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CONSTANT–COEFFICIENT DIFFERENTIAL–ALGEBRAIC OPERATORS AND THE KRONECKER FORM

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Abstract. We consider constant-coefficient differential-algebraic equations from an operator theoretic point of view. We show that the Kronecker form allows to determine the nullspace and range of the corresponding differential-algebraic operators. This yields simple matrix-theoretic characterizations of features like closed range and Fredholmness.

1. Introduction

The purpose of this article is to highlight properties of linear constant-coefficient differential-algebraic operators

$$\mathcal{T}_{(E,A)}: \quad x \mapsto \tfrac{\mathrm{d}}{\mathrm{d}t}Ex - Ax \tag{1}$$

on $L^p(0,T;\mathbb{K}^n)$, the space of p-integrable $\mathbb{K}^n$-valued functions on $(0,T)$, with $E,A \in \mathbb{K}^{k\times n}$, where $\mathbb{K}$ denotes the field of real or complex numbers. We show that many properties of such operators can be inferred from the algebraic theory of matrix pencils $sE-A$, i.e., first order matrix polynomials. In particular, we will see that important characteristics of operators (1) such as range and nullspace can be determined directly from the pencil's Kronecker form. The latter refers to a canonical form under the group action of multiplication from the left and right with constant invertible matrices, see Gantmacher's book [3]. This form allows for a blockdiagonal decomposition of (1) into certain prototypes of differential-algebraic operators, which can easily be analyzed. Naturally, (1) corresponds to the linear, time-invariant differential-algebraic equation (DAE)

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex = Ax + f, \tag{2}$$

with suitable initial conditions. DAEs appear in a wide range of fields, from electric circuit theory to economics, and multi-body systems in mechanics, and this type is subject of several textbooks [2, 6, 10]. One could be tempted to view DAEs only as a slight generalization of ordinary differential equations, but their intrinsic algebraic properties lead to very different behaviour – in theory as well as in numerical analysis,

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see e.g., [9]. An operator-theoretic viewpoint to DAEs is not entirely new, see [5,9]. In fact, recently März [9] already gave an overview on functional-analytic aspects of a class of differential-algebraic operators, but also stressed that *".. an adequate sophisticated functional-analytic characterization of DAEs has not been accomplished yet"*[1]. Her approach deals with time-varying DAEs of the form $\frac{\mathrm{d}}{\mathrm{d}t}E(t)x = A(t)x + f$ that are, loosely speaking, neither under- nor overdetermined. With our contribution we focus on the constant-coefficient case, but allowing for under- or overdetermined systems (which refers to the so-called "regularity" of the matrix pencil $sE - A$, see Definition 1). We will show that the stronger assumptions on the coefficients enables a rather elegant interplay between the theory of matrix pencils and the functional analytic properties of the operator (1). In particular, this approach gives rise to matrix-theoretic interpretations on certain properties of the DAE operator in (1), among them are simple characterizations of surjectivity, closed range, dense range, injectivity and Fredholmness.

In the following, $D(\mathcal{T})$ and $\operatorname{im}(\mathcal{T})$ will denote domain and range of a, possibly unbounded, linear operator $\mathcal{T} : D(\mathcal{T}) \subset X \to Y$ on Banach spaces X, Y. We use $\langle \cdot, \cdot \rangle_{X',X}$ for the duality brackets of the Banach space X, where we omit the subscripts, if clear from context. $\mathbb{K}[s]$ stands for the ring of polynomials with coefficients in $\mathbb{K}$. Furthermore, $\mathbf{Gl}_n(\mathbb{K})$ is the set of invertible matrices of size $n \times n$ and I_n, denote the identity matrix in $\mathbb{K}^{n \times n}$. By $0_{n,m}$, $n, m \in \mathbb{N} \cup \{0\}$ we refer to the zero matrix of size $n \times m$. Note that here we particularly allow for n, m being zero, cf. p.**??**. The symbols $\ell(\alpha), |\alpha|$ respectively stand for the length and the absolute value of a multi-index α. For an interval $J \subset \mathbb{R}$, $W^{m,p}(J;\mathbb{K}^n) \subset L^p(J;\mathbb{K}^n)$ denotes the usual Sobolev space of m-times (weakly) differentiable functions.

2. The DAE operator

Definition 1. For $E, A \in \mathbb{K}^{k \times n}$, the expression $sE - A \in \mathbb{K}[s]^{k \times n}$ is called a **matrix pencil**. Moreover, $\operatorname{rank}_{\mathbb{K}[s]}(sE - A)$ stands for the rank of the $\mathbb{K}[s]$-module spanned by the columns of $sE - A$. A pencil $sE - A$ is **regular**, if $n = k$ and $\operatorname{rank}_{\mathbb{K}[s]}(sE - A) = n$.

Note that $\operatorname{rank}_{\mathbb{K}[s]}(sE - A)$ coincides with the generic rank of the mapping $\lambda \mapsto \lambda E - A$.

Definition 2. Let $sE - A \in \mathbb{K}[s]^{k \times n}$, $T > 0$ and $p \in [1, \infty]$. The **DAE operator** $\mathcal{T}_{(E,A)}$ **on** L^p is defined by

$$\mathcal{T}_{(E,A)} : D(\mathcal{T}_{(E,A)}) \subset L^p([0,T];\mathbb{K}^n) \to L^p([0,T];\mathbb{K}^k), \quad x \mapsto \left(\tfrac{\mathrm{d}}{\mathrm{d}t}E - A\right)x, \tag{3a}$$

with domain

$$D(\mathcal{T}_{(E,A)}) = \left\{x \in L^p([0,T];\mathbb{K}^n) \mid Ex \in W^{1,p}([0,T];\mathbb{K}^k), (Ex)(0) = 0\right\}. \tag{3b}$$

[1]see R. März [9], p. 165.

Remark 1. For $p \in [1,\infty)$ it holds:

(a) The operator $\mathcal{T}_{(E,A)}$ is closed, which is linked to generalized Sobolev spaces of the form $W_E^{1,p} := \{x \in L^p \mid Ex \in W^{1,p}\}$, see also [5].

(b) As $D(\mathcal{T}_{(E,A)})$ is dense, the dual operator $\mathcal{T}'_{(E,A)}$ is well-defined [4, Thm. II.2.6].

Let us formulate our main result which relates algebraic characteristics of the pencil $sE - A$ to properties of the DAE operator.

Theorem 1. *Let $p \in [1,\infty)$ and consider the DAE operator in L^p associated to the pencil $sE - A \in \mathbb{K}[s]^{k\times n}$. The following statements hold:*

(i) $\ker(\mathcal{T}_{(E,A)}) = \{0\}$ *if, and only if,* $\operatorname{rank}_{\mathbb{K}[s]}(sE - A) = n$.

(ii) $\operatorname{im}(\mathcal{T}_{(E,A)}) \subset L^p([0,T];\mathbb{K}^k)$ *is dense if, and only if,* $\operatorname{rank}_{\mathbb{K}[s]}(sE - A) = k$.

(iii) $\operatorname{im}(\mathcal{T}_{(E,A)}) \subset L^p([0,T];\mathbb{K}^k)$ *is closed if, and only if,* $\operatorname{im} A \subset \operatorname{im} E + A \ker E$.

(iv) $sE - A$ *is regular if, and only if,* $\mathcal{T}_{(E,A)}$ *has trivial nullspace and dense range.*

(v) The following statements are equivalent:

a) $\mathcal{T}_{(E,A)}$ *is Fredholm (that is, both* $\dim\ker\mathcal{T}_{(E,A)}$ *and* $\operatorname{codim}\operatorname{im}\mathcal{T}_{(E,A)}$ *are finite);*

b) $\mathcal{T}_{(E,A)}$ *has a bounded inverse;*

c) $\operatorname{im} A \subset \operatorname{im} E + A \ker E$ *and* $sE - A$ *is regular.*

The dual of $\mathcal{T}_{(E,A)}$ is again a DAE operator as we show next. Similar results have been achieved in [5, 8, 9] for regular DAEs with varying coefficients. Further note that, the adjoints of a certain class of DAEs arising in optimal control has been considered in [7].

Proposition 1. *Let $sE - A \in \mathbb{K}[s]^{k\times n}$, $T > 0$, $p \in [1,\infty)$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$. Then the dual operator of $\mathcal{T}_{(E,A)}$ is given by*

$$\mathcal{T}'_{(E,A)} : \mathcal{D}(\mathcal{T}'_{(E,A)}) \subset L^q([0,T];\mathbb{K}^k) \to L^q([0,T];\mathbb{K}^n), \quad \mathcal{T}'_{(E,A)} z = -\tfrac{\mathrm{d}}{\mathrm{d}t} E^\top z - A^\top z, \tag{4a}$$

and

$$D(\mathcal{T}'_{(E,A)}) = \left\{ z \in L^q([0,T];\mathbb{K}^k) \mid E^\top z \in W^{1,q}([0,T];\mathbb{K}^n), (E^\top z)(T) = 0 \right\}. \tag{4b}$$

Proof. Define

$$D := \left\{ z \in L^q([0,T];\mathbb{K}^k) \mid E^\top z \in W^{1,q}([0,T];\mathbb{K}^n), (E^\top z)(T) = 0 \right\}.$$

First we show that $D(\mathcal{T}'_{(E,A)}) \subset D$. Let $z \in D(\mathcal{T}'_{(E,A)}) \subset L^q([0,T];\mathbb{K}^k)$. For all $\varphi \in C^\infty(\mathbb{R};\mathbb{K}^n)$ with $\operatorname{supp}\varphi \subset (0,T)$ it follows that $\varphi \in D(\mathcal{T}_{(E,A)})$ and

$$\begin{aligned}\langle(\mathcal{T}'_{(E,A)} + A^\top)z, \varphi\rangle &= \langle z, \mathcal{T}_{(E,A)}\varphi + A\varphi\rangle \\ &= \left\langle z, \tfrac{\mathrm{d}}{\mathrm{d}t}E\varphi\right\rangle = \left\langle E^\top z, \tfrac{\mathrm{d}}{\mathrm{d}t}\varphi\right\rangle\end{aligned}$$

By the definition of the weak derivative, this implies that $E^\top z \in W^{1,q}([0,T];\mathbb{K}^n)$ with

$$\tfrac{\mathrm{d}}{\mathrm{d}t}E^\top z = -(\mathcal{T}'_{(E,A)} + A^\top)z. \tag{5}$$

To see that an element $z \in D(\mathcal{T}'_{(E,A)})$ satisfies the boundary condition $(E^\top z)(T) = 0$, take $x \in C^\infty([0,T];K^n)$ with $x(0) = 0$ which is clearly an element of $D(\mathcal{T}_{(E,A)})$. A similar calculation and integration by parts lead to

$$\langle \mathcal{T}'_{(E,A)}z, x\rangle = (E^\top z)(T)^\top x(T) + \left\langle \left(-\tfrac{\mathrm{d}}{\mathrm{d}t}E^\top - A^\top\right) z, x\right\rangle.$$

By (5), we conclude that $(E^\top z)(T)^\top x(T) = 0$ and thus, $(E^\top z)(T) = 0$.

To show the converse, let $z \in D$ and $x \in C^\infty([0,T];\mathbb{K}^n)$ with $x(0) = 0$. Again by integration by parts,

$$\langle z, \mathcal{T}_{(E,A)}x\rangle = \left\langle \left(-\tfrac{\mathrm{d}}{\mathrm{d}t}E^\top - A^\top\right) z, x\right\rangle.$$

In order to conclude the proof, we claim that $\{x \in C^\infty([0,T];\mathbb{K}^n) \mid x(0) = 0\}$ is a dense subspace of $D(\mathcal{T}_{(E,A)})$ equipped with the graph norm induced by $\mathcal{T}_{(E,A)}$. Then the assertion follows from continuity of $\langle\cdot,\cdot\rangle$. To show the density, consider a subspace $S \subset \mathbb{K}^n$ such that $\mathbb{K}^n = \ker E \oplus S$. By the decomposition $x = x_1 + x_2$, with $x_1 \in \ker E$ and $x_2 \in S$, we obtain that

$$\psi : D(\mathcal{T}_{(E,A)}) \to L^p([0,T];\ker E) \times W^{1,p}_{(0)}([0,T];ES), \quad x \mapsto \psi(x) = (x_1, Ex_2)$$

is a topological isomorphism, where $W^{1,p}_{(0)}([0,T];ES)$ denotes the closed subspace of functions in $W^{1,p}([0,T];ES)$ that vanish at $t = 0$. Now the density follows by well-known properties of Sobolev spaces. □

Remark 2. (Dual DAE operator) Consider the **reflection operator**

$$R: \quad L^1_{\mathrm{loc}}(J;\mathbb{K}^n) \to L^1_{\mathrm{loc}}(-J;\mathbb{K}^n), \quad f(\cdot) \mapsto f(-\cdot),$$

and, for $a \in \mathbb{R}$, the **shift operator**

$$\tau_a: \quad L^1_{\mathrm{loc}}(J;\mathbb{K}^n) \to L^1_{\mathrm{loc}}(a + J;\mathbb{K}^n), \quad f(\cdot) \mapsto f(\cdot - a).$$

Then Prop. 1 states that for $p \in [1,\infty)$ it holds that

$$\mathcal{T}'_{(E,A)} = \tau_{-T}R\mathcal{T}_{E^\top,A^\top}R\tau_T. \tag{6}$$

For $n \in \mathbb{N}$ let $(X_k)_{k=1}^n$ and $(Y_k)_{k=1}^n$ be $\mathbb{K}$-vector spaces and let $A_k : X_k \to Y_k$ be linear mappings for $k \in \{1, ..., n\}$. We recall the **diagonal operator**

$$\text{diag}\ (A_1, \dots, A_n) : \bigtimes_{k=1}^{n} X_k \to \bigtimes_{k=1}^{n} Y_k, (x_1, \dots, x_n) \mapsto (A_1 x_1, \dots, A_n x_n).$$

Further note that, for $p \in \mathbb{N}$, the matrix $0_{m,0} \in \mathbb{K}^{m\times 0}$ stands for the unique linear mapping from $\{0\}$ to $\mathbb{K}^p$, whereas the matrix $0_{p,0} \in \mathbb{K}^{m\times 0}$ stands for the unique mapping from $\mathbb{K}^p$ to $\{0\}$. Hence the diagonal operator composed of $A \in \mathbb{K}^{m\times n}$ with $0_{0,p}$ ($0_{p,0}$) is the matrix formed by adding p zero columns (rows) to A, that is

$$\text{diag}\ (A, 0_{0,p}) = [A, 0_{m,p}] \in \mathbb{K}^{m\times(n+p)}, \quad \text{diag}\ (A, 0_{p,0}) = \begin{bmatrix} A \\ 0_{p,n} \end{bmatrix} \in \mathbb{K}^{m\times(n+p)}.$$

For $m \in \mathbb{N}$ we introduce the matrices

$$N_m = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \in \mathbb{K}^{m\times m}, \quad K_m = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix}, L_m = \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \mathbb{K}^{(m-1)\times m}.$$

Moreover, for multi-indices $\alpha \in \mathbb{N}^{\ell(\alpha)}, \beta \in \mathbb{N}^{\ell(\beta)}, \gamma \in \mathbb{N}^{\ell(\gamma)}$, we set

$$\begin{aligned} N_\alpha =& \text{diag}\ (N_{\alpha_1}, \dots, N_{\alpha_{\ell(\alpha)}}), \quad K_\beta = \text{diag}\ (K_{\beta_1}, \dots, K_{\beta_{\ell(\beta)}}), \\ L_\gamma =& \text{diag}\ (L_{\gamma_1}, \dots, K_{\gamma_{\ell(\gamma)}}). \end{aligned}$$

Next we present Kronecker's famous result on matrix pencils.

THEOREM 2. (Quasi-Kronecker form [3, Chap. XII]) *Let $sE - A \in \mathbb{K}[s]^{k\times n}$ be a matrix pencil. Then there exist $W \in \mathbf{Gl}_k(\mathbb{K})$, $V \in \mathbf{Gl}_n(\mathbb{K})$ and $A_1 \in \mathbb{K}^{n_1\times n_1}$ as well as multi-indices $\alpha \in \mathbb{N}^{\ell(\alpha)}, \beta \in \mathbb{N}^{\ell(\beta)}, \gamma \in \mathbb{N}^{\ell(\gamma)}$, such that*

$$W(sE - A)V = \begin{bmatrix} sI_{n_1} - A_1 & 0 & 0 & 0 \\ 0 & sN_\alpha - I_{|\alpha|} & 0 & 0 \\ 0 & 0 & sK_\beta - L_\beta & 0 \\ 0 & 0 & 0 & sK_\gamma^\top - L_\gamma^\top \end{bmatrix}. \tag{7}$$

*The multi-indices α, β, γ are uniquely determined by $sE - A$. Further, the matrix A_1 is unique up to similarity. The right hand side of (7) is called **quasi-Kronecker form** of the pencil $sE - A$.*

REMARK 3. a) If, additionally, the matrix A_1 in (7) is in Jordan canonical form, then (7) is called **Kronecker form**. Note that, in contrast to the quasi-Kronecker form, the Kronecker form of a real matrix pencil is not necessarily real.

b) The numbers $\ell(\beta)$ and $\ell(\gamma)$ respectively express the column and row rank deficiency of $sE - A \in \mathbb{K}[s]$. That is, $\ell(\beta) = n - r$ $\ell(\gamma) = k - r$ for $r = \text{rank}_{\mathbb{K}[s]}(sE - A)$.

Thm. 2 allows to block-diagonalize DAE operators. Namely, for $W \in \mathbf{Gl}_k(\mathbb{K})$, $V \in \mathbf{Gl}_n(\mathbb{K})$ leading to quasi-Kronecker form (7), we have

$$\mathcal{T}_{WEV,WAV} = \operatorname{diag}\left(\mathcal{T}_{I_{n_1},A_1},\ \operatorname*{diag}_{i=1}^{\ell(\alpha)}(\mathcal{T}_{N_{\alpha_i},I_{\alpha_i}}),\ \operatorname*{diag}_{i=1}^{\ell(\beta)}(\mathcal{T}_{K_{\beta_i},L_{\beta_i}}),\ \operatorname*{diag}_{i=1}^{\ell(\gamma)}(\mathcal{T}_{K^\top_{\gamma_i},L^\top_{\gamma_i}})\right). \quad (8)$$

This block diagonalization gives rise to the fact that $D(\mathcal{T}_{(E,A)}), \ker(\mathcal{T}_{(E,A)}), \operatorname{im}(\mathcal{T}_{(E,A)})$ can be expressed by Cartesian products, i.e.,

$$\begin{aligned}
D(\mathcal{T}_{(E,A)}) =& V^{-1}\left(D(\mathcal{T}_{I_{n_1},A_1}) \times D(\mathcal{T}_{N_\alpha,I_\alpha}) \times D(\mathcal{T}_{K_\beta,L_\beta}) \times D(\mathcal{T}_{K^\top_\gamma,L^\top_\gamma})\right),\\
\ker(\mathcal{T}_{(E,A)}) =& V^{-1}\left(\ker(\mathcal{T}_{I_{n_1},A_1}) \times \ker(\mathcal{T}_{N_\alpha,I_\alpha}) \times \ker(\mathcal{T}_{K_\beta,L_\beta}) \times \ker(\mathcal{T}_{K^\top_\gamma,L^\top_\gamma})\right),\\
\operatorname{im}(\mathcal{T}_{(E,A)}) =& W^{-1}\left(\operatorname{im}(\mathcal{T}_{I_{n_1},A_1}) \times \operatorname{im}(\mathcal{T}_{N_\alpha,I_\alpha}) \times \operatorname{im}(\mathcal{T}_{K_\beta,L_\beta}) \times \operatorname{im}(\mathcal{T}_{K^\top_\gamma,L^\top_\gamma})\right)
\end{aligned}$$

with, further

$$\begin{aligned}
D(\mathcal{T}_{N_\alpha,I_\alpha}) &= \bigtimes_{i=1}^{\ell(\alpha)} D(\mathcal{T}_{N_{\alpha_i},I_{\alpha_i}}), \quad D(\mathcal{T}_{K_\beta,I_\beta}) = \bigtimes_{i=1}^{\ell(\beta)} D(\mathcal{T}_{K_{\beta_i},L_{\beta_i}})\\
D(\mathcal{T}_{K^\top_\gamma,L^\top_\gamma}) &= \bigtimes_{i=1}^{\ell(\gamma)} D(\mathcal{T}_{K_{\gamma_i},L_{\gamma_i}}).
\end{aligned}$$

and analogous representations of the nullspaces and ranges of $\mathcal{T}_{N_\alpha,I_\alpha}$, $\mathcal{T}_{K_\beta,L_\beta}$ and $\mathcal{T}_{K^\top_\gamma,L^\top_\gamma}$ as Cartesian products.

The previous findings yield that the specification of nullspace and range of a DAE operator leads to the determination of nullspace and range of the prototypes $\mathcal{T}_{I_m,A}$, $\mathcal{T}_{N_m,I_m}$, $\mathcal{T}_{K_m,L_m}$ and $\mathcal{T}_{K^\top_m,L^\top_m}$.

PROPOSITION 2. *Let $m \in \mathbb{N}$ and $A \in \mathbb{K}^{m\times m}$. Then*

(i) $\ker(\mathcal{T}_{I_m,A}) = \{0\}$*;*

(ii) $\ker(\mathcal{T}_{N_m,I_m}) = \{0\}$*;*

(iii) $\ker(\mathcal{T}_{K_m,L_m}) = \left\{\begin{pmatrix} y^{(m-1)} \\ \vdots \\ y\end{pmatrix} \mid y \in W^{m-1,p}([0,T];\mathbb{K}), y(0) = \ldots = y^{(m-2)}(0) = 0\right\}$*;*

(iv) $\ker(\mathcal{T}_{K^\top_m,L^\top_m}) = \{0\}$*.*

Proof. Denote the i-th component of x by x_i.

(i) Let $x \in \ker(\mathcal{T}_{I_m,A})$. Then $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax$ with $x(0) = 0$, whence $x = 0$.

(ii) Let $x \in \ker(\mathcal{T}_{N_m,I_m})$. Then $x_m = 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}x_i = x_{i-1}$ for all $i \in \{2, \ldots, m\}$. A successive insertion gives $x_m = x_{m-1} = \ldots = x_1 = 0$, and thus $x = 0$.

(iii) To show the inclusion "$\supset$", assume that $y \in W^{m-1,p}([0,T];\mathbb{K})$ with vanishing first $m-2$ derivatives at zero. Consider $x \in L^p([0,T];\mathbb{K}^m)$ with $x_i = y^{(m-i)}$ for all $i \in \{1,\dots,m\}$. A simple calculation shows that $K_m x \in W^{1,p}([0,T];\mathbb{K}^{m-1})$ with $(K_m x)(0) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}t} K_m x = L_m x$, that is $x \in \ker(\mathcal{T}_{K_m,L_m})$.

To prove "$\subset$", consider $x \in \ker(\mathcal{T}_{K_m,L_m})$. Then $\frac{\mathrm{d}}{\mathrm{d}t} x_i = x_{i-1}$ for all $i \in \{2,\dots,m\}$. Hence, for $y := x_m$, we have $x_i = y^{(m-i)}$ for all $i \in \{1,\dots,m\}$, which further leads to $y \in W^{m-1,p}([0,T];\mathbb{K})$. Moreover, by $(K_m x)(0) = 0$, we obtain that $x_1(0) = \ldots = x_{m-1}(0) = 0$, whence $y(0) = \ldots = y^{(m-2)}(0) = 0$.

(iv) Assume that $x \in \ker(\mathcal{T}_{K_m^\top,L_m^\top})$, then $0 = x_1$, $\frac{\mathrm{d}}{\mathrm{d}t} x_{i-1} = x_i$ for all $i \in \{2,\dots,m-1\}$, and $\frac{\mathrm{d}}{\mathrm{d}t} x_{i-1} = 0$. Again, a successive insertion leads to $x = 0$.

□

Corollary 1. *Let $sE - A \in \mathbb{K}[s]^{k\times n}$ be a matrix pencil. With the notation of Thm. 2 it holds that*

$$\ker(\mathcal{T}_{(E,A)}) = V^{-1}\left(\{0\} \times \{0\} \times \bigtimes_{i=1}^{\ell(\beta)} \ker(\mathcal{T}_{K_{\beta_i},L_{\beta_i}}) \times \{0\}\right)$$

where, for $i \in \{1,\dots,\ell(\beta)\}$, the spaces $\ker(\mathcal{T}_{K_{\beta_i},L_{\beta_i}})$ are given by the expressions in Prop. 2 (iii).

We now characterize the ranges of $\mathcal{T}_{I_m,A}$, $\mathcal{T}_{N_m,I_m}$, $\mathcal{T}_{K_m,L_m}$ and $\mathcal{T}_{K_m^\top,L_m^\top}$.

Proposition 3. *For $m \in \mathbb{N}$ and $A \in \mathbb{K}^{m\times m}$ holds*

(i) $\operatorname{im}(\mathcal{T}_{I_m,A}) = L^p([0,T];\mathbb{K}^m)$;

(ii) $$\operatorname{im}(\mathcal{T}_{N_m,I_m}) = \left\{ \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \in \bigtimes_{i=0}^{m-1} W^{i,p}([0,T];\mathbb{K}) \mid \sum_{j=i}^{m} f_j^{(j-i)}(0) = 0 \ \forall i \in \{2,\dots,m\} \right\};$$

(iii) $\operatorname{im}(\mathcal{T}_{K_m,L_m}) = L^p([0,T];\mathbb{K}^{m-1})$;

(iv) $\operatorname{im}(\mathcal{T}_{K_m^\top,L_m^\top}) =$

$$\left\{ \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \in \bigtimes_{i=1}^{m} W^{m-i,p}([0,T];\mathbb{K}) \mid \sum_{i=0}^{m-1} f_{m-i}^{(i)} = 0 \wedge \sum_{j=0}^{i-1} f_{i-j}^{(j)}(0) = 0 \ \forall i \in \{2,\dots,m-1\} \right\}.$$

Furthermore, for $p \in [1,\infty)$, on has that $\operatorname{im}(\mathcal{T}_{N_m,I_m})$ is dense in $L^p([0,T];\mathbb{K}^m)$ and $\operatorname{im}(\mathcal{T}_{K_m^\top,L_m^\top})$ is not dense in $L^p([0,T];\mathbb{K}^m)$.

Proof. (i) The result follows by the fact that for any $f \in L^p([0,T];\mathbb{K}^m)$, the ordinary differential equation $\frac{\mathrm{d}}{\mathrm{d}t} x - Ax = f$ with initial value $x(0) = 0$ has a solution $x \in W^{1,p}([0,T];\mathbb{K}^m)$.

(ii) We first show the inclusion "$\supset$": For $i \in \{1,\dots,m\}$, let $f_i \in W^{i-1,p}([0,T];\mathbb{K})$ be the i-th component of $f \in L^p([0,T];\mathbb{K})$ with, further $\sum_{j=i}^m f_j^{(j-i)}(0) = 0$ for all $i \in \{2,\dots,m\}$. Let

$$x_i = -\sum_{j=i}^{m} f_j^{(j-i)} \tag{9}$$

be the i-th component of $x \in L^p([0,T];\mathbb{K}^m)$. Then $N_m x \in W^{1,p}([0,T];\mathbb{K}^m)$, and a simple calculation shows that $(N_x)(0) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}t} N_m x - x = f$.
To prove "$\subset$", let $x \in D(\mathcal{T}_{N_m,I_m})$. Then the components of $f = \mathcal{T}_{N_m,I_m} x$ fulfill $\frac{\mathrm{d}}{\mathrm{d}t} x_i - x_{i-1} = f_{i-1}$ for all $i \in \{2,\dots,m\}$, $x_m = -f_m$ and $x_2(0) = \dots = x_m(0) = 0$. A successive insertion gives (9), whence we obtain $f_i \in W^{i,p}([0,T];\mathbb{K})$ as well as $\sum_{j=i}^m f_j^{(j-i)}(0) = 0$ for all $i \in \{1,\dots,m\}$.

(iii) In the following, we make use of the fact that the DAE $\frac{\mathrm{d}}{\mathrm{d}t} K_m x = L_m x + f$ can be rewritten as an ordinary differential equation $\frac{\mathrm{d}}{\mathrm{d}t}\tilde{x} = N_m^\top \tilde{x} + e_1 x_1 + f$, where $\tilde{x} \in W^{1,p}([0,T];\mathbb{K}^{m-1})$ is composed of the last $m-1$ components of x:
Let $f \in L^p([0,T];\mathbb{K}^{m-1})$. Let $\tilde{x} \in W^{1,p}([0,T];\mathbb{K}^{m-1})$ with $\tilde{x}(0) = 0$ be a solution of $\frac{\mathrm{d}}{\mathrm{d}t}\tilde{x} = N_m^\top \tilde{x} + f$. Then $x = \left(\begin{smallmatrix} 0 \\ \tilde{x} \end{smallmatrix}\right) \in L^p([0,T];\mathbb{K}^m)$ fulfills $x \in D(\mathcal{T}_{K_m,L_m})$ and $\frac{\mathrm{d}}{\mathrm{d}t} K_m x = L_m x + f$, i.e., $f \in \operatorname{im}(\mathcal{T}_{K_m,L_m})$.

(iv) To prove "$\supset$", let f be such that its components fulfill $f_i \in W^{m-i,p}([0,T];\mathbb{K})$ and $\sum_{j=0}^{i-1} f_{i-j}^{(j)}(0) = 0$ for $i = 2,\dots,m-1$ with, further, $\sum_{i=0}^{m-1} f_{m-i}^{(i)} = 0$. Now defining the i-th component of $x \in L^p([0,T];\mathbb{K})$ by

$$x_i = -\sum_{j=0}^{i-1} f_{i-j}^{(j)}, \tag{10}$$

we obtain $(K_m^\top x) \in W^{1,p}([0,T];\mathbb{K}^m)$ with $(K_m^\top x)(0) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}t} K_m^\top x = L_m^\top x + f$. In other words, we have $x \in D(\mathcal{T}_{K_m^\top,L_m^\top})$ and $f = \mathcal{T}_{K_m^\top,L_m^\top} x$.
To prove "$\subset$", let $x \in D(\mathcal{T}_{K_m^\top,L_m^\top})$ and $f = \mathcal{T}_{K_m^\top,L_m^\top} x$. Then the components fulfill $x_1 = -f_1$, $\frac{\mathrm{d}}{\mathrm{d}t} x_{i-1} - x_i = f_i$ for all $i \in \{2,\dots,m-1\}$, and $\frac{\mathrm{d}}{\mathrm{d}t} x_{m-1} = f_m$. The relation $(K_m^\top x)(0)$ further leads to $x(0) = 0$. A successive insertion gives (10) for all $i \in \{1,\dots,m-1\}$. Then $\frac{\mathrm{d}}{\mathrm{d}t} x_{m-1} = f_m$ leads to $\sum\limits_{i=0}^{m-1} f_{m-i}^{(i)} = 0$, and $x(0) = 0$ yields $\sum\limits_{j=i}^{m} f_j^{(j-i)}(0) = 0$ for all $i \in \{2,\dots,m\}$.

□

COROLLARY 2. *Let $sE - A \in \mathbb{K}[s]^{k\times n}$ be a matrix pencil. With the notation of Thm. 2 holds*

$$\begin{aligned}&\operatorname{im}(\mathcal{T}_{(E,A)})\\ &= W^{-1}\left(L^p([0,T];\mathbb{K}^{n_1}) \times \bigtimes_{i=1}^{\ell(\alpha)} \operatorname{im}(\mathcal{T}_{N_{\alpha_i},I_{\alpha_i}}) \times L^p([0,T];\mathbb{K}^{|\beta|-\ell(\beta)}) \times \bigtimes_{j=1}^{\ell(\gamma)} \operatorname{im}(\mathcal{T}_{K_{\gamma_j}^\top,L_{\gamma_j}^\top}) \right),\end{aligned}$$

where, for $i \in \{1,\dots,\ell(\alpha)\}$, $j \in \{1,\dots,\ell(\gamma)\}$, the spaces $\operatorname{im}(\mathcal{T}_{N_{\alpha_i},I_{\alpha_i}})$, $\operatorname{im}(\mathcal{T}_{K^\top_{\gamma_i},L^\top_{\gamma_i}})$ *are given by the expressions in Prop. 3 (ii)&(iv).*

REMARK 4. a) Let $m \in \mathbb{N}$ and $f \in \operatorname{im}(\mathcal{T}_{N_m,I_m})$. Prop. 3 (ii) implies that $N^k f \in W^{k,p}([0,T];\mathbb{K}^m)$ for all $k \in \{1,\dots,m\}$. The relation (9) further implies the well-known relation (cf. [2,3,6,10])

$$x = -\sum_{i=0}^{m-1} \tfrac{\mathrm{d}^i}{\mathrm{d}t^i} N_m^i f.$$

b) Assume that a square pencil $sE - A \in \mathbb{K}[s]^{n\times n}$ is given which is additionally **regular**, that is $\det(sE-A)$ is not the zero polynomial. Then Remark 3 b) implies that the Kronecker form reads

$$W(sE-A)V = \begin{bmatrix} sI_{n_1} - A_1 & 0 \\ 0 & sN_\alpha - I_{|\alpha|} \end{bmatrix}.$$

Further, by Prop. 2, we obtain that the nullspace of $\mathcal{T}_{(E,A)}$ is trivial. Further, we can infer from Propostion 3 that $\operatorname{im}\mathcal{T}_{(E,A)}$ is a dense subspace of $L^p([0,T];\mathbb{K}^n)$. Consequently, $\mathcal{T}^{-1}_{(E,A)} : \operatorname{im}(\mathcal{T}_{(E,A)}) \subset L^p([0,T];\mathbb{K}^n) \to L^p([0,T];\mathbb{K}^n)$ is a densely defined operator with dense range. Consider $f \in \operatorname{im}(\mathcal{T}_{(E,A)})$ and partition $\left(\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right) = Wf$ according to the block structure in (b). The variations of constants formula and the previous remark then lead to the well-known solution formula

$$\mathcal{T}^{-1}_{(E,A)} f = V \begin{pmatrix} \int_0^{\cdot} \exp(A_1(\cdot - \tau)) f_1(\tau)\mathrm{d}\tau \\ -\sum_{i=0}^{\nu-1} \tfrac{\mathrm{d}^i}{\mathrm{d}t^i} N^i f_2 \end{pmatrix}. \tag{11}$$

where $\nu = \max\alpha$, cf. [2,3,6,10]. In particular, $\mathcal{T}_{(E,A)}$ has a bounded inverse, if, and only if, $\nu = 1$ and $sE - A$ is regular.

Proof of Theorem 1. (i) From Remark 3 we have that $\operatorname{rank}_{\mathbb{K}[s]}(sE-A) = n$ if, and only if, for in the Kronecker form of $sE - A$ holds $\ell(\beta) = 0$. On the other hand, Cor. 1 implies that the latter is equivalent to $\ker(\mathcal{T}_{(E,A)}) = \{0\}$.

(ii) We will use the fact that $\operatorname{im}\mathcal{T}_{(E,A)}$ is dense in $L^p([0,T];\mathbb{K}^k)$, if, and only if, the nullspace of the dual operator $\mathcal{T}'_{(E,A)}$ is trivial, see e.g. [4, Thm. II.3.7]. By (6), we have $\ker\mathcal{T}'_{(E,A)} = \ker\mathcal{T}_{E^\top,A^\top} R\tau_T$, where R is the reflection operator and τ_T is the shift operator. Now using that $R\tau_T : L^p([0,T];\mathbb{K}^k) \to L^p([0,T];\mathbb{K}^k)$ is bijective, we obtain that $\operatorname{im}\mathcal{T}_{(E,A)}$ is dense in $L^p([0,T];\mathbb{K}^k)$, if, and only if, $\mathcal{T}_{E^\top,A^\top}$ has trivial nullspace. The latter assertion is, by (i), equivalent to $\operatorname{rank}_{\mathbb{K}[s]}(sE-A) = k$.

(iii) First note that $\operatorname{im}A \subset \operatorname{im}E + A\ker E$, if, and only if, for any $W \in \mathbf{Gl}_k(\mathbb{K})$, $V \in \mathbf{Gl}_n(\mathbb{K})$ holds $\operatorname{im}WAV \subset \operatorname{im}WEV + WAV\ker WEV$. Since, further,

$\mathrm{im}\mathcal{T}_{WEV,WAV} = W \cdot \mathrm{im}\mathcal{T}_{(E,A)}$, we obtain that $\mathrm{im}\mathcal{T}_{(E,A)}$ has closed range, if, and only if, $\mathrm{im}\mathcal{T}_{WEV,WAV}$ has closed range. As a consequence, we may assume without loss of generality that $sE - A$ is in Kronecker form. We can infer from Cor. 2 that $\mathrm{im}(\mathcal{T}_{(E,A)})$ is closed if, and only if, $\alpha_i = 1$ for all $i = 1, ..., \ell(\alpha)$ and $\gamma_i = 1$ for all $i = 1, ..., \ell(\gamma)$. On the other hand, taking a close look at the Kronecker form (7), we see that $\mathrm{im}A \subset \mathrm{im}E + A \ker E$ holds if and only if $\alpha_i = 1$ for all $i = 1, ..., \ell(\alpha)$ and $\gamma_i = 1$ for all $i = 1, ..., \ell(\gamma)$.

(iv) This follows by a combination of (i) and (ii).

(v) The implication "b) $\Rightarrow$ a)" is trivial.
"c) $\Rightarrow$ b)": Assume that $sE - A$ is regular and that $\mathrm{im}A \subset \mathrm{im}E + A \ker E$: Then a combination of (iii) and (iv) implies that $\mathcal{T}_{(E,A)}$ is bijective, whence it possesses a bounded inverse.
"a) $\Rightarrow$ c)": Assume that $\mathcal{T}_{(E,A)}$ is Fredholm. Since Fredholm operators have closed range [1, Lem. 4.38], we obtain from (iii) that $\mathrm{im}A \subset \mathrm{im}E + A \ker E$. If $\mathrm{rank}_{\mathbb{K}[s]} sE - A < n$, then a combination of Remark 3 b), Prop. 2 and Cor. 1 implies $\dim \ker \mathcal{T}_{(E,A)} = \infty$. On the other hand, if $\mathrm{rank}_{\mathbb{K}[s]} sE - A < k$, then Remark 3 b) together with Prop. 3 (iv) and Cor. 2 yields $\mathrm{codim}\ \mathrm{im}\mathcal{T}_{(E,A)} = \infty$, which is again a contradiction. Hence, $sE - A$ is regular.

□

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