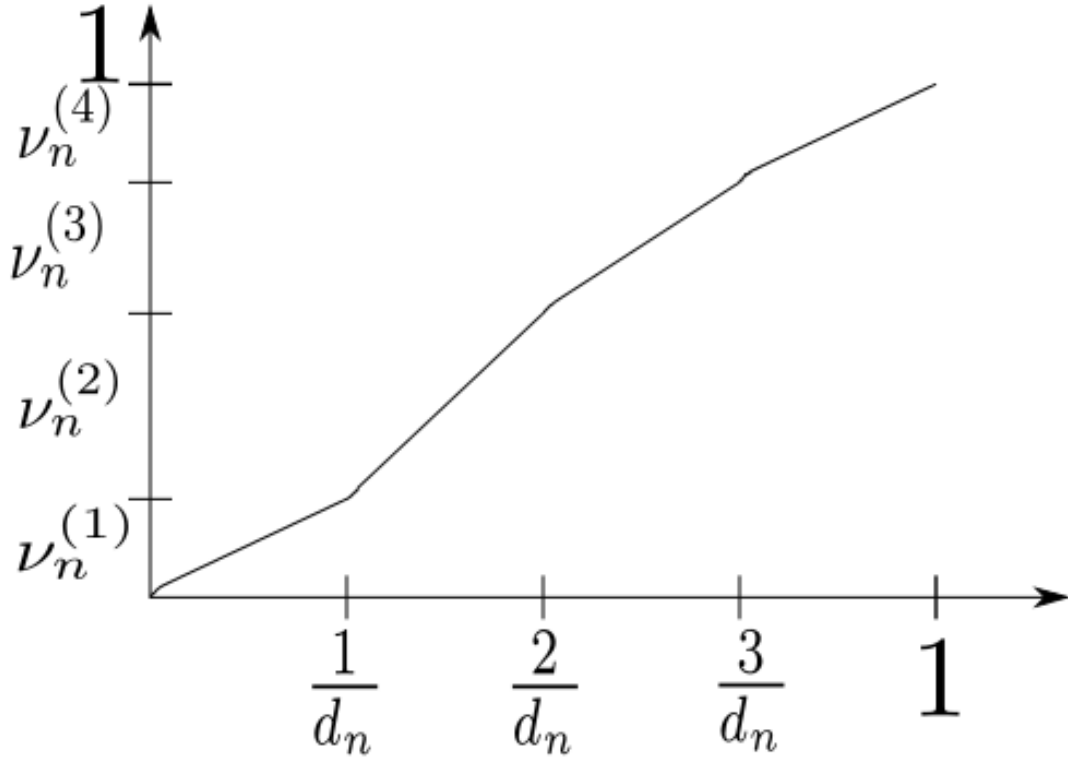
We present step n of the inductive process of our construction. Hence, we have already defined the orientation-preserving diffeomorphism $H_{n-1} = h_1 \circ \dots \circ h_{n-1}$ and the numbers $\alpha_{n-1} = \frac{p_{n-1}}{q_{n-1}} \in \mathbb{Q}$.

Let h_n be the lift of \tilde{h}_n by the cyclic q_n -fold covering map π_{q_n} such that $\text{Fix}(h_n) \neq \emptyset$ where the number q_n will be determined later (see Lemma 2.1). In particular, we have $h_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ h_n$ and for any $l \in \mathbb{N}$

$$(8) \quad |||h_n|||_l \leq C_{n,l} \cdot q_n^{l-1},$$

where the constant $C_{n,l}$ is independent of q_n . Then we can prove a statement analogous to Lemma 1.5:

Lemma 2.1. *Let $(l_n)_{n \in \mathbb{N}}$ be a strictly increasing sequence of natural numbers with $\sum_{n=1}^{\infty} \frac{1}{l_n} < \infty$ and C_{l_n} be the constants from [Ma12], Lemma 2.4. For any Liouvillean number α there are sequences $\alpha_n = \frac{p_n}{q_n}$ of rational numbers, such that the following conditions are satisfied:*

Figure 2: Qualitative shape of the function \tilde{h}_n .

1. For every $n \in \mathbb{N}$:

$$|\alpha - \alpha_n| < \frac{1}{2^{n+1} \cdot l_n \cdot C_{l_n} \cdot \|H_n\|_{l_n+1}^{l_n+1}}.$$

2. For every $n \in \mathbb{N}$:

$$|\alpha - \alpha_n| < \frac{1}{4 \cdot 2^{n+1} d_n q_n^2}.$$

As in the previous chapter we prove the denseness of constructed C^∞ -diffeomorphisms and observe that the sequence $(H_n^{-1})_{n \in \mathbb{N}}$ converges uniformly to a homeomorphism $H^{-1} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$. In the following, we will consider systems induced on a set A :

Definition 2. Let (X, \mathcal{B}, μ, T) be a non-singular system and $A \in \mathcal{B}$ be a set of positive measure. Then T induces a mapping T^A on A by $T^A(x) = T^{n(x)}(x)$ where $n(x)$ is the smallest positive integer n for which $T^n(x) \in A$. Then $(A, \mathcal{B}^A, \mu^A, T^A)$ is called the induced system where \mathcal{B}^A is the algebra of \mathcal{B} -subsets of A and μ^A is the normalized restriction of μ to \mathcal{B}^A .

On each interval $\Delta_1^{(i)} = \left[\frac{i}{d_1 q_1} + \frac{1}{2 \cdot 2^{d_1 q_1}}, \frac{i+1}{d_1 q_1} - \frac{1}{4 \cdot 2^{d_1 q_1}} \right] \subset \mathbb{S}^1$, $i = 0, \dots, d_1 q_1 - 1$, we choose $\lfloor \frac{q_2 \cdot (1 - \frac{3}{8})}{d_1 q_1} \rfloor$ intervals of type $\left[\frac{t}{q_2}, \frac{t+1}{q_2} \right]$ contained in $\Delta_1^{(i)}$ completely. In case $\alpha_2 < \alpha_1$ the right-

most of these contained in $\Delta_1^{(0)}$ is labelled by $B_1 = \left[\frac{t_1}{q_2}, \frac{t_1+1}{q_2} \right]$ (in case $\alpha_2 > \alpha_1$ the leftmost of these contained in $\Delta_1^{(0)}$ is labelled by B_1 analogously). The union of these chosen intervals $\left[\frac{t}{q_2}, \frac{t+1}{q_2} \right]$ will be denoted by F_1 and they will be numbered serially by the dynamical order determined by the induced map $f_1^{h_1(F_1)}$, the first one being $h_1(B_1)$. We note that there are $N_1 := \lfloor \frac{q_2 \cdot (1 - \frac{3}{4})}{d_1 q_1} \rfloor \cdot d_1 q_1$ numbers on the first level.

We proceed by an inductive process describing the constructions on the level $n \geq 2$ under the induction assumptions that $B_{n-1} = \left[\frac{t_{n-1}}{q_n}, \frac{t_{n-1}+1}{q_n} \right]$ is the base level of level $n-1$ and that there are

$$N_{n-1} = q_1 \cdot \prod_{i=1}^{n-1} \left\lfloor \frac{q_{i+1} \cdot \left(1 - \frac{3}{4 \cdot 2^i}\right)}{d_i q_i} \right\rfloor \cdot d_i$$

chosen intervals $\left(f_{n-1}^{H_{n-1}(F_{n-1})} \right)^l (H_{n-1}(B_{n-1}))$ in $H_{n-1}(F_{n-1})$.

We consider $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ with $\alpha_{n+1} > \alpha_n$ (the proof in the case $\alpha_{n+1} < \alpha_n$ is similar). Let $\Delta_n^{(t_{n-1}, i)} = \left[\frac{t_{n-1}}{q_n} + \frac{i}{d_n q_n} + \frac{1}{2 \cdot 2^n \cdot d_n q_n}, \frac{t_{n-1}}{q_n} + \frac{i+1}{d_n q_n} - \frac{1}{4 \cdot 2^n \cdot d_n q_n} \right] \subset B_{n-1}$ for $i = 0, \dots, d_n - 1$. In each of these we choose $\lfloor \frac{q_{n+1} \cdot (1 - \frac{3}{4 \cdot 2^n})}{d_n q_n} \rfloor$ intervals $\left[\frac{s}{q_{n+1}}, \frac{s+1}{q_{n+1}} \right]$ contained in it and denote their union by $F_n^{(1, i)}$. The collective union of these chosen $\frac{1}{q_{n+1}}$ -intervals is called $F_n^{(1)}$ and the leftmost of all of these is labelled by $B_n = \left[\frac{t_n}{q_{n+1}}, \frac{t_n+1}{q_{n+1}} \right]$.

On each of the other chosen intervals $\left[\frac{t}{q_n}, \frac{t+1}{q_n} \right] \subset F_{n-1} \setminus B_{n-1}$ we consider the intervals $\Delta_n^{(t, i)} = \left[\frac{t}{q_n} + \frac{i}{d_n q_n} + \frac{1}{4 \cdot 2^n \cdot d_n q_n}, \frac{t}{q_n} + \frac{i+1}{d_n q_n} \right]$ of size slightly bigger than $\Delta_n^{(t_{n-1}, i)}$ and on each of these we chose all possible intervals $\left[\frac{s}{q_{n+1}}, \frac{s+1}{q_{n+1}} \right]$ contained in it. Let $F_n^{(2)}$ denote their union. Moreover, we put $\tilde{F}_n = F_n^{(1)} \cup F_n^{(2)}$. Then we consider the iterates of $H_n(B_n)$ under $f_n^{H_n(\tilde{F}_n)}$. Put

$$l_0 = 0, \quad l_m = \min \left\{ l \geq l_{m-1} + N_{n-1} \mid \left(f_n^{H_n(\tilde{F}_n)} (H_n(B_n)) \right)^l \subset H_n(F_n^{(1)}) \right\} \text{ for } m \geq 1.$$

All the iterates $\left(f_n^{H_n(\tilde{F}_n)} \right)^i (H_n(B_n))$, $i = l_m, \dots, l_m + N_{n-1} - 1$ are numbered with “m” on the n -th level and their union is denoted by $B_m^{(n)}$. Note that $R_{\alpha_{n+1}}^l (B_n) \subset \Delta_n^{(t_{n-1}, i)}$ implies $R_{\alpha_{n+1}}^{l+k} (B_n) \subset F_n^{(2)} \cap \bigcup_{t \in \mathbb{Z}} \Delta_n^{(t, i)}$ for every $k = 1, \dots, N_{n-1} - 1$ because

$$N_{n-1} \cdot |\alpha_{n+1} - \alpha_n| \leq q_n \cdot 2 |\alpha - \alpha_n| \leq \frac{1}{4 \cdot 2^n d_n q_n}$$

due to Lemma 2.1. Hence, there are $\lfloor \frac{q_{n+1} \cdot (1 - \frac{3}{4 \cdot 2^n})}{d_n q_n} \rfloor \cdot d_n$ different numbers of the form l_m and we can choose

$$\left\lfloor \frac{q_{n+1} \cdot \left(1 - \frac{3}{4 \cdot 2^n}\right)}{d_n q_n} \right\rfloor \cdot d_n \cdot N_{n-1}$$

iterates of B_n from the family \tilde{F}_n . The complete union of chosen $\frac{1}{q_{n+1}}$ intervals is called F_n . More precisely, we introduce the clusters

$$\tilde{A}_i^{(n)} = \left\{ R_{\alpha_{n+1}}^l (I) \mid I \in F_n^{(1, i)}, l = 0, \dots, N_{n-1} - 1 \right\} \text{ and } A_i^{(n)} = H_n \left(\tilde{A}_i^{(n)} \right)$$

for $i = 0, \dots, d_n - 1$. Furthermore, let $\bar{A}_i^{(n)}$ denote the set of the corresponding numberings in F_n .

Lemma 2.2. *We have*

$$m\left(A_{i_n}^{(n)}\right) = q_1 \cdot \frac{\nu_n^{(i_n+1)}}{q_{n+1}} \cdot \prod_{i=1}^n \left\lfloor \frac{q_{i+1} \cdot \left(1 - \frac{3}{4 \cdot 2^i}\right)}{d_i \cdot q_i} \right\rfloor \cdot d_i.$$

Proof. In order to compute $m\left(A_i^{(n)}\right)$ we point out that

$$h_n \left(\left[\frac{t}{q_n} + \frac{i}{d_1 q_n} + \frac{1}{2 \cdot 2^n \cdot d_n q_n}, \frac{t}{q_n} + \frac{i+1}{d_n q_n} - \frac{1}{4 \cdot 2^n \cdot d_n q_n} \right] \right) \subset \left[\frac{t}{q_n}, \frac{t+1}{q_n} \right]$$

because h_n is a q_n -cyclic covering. For every $i = 0, \dots, d_1 - 1$ there are q_1 intervals of type $\left[\frac{t}{q_1} + \frac{i}{d_1 q_1}, \frac{t}{q_1} + \frac{i+1}{d_1 q_1} \right]$ and each domain $\left[\frac{t}{q_k} + \frac{i}{d_k q_k}, \frac{t}{q_k} + \frac{i+1}{d_k q_k} \right] \subset F_{k-1}$ contains $\left\lfloor \frac{q_{k+1} \cdot \left(1 - \frac{3}{4 \cdot 2^k}\right)}{d_k q_k} \right\rfloor$ many chosen intervals $\left[\frac{s}{q_{k+1}}, \frac{s+1}{q_{k+1}} \right]$. Everyone of these contains one domain of type

$$\left[\frac{s}{q_{k+1}} + \frac{i}{d_{k+1} q_{k+1}}, \frac{s}{q_{k+1}} + \frac{i+1}{d_{k+1} q_{k+1}} \right]$$

for each $i = 0, \dots, d_{k+1} - 1$. Since for any fixed i the iterates $\left(f_n^{H_n(\bar{F}_n)} \right)^k (H_n(I))$, $k = l_m, \dots, l_m + N_{n-1} - 1$ of $I \in F_n^{(1,i)}$ meet every occurring domain $\Delta_n^{(t,i)}$ exactly once, we get for $B_m^{(n)}$ in the situation of $R_{\alpha_{n+1}}^{l_m} |_{\bar{F}_n} (B_n) \subset F_n^{(1,i_n)}$:

$$\begin{aligned} & m\left(B_m^{(n)}\right) \\ &= \sum_{i_1=1}^{d_1} q_1 \sum_{i_2=1}^{d_2} \left\lfloor \frac{q_2 \cdot \left(1 - \frac{3}{4 \cdot 2}\right)}{d_1 q_1} \right\rfloor \dots \sum_{i_{n-1}=1}^{d_{n-1}} \left\lfloor \frac{q_n \cdot \left(1 - \frac{3}{4 \cdot 2^{n-1}}\right)}{d_{n-1} q_{n-1}} \right\rfloor \nu_1^{(i_1)} d_1 \cdot \nu_2^{(i_2)} d_2 \dots \nu_n^{(i_n+1)} d_n \cdot \frac{1}{q_{n+1}} \\ &= q_1 \left\lfloor \frac{q_2 \cdot \left(1 - \frac{3}{4 \cdot 2}\right)}{d_1 q_1} \right\rfloor \dots \left\lfloor \frac{q_n \cdot \left(1 - \frac{3}{4 \cdot 2^{n-1}}\right)}{d_{n-1} q_{n-1}} \right\rfloor d_1 \cdot d_2 \dots d_n \cdot \frac{\nu_n^{(i_n+1)}}{q_{n+1}} \end{aligned}$$

Then we have

$$m\left(A_{i_n}^{(n)}\right) = q_1 \left\lfloor \frac{q_2 \cdot \left(1 - \frac{3}{4 \cdot 2}\right)}{d_1 q_1} \right\rfloor \dots \left\lfloor \frac{q_n \cdot \left(1 - \frac{3}{4 \cdot 2^{n-1}}\right)}{d_{n-1} q_{n-1}} \right\rfloor d_1 \cdot d_2 \dots d_n \cdot \frac{\nu_n^{(i_n+1)}}{q_{n+1}} \cdot \left\lfloor \frac{q_{n+1} \cdot \left(1 - \frac{3}{4 \cdot 2^n}\right)}{d_n q_n} \right\rfloor$$

recalling that $\left\lfloor \frac{q_{n+1} \cdot \left(1 - \frac{3}{4 \cdot 2^n}\right)}{d_n q_n} \right\rfloor$ numbers belong to this cluster. \square

By this Lemma we get for the set $E_n = H_n(F_n)$

$$\begin{aligned} m(E_n) &= \sum_{i_n=0}^{d_n-1} m\left(A_{i_n}^{(n)}\right) \\ &= q_1 \left\lfloor \frac{q_2 \cdot \left(1 - \frac{3}{4 \cdot 2}\right)}{d_1 q_1} \right\rfloor \dots \left\lfloor \frac{q_n \cdot \left(1 - \frac{3}{4 \cdot 2^{n-1}}\right)}{d_{n-1} q_{n-1}} \right\rfloor d_1 \cdot d_2 \dots d_n \cdot \frac{1}{q_{n+1}} \cdot \left\lfloor \frac{q_{n+1} \cdot \left(1 - \frac{3}{4 \cdot 2^n}\right)}{d_n q_n} \right\rfloor \end{aligned}$$

In particular, we observe

$$m(E_n) \geq \prod_{i=1}^n \left(1 - \frac{1}{2^i}\right).$$

Moreover, we introduce the set

$$E = \bigcap_{n \in \mathbb{N}} E_n.$$

$F_n \subset F_{n-1}$ for every $n \in \mathbb{N}$ implies $E_n \subset E_{n-1}$ and we conclude

$$m(E) \geq \prod_{i=1}^{\infty} \left(1 - \frac{1}{2^i}\right) > 0.$$

By Dye's Theorem $(E, \mathcal{L}^E, m^E, f^E)$ is orbit equivalent to $(\mathbb{S}^1, \mathcal{L}, m, f)$ ([Ka79], Theorem 1.5), where \mathcal{L} is the σ -algebra of the Lebesgue measurable subsets of \mathbb{S}^1 . Hence, it is sufficient to examine the induced system:

Lemma 2.3. *The induced system $(E, \mathcal{L}^E, m^E, f^E)$ is orbit equivalent to the odometer of product type $\mathcal{O}((M_k d_k)_{k \in \mathbb{N}}, \{\nu_k^*\})$, where $M_k = \lfloor \frac{q_{k+1} \cdot (1 - \frac{3}{4 \cdot 2^k})}{d_k q_k} \rfloor$ and $\nu_k^*(t) = M_k^{-1} \nu_k(i)$ for $t \in \bar{A}_i^{(k)}$, $i = 0, \dots, d_k - 1$.*

Proof. The element $(t_k)_{k \in \mathbb{N}} \in \prod_{k \in \mathbb{N}} \{0, \dots, M_k d_k - 1\}$ corresponds to

$$\left\{ x \in E \mid x \in \bigcap_{k \in \mathbb{N}} H_k \left(B_{t_k}^{(k)} \right) \right\}.$$

The domains were introduced and labelled by numbers in such a way that f_n imitates the behaviour of the odometer up to level n and for the first q_{n+1} iterates. By the same calculations as above we get for $B_m^{(n)}$ in the situation of $R_{\alpha_{n+1}}^{l_m} |_{\bar{F}_n} (B_n) \subset F_n^{(1, i_n)}$:

$$\frac{m \left(H_n \left(B_m^{(n)} \right) \cap E \right)}{m(E)} = \frac{\nu_n^{(i_n+1)}}{\lfloor \frac{q_{n+1} \cdot (1 - \frac{3}{4 \cdot 2^n})}{d_n q_n} \rfloor}$$

and

$$\frac{m \left(A_{i_n}^{(n)} \cap E \right)}{m(E)} = \nu_n^{(i_n+1)} = \nu_n(\{i_n\}).$$

Moreover, we have for $B_{m_k}^{(k)}$ in the situation of $R_{\alpha_{k+1}}^{l_{m_k}} |_{\bar{F}_k} (B_k) \subset F_k^{(1, i_k)}$:

$$\frac{m \left(\bigcap_{k=1}^N H_k \left(B_{m_k}^{(k)} \right) \cap E \right)}{m(E)} = \prod_{k=1}^N \frac{\nu_k^{(i_k+1)}}{\lfloor \frac{q_{k+1} \cdot (1 - \frac{3}{4 \cdot 2^k})}{d_k q_k} \rfloor}.$$

□

Finally, we use the subsequent result on orbit equivalence of odometers stated in [Ka79], Theorem 1.8:

Lemma 2.4. *Let $\mathcal{O}((d_k)_{k \in \mathbb{N}}, \{\nu_k\}_{k \in \mathbb{N}})$ be an odometer of product type. For every $k \in \mathbb{N}$ let M_k be a positive integer and ν_k^* be a probability measure on $A = \{0, 1, \dots, M_k d_k - 1\}$ such that there exists a partition $A = \bigcup_j A_j$ where $\#A_j = M_k$ and $\nu_k^*(n) = M_k^{-1} \nu_k(j)$ for $n \in A_j$, $j = 0, \dots, d_k - 1$. Then $\mathcal{O}((M_k d_k)_{k \in \mathbb{N}}, \{\nu_k^*\}_{k \in \mathbb{N}})$ is orbit equivalent to $\mathcal{O}((d_k)_{k \in \mathbb{N}}, \{\nu_k\}_{k \in \mathbb{N}})$*

Hereby, we conclude that $(\mathbb{S}^1, \mathcal{L}, m, f)$ is orbit equivalent to $\mathcal{O}((d_k)_{k \in \mathbb{N}}, \{\nu_k\})$.

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