Spectral disjointness of powers of diffeomorphisms with arbitrary Liouvillean rotation behavior

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Abstract

We show that on any smooth compact connected manifold of dimension $m \geq 2$ admitting a smooth non-trivial circle action preserving a smooth volume ν the set of $(\kappa_1, \dots, \kappa_k)$-weakly mixing C^{∞}-diffeomorphisms is generic in $\mathcal{A}_{\alpha}(M) = \overline{\{h \circ S_{\alpha} \circ h^{-1} : h \in \text{Diff}^{\infty}(M, \nu)\}}^{C^{\infty}}$ for every Liouvillean number α, $k \in \mathbb{N}$ and specific tuples $(\kappa_1, \dots, \kappa_k) \in [0,1]^k$. In particular, these diffeomorphisms have spectrally disjoint powers. The proof is based on a quantitative version of the Approximation by Conjugation-method with explicitly defined conjugation maps and partitions.

1 Introduction

For a start, we recall that a dynamical system (X, T, ν) on a probability space (X, ν) is said to be weakly mixing if there is no nonconstant function $h \in L^2(X, \nu)$ such that $h(Tx) = \lambda \cdot h(x)$ for some $\lambda \in \mathbb{C}$. Equivalently there is an increasing sequence $(m_n)_{n \in \mathbb{N}}$ of natural numbers such that $\lim_{n \to \infty} |\nu(B \cap T^{-m_n}(A)) - \nu(A) \cdot \nu(B)| = 0$ for every pair of measurable sets $A, B \subseteq X$ (see [Skl67] or [AK70, Theorem 5.1]). A. Katok and A. Stepin introduced the more general notion of κ-weak mixing ([Ka03], [St87]):

Definition 1.1. An automorphism T of a Lebesgue probability space (X, μ) is said to be κ-weakly mixing, $\kappa \in [0,1]$, if there exists a strictly increasing sequence $(m_n)_{n \in \mathbb{N}}$ of natural numbers such that the weak convergence

$$U_T^{m_n} \longrightarrow_w (\kappa \cdot P_c + (1-\kappa) \cdot Id)$$

holds, where $U_T : L^2(X, \mu) \to L^2(X, \mu)$, $f \mapsto f \circ T$, is the Koopman-operator induced by T and P_c is the projection on the subspace of constants.

By [St87, Proposition 3.1.] we can characterize this property in geometric language: A transformation T is κ-weakly mixing if and only if there is an increasing sequence $(m_n)_{n \in \mathbb{N}}$ of natural numbers such that for all measurable sets A and B

$$\lim_{n \to \infty} \mu(A \cap T^{m_n} B) = \kappa \cdot \mu(A) \cdot \mu(B) + (1-\kappa) \cdot \mu(A \cap B)$$

We recognize that 0-weakly mixing corresponds to rigidity and 1-weakly mixing to the usual notion of weak mixing. So κ-weakly mixing interpolates between the notions of recurrence and weak

mixing. In fact for $\kappa > 0$ a κ-weakly mixing transformation has a continuous spectrum due to [St87, Proposition 3.4]. Thus it is weakly mixing.
The concept of κ-weakly mixing has implications on the spectral properties of the transformation (see [St87, Theorem 1]): If $\kappa \in (0,1)$ and T is a κ-weakly mixing transformation, then a measure σ_0 of maximal spectral type of the operator U_T on the orthogonal complement $H_0 \subset L^2(X,\mu)$ to the subspace of constants satisfies that it and its convolutions σ_0^k are pairwise mutually singular. This property is called *disjointness of convolutions*. It is linked to a conjecture of Kolmogorov respectively Rokhlin and Fomin (after verifying that the property held for all dynamical systems known at that time, especially large classes of systems of probabilistic origin like Gaussian ones), namely that every ergodic transformation possesses the so-called group property, i.e. the maximal spectral type σ is symmetric and dominates its square $\sigma * \sigma$. This conjecture is an analogue of the well-known group property of the set of eigenvalues of an ergodic automorphism and was proven to be false. Indeed, in [St66] Stepin gave the first example of a dynamical system without the group property. V.I. Oseledets constructed an analogous example with continuous spectrum ([Os69]). Later Stepin showed that for a generic transformation all convolutions σ^k, $k \in \mathbb{N}$, of the maximal spectral type σ on $L_0^2(X,\mu)$ are mutually singular (see [St87]) exploiting the concept of κ-weak mixing.
In [JL92] del Junco and Lemanczyk introduced the following strengthening of the notion of κ-weak mixing:

Definition 1.2. Let T be an automorphism of a Lebesgue probability space $(X,\mathcal{B},\mu)$ and $\kappa_1,\dots,\kappa_k \in [0,1]$. Then T is called $(\kappa_1,\dots,\kappa_k)$-weakly mixing, if there is an increasing sequence $(m_n)_{n\in\mathbb{N}}$ of natural numbers such that for each $i = 1,\dots,k$ and for all $A, B \in \mathcal{B}$

$$\lim_{n\to\infty} \mu\left(A \cap T^{i\cdot m_n} B\right) = \kappa_i \cdot \mu(A) \cdot \mu(B) + (1-\kappa_i) \cdot \mu(A \cap B).$$

In other words, each T^i is κ_i-weakly mixing along the common sequence $(m_n)_{n\in\mathbb{N}}$. With the aid of this concept del Junco and Lemanczyk were able to prove that for a generic automorphism for each $k(1),\dots,k(l) \in \mathbb{N}$, $k'(1),\dots,k'(l') \in \mathbb{N}$ the convolutions $\sigma_{T^{k(1)}} * \dots * \sigma_{T^{k(l)}}$ and $\sigma_{T^{k'(1)}} * \dots * \sigma_{T^{k'(l)}}$ are mutually singular unless $(k(1),\dots,k(l))$ is a permutation of $(k'(1),\dots,k'(l'))$ ([JL92, Theorem 1]). Hereby, they were also able to get a description of the centralizer of $T^{k_1} \times T^{k_2} \times \dots$ as well as to reproduce several counterexamples (like non-disjoint transformations that have no common factor or automorphisms with no roots) of Rudolph ([Ru79, section 4]) in a broader context.

An important question in Ergodic Theory (e. g. [OW91, p.89]) that dates back to the foundational paper [Ne32] of von Neumann asks

Question. *Are there smooth versions to the objects and concepts of abstract ergodic theory?*

By a smooth version we mean a smooth diffeomorphism of a compact manifold preserving a measure equivalent to the volume element. The only known restriction is due to A. G. Kushnirenko who proved that such a diffeomorphism must have finite entropy. On the other hand, there is a lack on general results on the smooth realization problem.

One of the most powerful tools of constructing smooth diffeomorphisms with prescribed ergodic or topological properties is the so-called *approximation by conjugation*-method (also known as the *AbC*-method or *Anosov-Katok*-method) developed by D. Anosov and A. Katok in [AK70]. In fact, on every smooth compact connected manifold M of dimension $m \geq 2$ admitting a non-trivial circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{S}^1}$ preserving a smooth volume ν this method enables the construction of smooth diffeomorphisms with specific ergodic properties (e. g. weakly mixing ones in [AK70, section 5]) or non-standard smooth realizations of measure-preserving systems (e. g. [AK70, section 6] and [FSW07]). These diffeomorphisms are constructed as limits of conjugates $f_n = H_n \circ S_{\alpha_{n+1}} \circ H_n^{-1}$, where $\alpha_{n+1} = \alpha_n + \frac{1}{k_n \cdot l_n \cdot q_n^2} \in \mathbb{Q}$, $H_n = H_{n-1} \circ h_n$ and h_n is a measure-preserving diffeomorphism satisfying $S_{\frac{1}{q_n}} \circ h_n = h_n \circ S_{\frac{1}{q_n}}$. In each step the conjugation map h_n and the parameter k_n are chosen such that the diffeomorphism f_n imitates the desired property with a certain precision. Then the parameter l_n is chosen large enough to guarantee closeness of f_n to f_{n-1} in the C^∞-topology and so the convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ to a limit diffeomorphism is provided. It is even possible to keep this limit diffeomorphism within any given C^∞-neighbourhood of the initial element S_{α_1} or, by applying a fixed diffeomorphism g first, of $g \circ S_{\alpha_1} \circ g^{-1}$. So the construction can be carried out in a neighbourhood of any diffeomorphism conjugate to an element of the action. Thus, $\mathcal{A}(M) = \overline{\{h \circ S_t \circ h^{-1} \,:\, t \in \mathbb{S}^1, h \in \text{Diff}^\infty(M,\nu)\}}^{C^\infty}$ is a natural space for the produced diffeomorphisms. Moreover, we will consider the restricted space $\mathcal{A}_\alpha(M) = \overline{\{h \circ S_\alpha \circ h^{-1} \,:\, h \in \text{Diff}^\infty(M,\nu)\}}^{C^\infty}$ for $\alpha \in \mathbb{S}^1$. Another feature of the AbC-method is the possibility to deduce statements on the genericity of the constructed properties in $\mathcal{A}(M)$ or $\mathcal{A}_\alpha(M)$: As mentioned above Anosov and Katok proved that the set of weakly mixing diffeomorphisms is generic in $\mathcal{A}(M)$ in the $C^\infty(M)$-topology. More specifically, for every Liouville number α the set of weakly mixing diffeomorphisms is generic in $\mathcal{A}_\alpha(M)$ ([FS05], [GKu]). See also [FK04] for more details and other results of this method.

Using a smooth variant of the method of approximation by periodic transformations Stepin constructed a smooth κ-weakly mixing diffeomorphism in [St87, section 4]. Another construction of a smooth diffeomorphisms without the group property even in the restricted space $\mathcal{A}_\alpha(M)$ for arbitrary Liouville number α was exhibited in [Ku16].

In this paper we construct the first smooth $(\kappa_1,\dots,\kappa_k)$-weakly mixing diffeomorphisms: For $k \in \mathbb{N}$ we define the set Π_k of tuples $(\kappa_1,\dots,\kappa_k) \in [0,1]^k$ satisfying $\kappa_1 < \kappa_2 < \dots < \kappa_k$, $\kappa_k > k \cdot (\kappa_2 - \kappa_1)$ as well as $\kappa_k - \kappa_{k-1} < \kappa_{k-1} - \kappa_{k-2} < \dots < \kappa_2 - \kappa_1$.

Theorem 1. *Let M be a smooth compact connected manifold of dimension $m \geq 2$ with a non-trivial circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$, $S_{t+1} = S_t$, preserving a smooth volume ν. Moreover, let $k \in \mathbb{N}$ and $(\kappa_1,\dots,\kappa_k) \in \Pi_k$. If $\alpha \in \mathbb{R}$ is Liouville, then the set of volume-preserving $(\kappa_1,\dots,\kappa_k)$-weakly mixing diffeomorphisms contains a dense G_δ-set in the C^∞-topology in $\mathcal{A}_\alpha(M)$.*

In particular, we get

Corollary 1. *If α is a Liouville number, then for a generic $T \in \mathcal{A}_\alpha(M)$ we have*

$$\sigma_{T^{k(1)}} * \dots * \sigma_{T^{k(l)}} \perp \sigma_{T^{k'(1)}} * \dots * \sigma_{T^{k'(l)}}$$

for every $k(1),\dots,k(l),k'(1),\dots,k'(l') \in \mathbb{N}$ unless $(k(1),\dots,k(l))$ is a permutation of $(k'(1),\dots,k'(l'))$. In particular, the powers of T are spectrally disjoint.

At this point, we recall that two automorphisms T and S are *spectrally disjoint* if the maximal spectral types σ_T and σ_T are mutually singular. In particular, spectral disjointness implies disjointness in the sense of Furstenberg ([Fu67]). Another motivation to study spectral disjointness of different powers deals with Sarnak's conjecture stating that for every homeomorphism T of a compact metric space X with topological entropy zero, any $\varphi \in C(X)$ and any $x \in X$ the sequence $(\varphi(T^n x))_{n\in\mathbb{N}}$ is orthogonal to the Möbius function, i. e. we have

$$\frac{1}{N}\sum_{n=1}^{N}\varphi(T^n x)\,\boldsymbol{\mu}(n) \to 0 \text{ as } N \to \infty, \tag{1}$$

where the Möbius function $\boldsymbol{\mu} : \mathbb{N} \to \mathbb{Z}$ is defined by $\boldsymbol{\mu}(1) = 1$, $\boldsymbol{\mu}(n) = 0$ for non-square-free positive integers and $\boldsymbol{\mu}(n) = \pm 1$ depending on the parity of the number of prime factors for the remaining positive integers (see [Sa], [ELR14]). In fact, it is shown in [BSZ12] that the spectral disjointness of different prime powers implies the validity of a generalized version of equation 1, where $\boldsymbol{\mu}$ can be replaced by any bounded multiplicative function of the positive integers.

Remark 1.3. By some modifications using the conjugation maps as in [GKu] and [Ku] we are even able to construct the $(\kappa_1, \dots, \kappa_k)$-weakly mixing diffeomorphisms in such a way that they preserve a measurable Riemannian metric and that their projectivized derivative extension is ergodic with respect to a measure in the projectivization of the tangent bundle which is absolutely continuous in the fibers.

Remark 1.4. Recently, great progress has been made in extending the approximation by conjugation-method to the real-analytic category in case of the torus $\mathbb{T}^m$, $m \geq 2$, in a series of papers ([Ba17], [Ku17], [BK]). All these constructions base on the concept of *block-slide type maps* on the torus and their sufficiently precise approximation by volume-preserving real-analytic diffeomorphisms. By this approach it is possible to adapt the constructions of this paper to show that for every $m \in \mathbb{N}$, $m \geq 2$, $k \in \mathbb{N}$, $\rho > 0$ and $(\kappa_1, \dots, \kappa_k) \in \Pi_k$ there are real-analytic $(\kappa_1, \dots, \kappa_k)$-weakly mixing diffeomorphisms $T \in \text{Diff}^{\omega}_{\rho}(\mathbb{T}^m, \mu)$. We are going to present the details in a forthcoming paper.

2 Preliminaries

We use the definitions and notations introduced in [Ku16, subsection 1.1].

2.1 First steps of the proof

First of all we show how constructions on $\mathbb{S}^1 \times [0,1]^{m-1}$ can be transfered to a general compact connected smooth manifold M with a non-trivial circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$, $S_{t+1} = S_t$. By [AK70, Proposition 2.1.] we can assume that 1 is the smallest positive number satisfying $S_t = \text{id}$. Hence, we can assume $\mathcal{S}$ to be effective. We denote the set of fixed points of $\mathcal{S}$ by F and for $q \in \mathbb{N}$ F_q is the set of fixed points of the map $S_{\frac{1}{q}}$. On the other hand, we consider $\mathbb{S}^1 \times [0,1]^{m-1}$ with Lebesgue measure μ. Furthermore let $\mathcal{R} = \{R_\alpha\}_{\alpha\in\mathbb{S}^1}$ be the standard action of $\mathbb{S}^1$ on $\mathbb{S}^1 \times [0,1]^{m-1}$, where the map R_α is given by $R_\alpha(\theta, r_1, ..., r_{m-1}) = (\theta + \alpha, r_1, ..., r_{m-1})$. Hereby we can formulate the following result (see [FSW07, Proposition 1]):

Proposition 2.1. *Let M be a m-dimensional smooth, compact and connected manifold admitting an effective circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$, $S_{t+1} = S_t$, preserving a smooth volume ν. Let $B := \partial M \cup F \cup \left(\bigcup_{q\geq 1} F_q\right)$. There exists a continuous surjective map $G : \mathbb{S}^1 \times [0,1]^{m-1} \to M$ with the following properties:*

1. *The restriction of* G *to* $\mathbb{S}^1 \times (0,1)^{m-1}$ *is a* C^∞*-diffeomorphic embedding.*

2. $\nu\left(G\left(\partial\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)\right)\right) = 0$

3. $G\left(\partial\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)\right) \supseteq B$

4. $G_*(\mu) = \nu$

5. $\mathcal{S} \circ G = G \circ \mathcal{R}$

By the same reasoning as in [FSW07, section 2.2] this proposition allows us to carry a construction from $\left(\mathbb{S}^1 \times [0,1]^{m-1}, \mathcal{R}, \mu\right)$ to the general case $(M, \mathcal{S}, \nu)$:
Suppose $f : \mathbb{S}^1 \times [0,1]^{m-1} \to \mathbb{S}^1 \times [0,1]^{m-1}$ is a $(\kappa_1, \dots, \kappa_k)$-weakly mixing diffeomorphism sufficiently close to R_α in the C^∞-topology obtained by $f = \lim_{n\to\infty} f_n$ with $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$, where $f_n = R_{\alpha_{n+1}}$ in a neighbourhood of the boundary (in Proposition 2.2 we will see that these conditions can be satisfied in the constructions of this article). Then we define a sequence of diffeomorphisms:

$$\tilde{f}_n : M \to M \qquad \tilde{f}_n(x) = \begin{cases} G \circ f_n \circ G^{-1}(x) & \text{if } x \in G\left(\mathbb{S}^1 \times (0,1)^{m-1}\right) \\ S_{\alpha_{n+1}}(x) & \text{if } x \in G\left(\partial\left(\mathbb{S}^1 \times (0,1)^{m-1}\right)\right) \end{cases}$$

Constituted in [FK04, section 5.1] (which bases upon [Ka79, Proposition 1.1]), this sequence is convergent in the C^∞-topology to the diffeomorphism

$$\tilde{f} : M \to M \qquad \tilde{f}(x) = \begin{cases} G \circ f \circ G^{-1}(x) & \text{if } x \in G\left(\mathbb{S}^1 \times (0,1)^{m-1}\right) \\ S_\alpha(x) & \text{if } x \in G\left(\partial\left(\mathbb{S}^1 \times (0,1)^{m-1}\right)\right) \end{cases}$$

provided the closeness from f to R_α in the C^∞-topology.
We observe that f and $\tilde{f}$ are metrically isomorphic. Then $\tilde{f}$ is $(\kappa_1, \dots, \kappa_k)$-weakly mixing because the $(\kappa_1, \dots, \kappa_k)$-weak mixing-property is invariant under isomorphisms.
Altogether the construction done in the case of $\left(\mathbb{S}^1 \times [0,1]^{m-1}, \mathcal{R}, \mu\right)$ is transfered to $(M, \mathcal{S}, \nu)$.
Hence it suffices to consider constructions on $M = \mathbb{S}^1 \times [0,1]^{m-1}$ with circle action $\mathcal{R}$ subsequently. In this case we will prove the following result:

Proposition 2.2. *For every* $k \in \mathbb{N}$, $(\kappa_1, \dots, \kappa_k) \in \Pi_k$ *and every Liouvillean number* α *there is a sequence* $(\alpha_n)_{n\in\mathbb{N}}$ *of rational numbers* $\alpha_n = \frac{p_n}{q_n}$ *satisfying* $\lim_{n\to\infty} |\alpha - \alpha_n| = 0$ *monotonically and sequences* $(g_n)_{n\in\mathbb{N}}$, $(\phi_n)_{n\in\mathbb{N}}$ *of measure-preserving diffeomorphisms satisfying* $g_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ g_n$ *as well as* $\phi_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ \phi_n$ *such that the diffeomorphisms* $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ *with* $H_n = h_1 \circ h_2 \circ ... \circ h_n$*, where* $h_n = g_n \circ \phi_n$*, coincide with* $R_{\alpha_{n+1}}$ *in a neighbourhood of the boundary, converge in the* $\text{Diff}^\infty(M)$*-topology and the diffeomorphism* $f = \lim_{n\to\infty} f_n$ *is* $(\kappa_1, \dots, \kappa_k)$*-weakly mixing and satisfies* $f \in \mathcal{A}_\alpha(M)$*.*
Furthermore for every $\varepsilon > 0$ *the parameters in the construction can be chosen in such a way that* $d_\infty(f, R_\alpha) < \varepsilon$*.*

2.2 Outline of the proof

The constructions are based on the "approximation by conjugation"-method developed by D.V. Anosov and A. Katok in [AK70]. Here one constructs successively a sequence of volume-preserving diffeomorphisms $f_n = H_n \circ S_{\alpha_{n+1}} \circ H_n^{-1}$, where the conjugation maps $H_n = h_1 \circ ... \circ h_n$ and the rational numbers $\alpha_n = \frac{p_n}{q_n}$ are chosen in such a way that the functions f_n converge to a diffeomorphism f with the aimed properties.
First of all, we will define the conjugation map h_n as a composition of two volume-preserving diffeomorphisms $h_n = g_n \circ \phi_n$. Here, g_n is the shear $g_n(\theta, r_1, \ldots, r_{m-1}) = (\theta + nq_n \cdot r_1, r_1, \ldots, r_{m-1})$ and ϕ_n is a step-by-step defined smooth volume-preserving diffeomorphism. Its construction in section 3.1 bases on maps of the form $\phi_\lambda^{(j)} = C_\lambda^{-1} \circ \varphi_{1,j} \circ C_\lambda$ with C_λ being a stretching by $\lambda \in \mathbb{N}$ in the first coordinate and $\varphi_{1,j}$ a "quasi-rotation", i.e. a rotation by $\frac{\pi}{2}$ in the x_1-x_j-coordinates on large part of the domain. In fact, ϕ_n will be of the form $\phi_{\lambda_m}^{(m)} \circ \cdots \circ \phi_{\lambda_2}^{(2)}$ with explicitly chosen parameters $\lambda_j \in \mathbb{N}$, $\lambda_j < \lambda_{j+1}$, on the different sections. Moreover, we define a sequence of partial partitions η_n whose elements have such a small diameter that even the image under $H_{n-1} \circ g_n$ converges to the decomposition into points. We will prove the $(\kappa_1, \ldots, \kappa_k)$-weak mixing property on those partition elements.
Like the criteria for weak mixing in [FS05], [GKu] and [Ku] we use the concept of "almost uniform distribution". Descriptively, a set of small diameter is "almost uniformly distributed" under a diffeomorphism Φ if it is mapped to a set of small width in the θ-coordinate and almost full length in the $r_1, \ldots, r_{m-1}$-coordinates in an uniform way (see Definition 4.1 for the precise definition). In our case $\Phi_n = \phi_n \circ R^{m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}$ with a specific sequence $(m_n)_{n\in\mathbb{N}}$ of natural numbers such that $R^{m_n}_{\alpha_{n+1}}$ causes a translation by $\frac{1}{nq_n}$ to the adjacent domain of definition of the map ϕ_n (see section 4). In Lemma 4.5 we make the key observation that

$$\phi_{\mu_m}^{(m)} \circ \cdots \circ \phi_{\mu_2}^{(2)} \circ R^{i \cdot m_n}_{\alpha_{n+1}} \circ \left(\phi_{\lambda_2}^{(2)}\right)^{-1} \circ \left(\phi_{\lambda_m}^{(m)}\right)^{-1}$$

almost uniformly distributes an element of the partition η_n if $\mu_j > \lambda_j$ for each $j = 2, \ldots, m$. On the other hand, we observe that Φ_n^i acts approximately as the translation by $R_{\frac{i}{nq_n}}$ if $\mu_j = \lambda_j$. On this account, the choice of parameters λ_j in the definition of ϕ_n is exactly done in such a way that after a translation by $R_{\frac{i}{nq_n}}$ we have an increase of the λ-values on a portion of about κ_i of the partition element (see Lemma 4.6). Thus, we will have "almost uniform distribution" under Φ_n^i on a portion of about κ_i of the partition element. The subsequent application of the shear g_n will cause that this is almost uniformly distributed on the whole manifold $\mathbb{S}^1 \times [0,1]^{m-1}$. In section 5 we will establish the $(\kappa_1, \ldots, \kappa_k)$-weak mixing property in our construction.
In section 6 we will show convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{A}_\alpha$ for a given Liouville number α by the same approach as in [FS05]. For this purpose, we have to estimate the norms $|||H_n|||_k$ very carefully.

3 Explicit constructions

We fix $(\kappa_1, \ldots, \kappa_k) \in \Pi_k$ and an arbitrary Liouvillean number $\alpha \in \mathbb{S}^1$. We also introduce the notation $\tilde{\beta}_1 = \kappa_1$, $\tilde{\beta}_i = \kappa_i - \kappa_{i-1}$ for $i = 2, \ldots, k$. Obviously, $\sum_{i=1}^d \tilde{\beta}_i = \kappa_d$. In particular, we have

$\sum_{i=1}^{k}\tilde{\beta}_i = \kappa_k \leq 1$. By the requirements on tuples in Π_k we also have

$$\sum_{i=1}^{k}\tilde{\beta}_i > k\cdot\tilde{\beta}_2,\ \tilde{\beta}_i > \tilde{\beta}_{i+1} \text{ for every } i=2,\dots,k-1.$$

Moreover, let $\left(\tilde{\beta}_{i,n}\right)_{n\in\mathbb{N}}$ be a sequence of rational numbers $\tilde{\beta}_{i,n} = \frac{c_{i,n}}{d_n}$ satisfying $\tilde{\beta}_{i,n} \to \tilde{\beta}_i$ as $n\to\infty$ as well as the relations

$$\sum_{i=1}^{k}\tilde{\beta}_{i,n} > k\cdot\tilde{\beta}_{2,n},\ \tilde{\beta}_{i,n} > \tilde{\beta}_{i+1,n} \text{ for every } i=2,\dots,k-1,\ \sum_{i=1}^{k}\tilde{\beta}_{i,n}\leq 1.$$

With the aid of these we also introduce the numbers $u_{i,n}\in\mathbb{Z}$, $i=0,\dots,k-1$ such that

$$u_{i,n} = d_n\cdot\left(\tilde{\beta}_{2,n}-\tilde{\beta}_{2+i,n}\right) \text{ for } i=0,\dots,k-2,\quad u_{k-1,n} = d_n\cdot\tilde{\beta}_{2,n} = c_{2,n}. \tag{2}$$

In particular, we have $u_{0,n}=0$ and $u_{i+1,n} > u_{i,n}$.

Finally, we also introduce the numbers

$$\kappa_{i,n} = \sum_{d=1}^{i}\tilde{\beta}_{d,n} \qquad \text{for } i=1,\dots,k. \tag{3}$$

Since $\tilde{\beta}_{d,n}\to\tilde{\beta}_d$ for $n\to\infty$, we also have $\kappa_{i,n}\to\kappa_i$.

3.1 The conjugation map ϕ_n

The construction of the conjugation map ϕ_n bases on the following "pseudo-rotations" inspired by [FS05].

Lemma 3.1. *For every $\varepsilon\in\left(0,\frac{1}{4}\right)$ and every $i,j\in\{1,...,m\}$ there exists a smooth measure-preserving diffeomorphism $\varphi_{\varepsilon,i,j}$ on $\mathbb{R}^m$, which is the rotation in the x_i-x_j-plane by $\pi/2$ about the point $\left(\frac{1}{2},...,\frac{1}{2}\right)\in\mathbb{R}^m$ on $[2\varepsilon,1-2\varepsilon]^m$ and coincides with the identity outside of $[\varepsilon,1-\varepsilon]^m$.*

Proof. See [Ku16, Lemma 4.1]. □

Furthermore, for $\lambda\in\mathbb{N}$ we define the maps $C_\lambda(x_1,x_2,...,x_m) = (\lambda\cdot x_1,x_2,...,x_m)$. Using these maps we build the smooth measure-preserving diffeomorphism

$$\phi_{\lambda,\varepsilon,j}:\left[0,\frac{1}{\lambda}\right]\times[0,1]^{m-1}\to\left[0,\frac{1}{\lambda}\right]\times[0,1]^{m-1},\quad \phi_{\lambda,\varepsilon,j} = C_\lambda^{-1}\circ\varphi_{\varepsilon,1,j}\circ C_\lambda$$

Afterwards $\phi_{\lambda,\varepsilon,j}$ is extended to a diffeomorphism on $\mathbb{S}^1\times[0,1]^{m-1}$ by the description

$$\phi_{\lambda,\varepsilon,j}\left(x_1+\frac{1}{\lambda},x_2,...,x_m\right) = \left(\frac{1}{\lambda},0,...,0\right)+\phi_{\lambda,\varepsilon,j}(x_1,x_2,...,x_m).$$

For convenience we will use the notation $\phi_\lambda^{(j)} = \phi_{\lambda,\frac{1}{40n^4},j}$. By construction the map $\phi_{\lambda,\varepsilon,j}$ satisfies the following properties: This map satisfies the following properties:

Proposition 3.2. *Let $j \in \{2,...,m\}$, $\varepsilon \in \left(0, \frac{1}{4}\right)$ and $\lambda \in \mathbb{N}$. Moreover, let $t_0 \in \mathbb{Z}$, $\mu_s \in \mathbb{N}$, $t_s \in \mathbb{Z}$, $\lceil 2\varepsilon\mu_s \rceil \leq t_s < \mu_s - \lceil 2\varepsilon\mu_s \rceil$, for $s = 1,...,m$. Then we have*

1.

$$\phi^{-1}_{\lambda,\varepsilon,j}\left(\left[\frac{t_0}{\lambda} + \frac{t_1}{\lambda\cdot\mu_1}, \frac{t_0}{\lambda} + \frac{t_1+1}{\lambda\cdot\mu_1}\right] \times \prod_{s=2}^{m}\left[\frac{t_s}{\mu_s}, \frac{t_s+1}{\mu_s}\right]\right)$$
$$= \left[\frac{t_0}{\lambda} + \frac{t_j}{\lambda\mu_j}, \frac{t_0}{\lambda} + \frac{t_j+1}{\lambda\mu_j}\right] \times \prod_{s=2}^{j-1}\left[\frac{t_s}{\mu_s}, \frac{t_s+1}{\mu_s}\right] \times \left[1 - \frac{t_1+1}{\mu_1}, 1 - \frac{t_1}{\mu_1}\right] \times \prod_{s=j+1}^{m}\left[\frac{t_s}{\mu_s}, \frac{t_s+1}{\mu_s}\right]$$

2.

$$\phi_{\lambda,\varepsilon,j}\left(\left[\frac{t_0}{\lambda} + \frac{t_1}{\lambda\cdot\mu_1}, \frac{t_0}{\lambda} + \frac{t_1+1}{\lambda\cdot\mu_1}\right] \times \prod_{s=2}^{m}\left[\frac{t_s}{\mu_s}, \frac{t_s+1}{\mu_s}\right]\right)$$
$$= \left[\frac{t_0+1}{\lambda} - \frac{t_j+1}{\lambda\mu_j}, \frac{t_0+1}{\lambda} - \frac{t_j}{\lambda\mu_j}\right] \times \prod_{s=2}^{j-1}\left[\frac{t_s}{\mu_s}, \frac{t_s+1}{\mu_s}\right] \times \left[\frac{t_1}{\mu_1}, \frac{t_1+1}{\mu_1}\right] \times \prod_{s=j+1}^{m}\left[\frac{t_s}{\mu_s}, \frac{t_s+1}{\mu_s}\right]$$

In particular, we get

$$\phi_{\lambda,\varepsilon,j}\left(\left[\frac{t_0+2\varepsilon}{\lambda}, \frac{t_0+1-2\varepsilon}{\lambda}\right] \times \prod_{s=2}^{m}\left[\frac{t_s}{\mu_s}, \frac{t_s+1}{\mu_s}\right]\right)$$
$$= \left[\frac{t_0+1}{\lambda} - \frac{t_j+1}{\lambda\mu_j}, \frac{t_0+1}{\lambda} - \frac{t_j}{\lambda\mu_j}\right] \times \prod_{s=2}^{j-1}\left[\frac{t_s}{\mu_s}, \frac{t_s+1}{\mu_s}\right] \times [2\varepsilon, 1-2\varepsilon] \times \prod_{s=j+1}^{m}\left[\frac{t_s}{\mu_s}, \frac{t_s+1}{\mu_s}\right]$$

We start to define the diffeomorphism ϕ_n on the sector $\left[\frac{l}{n\cdot q_n} + \frac{v}{n^2\cdot q_n^2}, \frac{l}{n\cdot q_n} + \frac{v}{n^2\cdot q_n^2}\right] \times [0,1]^{m-1}$ for each $l, v \in \mathbb{Z}$, $0 \leq l < n$, $0 \leq v < nq_n$.
In a first step, we consider domains of the form

$$\Delta_{l,v,s,u} =$$
$$\left[\frac{l}{nq_n} + \frac{v}{n^2\cdot q_n^2} + \frac{s\cdot c_{2,n}}{d_n\cdot n^2\cdot q_n^2} + \frac{u}{d_n\cdot n^2\cdot q_n^2}, \frac{l}{nq_n} + \frac{v}{n^2\cdot q_n^2} + \frac{s\cdot c_{2,n}}{d_n\cdot n^2\cdot q_n^2} + \frac{u+1}{d_n\cdot n^2\cdot q_n^2}\right] \times [0,1]^{m-1},$$

where $s, u \in \mathbb{Z}$, $0 \leq s < k$ and $0 \leq u < c_{2,n}$.
We define $\tilde{s} \equiv -l \mod k$. Then for every $s \in \mathbb{Z}$, $0 \leq s < k$, there is a unique $t \in \{0, 1, \dots, k-1\}$ such that $s \equiv \tilde{s} + t \mod k$. Recall the numbers $u_{i,n}$, $0 \leq i < k-1$, from equation 2.

- If the number $u \in \mathbb{Z}$, $0 \leq u < c_{2,n}$, satisfies $u_{i,n} \leq u < u_{i+1,n}$ with $i \geq t-1$, then we put
$$\phi_n = \phi^{(m)}_{d_n\cdot(nq_n)^{2\cdot(m-1)\cdot(l+k-t)}} \circ \dots \circ \phi^{(3)}_{d_n\cdot(nq_n)^{2\cdot 2\cdot(l+k-t)}} \circ \phi^{(2)}_{d_n\cdot(nq_n)^{2\cdot(l+k-t)}}$$
on the domain $\Delta_{l,v,s,u}$.
- If the number $u \in \mathbb{Z}$, $0 \leq u < c_{2,n}$, satisfies $u_{i,n} \leq u < u_{i+1,n}$ with $i < t-1$, then we put
$$\phi_n = \phi^{(m)}_{d_n\cdot(nq_n)^{2\cdot(m-1)\cdot(l+k)}} \circ \dots \circ \phi^{(3)}_{d_n\cdot(nq_n)^{2\cdot 2\cdot(l+k)}} \circ \phi^{(2)}_{d_n\cdot(nq_n)^{2\cdot(l+k)}}$$
on the domain $\Delta_{l,v,s,u}$.

In the next step we recall the requirements $\sum_{i=1}^{k} \tilde{\beta}_{i,n} > k \cdot \tilde{\beta}_{2,n}$ and $\sum_{i=1}^{k} \tilde{\beta}_{i,n} \leq 1$. Then we put

$$\phi_n = \phi^{(m)}_{d_n \cdot (nq_n)^{2 \cdot (m-1) \cdot (l+k)}} \circ \cdots \circ \phi^{(3)}_{d_n \cdot (nq_n)^{2 \cdot 2 \cdot (l+k)}} \circ \phi^{(2)}_{d_n \cdot (nq_n)^{2 \cdot (l+k)}}$$

on the domain

$$\left[\frac{l}{n \cdot q_n} + \frac{v}{n^2 \cdot q_n^2} + \frac{k \cdot c_{2,n}}{d_n \cdot n^2 \cdot q_n^2}, \frac{l}{n \cdot q_n} + \frac{v}{n^2 \cdot q_n^2} + \frac{\sum_{i=1}^{k} c_{i,n}}{d_n \cdot n^2 \cdot q_n^2} \right] \times [0,1]^{m-1}.$$

Finally, we put

$$\phi_n = \mathrm{id}$$

on the domain $\left[\frac{l}{n \cdot q_n} + \frac{v}{n^2 \cdot q_n^2} + \frac{\sum_{i=1}^{k} c_{i,n}}{d_n \cdot n^2 \cdot q_n^2}, \frac{l}{n \cdot q_n} + \frac{v+1}{n^2 \cdot q_n^2} \right] \times [0,1]^{m-1}$.

This is a smooth map because ϕ_n coincides with the identity in a neighbourhood of the different sections.

Hereby, we have defined the diffeomorphism ϕ_n on the fundamental sector $\left[0, \frac{1}{q_n}\right] \times [0,1]^{m-1}$. Now we extend ϕ_n to a diffeomorphism on $\mathbb{S}^1 \times [0,1]^{m-1}$ using the description $\phi_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ \phi_n$.

Example 3.3. Let $\kappa_1 = \frac{1}{2}$, $\kappa_2 = \frac{3}{4}$ and $\kappa_3 = \frac{7}{8}$. Since $\kappa_3 - \kappa_2 = \frac{1}{8} < \frac{1}{4} = \kappa_2 - \kappa_1$ and $3 \cdot (\kappa_2 - \kappa_1) = \frac{3}{4} < \frac{7}{8} = \kappa_3$ we have $(\kappa_1, \kappa_2, \kappa_3) \in \Pi_3$. In Figure 1 we list the powers γ of $\phi^{(2)}_{d_n \cdot (nq_n)^{\gamma}}$ on a domain $\left[\frac{l}{n \cdot q_n} + \frac{v}{n^2 \cdot q_n^2}, \frac{l}{n \cdot q_n} + \frac{v+1}{n^2 \cdot q_n^2} \right] \times [0,1]$ for different values of $l \in \mathbb{Z}$.

		$\frac{1}{8}$	$\frac{1}{4}$		$\frac{1}{2}$		$\frac{3}{4}$	
$l=0$	$2\cdot3$	$2\cdot3$	$2\cdot2$	$2\cdot2$	$2\cdot3$	$2\cdot1$	$2\cdot3$	id
$l=1$	$2\cdot3$	$2\cdot3$	$2\cdot4$	$2\cdot2$	$2\cdot4$	$2\cdot4$	$2\cdot4$	id
$l=2$	$2\cdot5$	$2\cdot3$	$2\cdot5$	$2\cdot5$	$2\cdot4$	$2\cdot4$	$2\cdot5$	id
$l=3$	$2\cdot6$	$2\cdot6$	$2\cdot5$	$2\cdot5$	$2\cdot6$	$2\cdot4$	$2\cdot6$	id
$l=4$	$2\cdot6$	$2\cdot6$	$2\cdot7$	$2\cdot5$	$2\cdot7$	$2\cdot7$	$2\cdot7$	id

Figure 1: List of powers γ of $\phi^{(2)}_{d_n \cdot (nq_n)^{\gamma}}$ on a domain $\left[\frac{l}{n \cdot q_n} + \frac{v}{n^2 \cdot q_n^2}, \frac{l}{n \cdot q_n} + \frac{v+1}{n^2 \cdot q_n^2} \right] \times [0,1]$ for different values of $l \in \mathbb{Z}$. In the horizontal direction we have the portions of the length $\frac{1}{n^2 \cdot q_n^2}$ of the domain.

3.2 The conjugation map h_n

We define the conjugation map as the composition

$$h_n = g_n \circ \phi_n,$$

where

$$g_n(\theta, r_1, \dots, r_{m-1}) = (\theta + nq_n \cdot r_1, r_1, \dots, r_{m-1}).$$

3.3 Partial partition η_n

In this subsection we define the announced sequence of partial partitions $(\eta_n)_{n\in\mathbb{N}}$ of $M = \mathbb{S}^1 \times [0,1]^{m-1}$.

Remark 3.4. For convenience we will use the notation $\prod_{i=2}^{m}[a_i, b_i]$ for $[a_2, b_2] \times ... \times [a_m, b_m]$

Initially η_n will be constructed on the fundamental sector $\left[0, \frac{1}{q_n}\right] \times [0,1]^{m-1}$. We start by considering the following sets: In the θ-coordinate we define

$$\tilde{I}_{l,v,u,t_1^{(2)},\dots,t_1^{(m\cdot(n-1+k))}} :=$$
$$\left[\frac{l}{nq_n} + \frac{v}{(nq_n)^2} + \frac{u}{d_n \cdot (nq_n)^2} + \frac{t_1^{(2)}}{d_n \cdot (nq_n)^{2\cdot 2}} + ... + \frac{t_1^{(m\cdot(n-1+k))}}{d_n \cdot (nq_n)^{2m\cdot(n-1+k)}} + \frac{1}{10n^4 \cdot d_n \cdot (nq_n)^{2m\cdot(n-1+k)}},\right.$$
$$\left.\frac{l}{nq_n} + \frac{v}{(nq_n)^2} + \frac{u}{d_n \cdot (nq_n)^2} + \frac{t_1^{(2)}}{d_n \cdot (nq_n)^{2\cdot 2}} + ... + \frac{t_1^{(m\cdot(n-1+k))}+1}{d_n \cdot (nq_n)^{2m\cdot(n-1+k)}} - \frac{1}{10n^4 \cdot d_n \cdot (nq_n)^{2m\cdot(n-1+k)}}\right]$$

and

$$\tilde{I}^1_{l,v} := \left[\frac{l}{n\cdot q_n} + \frac{v + \sum_{d=1}^{k}\tilde{\beta}_{d,n}}{n^2\cdot q_n^2} + \frac{1-\sum_{d=1}^{k}\tilde{\beta}_{d,n}}{2\cdot n^2 \cdot q_n^2}\cdot\left(1 - \left(1 - \frac{1}{5n^4}\right)^{m\cdot(n-1+k)-1}\right),\right.$$
$$\left.\frac{l}{n\cdot q_n} + \frac{v+1}{n^2\cdot q_n^2} - \frac{1-\sum_{d=1}^{k}\tilde{\beta}_{d,n}}{2\cdot n^2 \cdot q_n^2}\cdot\left(1 - \left(1 - \frac{1}{5n^4}\right)^{m\cdot(n-1+k)-1}\right)\right].$$

In the $\vec{r}$-coordinates we define

$$W_{j_2^{(1)},j_2^{(2)},t_2^{(2)},\dots,t_2^{(n-1+k)},j_3,t_3^{(2)},\dots,t_3^{(n-1+k)},\dots,j_m,t_m^{(2)},\dots,t_m^{(n-1+k)}} :=$$
$$\left[\frac{j_2^{(1)}}{n\cdot q_n^2} + \frac{j_2^{(2)}}{n^2\cdot q_n^2} + \frac{t_2^{(2)}}{(nq_n)^{2\cdot 2}} + ... + \frac{t_2^{(n-1+k)}}{(nq_n)^{2\cdot(n-1+k)}} + \frac{1}{10n^4\cdot(nq_n)^{2\cdot(n-1+k)}},\right.$$
$$\left.\frac{j_2^{(1)}}{n\cdot q_n^2} + \frac{j_2^{(2)}}{n^2\cdot q_n^2} + \frac{t_2^{(2)}}{(nq_n)^{2\cdot 2}} + ... + \frac{t_2^{(n-1+k)}+1}{(nq_n)^{2\cdot(n-1+k)}} - \frac{1}{10n^4\cdot(nq_n)^{2\cdot(n-1+k)}}\right]$$
$$\times \prod_{i=3}^{m}\left[\frac{j_i}{n^2\cdot q_n^2} + \frac{t_i^{(2)}}{(nq_n)^{2\cdot 2}} + ... + \frac{t_i^{(n-1+k)}}{(nq_n)^{2\cdot(n-1+k)}} + \frac{1}{10n^4\cdot(nq_n)^{2\cdot(n-1+k)}},\right.$$
$$\left.\frac{j_i+1}{n^2\cdot q_n^2} + \frac{t_i^{(2)}}{(nq_n)^{2\cdot 2}} + ... + \frac{t_i^{(n-1+k)}+1}{(nq_n)^{2\cdot(n-1+k)}} - \frac{1}{10n^4\cdot(nq_n)^{2\cdot(n-1+k)}}\right]$$

In case of $l, v \in \mathbb{Z}$, $0 \leq l < n$ and $0 \leq v < nq_n$, we define

$$I_{l,v,j_2^{(1)},j_2^{(2)},j_3,\dots,j_m} := \bigcup \left(\tilde{I}_{l,v,u,t_1^{(2)},\dots,t_1^{(m\cdot(n-1+k))}} \cup \tilde{I}^1_{l,v} \right) \times W_{j_2^{(1)},j_2^{(2)},t_2^{(2)},\dots,t_2^{(n-1+k)},j_3,t_3^{(2)},\dots,t_3^{(n-1+k)},\dots,j_m,t_m^{(2)},\dots,t_m^{(n-1+k)}},$$

where the union is taken over

- $u \in \mathbb{Z}$, $0 \leq u < \sum_{d=1}^{k} c_{d,n}$
- $t_1^{(s)} \in \mathbb{Z}$, $\left\lceil \frac{q_n^2}{10n^2} \right\rceil \leq t_1^{(s)} \leq n^2 q_n^2 - \left\lceil \frac{q_n^2}{10n^2} \right\rceil - 1$, for $s = 2, ..., m \cdot (n-1+k)$
- $t_i^{(s)} \in \mathbb{Z}$, $\left\lceil \frac{q_n^2}{10n^2} \right\rceil \leq t_i^{(s)} \leq n^2 q_n^2 - \left\lceil \frac{q_n^2}{10n^2} \right\rceil - 1$, for $s = 2, ..., n-1+k$ and $i = 2, \dots, m$.

In Lemma 6.9 we will choose q_n as a multiple of $10n^2$. In particular, $\left\lceil \frac{q_n^2}{10n^2} \right\rceil = \frac{q_n^2}{10n^2}$.

Remark 3.5. Descriptively $I_{l,v,j_2^{(1)},j_2^{(2)},j_3,\dots,j_m}$ is the cube

$$\left[\frac{l}{nq_n} + \frac{v}{n^2 \cdot q_n^2}, \frac{l}{nq_n} + \frac{v+1}{n^2 \cdot q_n^2} \right] \times \left[\frac{j_2^{(1)}}{n \cdot q_n} + \frac{j_2^{(2)}}{n^2 \cdot q_n^2}, \frac{j_2^{(1)}}{n \cdot q_n} + \frac{j_2^{(2)}+1}{n^2 \cdot q_n^2} \right] \times \prod_{i=3}^{m} \left[\frac{j_i}{n^2 \cdot q_n^2}, \frac{j_i+1}{n^2 \cdot q_n^2} \right]$$

with some holes. These holds are inserted in order to guarantee that $I_{l,v,j_2^{(1)},j_2^{(2)},j_3,\dots,j_m}$ belongs to the "good domain" of ϕ_n^{-1} and also of ϕ_n after a translation by $\frac{i}{nq_n}$ on the θ-axis. This property will be exploited in the proof of Lemma 4.5.

With the aid of these sets we define our partial partition η_n:

- On $\left[\frac{l}{n \cdot q_n}, \frac{l+1}{n \cdot q_n} \right] \times [0,1]^{m-1}$, $0 \leq l < n-k$, the partial partition η_n consists of all such sets $I_{l,v,j_2^{(1)},j_2^{(2)},j_3,\dots,j_m}$, at which $j_i \in \mathbb{Z}$ and $\left\lceil \frac{q_n^2}{10n^2} \right\rceil \leq j_i \leq n^2 q_n^2 - \left\lceil \frac{q_n^2}{10n^2} \right\rceil - 1$ for $i = 3, ..., m$ and $v \in \mathbb{Z}$, $0 \leq v \leq n \cdot q_n - 1$, as well as $j_2^{(1)} \in \mathbb{Z}$, $\left\lceil \frac{q_n^2}{10n^3} \right\rceil \leq j_2^{(1)} \leq nq_n^2 - \left\lceil \frac{q_n^2}{10n^3} \right\rceil - 1$ as well as $j_2^{(2)} \in \mathbb{Z}$, $0 \leq j_2^{(2)} \leq n-1$ apart from those $j_2^{(2)}$ satisfying

$$l + j_2^{(2)} \equiv n - s \mod n \quad \text{for} \quad s = 1, 2, \dots, k+1.$$

- On $\left[\frac{n-k}{n \cdot q_n}, \frac{1}{q_n} \right] \times [0,1]^{m-1}$ there are no elements of the partial partition η_n.

As the image under R_{a/q_n} with $a \in \mathbb{Z}$ this partial partition of $\left[0, \frac{1}{q_n}\right] \times [0,1]^{m-1}$ is extended to a partial partition of $\mathbb{S}^1 \times [0,1]^{m-1}$.

Remark 3.6. By construction this sequence of partial partitions converges to the decomposition into points.

Remark 3.7. In the following we will often write a partition element in the comprehensive form $\hat{I}_n = \bigcup_{j=0}^{N} I_j \times W$, at which $W = \pi_{\vec{r}} \left(\hat{I}_n \right)$ is the $m-1$-dimensional projection of $\hat{I}_n$ in the $r_1, \dots, r_{m-1}$-coordinates and we have the following sections I_j on the θ-axis: If $\phi_n = \phi^{(m)}_{d_n \cdot (nq_n)^{2 \cdot (m-1) \cdot T}} \circ \dots \circ \phi^{(3)}_{d_n \cdot (nq_n)^{2 \cdot 2 \cdot T}} \circ \phi^{(2)}_{d_n \cdot (nq_n)^{2 \cdot T}}$ on $\left[\frac{l}{nq_n} + \frac{v}{n^2 \cdot q_n^2} + \frac{u}{d_n \cdot n^2 \cdot q_n^2}, \frac{l}{nq_n} + \frac{v}{n^2 \cdot q_n^2} + \frac{u+1}{d_n \cdot n^2 \cdot q_n^2} \right] \times$

$[0,1]^{m-1}$, then sections on this domain are given by $I_j = \bigcup \tilde{I}_{l,v,u,t_1^{(2)},\dots,t_1^{(m\cdot(n-1+k)}}$, where the union is taken over the allowed values of $t_1^{(T+1)},\dots,t_1^{(m\cdot(n-1+k))}$.
Finally, I_N corresponds to the section $\tilde{I}^1_{l,v}$, i. e.

$$\left[\frac{l}{n\cdot q_n}+\frac{v+\sum_{d=1}^{k}\tilde{\beta}_{d,n}}{n^2\cdot q_n^2}+\frac{1-\sum_{d=1}^{k}\tilde{\beta}_{d,n}}{2\cdot n^2\cdot q_n^2}\cdot\left(1-\left(1-\frac{1}{5n^4}\right)^{m\cdot(n-1+k)-1}\right),\right.$$
$$\left.\frac{l}{n\cdot q_n}+\frac{v+1}{n^2\cdot q_n^2}-\frac{1-\sum_{d=1}^{k}\tilde{\beta}_{d,n}}{2\cdot n^2\cdot q_n^2}\cdot\left(1-\left(1-\frac{1}{5n^4}\right)^{m\cdot(n-1+k)-1}\right)\right]$$

Note that the partition elements are constructed in such a way that

$$\mu\left(I_N\times W\right)=\left(1-\sum_{d=1}^{k}\tilde{\beta}_{d,n}\right)\cdot\mu\left(\hat{I}_n\right).$$

Remark 3.8. For an element $\hat{I}_n = \bigcup_{j=0}^{N} I_j \times W$ of the partition η_n we observe:

$$\phi_n|_{I_N\times W}=\mathrm{id}$$

Remark 3.9. The additional restrictions on $j_2^{(2)}$ will be helpful in Remark 5.5.

4 $(\gamma,\delta,s,\epsilon)$-distribution

We introduce the central notion in the proof of the criterion for $(\kappa_1,\dots,\kappa_k)$-weak mixing deduced in the next section:

Definition 4.1. Let $\Phi : M \to M$ be a diffeomorphism. We say Φ $(\gamma,\delta,s_1,s,\epsilon)$-distributes a set $\hat{I}$, if the following properties are satisfied:

- $\Phi\left(\hat{I}\right)$ is contained in a set of the form $[c,c+\gamma]\times[\delta,1-\delta]^{m-1}$ for some $c\in\mathbb{S}^1$.
- For every $(m-1)$-dimensional cuboid $\tilde{J}\subseteq J$ of r_1-length at least s_1 and of side length s in the $r_2,\dots,r_{m-1}$-coordinates it holds:

  $$\left|\mu\left(\hat{I}\cap\Phi^{-1}\left(\mathbb{S}^1\times\tilde{J}\right)\right)-\mu\left(\hat{I}\right)\cdot\mu^{(m-1)}\left(\tilde{J}\right)\right|\leq\epsilon\cdot\mu\left(\hat{I}\right)\cdot\mu^{(m-1)}\left(\tilde{J}\right).$$

  where $\mu^{(m-1)}$ is the Lebesgue measure on $[0,1]^{m-1}$.

Remark 4.2. Inspired by [FS05] we will call the second property "almost uniform distribution" of $\hat{I}$ in the $r_1,..,r_{m-1}$-coordinates.

In the next step we define the sequence of natural numbers $(m_n)_{n\in\mathbb{N}}$:

$$m_n=\min\left\{m\leq q_{n+1}\quad:\quad m\in\mathbb{N},\ \inf_{k\in\mathbb{Z}}\left|m\cdot\frac{p_{n+1}}{q_{n+1}}-\frac{1}{n\cdot q_n}+k\right|\leq\frac{10n^2}{q_{n+1}}\right\}$$

Lemma 4.3. *The set* $M_n := \left\{ m \leq q_{n+1} \quad : \quad m \in \mathbb{N}, \quad \inf_{k\in\mathbb{Z}} \left| m \cdot \frac{p_{n+1}}{q_{n+1}} - \frac{1}{n \cdot q_n} + k \right| \leq \frac{10n^2}{q_{n+1}} \right\}$ *is nonempty for every* $n \in \mathbb{N}$*, i.e.* m_n *exists.*

Proof. In Lemma 6.9 we will construct the sequence $\alpha_n = \frac{p_n}{q_n}$ in such a way, that $q_n = 10n^2 \cdot \tilde{q}_n$ and $p_n = 10n^2 \cdot \tilde{p}_n$ with $\tilde{p}_n, \tilde{q}_n$ relatively prime. Therefore the set $\left\{ j \cdot \frac{p_{n+1}}{q_{n+1}} \ : \ j = 1, 2, ..., q_{n+1} \right\}$ contains $\frac{q_{n+1}}{10n^2}$ different equally distributed points on $\mathbb{S}^1$. Hence for every $x \in \mathbb{S}^1$ there is a $j \in \{1, ..., q_{n+1}\}$, such that $\inf_{k\in\mathbb{Z}} \left| x - j \cdot \frac{p_{n+1}}{q_{n+1}} + k \right| \leq \frac{10n^2}{q_{n+1}}$. In particular this is true for $x = \frac{1}{n \cdot q_n}$. □

Remark 4.4. We define

$$a_n = \left(m_n \cdot \frac{p_{n+1}}{q_{n+1}} - \frac{1}{n \cdot q_n} \right) \mod 1$$

By the above construction of m_n it holds: $|a_n| \leq \frac{10n^2}{q_{n+1}}$. In Lemma 6.9 we will see that it is possible to choose $q_{n+1} \geq 200 \cdot n^2 \cdot k \cdot d_n^2 \cdot n^{4\cdot(n+k)} \cdot q_n^{4\cdot(n-1+k)}$. Thus we get:

$$|a_n| \leq \frac{1}{20n^4 \cdot k \cdot d_n^2 \cdot (n \cdot q_n)^{4\cdot(n-1+k)}}.$$

By this choice of the number m_n, $R^{m_n}_{\alpha_{n+1}}$ causes a translation to the adjacent $\frac{1}{nq_n}$-domain of definition of the map ϕ_n. On such a domain the elements $\hat{I}_n$ of the partial partition η_n are positioned in such a way that all $\varphi^{-1}_{\varepsilon,1,j}$ act as the particular rotations. On the adjacent section, the stretching parameters λ_j in $\phi^{(j)}_{\lambda_j}$ are chosen so that either ϕ_n is of the same form as before or ϕ_n maps $R^{m_n}_{\alpha_{n+1}} \circ \phi_n^{-1} \left(\hat{I}_n \right)$ to a set of almost full length in the $r_1, ..., r_{m-1}$-coordinates. We make this precise in the subsequent lemma.

Lemma 4.5. *We consider a set* I_n *belonging to a partition element of* η_n *of the form*

$$\bigcup \left[\frac{t_1^{(T+1)}}{d_n \cdot (n \cdot q_n)^{2\cdot(T+1)}} + ... + \frac{t_1^{(m\cdot(n-1+k))}}{d_n \cdot (n \cdot q_n)^{2\cdot m\cdot(n-1+k)}} + \frac{1}{10n^4 \cdot d_n \cdot (n \cdot q_n)^{2\cdot m\cdot(n-1+k)}}, \right.$$
$$\left. \frac{t_1^{(T+1)}}{d_n \cdot (n \cdot q_n)^{2\cdot(T+1)}} + ... + \frac{t_1^{(m\cdot(n-1+k))} + 1}{d_n \cdot (n \cdot q_n)^{2\cdot m\cdot(n-1+k)}} - \frac{1}{10n^4 \cdot d_n \cdot (n \cdot q_n)^{2\cdot m\cdot(n-1+k)}} \right]$$
$$\times \prod_{i=2}^{m} \left[\frac{j_i}{(nq_n)^2} + \frac{t_i^{(2)}}{(nq_n)^{2\cdot 2}} + ... + \frac{t_i^{(n-1+k)}}{(nq_n)^{2\cdot(n-1+k)}} + \frac{1}{10n^4 \cdot (nq_n)^{2\cdot(n-1+k)}}, \right.$$
$$\left. \frac{j_i}{(nq_n)^2} + \frac{t_i^{(2)}}{(nq_n)^{2\cdot 2}} + ... + \frac{t_i^{(n-1+k)} + 1}{(nq_n)^{2\cdot(n-1+k)}} - \frac{1}{10n^4 \cdot (nq_n)^{2\cdot(n-1+k)}} \right],$$

where the union is taken over all occurring $t_i^{(j)}$*, and assume that*

$$\phi_n = \phi^{(m)}_{d_n\cdot(nq_n)^{2\cdot(m-1)\cdot T}} \circ \cdots \circ \phi^{(3)}_{d_n\cdot(nq_n)^{2\cdot 2\cdot T}} \circ \phi^{(2)}_{d_n\cdot(nq_n)^{2\cdot T}}$$

on it. After an application of $R^{l\cdot m_n}_{\alpha_{n+1}}$ *with some* $l \in \{1, \ldots, k\}$ *the image* $R^{l\cdot m_n}_{\alpha_{n+1}} \circ \phi_n^{-1} (I_n)$ *lies in a domain where*

$$\phi_n = \phi^{(m)}_{d_n\cdot(nq_n)^{2\cdot(m-1)\cdot U}} \circ \cdots \circ \phi^{(3)}_{d_n\cdot(nq_n)^{2\cdot 2\cdot U}} \circ \phi^{(2)}_{d_n\cdot(nq_n)^{2\cdot U}}.$$

1. *If $U=T$, then $\phi_n \circ R^{l\cdot m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}(I_n)$ is contained in the cube*

$$\left[\frac{l}{nq_n}, \frac{l}{nq_n} + \frac{1}{(nq_n)^2}\right] \times \prod_{i=2}^{m} \left[\frac{j_i}{n^2 \cdot q_n^2}, \frac{j_i+1}{n^2 \cdot q_n^2}\right].$$

   In particular, $\pi_{\vec{r}}\left(\phi_n \circ R^{l\cdot m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}(I_n)\right)$ is contained in the same $\frac{1}{(nq_n)^2}$-cube as $\pi_{\vec{r}}(I_n)$.

2. *If $U>T$, then $\phi_n \circ R^{l\cdot m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}$ is $\left(\frac{1}{d_n\cdot(nq_n)^2}, \frac{1}{10n^4}, \frac{1}{nq_n^2}, \frac{1}{q_n}, \frac{3}{n}\right)$-distributing the set I_n.*

Proof. When applying the map ϕ_n^{-1} we observe that the set is positioned in such a way that all the occurring maps $\varphi^{-1}_{\varepsilon,1,j}$ act as the respective rotations. Then we compute $\phi_n^{-1}(I_n)$ with the aid of Proposition 3.2:

$$\bigcup \Bigg[\frac{j_2}{d_n\cdot(nq_n)^{2(T+1)}} + \frac{t_2^{(2)}}{d_n\cdot(nq_n)^{2(T+2)}} + \dots + \frac{t_2^{(n-1+k)}}{d_n\cdot(nq_n)^{2(n-1+k+T)}} + \frac{1}{10n^4\cdot d_n\cdot(nq_n)^{2(n-1+k+T)}},$$
$$\frac{j_2}{d_n\cdot(nq_n)^{2(T+1)}} + \frac{t_2^{(2)}}{d_n\cdot(nq_n)^{2(T+2)}} + \dots + \frac{t_2^{(n-1+k)}+1}{d_n\cdot(nq_n)^{2(n-1+k+T)}} - \frac{1}{10n^4\cdot d_n\cdot(nq_n)^{2(n-1+k+T)}}\Bigg]$$
$$\times \prod_{i=2}^{m-1}\Bigg[1 - \frac{t_1^{((i-1)T+1)}}{(nq_n)^2} - \dots - \frac{t_1^{(iT)}}{(nq_n)^{2T}} - \frac{j_{i+1}}{(nq_n)^{2\cdot(T+1)}} - \frac{t_{i+1}^{(2)}}{(nq_n)^{2\cdot(T+2)}} - \dots - \frac{t_{i+1}^{(n-1+k)}+1}{(nq_n)^{2(n-1+k+T)}}$$
$$+ \frac{1}{10n^4\cdot(nq_n)^{2\cdot(n-1+k+T)}},$$
$$1 - \frac{t_1^{((i-1)T+1)}}{(nq_n)^2} - \dots - \frac{t_{i+1}^{(n-1+k)}}{(nq_n)^{2\cdot(n-1+k+T)}} - \frac{1}{10n^4\cdot(nq_n)^{2\cdot(n-1+k+T)}}\Bigg]$$
$$\times \Bigg[1 - \frac{t_1^{((m-1)\cdot T+1)}}{(nq_n)^2} - \dots - \frac{t_1^{(m\cdot(n-1+k))}+1}{(nq_n)^{2\cdot m\cdot(n-1+k)-2\cdot(m-1)\cdot T}} + \frac{1}{10n^4\cdot(nq_n)^{2\cdot m\cdot(n-1+k)-2\cdot(m-1)\cdot T}},$$
$$1 - \frac{t_1^{((m-1)\cdot T+1)}}{(nq_n)^2} - \dots - \frac{t_1^{(m\cdot(n-1+k))}}{(nq_n)^{2\cdot m\cdot(n-1+k)-2\cdot(m-1)\cdot T}} - \frac{1}{10n^4\cdot(nq_n)^{2\cdot m\cdot(n-1+k)-2\cdot(m-1)\cdot T}}\Bigg].$$

By our choice of the number m_n the subsequent application of $R^{l\cdot m_n}_{\alpha_{n+1}}$ yields a shift by $\frac{l}{nq_n} + la_n$ on the θ-axis, at which a_n is the "error term" introduced in Remark 4.4. With the aid of the bound on la_n from Remark 4.4 we can compute the image of I_n under $\Phi_n := \phi_n \circ R^{l\cdot m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}$. In the first case (i. e. $U=T$) we get

$$\bigcup\left[\frac{l}{nq_n}+\frac{t_1^{(T+1)}}{d_n\cdot(n\cdot q_n)^{2\cdot(T+1)}}+...+\frac{t_1^{(m\cdot(n-1+k))}}{d_n\cdot(n\cdot q_n)^{2\cdot m\cdot(n-1+k)}}+\frac{1}{10n^4\cdot d_n\cdot(n\cdot q_n)^{2\cdot m\cdot(n-1+k)}},\right.$$

$$\left.\frac{l}{nq_n}+\frac{t_1^{(T+1)}}{d_n\cdot(n\cdot q_n)^{2\cdot(T+1)}}+...+\frac{t_1^{(m\cdot(n-1+k))}+1}{d_n\cdot(n\cdot q_n)^{2\cdot m\cdot(n-1+k)}}-\frac{1}{10n^4\cdot d_n\cdot(n\cdot q_n)^{2\cdot m\cdot(n-1+k)}}\right]$$

$$\times\left[\frac{j_2}{(nq_n)^2}+\frac{t_2^{(2)}}{(nq_n)^{2\cdot 2}}+...+\frac{t_2^{(n-1+k)}}{(nq_n)^{2\cdot(n-1+k)}}+\frac{1}{10n^4\cdot(nq_n)^{2\cdot(n-1+k)}}+d_n\cdot(nq_n)^{2T}\cdot la_n,\right.$$

$$\left.\frac{j_2}{(nq_n)^2}+\frac{t_2^{(2)}}{(nq_n)^{2\cdot 2}}+...+\frac{t_2^{(n-1+k)}+1}{(nq_n)^{2\cdot(n-1+k)}}-\frac{1}{10n^4\cdot(nq_n)^{2\cdot(n-1+k)}}+d_n\cdot(nq_n)^{2T}\cdot la_n\right]$$

$$\times\prod_{i=3}^{m}\left[\frac{j_i}{(nq_n)^2}+\frac{t_i^{(2)}}{(nq_n)^{2\cdot 2}}+...+\frac{t_i^{(n-1+k)}}{(nq_n)^{2\cdot(n-1+k)}}+\frac{1}{10n^4\cdot(nq_n)^{2\cdot(n-1+k)}},\right.$$

$$\left.\frac{j_i}{(nq_n)^2}+\frac{t_i^{(2)}}{(nq_n)^{2\cdot 2}}+...+\frac{t_i^{(n-1+k)}+1}{(nq_n)^{2\cdot(n-1+k)}}-\frac{1}{10n^4\cdot(nq_n)^{2\cdot(n-1+k)}}\right],$$

In the second case $U=T+S>T$ we calculate $\phi_n\circ R_{\alpha_{n+1}}^{l\cdot m_n}\circ\phi_n^{-1}(I_n)$ to

$$\bigcup\left[\frac{l}{nq_n}+\frac{j_2}{d_n\cdot(nq_n)^{2\cdot(T+1)}}+\frac{t_2^{(2)}}{d_n\cdot(nq_n)^{2\cdot(T+2)}}+...+\frac{t_2^{(S)}}{d_n\cdot(nq_n)^{2U}}+\frac{t_1^{(T+1)}}{d_n\cdot(nq_n)^{2\cdot(U+1)}}+...\right.$$

$$+\frac{t_1^{(2T)}}{d_n\cdot(nq_n)^{2\cdot(U+T)}}+\frac{j_3}{d_n\cdot(nq_n)^{2\cdot(U+T+1)}}+\frac{t_3^{(2)}}{d_n\cdot(nq_n)^{2\cdot(U+T+2)}}+...+\frac{t_3^{(S)}}{d_n\cdot(nq_n)^{2\cdot 2U}}$$

$$+\frac{t_1^{(2T+1)}}{d_n\cdot(nq_n)^{2\cdot(2U+1)}}+...+\frac{t_1^{(3T)}}{d_n\cdot(nq_n)^{2\cdot(2U+T)}}+\frac{j_4}{d_n\cdot(nq_n)^{2\cdot(2U+T+1)}}+...+\frac{t_1^{((m-1)\cdot T+1)}}{d_n\cdot(nq_n)^{2((m-1)\cdot U+1)}}$$

$$+...+\frac{t_1^{(m\cdot(n-1+k))}}{d_n\cdot(nq_n)^{2\cdot m\cdot(n-1+k)+2\cdot(m-1)\cdot(U-T)}}+\frac{1}{10n^4\cdot d_n\cdot(nq_n)^{2m\cdot(n-1+k)+2\cdot(m-1)\cdot(U-T)}},$$

$$\left.\frac{l}{nq_n}+\frac{j_2}{d_n\cdot(nq_n)^{2\cdot(T+1)}}+...+\frac{t_1^{(m\cdot(n-1+k))}+1-\frac{1}{10n^4}}{d_n\cdot(nq_n)^{2\cdot m\cdot(n-1+k)+2\cdot(m-1)\cdot(U-T)}}\right]$$

$$\times\left[\frac{t_2^{(S+1)}}{(nq_n)^2}+...+\frac{t_2^{((n-1+k)}}{(nq_n)^{2\cdot(n-1+k-S)}}+\frac{1}{10n^4\cdot(nq_n)^{2\cdot(n-1+k-S)}}+d_n\cdot(nq_n)^{2U}\cdot la_n,\right.$$

$$\left.\frac{t_2^{(S+1)}}{(nq_n)^2}+...+\frac{t_2^{((n-1+k)}+1}{(nq_n)^{2\cdot(n-1+k-S)}}-\frac{1}{10n^4\cdot(nq_n)^{2\cdot(n-1+k-S)}}+d_n\cdot(nq_n)^{2U}\cdot la_n\right]$$

$$\times\prod_{i=3}^{m}\left[\frac{t_i^{(S+1)}}{(nq_n)^2}+...+\frac{t_i^{(n-1+k)}}{(nq_n)^{2\cdot(n-1+k-S)}}+\frac{1}{10n^4\cdot(nq_n)^{2\cdot(n-1+k-S)}},\right.$$

$$\left.\frac{t_i^{(S+1)}}{(nq_n)^2}+...+\frac{t_i^{(n-1+k)}+1}{(nq_n)^{2\cdot(n-1+k-S)}}-\frac{1}{10n^4\cdot(nq_n)^{2\cdot(n-1+k-S)}}\right].$$

Thus such a set $\Phi_n(I_n)$ has a θ-width of at most $\frac{1}{d_n \cdot (n \cdot q_n)^{2\cdot(T+1)}}$.

Let $J = \prod_{i=2}^{m} \left[\frac{1}{10n^4}, 1 - \frac{1}{10n^4}\right]$ and $\tilde{J} \subset J$ be any $(m-1)$-dimensional cuboid of r_1-length $l_1 \geq \frac{1}{nq_n^2}$ and of side length q_n^{-1} in the $r_2, \dots, r_{m-1}$-coordinates. In each of the coordinates $r_2, \dots, r_{m-1}$, $\tilde{J}$ contains at least $\frac{\left((nq_n)^2 \cdot \left(1 - \frac{1}{10n^4}\right)\right)^{n-1+k-S}}{q_n}$ and at most $\frac{(nq_n)^{2\cdot(n-1+k-S)}}{q_n}$ intervals of the form

$$\left[\frac{t_i^{(S+1)}}{(nq_n)^2} + \dots + \frac{t_i^{(n-1+k)}}{(nq_n)^{2\cdot(n-1+k-S)}} + \frac{1}{10n^4 \cdot (nq_n)^{2\cdot(n-1+k-S)}}, \right.$$
$$\left. \frac{t_i^{(S+1)}}{(nq_n)^2} + \dots + \frac{t_i^{(n-1+k)} + 1}{(nq_n)^{2\cdot(n-1+k-S)}} - \frac{1}{10n^4 \cdot (nq_n)^{2\cdot(n-1+k-S)}}\right].$$

while in the r_1-coordinate $\tilde{J}$ contains at least $\left(\lfloor l_1 (nq_n)^2 \rfloor - 2\right) \cdot \left((nq_n)^2 \cdot \left(1 - \frac{1}{10n^4}\right)\right)^{n-2+k-S}$ and at most $l_1 \cdot (nq_n)^{2\cdot(n-1+k-S)}$ such intervals. Hereby, we estimate

$$\begin{aligned}
&\mu\left(\Phi_n(I_n) \cap \mathbb{S}^1 \times \tilde{J}\right) \\
\geq &\frac{l_1}{d_n \cdot (nq_n)^{2\cdot(T+m-1)+m-2}} \cdot \left(1 - \frac{3}{l_1 (nq_n)^2}\right) \cdot \left(1 - \frac{1}{10n^4}\right)^{(m-1)\cdot(n-1+k-S)-1} \cdot \left(1 - \frac{1}{5n^4}\right)^{(m-1)\cdot(S-1)+m\cdot(n+k)-T} \\
\geq &\left(1 - \frac{3}{n}\right) \cdot \mu(I_n) \cdot \mu^{(m-1)}\left(\tilde{J}\right)
\end{aligned}$$

exploiting $l_n \geq \frac{1}{nq_n^2}$ and

$$\mu(I_n) = \frac{1}{d_n \cdot (nq_n)^{2\cdot(T+m-1)}} \cdot \left(1 - \frac{1}{5n^4}\right)^{m\cdot(n+k)-T+(m-1)\cdot(n+k-2)}.$$

On the other hand, we have

$$\begin{aligned}
\mu\left(\Phi_n(I_n) \cap \mathbb{S}^1 \times \tilde{J}\right) &\leq \frac{1}{d_n \cdot (nq_n)^{2\cdot(T+m-1)+m-2}} \cdot l_1 \cdot \left(1 - \frac{1}{5n^4}\right)^{m\cdot(n+k)-T+(m-1)\cdot(S-1)} \\
&\leq \left(1 + \frac{1}{n}\right) \cdot \mu(I_n) \cdot \mu^{(m-1)}\left(\tilde{J}\right)
\end{aligned}$$

for n sufficiently large. Hence, the properties of a $\left(\frac{1}{d_n \cdot (nq_n)^2}, \frac{1}{10n^4}, \frac{1}{nq_n^2}, q_n^{-1}, \frac{3}{n}\right)$-distribution are fulfilled. □

The previous lemma shows the significance of an understanding of how the (nq_n)-powers in the definition of ϕ_n evolve while passing from $\left[\frac{l}{nq_n}, \frac{l+1}{nq_n}\right] \times [0,1]^{m-1}$ to $\left[\frac{l+i}{nq_n}, \frac{l+i+1}{nq_n}\right] \times [0,1]^{m-1}$:

Lemma 4.6. *Let $l, v, i \in \mathbb{Z}$, $0 \leq l < n-k$, $0 \leq v < nq_n$ and $0 < i \leq k$. On every section $\left[\frac{l+i}{n \cdot q_n} + \frac{v}{n^2 \cdot q_n^2}, \frac{l+i}{n \cdot q_n} + \frac{v+1}{n^2 \cdot q_n^2}\right] \times [0,1]^{m-1}$ the ratio of domains, where the (nq_n)-power in the definition of ϕ_n is greater than the (nq_n)-power of ϕ_n in the corresponding domain in $\left[\frac{l}{n \cdot q_n} + \frac{v}{n^2 \cdot q_n^2}, \frac{l}{n \cdot q_n} + \frac{v+1}{n^2 \cdot q_n^2}\right] \times [0,1]^{m-1}$, is equal to $\sum_{d=1}^{i} \tilde{\beta}_{d,n} = \kappa_{i,n}$.*

Proof. First of all, we notice that there is no decline of (nq_n)-powers when passing from $\left[\frac{l}{nq_n}, \frac{l+1}{nq_n}\right] \times [0,1]^{m-1}$ to $\left[\frac{l+i}{nq_n}, \frac{l+i+1}{nq_n}\right] \times [0,1]^{m-1}$. Obviously, on the domains of the form

$$\left[\frac{l}{n \cdot q_n} + \frac{v}{n^2 \cdot q_n^2} + \frac{k \cdot c_{2,n}}{d_n \cdot n^2 \cdot q_n^2}, \frac{l}{n \cdot q_n} + \frac{v}{n^2 \cdot q_n^2} + \frac{\sum_{d=1}^{k} c_{d,n}}{d_n \cdot n^2 \cdot q_n^2}\right] \times [0,1]^{m-1},$$

i. e. on a corresponding θ-length of $\frac{\sum_{d=1}^{k} \tilde{\beta}_{d,n} - k \cdot \tilde{\beta}_{2,n}}{n^2 q_n^2}$, we always have an increasing (nq_n)-power while passing from l to $l+i$ for any $i \geq 1$.

A more careful analysis has to be executed on the domains of the form

$$\left[\frac{l}{n \cdot q_n} + \frac{v}{n^2 \cdot q_n^2}, \frac{l}{n \cdot q_n} + \frac{v}{n^2 \cdot q_n^2} + \frac{k \cdot c_{2,n}}{d_n \cdot n^2 \cdot q_n^2}\right] \times [0,1]^{m-1}$$

in the definition of ϕ_n: When passing from l to $l+i$ the number $\tilde{s}$ changes to $\tilde{s} - i \mod k$. Accordingly for each $s \in \{0, 1, \dots, k-1\}$ the corresponding number t_s is changed to $t_s + i \mod k$. On the one hand, we observe that on the domains $\Delta_{l+i,v,s,u}$ with $u < u_{t_s+i-1,n}$ the (nq_n)-power is larger than on $\Delta_{l,v,s,u}$. On the other hand, the (nq_n)-power remains the same on domains of the form $\Delta_{l+i,v,s,u}$ for $u \geq u_{t_s+i-1,n}$ due to $l+i+k-(t_s+i) = l+k-t_s$. Hence, this yields an increased (nq_n)-power on a corresponding θ-length of $\sum_{t_s=0}^{k-1} \frac{u_{t_s+i-1,n}}{d_n \cdot n^2 \cdot q_n^2}$, where we use the convention $u_{j,n} = c_{2,n}$ for $j \geq k$. By definition of the numbers $u_{i,n}$ in equation 2 we get

$$\sum_{t_s=0}^{k-1} \frac{u_{t_s+i-1,n}}{d_n \cdot n^2 \cdot q_n^2} = \frac{k \cdot \tilde{\beta}_{2,n} - \sum_{d=i+1}^{k} \tilde{\beta}_{d,n}}{n^2 \cdot q_n^2}$$

in case of $i < k$. In case of $i = k$ the fraction is equal to $\frac{k \cdot \tilde{\beta}_{2,n}}{n^2 \cdot q_n^2}$.

Altogether the (nq_n)-power in the definition of ϕ_n has increased on a corresponding θ-length of $\frac{1}{n^2 q_n^2} \sum_{d=1}^{i} \tilde{\beta}_{d,n} = \frac{1}{n^2 q_n^2} \kappa_{i,n}$ using equation 3. □

Remark 4.7. For a partition element $\hat{I}_n = \bigcup_{j=0}^{N} I_j \times W$ we introduce subsets N_i, $i = 1, \dots, k$ of the set of indices $\{0, 1, \dots, N-1\}$ in the following way: $j \in N_i$ if the (nq_n)-power of ϕ_n on $R_{\frac{i}{nq_n}}(I_j \times W)$ is larger than the (nq_n)-power of ϕ_n on $I_j \times W$.
By the previous Lemma and the shape of partition elements in η_n we have

$$\mu\left(\bigcup_{j \in N_i} I_j \times W\right) = \kappa_{i,n} \cdot \mu\left(\hat{I}_n\right).$$

5 Criterion for $(\kappa_1, \dots, \kappa_k)$-weak mixing

In this section we will prove a criterion for $(\kappa_1, \dots, \kappa_k)$-weak mixing on $M = \mathbb{S}^1 \times [0,1]^{m-1}$ in the setting of the beforehand constructions. For the derivation we need a couple of lemmas. At first we examine a sequence of partial partitions that will be used in the criterion:

Lemma 5.1. *Consider the sequence of partial partitions* $(\eta_n)_{n\in\mathbb{N}}$ *constructed in section 3.3 and the diffeomorphisms* g_n *from chapter 3.2. Furthermore let* $(H_n)_{n\in\mathbb{N}}$ *be a sequence of measure-preserving smooth diffeomorphisms satisfying* $\|DH_{n-1}\| \leq \frac{\ln(q_n)}{n}$ *for every* $n \in \mathbb{N}$ *and define the partial partitions* $\nu_n = \left\{\Gamma_n = H_{n-1} \circ g_n\left(\hat{I}_n\right) \ : \ \hat{I}_n \in \eta_n\right\}$. *Then we get* $\nu_n \to \varepsilon$.

Proof. By construction $\eta_n = \left\{\hat{I}_n^i : i \in \Lambda_n\right\}$, where Λ_n is a countable set of indices. Because of $\eta_n \to \varepsilon$ it holds $\lim_{n\to\infty} \mu\left(\bigcup_{i\in\Lambda_n} \hat{I}_n^i\right) = 1$. Since $H_{n-1} \circ g_n$ is measure-preserving we conclude:

$$\lim_{n\to\infty} \mu\left(\bigcup_{i\in\Lambda_n} \Gamma_n^i\right) = \lim_{n\to\infty} \mu\left(\bigcup_{i\in\Lambda_n} H_{n-1} \circ g_n\left(\hat{I}_n^i\right)\right) = \lim_{n\to\infty} \mu\left(H_{n-1} \circ g_n\left(\bigcup_{i\in\Lambda_n} \hat{I}_n^i\right)\right) = 1.$$

For any m-dimensional cube with sidelength l_n it holds: $\mathrm{diam}(W_n) = \sqrt{m} \cdot l_n$. Because every element of the partition η_n is contained in a cube of side length $\frac{1}{n^2 \cdot q_n^2}$ it follows for every $i \in \Lambda_n$: $\mathrm{diam}\left(\hat{I}_n^i\right) \leq \sqrt{m} \cdot \frac{1}{n^2 \cdot q_n^2}$. Hence, we have for every $\Gamma_n^i = H_{n-1} \circ g_n\left(\hat{I}_n^i\right)$:

$$\mathrm{diam}\left(\Gamma_n^i\right) \leq \|DH_{n-1}\|_0 \cdot \|Dg_n\|_0 \cdot \frac{\sqrt{m}}{n^2 \cdot q_n^2} \leq \frac{\ln(q_n)}{n} \cdot n \cdot q_n \cdot \frac{\sqrt{m}}{n^2 \cdot q_n^2} \leq \sqrt{m} \cdot \frac{\ln(q_n)}{n^2 \cdot q_n}.$$

We conclude $\lim_{n\to\infty} \mathrm{diam}\left(\Gamma_n^i\right) = 0$ and consequently $\nu_n \to \varepsilon$. □

In the following the Lebesgue measures on $\mathbb{S}^1$, $[0,1]^{m-2}$, $[0,1]^{m-1}$ are denoted by $\tilde{\lambda}$, $\mu^{(m-2)}$ and $\tilde{\mu}$ respectively. The next technical result is needed in the proof of Lemma 5.3.

Lemma 5.2. *Given an interval* K *on the* r_1*-axis and a* $(m-2)$*-dimensional interval* Z *in* $(r_2, ..., r_{m-1})$ $K_{c,\gamma}$ *denotes the cuboid* $[c, c+\gamma] \times K \times Z$ *for some* $\gamma > 0$. *We consider the diffeomorphism* $\tilde{g}_b : M \to M$, $\tilde{g}_b(\theta, r_1, \ldots, r_{m-1}) = (\theta + b \cdot r_1, r_1, \ldots, r_{m-1})$ *with some* $b \in \mathbb{N}$ *and an interval* $L = [l_1, l_2]$ *of* $\mathbb{S}^1$.
If $b \cdot \lambda(K) > 2$, *then for the set* $Q := \pi_{\vec{r}}\left(K_{c,\gamma} \cap \tilde{g}_b^{-1}(L \times K \times Z)\right)$ *we have:*

$$\left|\tilde{\mu}(Q) - \lambda(K) \cdot \tilde{\lambda}(L) \cdot \mu^{(m-2)}(Z)\right| \leq \left(\frac{2}{b} \cdot \tilde{\lambda}(L) + \frac{2 \cdot \gamma}{b} + \gamma \cdot \lambda(K)\right) \cdot \mu^{(m-2)}(Z)$$

Proof. We consider the set:

$$\begin{aligned} Q_b &:= \pi_{\vec{r}}\left(K_{c,\gamma} \cap \tilde{g}_b^{-1}(L \times K \times Z)\right) \\ &= \{(r_1, r_2, ..., r_{m-1}) \in K \times Z \ : \ (\theta + b \cdot r_1, \vec{r}) \in L \times K \times Z, \theta \in [c, c+\gamma]\} \\ &= \{(r_1, r_2, ..., r_{m-1}) \in K \times Z \ : \ b \cdot r_1 \in [l_1 - c - \gamma, l_2 - c] \mod 1\} \end{aligned}$$

The interval $b \cdot K$ seen as an interval in $\mathbb{R}$ does not intersect more than $b \cdot \lambda(K) + 2$ and not less than $b \cdot \lambda(K) - 2$ intervals of the form $[i, i+1]$ with $i \in \mathbb{Z}$. Therefore we compute on the one side:

$$\begin{aligned} \tilde{\mu}(Q) &\leq (b \cdot \lambda(K) + 2) \cdot \frac{l_2 - (l_1 - \gamma)}{b} \cdot \mu^{(m-2)}(Z) \\ &= \left(\lambda(K) \cdot \tilde{\lambda}(L) + 2 \cdot \frac{\tilde{\lambda}(L)}{b} + \lambda(K) \cdot \gamma + \frac{2 \cdot \gamma}{b}\right) \cdot \mu^{(m-2)}(Z) \end{aligned}$$

and on the other side

$$\begin{aligned}\tilde{\mu}(Q) &\geq (b\cdot\lambda(K)-2)\cdot\frac{l_2-(l_1-\gamma)}{b}\cdot\mu^{(m-2)}(Z)\\ &=\left(\lambda(K)\cdot\tilde{\lambda}(L)-2\cdot\frac{\tilde{\lambda}(L)}{b}+\lambda(K)\cdot\gamma-\frac{2\cdot\gamma}{b}\right)\cdot\mu^{(m-2)}(Z).\end{aligned}$$

Both equations together yield:

$$\left|\tilde{\mu}(Q)-\lambda(K)\cdot\tilde{\lambda}(L)\cdot\mu^{(m-2)}(Z)-\gamma\cdot\lambda(K)\cdot\mu^{(m-2)}(Z)\right|\leq\left(\frac{2}{b}\cdot\tilde{\lambda}(L)+\frac{2\cdot\gamma}{b}\right)\cdot\mu^{(m-2)}(Z).$$

The claim follows because

$$\begin{aligned}&\left|\tilde{\mu}(Q)-\lambda(K)\cdot\tilde{\lambda}(L)\cdot\mu^{(m-2)}(Z)\right|-\gamma\cdot\lambda(K)\cdot\mu^{(m-2)}(Z)\\ &\leq\left|\tilde{\mu}(Q)-\lambda(K)\cdot\tilde{\lambda}(L)\cdot\mu^{(m-2)}(Z)-\gamma\cdot\lambda(K)\cdot\mu^{(m-2)}(Z)\right|.\end{aligned}$$

□

Lemma 5.3. *Let n be sufficiently large, g_n as in section 3.2 and $\hat{I}_n=\bigcup_{j=0}^{N}I_j\times W\in\eta_n$, where η_n is the partial partition constructed in section 3.3. For the diffeomorphism ϕ_n constructed in section 3.1 and m_n as in chapter 4 we consider $\Phi_n=\phi_n\circ R_{\alpha_{n+1}}^{i\cdot m_n}\circ\phi_n^{-1}$ for some $i\in\{1,\dots,k\}$. We assume that Φ_n $\left(\frac{1}{d_n\cdot(nq_n)^2},\frac{1}{10n^4},\frac{1}{nq_n^2},q_n^{-1},\frac{3}{n}\right)$-uniformly distributes $I_j\times W$ and denote $\left[\frac{1}{10n^4},1-\frac{1}{10n^4}\right]^{m-1}$ by J_n. Then for every m-dimensional cube S of side length q_n^{-1} lying in $\mathbb{S}^1\times J_n$ we get*

$$\left|\mu\left(I_j\times W\cap\Phi_n^{-1}\circ g_n^{-1}(S)\right)-\mu(I_j\times W)\cdot\mu(S)\right|\leq\frac{22}{n}\cdot\mu(I_j\times W)\cdot\mu(S)\tag{4}$$

In other words this Lemma tells us that parts of a partition element contained in the "distribution part" of Φ_n are "almost uniformly distributed" under $g_n\circ\Phi_n$ on the whole manifold $M=\mathbb{S}^1\times[0,1]^{m-1}$.

Proof. Let S be a m-dimensional cube with sidelength q_n^{-1} lying in $\mathbb{S}^1\times J_n$. Furthermore we denote:

$$S_\theta=\pi_\theta(S)\qquad S_{r_1}=\pi_{r_1}(S)\qquad S_{\tilde{\vec{r}}}=\pi_{(r_2,\dots,r_{m-1})}(S)\qquad S_r=S_{r_1}\times S_{\tilde{\vec{r}}}=\pi_{\vec{r}}(S).$$

Obviously: $\tilde{\lambda}(S_\theta)=\lambda(S_{r_1})=q_n^{-1}$ and $\tilde{\lambda}(S_\theta)\cdot\lambda(S_{r_1})\cdot\mu^{(m-2)}\left(S_{\tilde{\vec{r}}}\right)=\mu(S)=q_n^{-m}$.
By assumption Φ_n $\left(\frac{1}{d_n\cdot(nq_n)^2},\frac{1}{10n^4},\frac{1}{nq_n^2},q_n^{-1},\frac{3}{n}\right)$-distributes $I_j\times W$, in particular $\Phi_n(I_j\times W)\subseteq[c,c+\gamma]\times J_n$ for some $c\in\mathbb{S}^1$ and some $\gamma\leq\frac{1}{d_n\cdot(n\cdot q_n)^2}$. By construction of the map g_n it holds: $\Phi_n(I_j\times W)\cap g_n^{-1}(S)\subseteq[c,c+\gamma]\times S_r=:K_{c,\gamma}$.
Since $2\gamma\leq\frac{2}{d_n\cdot(n\cdot q_n)^2}<q_n^{-1}$, we can define a cuboid $S_1\subseteq S$, where $S_1:=[s_1+\gamma,s_2-\gamma]\times S_r$ using the notation $S_\theta=[s_1,s_2]$. We examine the two sets

$$Q:=\pi_{\vec{r}}\left(K_{c,\gamma}\cap g_n^{-1}(S_\theta\times S_r)\right)\qquad Q_1:=\pi_{\vec{r}}\left(K_{c,\gamma}\cap g_n^{-1}([s_1+\gamma,s_2-\gamma]\times S_r)\right)$$

As seen above $\Phi_n(I_j\times W)\cap g_n^{-1}(S)\subseteq\Phi_n(I_j\times W)\cap g_n^{-1}(S)\cap K_{c,\gamma}$, which implies the inclusion $\Phi_n(I_j\times W)\cap g_n^{-1}(S)\subseteq\Phi_n(I_j\times W)\cap\left(\mathbb{S}^1\times Q\right)$.

Claim: On the other hand: $\Phi_n(I_j \times W) \cap (\mathbb{S}^1 \times Q_1) \subseteq \Phi_n(I_j \times W) \cap g_n^{-1}(S)$.
Proof of the claim: For $(\theta, \vec{r}) \in \Phi_n(I_j \times W) \cap (\mathbb{S}^1 \times Q_1)$ arbitrary it holds $(\theta, \vec{r}) \in \Phi_n(I_j \times W)$, i.e. $\theta \in [c, c+\gamma]$, and $\vec{r} \in \pi_{\vec{r}}\left(K_{c,\gamma} \cap g_n^{-1}([s_1+\gamma, s_2-\gamma] \times S_r)\right)$. This implies the existence of $\bar{\theta} \in [c, c+\gamma]$ satisfying $(\bar{\theta}, \vec{r}) \in K_{c,\gamma} \cap g_n^{-1}(S_1)$. Hence, there are $\beta \in [s_1+\gamma, s_2-\gamma]$ and $\vec{r}_1 \in S_r$, such that $g_n(\bar{\theta}, \vec{r}) = (\beta, \vec{r}_1)$. Since g_n maps sets of the form $I \times \vec{r}$ to a set $\tilde{I} \times \vec{r}$, where $I, \tilde{I} \subset \mathbb{S}^1$ are intervals of the same length, and $|\theta - \bar{\theta}| \le \gamma$, we have $g_n(\theta, \vec{r}) = (\bar{\beta}, \vec{r})$ for some $\bar{\beta} \in [s_1, s_2]$. So $(\theta, \vec{r}) \in \Phi_n(I_j \times W) \cap g_n^{-1}(S)$. □
Altogether the following inclusions are true:

$$\Phi_n(I_j \times W) \cap (\mathbb{S}^1 \times Q_1) \subseteq \Phi_n(I_j \times W) \cap g_n^{-1}(S) \subseteq \Phi_n(I_j \times W) \cap (\mathbb{S}^1 \times Q)$$

Thus we obtain:

$$\begin{aligned} &\left|\mu\left(I_j \times W \cap \Phi_n^{-1}\left(g_n^{-1}(S)\right)\right) - \mu(I_j \times W) \cdot \mu(S)\right| \\ \le \max\Bigg(&\left|\mu\left(I_j \times W \cap \Phi_n^{-1}\left(\mathbb{S}^1 \times Q\right)\right) - \mu(I_j \times W) \cdot \mu(S)\right|, \\ &\left|\mu\left(I_j \times W \cap \Phi_n^{-1}\left(\mathbb{S}^1 \times Q_1\right)\right) - \mu(I_j \times W) \cdot \mu(S)\right| \Bigg) \end{aligned} \tag{5}$$

We want to apply Lemma 5.2 for $K = S_{r_1}$, $L = S_\theta$, $Z = S_{\tilde{\vec{r}}}$ and $b = n \cdot q_n$ (note that for $n > 2$: $b \cdot \lambda(K) = n \cdot q_n \cdot q_n^{-1} = n > 2$):

$$\begin{aligned} |\tilde{\mu}(Q) - \mu(S)| &\le \left(\frac{2}{n \cdot q_n} \cdot \tilde{\lambda}(S_\theta) + \frac{2\gamma}{n \cdot q_n} + \gamma \cdot \lambda(S_{r_1})\right) \cdot \mu^{(m-2)}\left(S_{\tilde{\vec{r}}}\right) \\ &\le \left(\frac{2}{nq_n} \cdot \tilde{\lambda}(S_\theta) + \frac{4}{(nq_n)^3} + \frac{2 \cdot \lambda(S_{r_1})}{(nq_n)^2}\right) \cdot \mu^{(m-2)}\left(S_{\tilde{\vec{r}}}\right) \\ &\le \frac{14}{n} \cdot \mu(S). \end{aligned}$$

In particular we receive from this estimate: $\frac{14}{n} \cdot \mu(S) \ge \tilde{\mu}(Q) - \mu(S)$, hence we have $\tilde{\mu}(Q) \le \left(1 + \frac{14}{n}\right) \cdot \mu(S) \le 2 \cdot \mu(S)$ for n sufficiently large.
Analogously we obtain: $|\tilde{\mu}(Q_1) - \mu(S_1)| \le \frac{14}{n} \cdot \mu(S)$ as well as $\tilde{\mu}(Q_1) \le 2 \cdot \mu(S)$.
Since Q as well as Q_1 are a finite union of disjoint $(m-1)$-dimensional intervals contained in J_n of r_1-length at least $\frac{1}{nq_n^2}$ as well as r_i-length $\frac{1}{q_n}$ for $i \ge 2$ and Φ_n $\left(\frac{1}{d_n \cdot (nq_n)^2}, \frac{1}{10n^4}, \frac{1}{nq_n^2}, \frac{1}{q_n}, \frac{3}{n}\right)$-distributes the interval $I_j \times W$, we get:

$$\left|\mu\left(I_j \times W \cap \Phi_n^{-1}\left(\mathbb{S}^1 \times Q\right)\right) - \mu(I_j \times W)\,\tilde{\mu}(Q)\right| \le \frac{3}{n}\mu(I_j \times W) \cdot \tilde{\mu}(Q) \le \frac{6}{n}\mu(I_j \times W) \cdot \mu(S)$$

as well as

$$\left|\mu\left(I_j \times W \cap \Phi_n^{-1}\left(\mathbb{S}^1 \times Q_1\right)\right) - \mu(I_j \times W)\,\tilde{\mu}(Q_1)\right| \le \frac{3}{n}\mu(I_j \times W)\,\tilde{\mu}(Q_1) \le \frac{6}{n}\mu(I_j \times W) \cdot \mu(S).$$

Now we can proceed

$$\begin{aligned}
&\left|\mu\left(I_j\times W\cap\Phi_n^{-1}\left(\mathbb{S}^1\times Q\right)\right)-\mu\left(I_j\times W\right)\cdot\mu\left(S\right)\right|\\
&\leq\left|\mu\left(I_j\times W\cap\Phi_n^{-1}\left(\mathbb{S}^1\times Q\right)\right)-\mu\left(I_j\times W\right)\cdot\tilde{\mu}\left(Q\right)\right|+\mu\left(I_j\times W\right)\cdot\left|\tilde{\mu}\left(Q\right)-\mu\left(S\right)\right|\\
&\leq\frac{6}{n}\cdot\mu\left(I_j\times W\right)\cdot\mu\left(S\right)+\mu\left(I_j\times W\right)\cdot\frac{14}{n}\cdot\mu\left(S\right)=\frac{20}{n}\cdot\mu\left(I_j\times W\right)\cdot\mu\left(S\right).
\end{aligned}$$

Noting that $\mu\left(S_1\right)=\mu\left(S\right)-2\gamma\cdot\tilde{\mu}\left(S_r\right)$ and so $\mu\left(S\right)-\mu\left(S_1\right)\leq 2\cdot\frac{1}{n\cdot q_n}\cdot\tilde{\mu}\left(S_r\right)\leq\frac{2}{n}\cdot\mu\left(S\right)$ we obtain in the same way as above:

$$\left|\mu\left(I_j\times W\cap\Phi_n^{-1}\left(\mathbb{S}^1\times Q_1\right)\right)-\mu\left(I_j\times W\right)\cdot\mu\left(S\right)\right|\leq\frac{22}{n}\cdot\mu\left(I_j\times W\right)\cdot\mu\left(S\right).$$

Using equation 5 this yields:

$$\left|\mu\left(I_j\times W\cap\Phi_n^{-1}\left(g_n^{-1}\left(S\right)\right)\right)-\mu\left(I_j\times W\right)\cdot\mu\left(S\right)\right|\leq\frac{22}{n}\cdot\mu\left(I_j\times W\right)\cdot\mu\left(S\right).$$

□

In the proof of the criterion we will consider specific cubes:

Definition 5.4. By $S_{j_1,\dots,j_m}$ we denote the cube $\prod_{i=1}^{m}\left[\frac{j_i}{q_n},\frac{j_i+1}{q_n}\right]$. Let $\tilde{\mathfrak{S}}_n$ be the family of cubes $S_{j_1,\dots,j_m}$ satisfying $0\leq j_1\leq q_n-1$ as well as $\left\lceil\frac{q_n}{10n^4}\right\rceil\leq j_i\leq q_n-\left\lceil\frac{q_n}{10n^4}\right\rceil-1$ for $i=2,...,m$. Then $\mathfrak{S}_n$ is defined to be the partial partition $\mathfrak{S}_n:=\left\{H_{n-1}\left(S_n\right)\ :\ S_n\in\tilde{\mathfrak{S}}_n\right\}$.

Remark 5.5. A partition element $\hat{I}_n=I_{l,v,j_2^{(1)},j_2^{(2)},j_3,\dots,j_m}$ is contained in a cube

$$\left[\frac{s}{q_n}+\frac{l}{nq_n}+\frac{v}{\left(nq_n\right)^2},\frac{s}{q_n}+\frac{l}{nq_n}+\frac{v+1}{\left(nq_n\right)^2}\right]\times\left[\frac{j_2^{(1)}}{nq_n^2}+\frac{j_2^{(2)}}{n^2\cdot q_n^2},\frac{j_2^{(1)}}{nq_n^2}+\frac{j_2^{(2)}+1}{n^2\cdot q_n^2}\right]\times\prod_{i=3}^{m}\left[\frac{j_i}{\left(nq_n\right)^2},\frac{j_i+1}{\left(nq_n\right)^2}\right]$$

Thus $g_n\left(\hat{I}_n\right)$ is contained in

$$\begin{aligned}
&\left[\frac{s+j_2^{(1)}}{q_n}+\frac{l+j_2^{(2)}}{n\cdot q_n}+\frac{v}{n^2\cdot q_n^2},\frac{s+j_2^{(1)}}{q_n}+\frac{l+j_2^{(2)}+1}{n\cdot q_n}+\frac{v+1}{n^2\cdot q_n^2}\right]\\
&\times\left[\frac{j_2^{(1)}}{n\cdot q_n^2}+\frac{j_2^{(2)}}{n^2\cdot q_n^2},\frac{j_2^{(1)}}{n\cdot q_n^2}+\frac{j_2^{(2)}+1}{n^2\cdot q_n^2}\right]\times\prod_{i=3}^{m}\left[\frac{j_i}{n^2\cdot q_n^2},\frac{j_i+1}{n^2\cdot q_n^2}\right].
\end{aligned}$$

So we observe that $g_n\left(\hat{I}_n\right)$ is contained in a cube $S_n\in\tilde{\mathfrak{S}}_n$ completely or both have an empty intersection due to the restrictions on $j_2^{(2)}$.
By the same reasoning we have $\mu\left(R_{\alpha_{n+1}}^{i\cdot m_n}\left(\hat{I}_n\right)\cap g_n^{-1}\left(S_n\right)\right)=\mu\left(\hat{I}_n\cap g_n^{-1}\left(S_n\right)\right)$ for every $\hat{I}_n\in\eta_n$, $i\in\{1,\dots,k\}$ and $S_n\in\tilde{\mathfrak{S}}_n$. With the aid of Lemma 4.5 and the bounds on a_n from Remark 4.4 this also yields

$$\mu\left(\phi_n\circ R_{\alpha_{n+1}}^{i\cdot m_n}\circ\phi_n^{-1}\left(I_j\times W\right)\cap g_n^{-1}\left(S_n\right)\right)=\mu\left(I_j\times W\cap g_n^{-1}\left(S_n\right)\right)$$

for every $j\notin N_i$.

Remark 5.6. Under the condition $\|DH_{n-1}\|_0 \leq \frac{\ln(q_n)}{n}$ we have

$$\operatorname{diam}\left(H_{n-1}\left(S_n\right)\right) \leq \frac{\ln\left(q_n\right)}{n} \cdot \frac{\sqrt{m}}{q_n} \rightarrow 0$$

as $n \rightarrow \infty$. So we have $\mathfrak{S}_n \rightarrow \varepsilon$.

We investigate $(\kappa_{1,n}, \ldots, \kappa_{k,n})$-weak mixing of f_n on such sets $A_n \in \mathfrak{S}_n$ and partition elements $\Gamma_n \in \nu_n$:

Lemma 5.7. *Let $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ and the sequence $(m_n)_{n\in\mathbb{N}}$ be constructed as in the previous sections. For $i = 1, \ldots, k$, every $A_n \in \mathfrak{S}_n$ and $\Gamma_n \in \nu_n$ we have:*

$$\left|\mu\left(\Gamma_n \cap f_n^{-i\cdot m_n}\left(A_n\right)\right) - \kappa_{i,n} \cdot \mu\left(\Gamma_n\right) \cdot \mu\left(A_n\right) - \left(1-\kappa_{i,n}\right) \cdot \mu\left(\Gamma_n \cap A_n\right)\right| < \frac{22}{n} \cdot \mu\left(\Gamma_n\right) \cdot \mu\left(A_n\right).$$

Proof. We write $A_n = H_{n-1}\left(S_n\right)$ and $\Gamma_n = H_{n-1} \circ g_n\left(\hat{I}_n\right)$, at which $\hat{I}_n = \bigcup_{j=0}^{N} I_j \times W$. Furthermore we note $f_n^{i\cdot m_n} = H_n \circ R_{\alpha_{n+1}}^{i\cdot m_n} \circ H_n^{-1} = H_{n-1} \circ g_n \circ \Phi_n^i \circ g_n^{-1} \circ H_{n-1}^{-1}$, where $\Phi_n = \phi_n \circ R_{\alpha_{n+1}}^{m_n} \circ \phi_n^{-1}$. Then we calculate

$$\begin{aligned}
&\left|\mu\left(\Gamma_n \cap f_n^{-i\cdot m_n}\left(A_n\right)\right) - \kappa_{i,n} \cdot \mu\left(\Gamma_n\right) \cdot \mu\left(A_n\right) - \left(1-\kappa_{i,n}\right) \cdot \mu\left(\Gamma_n \cap A_n\right)\right| \\
=&\left|\mu\left(\hat{I}_n \cap \Phi_n^{-i} \circ g_n^{-1}\left(S_n\right)\right) - \kappa_{i,n} \cdot \mu\left(\hat{I}_n\right) \cdot \mu\left(S_n\right) - \left(1-\kappa_{i,n}\right) \cdot \mu\left(\hat{I}_n \cap g_n^{-1}\left(S_n\right)\right)\right|.
\end{aligned}$$

We recall that for every $i = 1, \ldots, k$:

$$\mu\left(\bigcup_{j\in N_i} I_j \times W\right) = \kappa_{i,n} \cdot \mu\left(\hat{I}_n\right)$$

(see Remark 4.7). Hereby, we conclude

$$\begin{aligned}
&\left|\mu\left(\Gamma_n \cap f_n^{-i\cdot m_n}\left(A_n\right)\right) - \kappa_{i,n} \cdot \mu\left(\Gamma_n\right) \cdot \mu\left(A_n\right) - \left(1-\kappa_{i,n}\right) \cdot \mu\left(\Gamma_n \cap A_n\right)\right| \\
\leq&\left|\mu\left(\Phi_n^i\left(\bigcup_{j\notin N_i} I_j \times W\right) \cap g_n^{-1}\left(S_n\right)\right) - \left(1-\kappa_{i,n}\right) \cdot \mu\left(\hat{I}_n \cap g_n^{-1}\left(S_n\right)\right)\right| \\
&+ \sum_{j\in N_i}\left|\mu\left(I_j \times W \cap \Phi_n^{-i} \circ g_n^{-1}\left(S_n\right)\right) - \mu\left(I_j \times W\right) \cdot \mu\left(S_n\right)\right|.
\end{aligned}$$

We start to examine the first term. By Remarks 4.7 and 5.5 we have

$$\left(1-\kappa_{i,n}\right) \cdot \mu\left(\hat{I}_n \cap g_n^{-1}\left(S_n\right)\right) = \mu\left(\left(\bigcup_{j\notin N_i} I_j \times W\right) \cap g_n^{-1}\left(S_n\right)\right).$$

Additionally, Remark 5.5 tells us that $\mu\left(I_j \times W \cap g_n^{-1}\left(S_n\right)\right) = \mu\left(\Phi_n^{-i}\left(I_j \times W\right) \cap g_n^{-1}\left(S_n\right)\right)$ for each $j \notin N_i$. Hence, the first term is equal to 0.

In the next step, we examine the second term. We note that the cube S_n is contained in $\mathbb{S}^1 \times \left[\frac{1}{10n^4}, 1 - \frac{1}{10n^4}\right]^{m-1}$. So we can apply Lemma 5.3 for $j \in N_i$:

$$\begin{aligned}\sum_{j \in N_i} \left|\mu\left(I_j \times W \cap \Phi_n^{-i} \circ g_n^{-1}\left(S_n\right)\right) - \mu\left(I_j \times W\right) \cdot \mu\left(S_n\right)\right| &\leq \sum_{j \in N_i} \frac{22}{n} \cdot \mu\left(I_j \times W\right) \cdot \mu\left(S_n\right) \\ &\leq \frac{22}{n} \cdot \mu\left(\hat{I}_n\right) \cdot \mu\left(S_n\right).\end{aligned}$$

□

Now we are able to prove the aimed criterion for $(\kappa_1, \ldots, \kappa_k)$-weak mixing.

Proposition 5.8 (Criterion for $(\kappa_1, \ldots, \kappa_k)$-weak mixing)**.** *Let $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ and the sequence $(m_n)_{n \in \mathbb{N}}$ be constructed as in the previous sections. Suppose additionally that for every $n \in \mathbb{N}$ $d_0\left(f^{i \cdot m_n}, f_n^{i \cdot m_n}\right) < \frac{1}{2^n}$ for $i = 1, \ldots, k$, $\|DH_{n-1}\|_0 \leq \frac{\ln(q_n)}{n}$ and that the limit $f = \lim_{n \to \infty} f_n$ exists. Then f is $(\kappa_1, \ldots, \kappa_k)$-weakly mixing.*

Proof. Since every measurable set in $M = \mathbb{S}^1 \times [0,1]^{m-1}$ can be approximated by a countable disjoint union of cubes in $\mathbb{S}^1 \times (0,1)^{m-1}$ in arbitrary precision, we only have to prove the $(\kappa_1, \ldots, \kappa_k)$-weak mixing property in case that A and B are m-dimensional cubes in $\mathbb{S}^1 \times (0,1)^{m-1}$. So let $A, B \subseteq \mathbb{S}^1 \times (0,1)^{m-1}$ be m-dimensional cubes and $\varepsilon > 0$ be given.
According to Lemma 5.1 and Remark 5.6 respectively the partial partitions $\nu_n := H_{n-1} \circ g_n\left(\eta_n\right)$ and $\mathfrak{S}_n$ converge to the decomposition into points. Thus we can approximate B by a countable disjoint union of sets $\Gamma_n \in \nu_n$ and A by a countable disjoint union of sets $C_n \in \mathfrak{S}_n$ in given precision, when n is chosen big enough. Consequently for n sufficiently large there are sets $B_1 = \dot{\bigcup}_{i \in \Sigma_n^1} \Gamma_n^i$, $B_2 = \dot{\bigcup}_{i \in \Sigma_n^2} \Gamma_n^i$ with countable sets Σ_n^1, Σ_n^2 of indices satisfying $B_1 \subseteq B \subseteq B_2$ and $\mu\left(B \triangle B_j\right) \leq \varepsilon \cdot \mu(A) \cdot \mu\left(B\right)$ for $j = 1, 2$. Furthermore, there are sets $A_1 = \dot{\bigcup}_{i \in \Sigma_n^3} C_n^i$, $A_2 = \dot{\bigcup}_{i \in \Sigma_n^4} C_n^i$ with countable sets Σ_n^3, Σ_n^4 of indices satisfying $A_1 \subseteq A \subseteq A_2$, $|\mu(A) - \mu(A_j)| \leq \varepsilon \cdot \mu(A) \cdot \mu\left(B\right)$ as well as $\operatorname{dist}(\partial A, \partial A_j) > \frac{1}{2^n}$ for $j = 1, 2$, if n is chosen sufficiently large. Because of $d_0\left(f^{i \cdot m_n}, f_n^{i \cdot m_n}\right) < \frac{1}{2^n}$ for $i = 1, \ldots, k$ the following relations are true:

$$\begin{aligned} f_n^{i \cdot m_n}(x) \in A_1 &\Longrightarrow f^{i \cdot m_n}(x) \in A \\ f^{i \cdot m_n}(x) \in A &\Longrightarrow f_n^{i \cdot m_n}(x) \in A_2 \end{aligned}$$

Thus:

$$\begin{aligned}\mu\left(B_1 \cap f_n^{-i \cdot m_n}\left(A_1\right)\right) \leq \mu\left(B \cap f_n^{-i \cdot m_n}\left(A_1\right)\right) &\leq \mu\left(B \cap f^{-i \cdot m_n}\left(A\right)\right) \\ &\leq \mu\left(B \cap f_n^{-i \cdot m_n}\left(A_2\right)\right) \leq \mu\left(B_2 \cap f_n^{-i \cdot m_n}\left(A_2\right)\right)\end{aligned}$$

Additionally we choose n such that $\frac{22}{n} < \varepsilon$ as well as $|\kappa_i - \kappa_{i,n}| < \varepsilon \cdot \mu\left(A\right) \cdot \mu\left(B\right)$ hold. We can apply Lemma 5.7 on the sets A_2 and B_2. Therewith we obtain the following estimate from above for every $i \in \{1, \dots, k\}$:

$$\begin{aligned}
&\mu\left(B \cap f^{-i \cdot m_n}(A)\right) - \kappa_i \cdot \mu(A) \cdot \mu(B) - (1-\kappa_i) \cdot \mu(A \cap B) \\
\leq &\mu\left(B_2 \cap f_n^{-i \cdot m_n}(A_2)\right) - \kappa_{i,n} \cdot \mu(A) \cdot \mu(B) - (1-\kappa_{i,n}) \cdot \mu(A \cap B) + 2 \cdot |\kappa_i - \kappa_{i,n}| \\
\leq &\varepsilon \cdot \mu(A_2) \cdot \mu(B_2) + \kappa_{i,n} \cdot \mu(A_2) \cdot \mu(B_2) + (1-\kappa_{i,n}) \cdot \mu(A_2 \cap B_2) - \kappa_{i,n} \cdot \mu(A) \cdot \mu(B) \\
&- (1-\kappa_{i,n}) \cdot \mu(A \cap B) + 2 \cdot \varepsilon \cdot \mu(A) \cdot \mu(B) \\
\leq &\varepsilon \cdot (\mu(A) + \mu(A \triangle A_2)) \cdot (\mu(B) + \mu(B \triangle B_2)) + \kappa_{i,n} \cdot 3 \cdot \varepsilon \cdot \mu(A) \cdot \mu(B) \\
&+ (1-\kappa_{i,n}) \cdot (\mu(A \triangle A_2) + \mu(B \triangle B_2)) + 2 \cdot \varepsilon \cdot \mu(A) \cdot \mu(B) \\
\leq &8 \cdot \varepsilon \cdot \mu(A) \cdot \mu(B).
\end{aligned}$$

Analogously we estimate:

$$\mu\left(B \cap f^{-i \cdot m_n}(A)\right) - \kappa_i \cdot \mu(A) \cdot \mu(B) - (1-\kappa_i) \cdot \mu(A \cap B) \geq -8 \cdot \varepsilon \cdot \mu(A) \cdot \mu(B).$$

Both estimates combined enable us to conclude:

$$\left|\mu\left(B \cap f^{-i \cdot m_n}(A)\right) - \kappa_i \cdot \mu(A) \cdot \mu(B) - (1-\kappa_i) \cdot \mu(A \cap B)\right| \leq 8 \cdot \varepsilon \cdot \mu(A) \cdot \mu(B).$$

Since ε can be chosen arbitrarily small, the $(\kappa_1, \dots, \kappa_k)$-weak mixing property is proven. □

6 Convergence of $(f_n)_{n \in \mathbb{N}}$ in $\mathrm{Diff}^\infty(M)$

In the following we show that the sequence of constructed measure-preserving smooth diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ converges. For this purpose, we need a couple of results concerning the conjugation maps.

6.1 Properties of the conjugation maps ϕ_n and H_n

In this subsection we want to find estimates on the norms $|||H_n|||_k$. For this we will need the next technical result which is an application of the chain rule:

Lemma 6.1. *Let $\phi := \phi^{(m)}_{\lambda_m} \circ ... \circ \phi^{(2)}_{\lambda_2}$, $j \in \{1, ..., m\}$ and $k \in \mathbb{N}$. For any multiindex $\vec{a}$ with $|\vec{a}| = k$ the partial derivative $D_{\vec{a}}[\phi]_j$ consists of a sum of products of at most $(m-1) \cdot k$ terms of the following form*

$$D_{\vec{b}}\left(\left[\phi^{(i)}_{\lambda_i}\right]_l\right) \circ \phi^{(i-1)}_{\lambda_{i-1}} \circ ... \circ \phi^{(2)}_{\lambda_2}$$

where $l \in \{1, ..., m\}$, $i \in \{2, ..., m\}$ and $\vec{b}$ is a multiindex with $\left|\vec{b}\right| \leq k$.

In the same way we obtain a similar statement holding for the inverses:

Lemma 6.2. *Let $\psi := \left(\phi^{(2)}_{\lambda_2}\right)^{-1} \circ ... \circ \left(\phi^{(m)}_{\lambda_m}\right)^{-1}$, $j \in \{1, ..., m\}$ and $k \in \mathbb{N}$. For any multiindex $\vec{a}$ with $|\vec{a}| = k$ the partial derivative $D_{\vec{a}}[\psi]_j$ consists of a sum of products of at most $(m-1) \cdot k$ terms of the following form*

$$D_{\vec{b}}\left(\left[\left(\phi^{(i)}_{\lambda_i}\right)^{-1}\right]_l\right) \circ \left(\phi^{(i+1)}_{\lambda_{i+1}}\right)^{-1} \circ ... \circ \left(\phi^{(m)}_{\lambda_m}\right)^{-1}$$

where $l \in \{1, ..., m\}$, $i \in \{2, ..., m\}$ and $\vec{b}$ is a multiindex with $\left|\vec{b}\right| \leq k$.

With the aid of these technical results we can prove an estimate on the norms of the map ϕ_n:

Lemma 6.3. *For every $l \in \mathbb{N}$ it holds*

$$|||\phi_n|||_l \leq C \cdot q_n^{2\cdot(m-1)^2\cdot l\cdot(n-1+k)}$$

where C is a constant depending on m, l and n, but is independent of q_n.

Proof. First of all we consider the map $\tilde{\phi}_\lambda := \phi_{\lambda,\varepsilon,j} = C_\lambda^{-1} \circ \varphi_{\varepsilon,1,j} \circ C_\lambda$ introduced in subsection 3.1:

$$\tilde{\phi}_\lambda (x_1, ..., x_m) = \left(\frac{1}{\lambda} [\varphi_\varepsilon]_1 (\lambda x_1, x_2, ..., x_m), [\varphi_\varepsilon]_2 (\lambda x_1, x_2, ..., x_m), ..., [\varphi_\varepsilon]_m (\lambda x_1, x_2, ..., x_m) \right)$$

Let $l \in \mathbb{N}$. We compute for a multiindex $\vec{a}$ with $0 \leq |\vec{a}| \leq l$: $\|D_{\vec{a}} [\phi_\lambda]_1\|_0 \leq \lambda^{l-1} \cdot |||\varphi_\varepsilon|||_l$ and for $r \in \{2, ..., m\}$: $\|D_{\vec{a}} [\phi_\lambda]_r\|_0 \leq \lambda^l \cdot |||\varphi_\varepsilon|||_l$. Hereby we estimate $\|D_{\vec{a}} [\phi_\lambda]_r\|_0 \leq C \cdot \lambda^l$ and analogously $\left\|D_{\vec{a}} \left[\phi_\lambda^{-1}\right]_r\right\|_0 \leq C \cdot \lambda^l$ for a constant independent of λ. In conclusion this yields $|||\phi_\lambda|||_l \leq C \cdot \lambda^l$.
In the next step we consider $\phi := \phi_{\lambda_m}^{(m)} \circ ... \circ \phi_{\lambda_2}^{(2)}$. Let $\lambda_{max} := \max\{\lambda_2, ..., \lambda_m\}$. Inductively we will show $|||\phi|||_l \leq \tilde{C} \cdot \lambda_{max}^{(m-1)\cdot l}$ for every $l \in \mathbb{N}$, where $\tilde{C}$ is a constant independent of λ_i.
Start: $l = 1$. Let $l \in \{1, ..., m\}$ be arbitrary. By Lemma 6.1 a partial derivative of $[\phi]_l$ of first order consists of a sum of products of at most $m-1$ first order partial derivatives of functions $\phi_{\lambda_j}^{(j)}$. Therewith we obtain using $|||\phi_{\lambda,\varepsilon}^{(j)}|||_1 \leq C \cdot \lambda_{max}$ the estimate $\|D_i [\phi]_l\|_0 \leq C_1 \cdot \lambda_{max}^{m-1}$ for every $i \in \{1, ..., m\}$, where C_1 is a constant independent of λ.
With the aid of Lemma 6.2 we obtain the same statement for $\phi^{-1} = \left(\phi_{\lambda_2}^{(2)}\right)^{-1} \circ ... \circ \left(\phi_{\lambda_m}^{(m)}\right)^{-1}$. Hence we conclude: $|||\phi|||_1 \leq \tilde{C}_1 \cdot \lambda_{max}^{m-1}$.
Assumption: The claim is true for $l \in \mathbb{N}$.
Induction step $l \to l+1$: In the proof of Lemma 6.1 one observes that at the transition $l \to l+1$ in the product of at most $(m-1)\cdot l$ terms of the form $D_{\vec{b}}\left(\left[\phi_{\lambda_i}^{(i)}\right]_l\right) \circ \phi_{\lambda_{i-1}}^{(i-1)} \circ ... \circ \phi_{\lambda_2}^{(2)}$ one is replaced by a product of a term $\left(D_j D_{\vec{b}} \left[\phi_{\lambda_i}^{(i)}\right]_l\right) \circ \phi_{\lambda_{i-1}}^{(i-1)} \circ ... \circ \phi_{\lambda_2}^{(2)}$ with $j \in \{1, ..., m\}$ and at most $m-2$ partial derivatives of first order. Because of $|||\phi_{\lambda_i}^{(i)}|||_{l+1} \leq C \cdot \lambda_{max}^{l+1}$ and $|||\phi_{\lambda_j}^{(j)}|||_1 \leq C \cdot \lambda_{max}$ the λ_{max}-exponent increases by at most $1 + (m-2)\cdot 1 = m-1$.
In the same spirit one uses the proof of Lemma 6.2 to show that also in case of ϕ^{-1} the λ_{max}-exponent increases by at most $m-1$.
Using the assumption we conclude

$$|||\phi|||_{l+1} \leq \hat{C} \cdot \lambda_{max}^{l\cdot(m-1)+m-1} = \hat{C} \cdot \lambda_{max}^{(l+1)\cdot(m-1)}.$$

So the proof by induction is completed.

In the setting of our explicit construction of the map ϕ_n in section 3.1 we have $\varepsilon = \frac{1}{40\cdot n^4}$ and $\lambda_{max} = d_n \cdot (n \cdot q_n)^{2\cdot(m-1)\cdot(n-1+k)}$. Thus:

$$\begin{aligned} |||\phi_n|||_l &\leq \tilde{C}(m,l,n) \cdot \left(d_n \cdot (n\cdot q)^{2\cdot(m-1)\cdot(n-1+k)}\right)^{(m-1)\cdot l} \\ &\leq C(m,l,n) \cdot q_n^{2\cdot(m-1)^2\cdot l\cdot(n-1+k)} \end{aligned}$$

where $C(m,l,n)$ is a constant independent of q_n. □

In the next step we consider the map $h_n = g_n \circ \phi_n$ defined in section 3.2:

Lemma 6.4. *For every $l \in \mathbb{N}$ it holds:*

$$|||h_n|||_l \leq \bar{C} \cdot q_n^{2\cdot(m-1)^2\cdot l\cdot(n+k)}$$

where $\bar{C}$ is a constant depending on m, l and n, but is independent of q_n.

Proof. By definition of the map $h_n = g_n \circ \phi_n$ in section 3.2 we have:

$$\begin{aligned} & h_n(x_1,...,x_m) = g_n \circ \phi_n(x_1,...,x_m) \\ & = \left([\phi_n(x_1,...,x_m)]_1 + n\cdot q_n \cdot [\phi_n(x_1,...,x_m)]_2, [\phi_n(x_1,...,x_m)]_2, ..., [\phi_n(x_1,...,x_m)]_m\right) \end{aligned}$$

and

$$\begin{aligned} & h_n^{-1}(x_1,...,x_m) = \phi_n^{-1} \circ g_n^{-1}(x_1,...,x_m) \\ & = \left(\left[\phi_n^{-1}(x_1 - n\cdot q_n \cdot x_2, x_2, ..., x_m)\right]_1, ..., \left[\phi_n^{-1}(x_1 - n\cdot q_n\cdot x_2, x_2, ..., x_m)\right]_m\right) \end{aligned}$$

We estimate:

$$|||h_n|||_l \leq 2\cdot(n\cdot q_n)^l \cdot |||\phi_n|||_l \leq \bar{C}(m,l,n)\cdot q_n^l \cdot q_n^{2\cdot(m-1)^2\cdot l\cdot(n-1+k)} \leq \bar{C}(m,l,n)\cdot q_n^{2\cdot(m-1)^2\cdot l\cdot(n+k)}$$

with a constant $\bar{C}(m,l,n)$ independent of q_n. □

Remark 6.5. In the proof of the following Lemma we will use the formula of Faà di Bruno in several variables (e. g. [CS96]). For this we introduce an ordering on $\mathbb{N}_0^d$: For multiindices $\vec{\mu} = (\mu_1, ..., \mu_d)$ and $\vec{\nu} = (\nu_1, ..., \nu_d)$ in $\mathbb{N}_0^d$ we will write $\vec{\mu} \prec \vec{\nu}$, if one of the following properties is satisfied:

1. $|\vec{\mu}| < |\vec{\nu}|$, where $|\vec{\mu}| = \sum_{i=1}^d \mu_i$.
2. $|\vec{\mu}| = |\vec{\nu}|$ and $\mu_1 < \nu_1$
3. $|\vec{\mu}| = |\vec{\nu}|$, $\mu_i = \nu_i$ for $1 \leq i \leq k$ and $\mu_{k+1} < \nu_{k+1}$ for a $1 \leq k < d$

Additionally we will use these notations:

- For $\vec{\nu} = (\nu_1, ..., \nu_d) \in \mathbb{N}_0^d$:

$$\vec{\nu}! = \prod_{i=1}^{d} \nu_i!$$

- For $\vec{\nu} = (\nu_1, ..., \nu_d) \in \mathbb{N}_0^d$ and $\vec{z} = (z_1, ..., z_d) \in \mathbb{R}^d$:

$$\vec{z}^{\,\vec{\nu}} = \prod_{i=1}^{d} z_i^{\nu_i}$$

Then we get for the composition $h(x_1,...,x_d) := f\left(g^{(1)}(x_1,...,x_d), ..., g^{(m)}(x_1,...,x_d)\right)$ with sufficiently differentiable functions $f : \mathbb{R}^m \to \mathbb{R}$, $g^{(i)} : \mathbb{R}^d \to \mathbb{R}$ and a multiindex $\vec{\nu} \in \mathbb{N}_0^d$ with $|\vec{\nu}| = n$:

$$D_{\vec{\nu}} h = \sum_{\vec{\lambda}\in\mathbb{N}_0^m \text{ with } 1\leq|\vec{\lambda}|\leq n} D_{\vec{\lambda}} f \cdot \sum_{s=1}^{n} \sum_{p_s(\vec{\nu},\vec{\lambda})} \vec{\nu}! \cdot \prod_{j=1}^{s} \frac{\left[D_{\vec{l}_j}\vec{g}\right]^{\vec{k}_j}}{\vec{k}_j! \cdot \left(\vec{l}_j!\right)^{|\vec{k}_j|}}$$

Hereby $\left[D_{\vec{l}_j}\vec{g}\right]$ denotes $\left(D_{\vec{l}_j}g^{(1)},...,D_{\vec{l}_j}g^{(m)}\right)$ and

$p_s\left(\vec{\nu},\vec{\lambda}\right) :=$

$$\left\{\left(\vec{k}_1,...,\vec{k}_s,\vec{l}_1,...,\vec{l}_s\right) : \vec{k}_i \in \mathbb{N}_0^m, \left|\vec{k}_i\right| > 0, \vec{l}_i \in \mathbb{N}_0^d, 0 \prec \vec{l}_1 \prec ... \prec \vec{l}_s, \sum_{i=1}^{s}\vec{k}_i = \vec{\lambda} \text{ and } \sum_{i=1}^{s}\left|\vec{k}_i\right| \cdot \vec{l}_i = \vec{\nu}\right\}$$

Finally we are able to prove an estimate on the norms of the map H_n:

Lemma 6.6. *For every $l \in \mathbb{N}$ we get:*

$$|||H_n|||_l \leq \check{C} \cdot q_n^{2\cdot(m-1)^2\cdot l\cdot(n+k)}$$

where $\check{C}$ is a constant depending solely on m, l, n and H_{n-1}. Since H_{n-1} is independent of q_n in particular, the same is true for $\check{C}$.

Proof. Let $l \in \mathbb{N}$, $r \in \{1,...,m\}$ and $\vec{\nu} \in \mathbb{N}_0^m$ be a multiindex with $|\vec{\nu}| = l$. By applying the before mentioned formula of Faà di Bruno we estimate:

$$\begin{aligned}\left\|D_{\vec{\nu}}\left[H_n\right]_r\right\|_0 &= \left\|D_{\vec{\nu}}\left[H_{n-1}\circ h_n\right]_r\right\|_0 \\ &\leq \sum_{\vec{\lambda}\in\mathbb{N}_0^m \text{ with } 1\leq\left|\vec{\lambda}\right|\leq l}\left\|D_{\vec{\lambda}}\left[H_{n-1}\right]_r\right\|_0 \cdot \sum_{s=1}^{l}\sum_{p_s\left(\vec{\nu},\vec{\lambda}\right)}\vec{\nu}! \cdot \prod_{j=1}^{s}\frac{|||h_n|||_{\left|\vec{l}_j\right|}^{\left|\vec{k}_j\right|}}{\vec{k}_j! \cdot \left(\vec{l}_j!\right)^{\left|\vec{k}_j\right|}}.\end{aligned}$$

By definition of the set $p_s\left(\vec{\nu},\vec{\lambda}\right)$ we have $\sum_{i=1}^{s}\left|\vec{k}_i\right| \cdot \vec{l}_i = \vec{\nu}$. Hence:

$$l = |\vec{\nu}| = \left|\sum_{i=1}^{s}\left|\vec{k}_i\right| \cdot \vec{l}_i\right| = \sum_{t=1}^{m}\left(\sum_{i=1}^{s}\left|\vec{k}_i\right| \cdot \vec{l}_i\right)_t = \sum_{t=1}^{m}\sum_{i=1}^{s}\left|\vec{k}_i\right| \cdot \vec{l}_{i_t} = \sum_{i=1}^{s}\left|\vec{k}_i\right| \cdot \left(\sum_{t=1}^{m}\vec{l}_{i_t}\right) = \sum_{i=1}^{s}\left|\vec{k}_i\right| \cdot \left|\vec{l}_i\right|$$

Hereby, we compute using Lemma 6.4: $\prod_{j=1}^{s}|||h_n|||_{\left|\vec{l}_j\right|}^{\left|\vec{k}_j\right|} \leq \hat{C} \cdot q_n^{2\cdot(m-1)^2\cdot l\cdot(n+k)}$, where $\hat{C}$ is a constant independent of q_n. Since H_{n-1} is independent of q_n, we conclude:

$$\left\|D_{\vec{\nu}}\left[H_n\right]_r\right\|_0 \leq \check{C} \cdot q_n^{2\cdot(m-1)^2\cdot l\cdot(n+k)},$$

where $\check{C}$ is a constant independent of q_n.
In the same way we prove an analogous estimate on $\left\|D_{\vec{\nu}}\left[H_n^{-1}\right]_r\right\|_0$ and verify the claim. □

In particular we see that this norm can be estimated by a power of q_n.

6.2 Proof of convergence

For the proof of the convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ in the Diff$^\infty(M)$-topology the next result, that can be found in [FSW07, Lemma 4] is very useful.

Lemma 6.7. *Let $k \in \mathbb{N}_0$ and h be a C^∞-diffeomorphism on M. Then we get for every $\alpha, \beta \in \mathbb{R}$:*

$$d_k\left(h \circ R_\alpha \circ h^{-1}, h \circ R_\beta \circ h^{-1}\right) \leq C_k \cdot |||h|||_{k+1}^{k+1} \cdot |\alpha - \beta|,$$

where the constant C_k depends solely on k and m. In particular $C_0 = 1$.

In the following Lemma we state that under some assumptions on the sequence $(\alpha_n)_{n\in\mathbb{N}}$ the sequence $(f_n)_{n\in\mathbb{N}}$ converges to $f \in \mathcal{A}_\alpha(M)$ in the $\text{Diff}^\infty(M)$-topology. Afterwards we will show that we can fulfil these conditions (see Lemma 6.9).

Lemma 6.8. *Let $\varepsilon > 0$ be arbitrary and $(k_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers satisfying $\sum_{n=1}^{\infty} \frac{1}{k_n} < \varepsilon$. Furthermore we assume that in our constructions the following conditions are fulfilled:*

$$|\alpha - \alpha_1| < \varepsilon \qquad \text{and} \qquad |\alpha - \alpha_n| \leq \frac{1}{2 \cdot k_n \cdot C_{k_n} \cdot |||H_n|||_{k_n+1}^{k_n+1}} \text{ for every } n \in \mathbb{N}$$

where C_{k_n} are the constants from Lemma 6.7.

1. *Then the sequence of diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ converges in the $\text{Diff}^\infty(M)$-topology to a measure-preserving smooth diffeomorphism f, for which $d_\infty(f, R_\alpha) < 3 \cdot \varepsilon$ holds.*
2. *Also the sequence of diffeomorphisms $\hat{f}_n = H_n \circ R_\alpha \circ H_n^{-1} \in \mathcal{A}_\alpha(M)$ converges to f in the $\text{Diff}^\infty(M)$-topology. Hence $f \in \mathcal{A}_\alpha(M)$.*

Proof. See [Ku16, Lemma 5.8.]. □

As announced we show that we can satisfy the conditions from Lemma 6.8 in our constructions:

Lemma 6.9. *Let $(k_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers with $\sum_{n=1}^{\infty} \frac{1}{k_n} < \infty$ and C_{k_n} be the constants from Lemma 6.7. For any Liouvillean number α there exists a sequence $\alpha_n = \frac{p_n}{q_n}$ of rational numbers with $10n^2$ divides q_n, such that our conjugation maps H_n constructed in section 3.1 and 3.2 fulfil the following conditions:*

1. *For every $n \in \mathbb{N}$:*
$$|\alpha - \alpha_n| < \frac{1}{2 \cdot k_n \cdot C_{k_n} \cdot |||H_n|||_{k_n+1}^{k_n+1}}$$
2. *For every $n \in \mathbb{N}$:*
$$|\alpha - \alpha_n| < \frac{1}{2^{n+1} \cdot k \cdot q_n \cdot |||H_n|||_1}$$
3. *For every $n \in \mathbb{N}$*
$$\|DH_{n-1}\|_0 < \frac{\ln(q_n)}{n}$$

Proof. In Lemma 6.6 we saw $|||H_n|||_{k_n+1} \leq \breve{C}_n \cdot q_n^{2\cdot(m-1)^2\cdot(n+k)\cdot(k_n+1)}$, where the constant $\breve{C}_n$ was independent of q_n. Thus we can choose $q_n \geq \breve{C}_n$ for every $n \in \mathbb{N}$. Hence we obtain: $|||H_n|||_{k_n+1} \leq q_n^{2\cdot m^2\cdot(n+k)\cdot(k_n+1)}$.
Besides $q_n \geq \breve{C}_n$ we keep the condition $q_n \geq 200 \cdot (n-1)^2 \cdot k \cdot d_{n-1}^2 \cdot (n-1)^{4\cdot(n-1+k)} \cdot q_{n-1}^{4\cdot(n-2+k)}$ from Remark 4.4 in mind. Furthermore we can demand $\|DH_{n-1}\|_0 < \frac{\ln(q_n)}{n}$ from q_n because H_{n-1}

is independent of q_n. Since α is a Liouvillean number, we find a sequence of rational numbers $\tilde{\alpha}_n = \frac{\tilde{p}_n}{\tilde{q}_n}$, $\tilde{p}_n, \tilde{q}_n$ relatively prime, under the above restrictions (formulated for $\tilde{q}_n$) satisfying:

$$|\alpha - \tilde{\alpha}_n| = \left|\alpha - \frac{\tilde{p}_n}{\tilde{q}_n}\right| < \frac{|\alpha - \alpha_{n-1}|}{2^{n+1} \cdot k \cdot k_n \cdot C_{k_n} \cdot (10n^2)^{1+2\cdot m^2 \cdot (n+k)\cdot(k_n+1)^2} \cdot \tilde{q}_n^{1+2\cdot m^2\cdot(n+k)\cdot(k_n+1)^2}}$$

Put $q_n \coloneqq 10n^2 \cdot \tilde{q}_n$ and $p_n \coloneqq 10n^2 \cdot \tilde{p}_n$. Then we obtain:

$$|\alpha - \alpha_n| < \frac{|\alpha - \alpha_{n-1}|}{2^{n+1} \cdot k \cdot k_n \cdot C_{k_n} \cdot q_n^{1+2\cdot m^2\cdot(n+k)\cdot(k_n+1)^2}}.$$

So we have $|\alpha - \alpha_n| \overset{n\to\infty}{\longrightarrow} 0$ monotonically. Because of $|||H_n|||_{k_n+1}^{k_n+1} \leq q_n^{2\cdot m^2\cdot(n+k)\cdot(k_n+1)^2}$ this yields: $|\alpha - \alpha_n| < \frac{1}{2^{n+1}\cdot k\cdot q_n\cdot k_n\cdot C_{k_n}\cdot |||H_n|||_{k_n+1}^{k_n+1}}$. Thus the first property of this Lemma is fulfilled.
Furthermore, we note $k_n \geq 1$ and $C_{k_n} \geq 1$ by Lemma 6.7. Thus $q_n \cdot k_n \cdot C_{k_n} \geq q_n$. Moreover, $|||H_n|||_1 \geq \|H_n\|_0 = 1$, because $H_n : \mathbb{S}^1 \times [0,1]^{m-1} \to \mathbb{S}^1 \times [0,1]^{m-1}$ is a diffeomorphism. Hence $|||H_n|||_{k_n+1}^{k_n+1} \geq |||H_n|||_1$. Altogether we conclude $2^{n+1} \cdot q_n \cdot k_n \cdot C_{k_n} \cdot |||H_n|||_{k_n+1}^{k_n+1} \geq 2^{n+1} \cdot q_n \cdot |||H_n|||_1$ and so:

$$|\alpha - \alpha_n| < \frac{1}{2^{n+1} \cdot k \cdot q_n \cdot k_n \cdot C_{k_n} \cdot |||H_n|||_{k_n+1}^{k_n+1}} \leq \frac{1}{2^{n+1} \cdot k \cdot q_n \cdot |||H_n|||_1}, \tag{6}$$

i.e. we verified the second property. □

Remark 6.10. Lemma 6.9 shows that the conditions of Lemma 6.8 are satisfied. Therefore our sequence of constructed diffeomorphisms f_n converges in the $\text{Diff}^\infty(M)$-topology to a diffeomorphism $f \in \mathcal{A}_\alpha(M)$.

To apply Proposition 5.8 we need another result:

Lemma 6.11. *Let $(\alpha_n)_{n\in\mathbb{N}}$ be constructed as in Lemma 6.9. Then it holds for every $n \in \mathbb{N}$, for every $i \in \mathbb{N}$, $i \leq k$, and for every $\tilde{m} \leq q_{n+1}$:*

$$d_0\left(f^{i\cdot\tilde{m}}, f_n^{i\cdot\tilde{m}}\right) \leq \frac{1}{2^n}$$

Proof. According to our construction it holds $h_n \circ R_{\alpha_n} = R_{\alpha_n} \circ h_n$ and hence

$$\begin{aligned} f_{n-1} = H_{n-1} \circ R_{\alpha_n} \circ H_{n-1}^{-1} &= H_{n-1} \circ R_{\alpha_n} \circ h_n \circ h_n^{-1} \circ H_{n-1}^{-1} \\ &= H_{n-1} \circ h_n \circ R_{\alpha_n} \circ h_n^{-1} \circ H_{n-1}^{-1} = H_n \circ R_{\alpha_n} \circ H_n^{-1}. \end{aligned}$$

Hereby and with the aid of Lemma 6.7 we compute:

$$d_0\left(f_j^{i\cdot\tilde{m}}, f_{j-1}^{i\cdot\tilde{m}}\right) = d_0\left(H_j \circ R_{i\cdot\tilde{m}\cdot\alpha_{j+1}} \circ H_j^{-1}, H_j \circ R_{i\cdot\tilde{m}\cdot\alpha_j} \circ H_i^{-1}\right) \leq |||H_j|||_1 \cdot i \cdot \tilde{m} \cdot 2 \cdot |\alpha - \alpha_j|.$$

Since $\tilde{m} \leq q_{n+1} \leq q_j$ we conclude for every $j > n$ using equation 6 :

$$d_0\left(f_j^{i\cdot\tilde{m}}, f_{j-1}^{i\cdot\tilde{m}}\right) \leq |||H_j|||_1 \cdot i \cdot \tilde{m} \cdot 2 \cdot |\alpha - \alpha_j| \leq |||H_j|||_1 \cdot i \cdot \tilde{m} \cdot 2 \cdot \frac{1}{2^{j+1} \cdot k \cdot q_j \cdot |||H_j|||_1} \leq \frac{i \cdot \tilde{m}}{k \cdot q_j} \cdot \frac{1}{2^j} \leq \frac{1}{2^j}.$$

Thus for every $\tilde{m} \leq q_{n+1}$ we get the claimed result:

$$d_0\left(f^{i\cdot\tilde{m}}, f_n^{i\cdot\tilde{m}}\right) = \lim_{k\to\infty} d_0\left(f_k^{i\cdot\tilde{m}}, f_n^{i\cdot\tilde{m}}\right) \leq \lim_{k\to\infty} \sum_{j=n+1}^{k} d_0\left(f_j^{i\cdot\tilde{m}}, f_{j-1}^{i\cdot\tilde{m}}\right) \leq \sum_{j=n+1}^{\infty} \frac{1}{2^j} = \left(\frac{1}{2}\right)^n.$$

□

Remark 6.12. Note that the sequence $(m_n)_{n\in\mathbb{N}}$ defined in section 4 meets the mentioned condition $m_n \leq q_{n+1}$ and hence 6.11 can be applied on it.

Concluding we have checked that all the assumptions of Proposition 5.8 are satisfied. Thus this criterion guarentees that the constructed diffeomorphism $f \in \mathcal{A}_\alpha(M)$ is $(\kappa_1,\dots,\kappa_k)$-weakly mixing. In addition, for every $\varepsilon > 0$ we can choose the parameters by Lemma 6.8 in such a way, that $d_\infty(f, R_\alpha) < \varepsilon$ holds.

In order to prove the genericity results in section 7 we have to compute the weak distance between f and f_n:

Lemma 6.13. *Let $\nu_n \coloneqq H_{n-1}\circ g_n(\eta_n)$ and $(m_n)_{n\in\mathbb{N}}$ be the sequence of natural numbers defined in section 4. Then we have for every $i \in \{1,\dots,k\}$*

$$\sum_{c\in\nu_n} \mu\left(f_n^{i\cdot m_n}(c) \triangle f^{i\cdot m_n}(c)\right) < \frac{1}{n}\cdot\frac{1}{n^{2m}\cdot q_n^{3m}}$$

Proof. At first we observe for $\Gamma_n = H_{n-1}\circ g_n\left(\hat{I}_n\right) \in \nu_n$

$$\mu\left(f_n^{i\cdot m_n}(\Gamma_n)\triangle f_{n+1}^{i\cdot m_n}(\Gamma_n)\right) = \mu\left(R_{\alpha_{n+1}}^{i\cdot m_n}\left(\phi_n^{-1}\left(\hat{I}_n\right)\right)\triangle h_{n+1}\circ R_{\alpha_{n+2}}^{i\cdot m_n}\circ h_{n+1}^{-1}\left(\phi_n^{-1}\left(\hat{I}_n\right)\right)\right)$$

and introduce the notation $b_{n,i} \coloneqq d_0\left(R_{\alpha_{n+1}}^{i\cdot m_n}, h_{n+1}\circ R_{\alpha_{n+2}}^{i\cdot m_n}\circ h_{n+1}^{-1}\right)$.
In case of $0 \leq l < n-k$ there are

$$n\cdot q_n\cdot\left(nq_n^2 - 2\cdot\left\lceil\frac{q_n^2}{10n^3}\right\rceil\right)\cdot(n-k-1)\cdot\left(n^2q_n^2 - 2\cdot\left\lceil\frac{q_n^2}{10n^2}\right\rceil\right)^{m-2} \leq (n\cdot q_n)^{2\cdot(m-1)+1}$$

elements of the partition η_n on $\left[\frac{l}{n\cdot q_n}, \frac{l+1}{n\cdot q_n}\right]\times[0,1]^{m-1}$. By the shape of elements in η_n and the calculations in the proof of Lemma 4.5 the image of such an element under $R_{\alpha_{n+1}}^{i\cdot m_n}\circ\phi_n^{-1}$ consists of

$$\begin{aligned}&1+\sum_{d=1}^{k} c_{d,n}\cdot\left(n^2q_n^2 - 2\cdot\left\lceil\frac{q_n^2}{10n^2}\right\rceil\right)^{m\cdot(n-1+k)-1}\cdot\left(n^2q_n^2 - 2\cdot\left\lceil\frac{q_n^2}{10n^2}\right\rceil\right)^{(m-1)\cdot(n-2+k)}\\ &\leq\sum_{d=1}^{k} c_{d,n}\cdot(nq_n)^{4\cdot m\cdot(n-1+k)}\end{aligned}$$

blocks.
On the “distribution part” such a block has sidelengths $\frac{1-\frac{1}{5n^4}}{d_n\cdot(n\cdot q_n)^{2\cdot(n-1+k+T)}}$ in the θ-coordinate, $\frac{1-\frac{1}{5n^4}}{(n\cdot q_n)^{2\cdot(n-1+k+T)}}$ in the coordinates $r_1,\dots,r_{m-2}$ as well as $\frac{1-\frac{1}{5n^4}}{(n\cdot q_n)^{2\cdot m\cdot(n-1+k)-2\cdot(m-1)\cdot T}}$ in the r_{m-1}-coordinate for some $T\in\{1,\dots,n-1\}$. The part of $h_{n+1}\circ R_{\alpha_{n+2}}^{i\cdot m_n}\circ h_{n+1}^{-1}\left(\phi_n^{-1}\left(\hat{I}_n\right)\right)$ corresponding to any block of $R_{\alpha_{n+1}}^{i\cdot m_n}\left(\phi_n^{-1}\left(\hat{I}_n\right)\right)$ surrounds a block of sidelengths $\frac{1-\frac{1}{5n^4}}{d_n\cdot(n\cdot q_n)^{2\cdot(n-1+k+T)}} - 2b_{n,i}$ in the θ-coordinate, $\frac{1-\frac{1}{5n^4}}{(n\cdot q_n)^{2\cdot(n-1+k+T)}} - 2b_{n,i}$ in the $r_1,\dots,r_{m-2}$-coordinates and $\frac{1-\frac{1}{5n^4}}{(n\cdot q_n)^{2\cdot m\cdot(n-1+k)-2\cdot(m-1)\cdot T}}$ −

$2b_{n,i}$ in the r_{m-1}-coordinate as well as it is contained in a block of sidelengths $\frac{1-\frac{1}{5n^4}}{d_n \cdot (n \cdot q_n)^{2\cdot(n-1+k+T)}} +$ $2b_{n,i}$ in the θ-coordinate, $\frac{1-\frac{1}{5n^4}}{(n \cdot q_n)^{2\cdot(n-1+k+T)}} + 2b_{n,i}$ in the $r_1, ..., r_{m-2}$-coordinates and $\frac{1-\frac{1}{5n^4}}{(n\cdot q_n)^{2\cdot m\cdot(n-1+k)-2\cdot(m-1)\cdot T}} +$ $2b_{n,i}$ in the r_{m-1}-coordinate.
Similar considerations apply on the block $R^{i\cdot m_n}_{\alpha_{n+1}} \circ \phi_n^{-1} (I_N \times W)$.
Since

$$\begin{aligned} b_{n,i} =& d_0 \left(R^{i\cdot m_n}_{\alpha_{n+1}}, h_{n+1} \circ R^{i\cdot m_n}_{\alpha_{n+2}} \circ h^{-1}_{n+1} \right) \leq |||h_{n+1}|||_1 \cdot k \cdot m_n \cdot |\alpha_{n+1} - \alpha_{n+2}| \\ <& |||H_{n+1}|||_1 \cdot q_{n+1} \cdot 2 \cdot |\alpha_{n+1} - \alpha| < \frac{1}{q_{n+1}^{2\cdot m^2 \cdot (n+k+1)\cdot k^2_{n+1}}} \end{aligned}$$

by Lemma 6.9, we conclude

$$\begin{aligned} \sum_{c \in \nu_n} \mu \left(f_n^{i\cdot m_n}(c) \triangle f_{n+1}^{i\cdot m_n}(c) \right) \leq& (n-k) \cdot q_n \cdot (n \cdot q_n)^{2\cdot(m-1)+1} \cdot \sum_{d=1}^{k} c_{d,n} \cdot (nq_n)^{4\cdot m\cdot(n-1+k)} \cdot 4 \cdot m \cdot b_{n,i} \\ <& \frac{1}{2n} \cdot \frac{1}{n^{2m} \cdot q_n^{3m}} \end{aligned}$$

This implies the claim. □

7 Proof of genericity

First of all, Proposition 2.2 yields the denseness of $(\kappa_1, \dots, \kappa_k)$-weakly mixing diffeomorphisms in $\mathcal{A}_\alpha(M)$:
Because of $\mathcal{A}_\alpha(M) = \overline{\{h \circ R_\alpha \circ h^{-1} : h \in \text{Diff}^\infty(M, \mu)\}}^{C^\infty}$ it is enough to show that for every diffeomorphism $h \in \text{Diff}^\infty(M,\mu)$ and every $\epsilon > 0$ there is a $(\kappa_1, \dots, \kappa_k)$-weakly mixing diffeomorphism $\tilde{f}$ such that $d_\infty\left(\tilde{f}, h \circ R_\alpha \circ h^{-1}\right) < \epsilon$. For this purpose, let $h \in \text{Diff}^\infty(M,\mu)$ and $\epsilon > 0$ be arbitrary. Since $\text{Diff}^\infty(M)$ is a Lie group, the conjugating map $g \mapsto h \circ g \circ h^{-1}$ is continuous with respect to the metric d_∞. Continuity in the point R_α yields the existence of $\delta > 0$, such that $d_\infty(g, R_\alpha) < \delta$ implies $d_\infty\left(h \circ g \circ h^{-1}, h \circ R_\alpha \circ h^{-1}\right) < \epsilon$. By Proposition 2.2 we can find a $(\kappa_1, \dots, \kappa_k)$-weakly mixing diffeomorphism f with $d_\infty(f, R_\alpha) < \delta$. Hence $\tilde{f} := h \circ f \circ h^{-1}$ satisfies $d_\infty\left(\tilde{f}, h \circ R_\alpha \circ h^{-1}\right) < \epsilon$. Note that $\tilde{f}$ is $(\kappa_1, \dots, \kappa_k)$-weakly mixing.

In order to prove the genericity statement in Theorem 1 we consider all sequences of constructed diffeomorphisms $(f_n)_{n\in\mathbb{N}}$ satisfying the requirements from the previous sections. Let $U_n(f_n)$ be the subsequent neighbourhood of the diffeomorphism f_n:

$$U_n(f_n) :=$$
$$\left\{ g \in \text{Diff}^\infty(M,\mu) : \ d_{k_{n+1}}(f_n, g) < \frac{2}{k_{n+1}}, \sum_{\Gamma_n \in \nu_n} \mu\left(g^{i\cdot m_n}(\Gamma_n) \triangle f_n^{i\cdot m_n}(\Gamma_n)\right) < \frac{1}{n} \cdot \frac{1}{n^{2m} \cdot q_n^{3m}} \text{ for } i = 1, \dots, k \right\}$$

By Θ_n we denote the union of all neighbourhoods $U_n(f_n)$ over all the n-th diffeomorphisms in the above mentioned sequences. Since the neighbourhoods $U_n(f_n)$ are open the sets Θ_n are open as well. Then

$$\Theta \coloneqq \bigcap_{n\in\mathbb{N}} \bigcup_{s\geq n} \Theta_s$$

is a G_δ-set as the countable intersection of open sets.

- For all the sequences $(f_n)_{n\in\mathbb{N}}$ the respective limit diffeomorphism $f \in \mathcal{A}_\alpha(M)$ belongs to Θ, because it belongs to $U_n(f_n)$ for every $n \in \mathbb{N}$ by construction (cf. Lemma 6.13). So Θ contains all the constructed diffeomorphisms with the aimed properties. Hence, it is dense in $\mathcal{A}_\alpha(M)$ due to the above considerations.

- In the next step we want to show that $f \in \Theta$ is a $(\kappa_1, \dots, \kappa_k)$-weakly mixing diffeomorphism: For any $f \in \bigcap_{n\in\mathbb{N}} \bigcup_{s\geq n} \Theta_s$ there is a sequence $(n_l)_{l\in\mathbb{N}}$ with $n_l \to \infty$ as $l \to \infty$, such that $f \in \Theta_{n_l}$. So there is a sequence $(f_{n_l})_{l\in\mathbb{N}}$ of diffeomorphisms, at which f_{n_l} is the n_l-th element of one of the above mentioned sequences of constructed diffeomorphisms, such that $f \in U_{n_l}(f_{n_l})$. We observe that $\nu_{n_l} \to \varepsilon$ as $l \to \infty$, where ν_{n_l} is the partition belonging to the diffeomorphism f_{n_l}. Moreover, we have

$$\begin{aligned}&\left|\mu\left(\Gamma_{n_l} \cap f^{-i\cdot m_{n_l}}\left(A_{n_l}\right)\right) - \kappa_{i,n_l} \cdot \mu\left(\Gamma_{n_l}\right) \cdot \mu\left(A_{n_l}\right) - \left(1-\kappa_{i,n_l}\right) \cdot \mu\left(\Gamma_{n_l} \cap A_{n_l}\right)\right| \\ < &\frac{23}{n_l} \cdot \mu\left(\Gamma_{n_l}\right) \cdot \mu\left(A_{n_l}\right)\end{aligned}$$

  for every $\Gamma_{n_l} \in \nu_{n_l}$ and $A_{n_l} \in \mathfrak{S}_{n_l}$ by the definition of the neighbourhoods $U_{n_l}(f_{n_l})$. Then we can conclude that f is $(\kappa_1, \dots, \kappa_k)$-weakly mixing.

Thus the set of $(\kappa_1, \dots, \kappa_k)$-weakly mixing diffeomorphisms in $\mathcal{A}_\alpha(M)$ contains a dense G_δ-set. Hence, Theorem 1 is deduced from Proposition 2.2.

8 Applications

As a first application of the $(\kappa_1, \dots, \kappa_k)$-weak mixing property we state [JL92, Lemma 1.3]:

Lemma 8.1. *Suppose that T is $(1-\kappa_1, \dots, 1-\kappa_K)$-weakly mixing with $\kappa_1, \dots, \kappa_K \in (0,1)$ and $(\log(\kappa_1), \dots, \log(\kappa_K)\}$ linearly independent over $\mathbb{Q}$. Then*

$$\sigma_{T^{k(1)}} * \dots * \sigma_{T^{k(l)}} \perp \sigma_{T^{k'(1)}} * \dots * \sigma_{T^{k'(l)}}$$

for all numbers $0 < k(1), \dots, k(l), k'(1), \dots, k'(l') \leq K$ unless $(k(1), \dots, k(l))$ is a permutation of $(k'(1), \dots, k'(l'))$.

Hereby, Corollary 1 follows from Theorem 1.

Moreover, we can follow the lines of [JL92] and [Ru79, Example 2] to get a smooth weakly mixing diffeomorphism with no measurable square root: Let $T \in \mathcal{A}_\alpha(M)$ be a diffeomorphism as constructed in Corollary 1 and $S = T \times T \times T$. By [JL92, Corollary 3] any measurable square root must have the form $\left(T^{l(1)} \times T^{l(2)} \times T^{l(3)}\right) U_\pi$ with $l(i) \in \mathbb{Z}$ and $\pi^2 = \text{id}$, where $U_\pi : M^3 \to M^3$ is defined by $U_\pi(x)(j) = x(\pi(j))$. Thus, for some i we would have $\pi(i) = i$ and then $2l(i) = 1$ which is impossible for $l(i) \in \mathbb{Z}$.

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