

Real-analytic diffeomorphisms with homogeneous spectrum and disjointness of convolutions

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Abstract

On any torus \mathbb{T}^d , $d \geq 2$, we prove the existence of a real-analytic diffeomorphism T with a good approximation of type $(h, h + 1)$, a maximal spectral type disjoint with its convolutions and a homogeneous spectrum of multiplicity two for the Cartesian square $T \times T$. The proof is based on a real-analytic version of the Approximation by Conjugation-method.

1 Introduction

One of the main problems in the spectral theory of dynamical systems at the interface of unitary operator theory and ergodic theory is the following question:

Question 1. *What are possible spectral properties for a Koopman operator associated with a measure-preserving transformation?*

More specifically, one can search for transformations possessing specific essential values \mathcal{M}_{U_T} of the spectral multiplicities:

Question 2. *Given a subset $E \subset \mathbb{N} \cup \{\infty\}$, is there an ergodic transformation T such that $\mathcal{M}_{U_T} = E$?*

These two problems are open and no restrictions (except for the obvious ones) are known. However, there is an impressive progress concerning Question 2 (see [Da13] for a survey on spectral multiplicities of ergodic actions) and there exist two standard points of view: to consider the spectrum of T (and in particular \mathcal{M}_{U_T}) either on $L^2(X, \mu)$ or on the orthogonal complement $L_0^2(X, \mu)$ of the constant functions. In [KL95] it was proved that all possible subsets of $\mathbb{N} \cup \{\infty\}$ can be realized as \mathcal{M}_{U_T} for some ergodic transformation T in the first case (since 1 is always an eigenvalue because of the constant functions, “possible” means any subset of $\mathbb{N} \cup \{\infty\}$ with 1 as an element). In the second case the Cartesian powers of a generic transformation provide a good opportunity for the construction of examples with the infimum of essential spectral multiplicities larger than 1. Although, it seems very unlikely that these Cartesian powers have finite maximal spectral multiplicity, this is the generic case: Independently Ageev and Ryzhikov proved the celebrated result, that for a generic automorphism T the Cartesian square $T \times T$ has homogeneous spectrum of multiplicity 2 (see [Ag99] resp. [Ry99a]). Ageev was even able to show that for the n -th power $T^n = T \times \dots \times T$ of a generic transformation T it holds $M(T^n) = \{n, n \cdot (n - 1), \dots, n!\}$ (cf. [Ag99, Theorem 2]). He also proved for every $n \in \mathbb{N}$ the existence of an ergodic transformation with homogeneous spectrum of multiplicity n in the orthogonal complement of the constant functions ([Ag05, Theorem 1]) solving

Rokhlin's problem on homogeneous spectrum in ergodic theory.

Another important question in Ergodic Theory asks

Question 3. *Are there smooth versions to the objects and concepts of abstract ergodic theory?*

With regard to the *spectral multiplicity problem* in Question 2 there are only few results in this direction. Explicitly, Danilenko asks which subsets $E \neq \{1\}$ admit a smooth ergodic transformation T with $\mathcal{M}_{U_T} = E$ ([Da13, section 10]). Blanchard and Lemanczyk showed that every set E containing 1 as well as $\text{lcm}(e_1, e_2)$ for $e_1, e_2 \in E$ is realizable as the set of essential spectral multiplicities for a Lebesgue measure-preserving analytic diffeomorphism of a finite dimensional torus ([BL93]). Recently, the second author proved that on any smooth compact connected manifold M of dimension $d \geq 2$ admitting a non-trivial circle action $\mathcal{S} = \{\phi^t\}_{t \in \mathbb{S}^1}$ the set of C^∞ -diffeomorphisms T , that have a homogeneous spectrum of multiplicity 2 for $T \times T$ and a maximal spectral type disjoint with its convolutions, is residual (i.e. it contains a dense G_δ -set) in $\mathcal{A}_\alpha(M) = \overline{\{h \circ \phi^\alpha \circ h^{-1} : h \in \text{Diff}^\infty(M, \mu)\}}^{C^\infty}$ for every Liouvillean number α ([Ku16]). Hereby, Problem 7.11. in [FK04] was answered affirmatively. We are able to prove an analogous result in the real-analytic category,

Theorem 1.1. *For any $\rho > 0$ and $d \geq 2$, there exist real-analytic diffeomorphisms $T \in \text{Diff}_\rho^\omega(\mathbb{T}^d, \mu)$ that have a maximal spectral type disjoint with its convolutions, a homogeneous spectrum of multiplicity 2 for $T \times T$ and admit a good approximation of type $(h, h + 1)$.*

The first part of this Theorem is linked to a conjecture of Kolmogorov respectively Rokhlin and Fomin (after verifying that the property held for all dynamical systems known at that time, especially large classes of systems of probabilistic origin like Gaussian ones), namely that every ergodic transformation possesses the so-called group property, i.e. the maximal spectral type σ is symmetric and dominates its square $\sigma * \sigma$. This conjecture is an analogue of the well-known group property of the set of eigenvalues of an ergodic automorphism and was proven to be false. Indeed, in [St66] A.M. Stepin gave the first example of a dynamical system without the group property. V.I. Oseledets constructed an analogous example with continuous spectrum ([Os69]). Later Stepin showed that for a generic transformation all convolutions σ_0^k , $k \in \mathbb{N}$, of the maximal spectral type σ_0 on $L_0^2(X, \mu)$ are mutually singular (see [St87]).

In general, one of the most powerful tools for finding answers to Question 3 and for constructing volume preserving C^∞ -diffeomorphisms with prescribed ergodic or topological properties on any compact connected manifold M of dimension $d \geq 2$ admitting a non-trivial circle action $\mathcal{S} = \{\phi^t\}_{t \in \mathbb{S}^1}$ is the so called *Approximation by Conjugation*-method developed by D.V. Anosov and A. Katok in their fundamental paper [AK70]. These diffeomorphisms are constructed as limits of conjugates $T_n = H_n^{-1} \circ \phi^{\alpha_{n+1}} \circ H_n$, where $\alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} = \alpha_n + \frac{1}{k_n \cdot l_n \cdot q_n^2} \in \mathbb{Q}$, $H_n = h_n \circ H_{n-1}$ and h_n is a measure-preserving diffeomorphism satisfying $\phi^{\alpha_n} \circ h_n = h_n \circ \phi^{\alpha_n}$. In each step the conjugation map h_n and the parameter l_n are chosen such that the diffeomorphism T_n imitates the desired property with a certain precision. Then the parameter k_n is chosen large enough to guarantee closeness of T_n to T_{n-1} in the C^∞ -topology and so the convergence of the sequence $(T_n)_{n \in \mathbb{N}}$ to a limit diffeomorphism is provided. This method enables the construction of smooth diffeomorphisms with specific ergodic properties (e.g. weak mixing ones in [AK70, section 5]) or non-standard smooth realizations of measure-preserving systems (e.g. [AK70, section 6] and [FSW07]). See also the very interesting survey article [FK04] for more details and other results of this method.

Unfortunately, there are great challenging differences in the real-analytic category as discussed in [FK04, section 7.1]: Since maps with very large derivatives in the real domain or its inverses are expected to have singularities in a small complex neighbourhood, for a real analytic family S_t , $0 \leq t \leq t_0$, $S_0 = \text{id}$, the family $h^{-1} \circ S_t \circ h$ is expected to have singularities very close to the real domain for any $t > 0$. So, the domain of analyticity for maps of our form $f_n = H_n^{-1} \circ \phi^{\alpha_{n+1}} \circ H_n$ will shrink at any step of the construction and the limit diffeomorphism will not be analytic. Thus, it is necessary to find conjugation maps of a special form which may be inverted more or less explicitly in such a way that one can guarantee analyticity of the map and its inverse in a large complex domain.

Recently, some progress has been made in applying the AbC (short for approximation by conjugation)-method in the real-analytic category on particular manifolds. Fayad and Katok constructed volume-preserving uniquely ergodic real-analytic diffeomorphisms on odd-dimensional spheres in [FK14]. In case of the torus \mathbb{T}^d , $d \geq 2$, the authors were able to reproduce several examples of smooth dynamical systems obtained by the AbC-scheme in the category of real-analytic diffeomorphisms in a series of papers ([Ba15], [Ku15], [BK], [Ba]). In particular, we construct minimal but not uniquely ergodic diffeomorphisms and nonstandard real-analytic realizations of toral translations. All these constructions base on the concept of *block-slide type maps* on the torus and their sufficiently precise approximation by measure preserving real-analytic diffeomorphisms. This approach is the important mechanism in the constructions of this paper as well. We emphasize that all constructions in this article are done on the torus and that real-analytic AbC constructions on arbitrary real-analytic manifolds continue to remain an intractable problem.

Outline of the paper

For a start we present several important definitions and concepts that will be used in this paper. In particular, we introduce the topology of real-analytic diffeomorphisms on the torus, the general AbC-scheme as well as block-slide type maps and their analytic approximations. In section 3 we give an overview of periodic approximation in ergodic theory. We have also included a survey section about spectral theory in Dynamical Systems. With the aid of this theoretical background we get a criterion for the proof of the main theorem in Proposition 5.1. In fact, this Proposition reduces the proof to the construction of a diffeomorphism admitting a good cyclic approximation and a good linked approximation of type $(h, h + 1)$. At this point, we will also sketch how to construct such a map. In particular, we describe the underlying combinatorics. In the rest of the paper, we present its construction via the concept of block-slide type maps and with explicitly defined tower elements in detail.

2 Preliminaries

Here we introduce the basic concepts and establish notations that we will use for the rest of this article.

For a natural number d , we will denote the d dimensional torus by $\mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d$. The standard Lebesgue measure on \mathbb{T}^d will be denoted by μ . We define ϕ , a measure preserving \mathbb{T}^1 action on the torus \mathbb{T}^d as follows:

$$\phi^t(x_1, \dots, x_d) = (x_1 + t, x_2, \dots, x_d). \quad (2.1)$$

2.1 The topology of real-analytic diffeomorphisms on the torus

We give a description of the space of diffeomorphisms that are interesting to us. Any real-analytic diffeomorphism on \mathbb{T}^d homotopic to the identity admits a lift to a map from \mathbb{R}^d to \mathbb{R}^d and has the following form

$$F(x_1, \dots, x_d) = (x_1 + f_1(x_1, \dots, x_d), \dots, x_d + f_d(x_1, \dots, x_d)), \quad (2.2)$$

where $f_i : \mathbb{R}^d \rightarrow \mathbb{R}$ are \mathbb{Z}^d -periodic real-analytic functions. Any real-analytic \mathbb{Z}^d -periodic function defined on \mathbb{R}^d can be extended to some complex neighbourhood ¹ of \mathbb{R}^d as a holomorphic (complex analytic) function. For a fixed $\rho > 0$, let

$$\Omega_\rho := \{(z_1, \dots, z_d) \in \mathbb{C}^d : |\operatorname{Im}(z_1)| < \rho, \dots, |\operatorname{Im}(z_d)| < \rho\} \quad (2.3)$$

and for a function f defined on this set, put

$$\|f\|_\rho := \sup_{(z_1, \dots, z_d) \in \Omega_\rho} |f(z_1, \dots, z_d)|. \quad (2.4)$$

We define $C_\rho^\omega(\mathbb{T}^d)$ to be the space of all \mathbb{Z}^d -periodic real-analytic functions on \mathbb{R}^d that extends to a holomorphic function on Ω_ρ and $\|f\|_\rho < \infty$.

We define, $\operatorname{Diff}_\rho^\omega(\mathbb{T}^d, \mu)$ to be the set of all measure-preserving real-analytic diffeomorphisms of \mathbb{T}^d homotopic to the identity, whose lift $F(x) = (x_1 + f_1(x), \dots, x_d + f_d(x))$ to \mathbb{R}^d satisfies $f_i \in C_\rho^\omega(\mathbb{T}^d)$ and we also require the lift $\tilde{F}(x) = (x_1 + \tilde{f}_1(x), \dots, x_d + \tilde{f}_d(x))$ of its inverse to \mathbb{R}^d to satisfies $\tilde{f}_i \in C_\rho^\omega(\mathbb{T}^d)$. The metric d in $\operatorname{Diff}_\rho^\omega(\mathbb{T}^d, \mu)$ is defined by

$$d_\rho(f, g) = \max\{\tilde{d}_\rho(f, g), \tilde{d}_\rho(f^{-1}, g^{-1})\}, \quad \text{where} \quad \tilde{d}_\rho(f, g) = \max_{i=1, \dots, d} \left\{ \inf_{n \in \mathbb{Z}} \|f_i - g_i + n\|_\rho \right\}.$$

Let $F = (F_1, \dots, F_d)$ be the lift of a diffeomorphism in $\operatorname{Diff}_\rho^\omega(\mathbb{T}^d, \mu)$, we define the norm of the total derivative

$$\|DF\|_\rho := \max_{\substack{i=1, \dots, d \\ j=1, \dots, d}} \left\| \frac{\partial F_i}{\partial x_j} \right\|_\rho.$$

Next, with some abuse of notation, we define the following two spaces

$$C_\infty^\omega(\mathbb{T}^d) := \bigcap_{n=1}^\infty C_n^\omega(\mathbb{T}^d), \quad (2.5)$$

$$\operatorname{Diff}_\infty^\omega(\mathbb{T}^d, \mu) := \bigcap_{n=1}^\infty \operatorname{Diff}_n^\omega(\mathbb{T}^d, \mu). \quad (2.6)$$

Note that the functions in 2.5 can be extended to \mathbb{C}^d as entire functions. We also note that $\operatorname{Diff}_\infty^\omega(\mathbb{T}^d, \mu)$ is closed under composition. To see this, let $f, g \in \operatorname{Diff}_\infty^\omega(\mathbb{T}^d, \mu)$ and F and G be their corresponding lifts. Then note that $F \circ G$ is the lift of $f \circ g$ (with $\pi : \mathbb{R}^2 \rightarrow \mathbb{T}^2$ as the natural projection, $\pi \circ F \circ G = f \circ \pi \circ G = f \circ g \circ \pi$). Now for the complexification of F and G note that the composition $F \circ G(z) = (z_1 + g_1(z) + f_1(G(z)), \dots, z_d + g_d(z) + f_d(G(z)))$. Since $g_i \in C_\infty^\omega(\mathbb{T}^d)$, we have for any ρ , $\sup_{z \in \Omega_\rho} |\operatorname{Im}(G(z))| \leq \max_i (\sup_{z \in \Omega_\rho} |\operatorname{Im}(z_i) + \operatorname{Im}(g_i(z))|) \leq \max_i (\sup_{z \in \Omega_\rho} |\operatorname{Im}(z_i)| + \sup_{z \in \Omega_\rho} |\operatorname{Im}(g_i(z))|) \leq \rho + \max_i (\sup_{z \in \Omega_\rho} |g_i(z)|) < \rho + \text{const} < \rho' < \infty$ for

¹we identify \mathbb{R}^d inside \mathbb{C}^d via the natural inclusion $(x_1, \dots, x_d) \mapsto (x_1 + i0, \dots, x_d + i0)$.

some ρ' . So, $\sup_{z \in \Omega_{\rho'}} |z_i + g_i(z) + f_i(G(z))| \leq |z_i| + |g_i(z)| + |f_i(G(z))| < \infty$ since $z \in \Omega_{\rho}$, $g_i \in C_{\infty}^{\omega}(\mathbb{T}^d)$ and $G(z) \in \Omega_{\rho'}$. An identical treatment gives the result for the inverse.

All intermediate diffeomorphisms constructed during the AbC method in this paper will belong to this category.²

This completes the description of the real-analytic topology necessary for our construction. Also throughout this paper, the word “diffeomorphism” will refer to a real-analytic diffeomorphism. Also, the word “real-analytic topology” will refer to the topology of $\text{Diff}_{\rho}^{\omega}(\mathbb{T}^d, \mu)$ described above. See [Sa03] for a more extensive treatment of these spaces.

2.2 Some partitions of the torus

First of all, we introduce the notion of a *partial partition* of a measure space (X, μ) , which is a pairwise disjoint countable collection of measurable subsets of X .

Definition 2.7. • A sequence of partial partitions ν_n converges to the decomposition into points if and only if for a given measurable set A and for every $n \in \mathbb{N}$ there exists a measurable set A_n , which is a union of elements of ν_n , such that $\lim_{n \rightarrow \infty} \mu(A \Delta A_n) = 0$. We often denote this by $\nu_n \rightarrow \varepsilon$.

- A partial partition ν is a *refinement* of a partial partition η if and only if for every $C \in \nu$ there exists a set $D \in \eta$ such that $C \subseteq D$. We write this as $\eta \leq \nu$.

Using the notion of a partition we can introduce the weak topology in the space of measure-preserving transformations on a Lebesgue space:

Definition 2.8. 1. For two measure-preserving transformations T, S and for a finite partition ξ the *weak distance* with respect to ξ is defined by $d(\xi, T, S) := \sum_{c \in \xi} \mu(T(c) \Delta S(c))$.

2. The base of neighbourhoods of T in the *weak topology* consists of the sets

$$W(T, \xi, \varepsilon) = \{S : d(\xi, T, S) < \varepsilon\},$$

where ξ is a finite partition and ε is a positive number.

There are some partitions of \mathbb{T}^d that are of special interest to us. They appear repeatedly in this article and we summarize them here.

Assume that we are given three natural numbers l, k, q and a function $a : \{0, 1, \dots, k-1\} \rightarrow \{0, 1, \dots, q-1\}$. We define the following three partitions of \mathbb{T}^d :

$$\mathcal{T}_q := \left\{ \Delta_{i,q} := \left[\frac{i}{q}, \frac{i+1}{q} \right) \times \mathbb{T}^{d-1} : i = 0, 1, \dots, q-1 \right\}, \quad (2.9)$$

$$\mathcal{G}_{l,q} := \left\{ \left[\frac{i_1}{lq}, \frac{i_1+1}{lq} \right) \times \left[\frac{i_2}{l}, \frac{i_2+1}{l} \right) \times \dots \times \left[\frac{i_d}{l}, \frac{i_d+1}{l} \right) : i_1 = 0, 1, \dots, lq-1, \right. \\ \left. (i_2, \dots, i_d) \in \{0, 1, \dots, l-1\}^{d-1} \right\}, \quad (2.10)$$

$$\mathcal{G}_{j,l,q} := \left\{ \left[\frac{i_1}{l^{d+1-j}q}, \frac{i_1+1}{l^{d+1-j}q} \right) \times \left[\frac{i_2}{l}, \frac{i_2+1}{l} \right) \times \dots \times \left[\frac{i_j}{l}, \frac{i_j+1}{l} \right) \times \mathbb{T}^{d-j} : i_1 = 0, 1, \dots, l^{d+1-j}q-1, \right.$$

²We note that the existence of such real-analytic functions whose complexification is entire or as in this case, the complexification of their lift is entire is central to a real-analytic AbC method. As of now we only know how to construct such functions on the torus, odd dimensional spheres and certain homogeneous spaces.

$$(i_2, \dots, i_{d-j+1}) \in \{0, 1, \dots, l-1\}^{j-1}, \quad (2.11)$$

$$\mathcal{R}_{a,k,q} := \left\{ R_{j,q} := \phi^{j/q} \left(\bigcup_{i=0}^{k-1} \Delta_{a(i)k+i,kq} \right), j = 0, \dots, q-1 \right\}. \quad (2.12)$$

We note ϕ^α acts on the partitions 2.9, 2.10, 2.11 and 2.12 as a permutation for any choice of p when $\alpha = p/q$.

2.3 Block-slide type maps and their real-analytic approximations

We recall that a *step function* on the unit interval is a finite linear combination of indicator functions on intervals. We define for $1 \leq i, j \leq d$ and $i \neq j$, the following piecewise continuous map on the d dimensional torus,

$$\mathfrak{h} : \mathbb{T}^d \rightarrow \mathbb{T}^d \text{ defined by } \mathfrak{h}(x_1, \dots, x_d) := (x_1, \dots, x_{i-1}, x_i + s(x_j) \pmod{1}, x_{i+1}, \dots, x_d), \quad (2.13)$$

where s is a step function on the unit interval. We refer to any finite composition of maps of the above kind as a *block-slide type of map* on the torus.

Inspired by [Ka73] the purpose of the section is to demonstrate that a block-slide type of map can be approximated extremely well by measure-preserving real-analytic diffeomorphisms outside a set of arbitrarily small measure. This can be achieved because step function can be approximated well by real-analytic functions whose complexification is entire.

Lemma 2.14. *Let k and N be two positive integer and $\beta = (\beta_0, \dots, \beta_{k-1}) \in [0, 1)^k$. Consider a step function of the form*

$$\tilde{s}_{\beta,N} : [0, 1) \rightarrow \mathbb{R} \quad \text{defined by} \quad \tilde{s}_{\beta,N}(x) = \sum_{i=0}^{kN-1} \tilde{\beta}_i \chi_{[\frac{i}{kN}, \frac{i+1}{kN})}(x).$$

Here $\tilde{\beta}_i := \beta_j$ where $j := i \pmod{k}$. Then, given any $\varepsilon > 0$ and $\delta > 0$, there exists a periodic real-analytic function $s_{\beta,N} : \mathbb{R} \rightarrow \mathbb{R}$ satisfying the following properties:

1. *Entirety:* The complexification of $s_{\beta,N}$ extends holomorphically to \mathbb{C} .
2. *Proximity criterion:* $s_{\beta,N}$ is L^1 -close to $\tilde{s}_{\beta,N}$. We can say more,

$$\sup_{x \in [0,1) \setminus F} |s_{\beta,N}(x) - \tilde{s}_{\beta,N}(x)| < \varepsilon. \quad (2.15)$$

3. *Periodicity:* $s_{\beta,N}$ is $1/N$ -periodic. More precisely, the complexification will satisfy,

$$s_{\beta,N}(z + n/N) = s_{\beta,N}(z) \quad \forall z \in \mathbb{C} \text{ and } n \in \mathbb{Z}. \quad (2.16)$$

Where $F = \bigcup_{i=0}^{kN-1} I_i \subset [0, 1)$ is a union of intervals centred around $\frac{i}{kN}$, $i = 1, \dots, kN-1$ and $I_0 = [0, \frac{\delta}{2kN}] \cup [1 - \frac{\delta}{2kN}, 1)$ and $\lambda(I_i) = \frac{\delta}{kN} \forall i$.

Proof. See [Ba15, Lemma 4.7] and [Ku15, Lemma 3.6]. □

Note that the condition 2.16 in particular implies

$$\sup_{z: \text{Im}(z) < \rho} s_{\beta, N}(z) < \infty \quad \forall \rho > 0.$$

Indeed, for any $\rho > 0$, put $\Omega'_\rho = \{z = x + iy : x \in [0, 1], |y| < \rho\}$ and note that entirety of $s_{\beta, N}$ combined with compactness of $\overline{\Omega'_\rho}$ implies $\sup_{z \in \Omega'_\rho} |s_{\beta, N}(z)| < C$ for some constant C . Periodicity of $s_{\alpha, N}$ in the real variable and the observation $\Omega_\rho = \cup_{n \in \mathbb{Z}} (\Omega'_\rho + n)$ implies that $\sup_{z \in \Omega_\rho} |s_{\beta, N}(z)| < C$. We have essentially concluded that $s_{\beta, N} \in C_\infty^\omega(\mathbb{T}^1)$.

Finally we piece together everything and demonstrate how a block-slide type of map on the torus can be approximated by a measure-preserving real-analytic diffeomorphism.

Proposition 2.17. *Let $\mathfrak{h} : \mathbb{T}^d \rightarrow \mathbb{T}^d$ be a block-slide type of map which commutes with $\phi^{1/q}$ for some natural number q . Then for any $\varepsilon > 0$ and $\delta > 0$, there exists a real-analytic diffeomorphism $h \in \text{Diff}_\infty^\omega(\mathbb{T}^d, \mu)$ satisfying the following conditions:*

1. *Proximity property: There exists a set $E \subset \mathbb{T}^d$ such that $\mu(E) < \delta$ and $\sup_{x \in \mathbb{T}^d \setminus E} \|h(x) - \mathfrak{h}(x)\| < \varepsilon$.*
2. *Commuting property: $h \circ \phi^{1/q} = \phi^{1/q} \circ h$.*

In this case we say the the diffeomorphism h is (ε, δ) -close to the block-slide type map \mathfrak{h} .

Proof. See [BK, Proposition 2.22] □

2.4 Real-analytic AbC-method

Our objective now is to recall the approximation by conjugation scheme developed by Anosov and Katok in [AK70]. Though we modify this scheme slightly to be more suitable for our purpose and fit the notations of our article we insist that the method presented here is almost identical to the original construction.

The AbC-method is an inductive process where a sequence of diffeomorphisms $T_n \in \text{Diff}_\infty^\omega(\mathbb{T}^d, \mu)$ is constructed inductively. The diffeomorphisms T_n converge to some diffeomorphism $T \in \text{Diff}_\rho^\omega(\mathbb{T}^d, \mu)$. Additionally T_n s are chosen carefully so that they satisfy some finite version of the desired property of T .

We now give an explicit description. At the beginning of the construction we fix a constant $\rho > 0$ and note that all parameters chosen will depend on this ρ .

Assume that the construction has been carried out up to the $n - 1$ th stage and we have the following information available to us:

1. We have sequences of natural numbers $\{p_m\}_{m=1}^n, \{q_m\}_{m=1}^n, \{k_m\}_{m=1}^{n-1}, \{l_m\}_{m=1}^{n-1}$, a sequence of functions $\{a_m : \{0, \dots, l_m - 1\} \rightarrow \{0, \dots, q_m - 1\}\}_{m=1}^{n-1}$ and sequences of numbers $\{\alpha_m\}_{m=1}^n, \{\varepsilon_m\}_{m=1}^{n-1}$. They satisfy the following conditions:

$$p_m = k_{m-1} l_{m-1} q_{m-1} p_{m-1} + 1, \quad q_m = k_{m-1} l_{m-1} q_{m-1}^2, \quad \alpha_m = \frac{p_m}{q_m}, \quad \varepsilon_m < 2^{-q_m}. \quad (2.18)$$

2. The sequence of diffeomorphisms $\{T_m\}_{m=1}^{n-1}$ is constructed as conjugates of a periodic translation. More precisely,

$$T_m := H_m^{-1} \circ \phi^{\alpha_{m+1}} \circ H_m, \quad H_m := h_m \circ H_{m-1}, \quad h_m \in \text{Diff}_\infty^\omega(\mathbb{T}^d, \mu). \quad (2.19)$$

The diffeomorphisms $\{h_m\}_{m=1}^{n-1}$ satisfy the following commuting condition:

$$h_m \circ \phi^{\alpha_m} = \phi^{\alpha_m} \circ h_m. \quad (2.20)$$

3. For $m = 1, \dots, n-1$, the diffeomorphism T_m preserves and permutes two sequences of partitions, namely, $H_m^{-1}\mathcal{R}_{a_{m+1}, l_{m+1}, q_{m+1}}$ and $\mathcal{F}_{q_{m+1}} := H_m^{-1}\mathcal{T}_{q_{m+1}}$.
4. For $m = 1, \dots, n-1$, $\mu(h_m^{-1}R_{i, q_m} \Delta_{i, q_m}) < \varepsilon_m$ for any $R_{i, q_m} \in \mathcal{R}_{a_m, l_m, q_m}$ and $\Delta_{i, q_m} \in \mathcal{T}_{q_m}$ with the same i .
5. For $m = 1, \dots, n-1$, $\text{diam}(\mathcal{F}_{q_{m+1}} \cap E_m) < \varepsilon_m^3$ for some measurable set E_m satisfying $\mu(E_m) > 1 - \varepsilon_m$. (Note that this means $\mathcal{F}_{q_{m+1}}$ is a generating but not necessarily monotonic sequence of partitions.)
6. For $m = 1, \dots, n-1$: $d_\rho(T_m, T_{m-1}) < \varepsilon_m$.

Now we show how to do the construction at the n th stage of this induction process. We proceed in the following order:

1. The number l_n has to be a large enough integer so that the following conditions are satisfied:

$$l_n > d \cdot n^2 \cdot \|DH_{n-1}^{-1}\|_0, \quad (2.21)$$

$$2 \cdot l_{n-1} \cdot q_n \text{ divides } l_n. \quad (2.22)$$

2. We choose l_n and our function $a_n : \{0, \dots, l_n - 1\} \rightarrow \{0, \dots, q_n - 1\}$. This choice will depend on the construction we are doing and the specific properties we are targeting to prove.
3. Find a block-slide type map $\mathfrak{h}_{a_n, l_n, q_n}$ which commutes with ϕ^{α_n} , maps the partition \mathcal{G}_{l_n, q_n} to $\mathcal{T}_{l_n^d, q_n}$ and it maps the partition \mathcal{T}_{q_n} to the partition $\mathcal{R}_{a_n, l_n, q_n}$.
4. Use Proposition 2.17 to construct h_n which is $(\varepsilon_n, \delta_n)$ close to $\mathfrak{h}_{a_n, l_n, q_n}$. Put E_n to be the error set in Proposition 2.17. In our constructions we will choose

$$\delta_n = \frac{1}{n \cdot q_n}, \quad \varepsilon_n = \frac{\delta_n}{4 \cdot l_n^d \cdot q_n^2}. \quad (2.23)$$

5. By choosing k_n sufficiently large we ensure that $|\alpha_{n+1} - \alpha_n|$ is small enough to guarantee $d_\rho(T_n, T_{n-1}) < \varepsilon_m$.

This completes the construction at the n th stage. Note that this way convergence of T_n to some $T \in \text{Diff}_\rho^\omega(\mathbb{T}^d, \mu)$ is guaranteed.

³ This means that the diameter of the intersection of any atom of \mathcal{F}_{q_m} and E_m is less than ε_m .

3 Periodic approximation in Ergodic Theory

In this section we provide a short introduction to the concept of periodic approximation in Ergodic Theory. A more comprehensive presentation can be found in [Ka03].

Let (X, μ) be a Lebesgue space. A *tower* t of height $h(t) = h$ is an ordered sequence of disjoint measurable sets $t = \{c_1, \dots, c_h\}$ of X having equal measure, which is denoted by $m(t)$. The sets c_i are called the *levels* of the tower, especially c_1 is the *base*. Associated with a tower there is a *cyclic permutation* σ sending c_1 to c_2 , c_2 to c_3, \dots and c_h to c_1 . Using the notion of a tower we can give the next definition:

Definition 3.1. A *periodic process* is a collection of disjoint towers covering the space X and the associated cyclic permutations together with an equivalence relation among these towers identifying their bases.

There are two partial partitions associated with a periodic process: The partition ξ into all levels of all towers and the partition η consisting of the union of bases of towers in each equivalence class and their images under the iterates of σ , where when we go beyond the height of a certain tower in the class we drop this tower and continue until the highest tower in the equivalence class has been exhausted. Obviously, we have $\eta \leq \xi$.

A sequence $(\xi_n, \eta_n, \sigma_n)$ of periodic processes is called *exhaustive* if $\eta_n \rightarrow \xi$. Such an exhaustive sequence of periodic processes is *consistent* if for every measurable subset $A \subseteq X$ and each union of sets in η_n approximating A (i. e. $\mu(A_n \Delta A) \rightarrow 0$ as $n \rightarrow \infty$) the sequence $\sigma_n(A_n)$ converges to a set B , i. e. $\mu(\sigma_n(A_n) \Delta B) \rightarrow 0$ as $n \rightarrow \infty$. Moreover, we will call a sequence of towers $t^{(n)}$ from the periodic process $(\xi_n, \eta_n, \sigma_n)$ *substantial* if there exists $r > 0$ such that $h(t^{(n)}) \cdot m(t^{(n)}) > r$ for every $n \in \mathbb{N}$.

Definition 3.2. Let $T : (X, \mu) \rightarrow (X, \mu)$ be a measure-preserving transformation. An exhaustive sequence of periodic processes $(\xi_n, \eta_n, \sigma_n)$ forms a *periodic approximation* of T if

$$d(\xi_n, T, \sigma_n) = \sum_{c \in \xi_n} \mu(T(c) \Delta \sigma_n(c)) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Given a sequence $g(n)$ of positive numbers we will say that the transformation T admits a periodic approximation with *speed* $g(n)$ if for a certain subsequence $(n_k)_{k \in \mathbb{N}}$ there exists an exhaustive sequence of periodic processes $(\xi_k, \eta_k, \sigma_k)$ such that $d(\xi_k, T, \sigma_k) < g(n_k)$.

In order to define the type of the periodic approximation we need the notion of equivalence for sequences of periodic processes:

Definition 3.3. Two sequences of periodic processes $P_n = (\xi_n, \eta_n, \sigma_n)$ and $P'_n = (\xi'_n, \eta'_n, \sigma'_n)$ are called *equivalent* if for every $n \in \mathbb{N}$ there is a bijective correspondence θ_n between subsets S_n and S'_n of the sets of towers of P_n respectively P'_n such that

- For $t \in S_n$: $h(\theta_n(t)) = h(t)$.
- $\sum_{t \in S_n} h(t) m(t) \rightarrow 1$ as $n \rightarrow \infty$.
- $\sum_{t \in S_n} h(t) \cdot |m(t) - m(\theta_n(t))| \rightarrow 0$ as $n \rightarrow \infty$.
- If two towers from S_n are equivalent in P_n , then their images under θ_n are equivalent in P'_n .

There are various types of approximation. We introduce the most important ones:

- Definition 3.4.**
1. A *cyclic process* is a periodic process which consists of a single tower of height h . An approximation by an exhaustive sequence of cyclic processes is called a cyclic approximation. More specifically we will refer to a cyclic approximation with speed $o\left(\frac{1}{h}\right)$ as a *good cyclic approximation*.
 2. An approximation generated by periodic processes equivalent to periodic processes consisting of two substantial towers whose heights differ by one is said to be of *type* $(h, h + 1)$. Equivalently the heights of the two towers t_1 and t_2 with base B_1 resp. B_2 are equal to h and $h + 1$ and for some $r > 0$ we have $\mu(B_1) > \frac{r}{h}$ as well as $\mu(B_2) > \frac{r}{h+1}$. We will call the approximation of type $(h, h + 1)$ with speed $o\left(\frac{1}{h}\right)$ *good* and with speed $o\left(\frac{1}{h \cdot (h+1)}\right)$ *excellent*.
 3. An approximation of type $(h, h + 1)$ will be called a *linked approximation of type* $(h, h + 1)$ if the two towers involved in the approximation are equivalent. This insures that the sequence of partitions η_n generated by the union of the bases of the two towers and the iterates of this set converges to the decomposition into points.

Remark 3.5. As noted in [Ry06] a good linked approximation of type $(h, h + 1)$ implies the convergence

$$U_T^{k \cdot (h+1)} \xrightarrow{w} r \cdot U_T^k + (1 - r) \cdot Id$$

in the weak operator topology for every $k \in \mathbb{N}$ and some $r \in (0, 1)$, where U_T is the Koopman operator of T (see section 4.1).

From the different types of approximations various ergodic properties can be derived. For example in [KS67, Corollary 2.1.] the subsequent Lemma is proven.

Lemma 3.6. *Let $T : (X, \mu) \rightarrow (X, \mu)$ be a measure-preserving transformation. If T admits a good cyclic approximation, then T is ergodic.*

In [KS70] Katok and Stepin proved the genericity of automorphisms having a continuous spectrum in the set of measure-preserving homeomorphisms (recall that a transformation has a continuous spectrum, i.e. the corresponding operator U_T in the space $L^2(M, \mu)$ has no eigenfunctions other than constants, if and only if it is weakly mixing). For this purpose, they deduced the following result:

Lemma 3.7 ([KS70], Theorem 5.1.). *Let $T : (X, \mu) \rightarrow (X, \mu)$ be a measure preserving transformation. If T is ergodic and admits a good approximation of type $(h, h + 1)$, then T has continuous spectrum.*

4 Spectral theory of dynamical systems

Besides the concept of periodic approximation we will need further mathematical tools. We refer to [Na98] and [Go99] for more details.

4.1 Spectral types

Let (X, μ) be a Lebesgue space and $T : (X, \mu) \rightarrow (X, \mu)$ be an automorphism. Then we define the induced *Koopman-operator* $U_T : L^2(X, \mu) \rightarrow L^2(X, \mu)$ by $U_T f = f \circ T$. Since

$$\langle U_T f, U_T g \rangle = \int_X f \circ T \cdot \overline{g \circ T} d\mu = \int_X f \cdot \bar{g} d\mu = \langle f, g \rangle \quad \text{for every } f, g \in L^2(X; \mu)$$

and $U_T^{-1} = U_{T^{-1}}$ this is an unitary operator on the Hilbert space $L^2(X, \mu)$.

Remark 4.1. If two measure-preserving dynamical systems (X_1, μ_1, T_1) and (X_2, μ_2, T_2) are metrically isomorphic, their isomorphism $h : X_1 \rightarrow X_2$ induces an isomorphism of Hilbert spaces $V_h : L^2(X_2, \mu_2) \rightarrow L^2(X_1, \mu_1)$ by $(V_h f) = f \circ h$. Then we have $U_{T_1} = V_h \circ U_{T_2} \circ V_h^{-1}$ and this relation is called *unitary equivalence* of operators. Hence, any invariant of unitary equivalence defines an invariant of isomorphisms. Such invariants are said to be *spectral invariants* or spectral properties.

Moreover, we note that 1 is always an eigenvalue of U_T because of the constant functions. So when we discuss the spectral properties of U_T we refer to its spectral properties that are restricted to the orthogonal complement of the constants. Hence, we consider the properties of U_T in the space $L_0^2(X, \mu)$ of all L^2 -functions with zero integral.

One of the important spectral invariants are the so-called *spectral measures*: Let $f \in L_0^2(X, \mu)$ and $Z(f) := \overline{\text{span}\{U_T^n f : n \in \mathbb{Z}\}}^{L_0^2(X, \mu)}$. Using Bochner's theorem one can prove the existence of a finite Borel measure σ_f defined on the unit circle \mathbb{S}^1 in the complex plane satisfying

$$\langle U_T^n f, f \rangle = \int_{\mathbb{S}^1} z^n d\sigma_f(z) \quad \text{for every } n \in \mathbb{Z}.$$

Then σ_f is called the spectral measure of f with respect to U_T .

Moreover, by the Hahn-Hellinger Theorem, there is a sequence of functions $f_n \in L_0^2(X, \mu)$, $n \in \mathbb{N}$, for which

$$L_0^2(X, \mu) = \bigoplus_{n \in \mathbb{N}} Z(f_n) \quad \text{and} \quad \sigma_{f_1} \gg \sigma_{f_2} \gg \dots$$

These measures are unique in the sense that for any other family of functions $g_n \in L_0^2(X, \mu)$, $n \in \mathbb{N}$, for which $L_0^2(X, \mu) = \bigoplus_{n \in \mathbb{N}} Z(g_n)$ and $\sigma_{g_1} \gg \sigma_{g_2} \gg \dots$ we have $\sigma_{f_n} \sim \sigma_{g_n}$ for every $n \in \mathbb{N}$.

Definition 4.2. The spectral type of σ_{f_1} is called the *maximal spectral type* σ of U_T .

According to this we say that U_T has a continuous spectrum if σ_{f_1} is a continuous measure and U_T has a discrete spectrum if σ_{f_1} is a discrete measure.

4.2 Spectral multiplicities

Besides the maximal spectral type an important characterization of U_T is the *multiplicity function* $M_{U_T} : \mathbb{S}^1 \rightarrow \mathbb{N} \cup \{\infty\}$, which is σ_{f_1} -almost everywhere defined by

$$M_{U_T}(z) = \sum_{i=1}^{\infty} \chi_{A_i}(z), \quad \text{where } A_i = \left\{ z \in \mathbb{S}^1 : \frac{d\sigma_{f_i}}{d\sigma_{f_1}}(z) > 0 \right\}.$$

Here $\frac{d\sigma_{f_i}}{d\sigma_{f_1}}$ is the Radon-Nikodym derivative of σ_{f_i} with respect to σ_{f_1} .

Using this multiplicity function we establish the set \mathcal{M}_{U_T} of *essential spectral multiplicities*, which

is the essential range of M_{U_T} with respect to σ_{f_1} . Then we define the *maximal spectral multiplicity* m_{U_T} as the essential supremum (with respect to σ_{f_1}) of \mathcal{M}_{U_T} .

Definition 4.3. U_T is said to have *homogeneous spectrum* of multiplicity m if $\mathcal{M}_{U_T} = \{m\}$. In particular, U_T has a *simple spectrum* if $\mathcal{M}_{U_T} = \{1\}$. In all other cases U_T has a non-simple spectrum.

In connection with the previous chapter 3 we state the following result ([KS67, Theorem 3.1.]):

Lemma 4.4. *Let T be an automorphism of a Lebesgue space. If T admits a cyclic approximation of speed $\frac{\theta}{h}$, where $\theta < \frac{1}{2}$, then the spectrum of U_T is simple.*

For automorphisms with simple spectrum we have the subsequent theorem of Ryzhikov ([Ry99b, Theorem 2.1.]):

Lemma 4.5. *Let (X, μ) be a Lebesgue probability space and $T : (X, \mu) \rightarrow (X, \mu)$ be an automorphism with simple spectrum. Suppose that the weak convergence*

$$U_T^{k_n} \longrightarrow_w (a \cdot U_T + (1 - a) \cdot Id)$$

holds for some $a \in (0, 1)$ and some strictly increasing sequence $(k_n)_{n \in \mathbb{N}}$ of natural numbers. Then the Cartesian square $T \times T$ has a homogeneous spectrum of multiplicity 2.

4.3 Disjointness of convolutions

In this section we study the convolutions of the maximal spectral type σ . Therefore, we state the definition of a *convolution of measures*:

Definition 4.6. Let G be a topological group and μ, ν finite Borel measures on G . Then their convolution $\mu * \nu$ is defined by

$$(\mu * \nu)(A) = \int \int 1_A(x \cdot y) d\mu(x) d\nu(y)$$

for each measurable set A of G .

If all the convolutions $\sigma^k = \sigma * \dots * \sigma$ for $k \in \mathbb{N}$ are pairwise mutually singular, one speaks about *disjointness of convolutions*. To guarantee this pairwise singularity of convolutions of the maximal spectral type of a measure-preserving transformation the following property is useful:

Definition 4.7. An automorphism T of a Lebesgue space (X, μ) is said to be κ -*weakly mixing*, $\kappa \in [0, 1]$, if there exists a strictly increasing sequence $(k_n)_{n \in \mathbb{N}}$ of natural numbers such that the weak convergence

$$U_T^{k_n} \longrightarrow_w (\kappa \cdot P_c + (1 - \kappa) \cdot Id)$$

holds, where P_c is the projection on the subspace of constants.

Remark 4.8. By [St87, Proposition 3.1.] we can characterise this property in geometric language: A transformation T is κ -weakly mixing if and only if there is an increasing sequence $(k_n)_{n \in \mathbb{N}}$ of natural numbers such that for all measurable sets A and B

$$\lim_{n \rightarrow \infty} \mu(A \cap T^{k_n} B) = \kappa \cdot \mu(A) \cdot \mu(B) + (1 - \kappa) \cdot \mu(A \cap B).$$

We recognize that 0-weak mixing corresponds to rigidity and 1-weak mixing to the usual notion of weak mixing.

As announced this property has connections with certain properties of the maximal spectral type (see [St87, Theorem 1]):

Lemma 4.9. *If the transformation T is κ -weakly mixing for some $0 < \kappa < 1$ and σ is the maximal spectral type for $U_T|_{L_0^2(X,\mu)}$, then σ and all its convolutions $\sigma^k = \sigma * \dots * \sigma$ are pairwise mutually singular.*

5 Criterion for the proof of Theorem 1.1

5.1 Statement of the criterion

As discussed in [Ku16, section 9] we can reduce the proof of the Theorem to Proposition 5.1.

Proposition 5.1. *Let $f \in \text{Diff}_\rho^s(\mathbb{T}^d, \mu)$ admit a good linked approximation of type $(h, h + 1)$ and a good cyclic approximation. Then f has a maximal spectral type disjoint with its convolutions and homogeneous spectrum of multiplicity 2 for its ergodic Cartesian square $f \times f$*

Proof. Indeed, such a constructed diffeomorphism has the following properties:

- Since f allows a good cyclic approximation, it has simple spectrum by Lemma 4.4. Furthermore, f admits a good linked approximation of type $(h, h + 1)$ and so we can use Remark 3.5 to obtain the weak convergence $U_f^{h+1} \rightarrow_w r \cdot U_f + (1 - r) \cdot Id$ for some $r \in (0, 1)$. Hence, we can apply Lemma 4.5 and conclude that $f \times f$ has homogeneous spectrum of multiplicity 2.
- With the aid of Lemma 3.6 and the good cyclic approximation of f we have that f is ergodic. Then we can exploit the good approximation of type $(h, h + 1)$ and Lemma 3.7 to see that f is even weakly mixing (hence, $f \times f$ is ergodic). Due to Remark 4.8 there exists a strictly increasing sequence $(k_n)_{n \in \mathbb{N}}$ of natural numbers such that the convergence $U_f^{k_n} \rightarrow_w P_c$ holds in the weak operator topology as $n \rightarrow \infty$. Along this sequence we have using Remark 3.5

$$U_f^{k_n \cdot (h+1)} \rightarrow_w (r \cdot P_c + (1 - r) \cdot Id)$$

for some $r \in (0, 1)$. Thus, f is κ -weakly mixing (with $\kappa = r \in (0, 1)$). Then the maximal spectral type σ of f is disjoint with its convolutions by Lemma 4.9.

□

5.2 Sketch of the construction

As announced we would like to sketch the construction of $T_n = H_n^{-1} \circ \phi^{\alpha_{n+1}} \circ H_n$ with $\alpha_{n+1} = \alpha_n + \frac{1}{k_n \cdot l_n \cdot q_n^2}$ and its combinatorics satisfying the requirements of Proposition 5.1.

First of all, we define $m_n := \frac{q_{n+1}}{2 \cdot q_n^2}$. Additionally, we introduce two sets $\tilde{c}_{0,j}^{(n)}$, $j = 1, 2$, in section 7.1. With the aid of these we define the bases $c_{0,j}^{(n)} = H_n^{-1} \left(\tilde{c}_{0,j}^{(n)} \right)$ as well as levels $T_n^i \left(c_{0,j}^{(n)} \right) = H_n^{-1} \circ \phi^{i \cdot \alpha_{n+1}} \left(\tilde{c}_{0,j}^{(n)} \right)$ of the two towers of heights m_n and $m_n + 1$ respectively. These towers will be used to prove that $T = \lim_{n \rightarrow \infty} T_n$ admits a good linked approximation of type $(h, h + 1)$. In particular, $\tilde{c}_{0,j}^{(n)}$ is built as a union of sets due to different reasons which we will explain in the following. First of all, the union over i_2, \dots, i_d is taken such that $\tilde{c}_{0,j}^{(n)}$ has almost full length in the

x_2, \dots, x_d -coordinates and at the same time is positioned in domains, where we can approximate our block slide type of map with a real-analytic volume-preserving diffeomorphism very well. This will be very useful in the calculations of the speed of approximation, especially in the proof of Lemma 7.8. In order to see that T admits an approximation of type $(h, h + 1)$ the so-called “ i_1 -stripes”, i. e. sets obtained by the union over i_2, \dots, i_d , are very important. These stripes are depicted in the schematic visualisation of the tower bases in Figure 2. In this connection we note that the number m_n is defined such that $m_n \cdot \alpha_{n+1} = \frac{r_n}{q_n} + \frac{1}{2q_n^2}$ with some $r_n \in \mathbb{N}$. Hence, under $\phi^{m_n \cdot \alpha_{n+1}}$ a “ i_1 -stripe” is mapped into another $\frac{1}{q_n}$ -sector and is shifted by $\frac{1}{2q_n^2}$. In order to get recurrence in the base we need another stripe to be positioned there, which will be fulfilled by our choice of the term $\frac{i_1 \cdot r_n}{q_n}$ and the union over i_1 in the definition of $\tilde{c}_{0,1}^{(n)}$. Similarly, we proceed to define $\tilde{c}_{0,2}^{(n)}$. So the sets $\tilde{c}_{0,j}^{(n)}$ will be defined in such a way that a great portion of $\phi^{m_n \cdot \alpha_{n+1}} \left(\tilde{c}_{0,1}^{(n)} \right)$ (resp. $\phi^{(m_n+1) \cdot \alpha_{n+1}} \left(\tilde{c}_{0,2}^{(n)} \right)$) is mapped back into $\tilde{c}_{0,1}^{(n)}$ (resp. $\tilde{c}_{0,2}^{(n)}$) for the prescribed height m_n (resp. $m_n + 1$) of the particular tower. Since both towers have to be substantial, the measure of each base element has to be about $\frac{1}{2m_n}$. Additionally, the tower levels have to be disjoint sets. We ensure this by taking a x_1 -width of $\frac{q_n}{q_{n+1}}$ (note that under $\phi^{\alpha_{n+1}}$ we return to the same $\frac{1}{q_n}$ -section after q_n steps and then we have a x_1 -deviation of $q_n \cdot (\alpha_{n+1} - \alpha_n) = \frac{q_n}{q_{n+1}}$). Moreover, the conjugation map h_n will be defined such that each image $h_n^{-1} \left(\tilde{c}_{0,j}^{(n)} \right)$ is contained in a cube of edge length $\frac{1}{l_n}$. Hereby, the diameter of $c_{0,j}^{(n)}$ is small due to condition 2.21. In Figure 1 we sketch this action of the conjugation map h_n . The proof that T admits a good cyclic approximation is considerably easier. Since $T_n^{q_{n+1}} = \text{id}$ we can use a subset with measure about $\frac{1}{q_{n+1}}$ of our tower base in the $(h, h + 1)$ -approximation.

6 Explicit constructions

In this section, we present the construction of the conjugation map h_n . We start by describing the combinatorial picture in subsection 6.1. In order to find a real-analytic conjugation map in subsection 6.2, we recall the result from [BK, section 5.1] that a permutation Π of the partition

$$\mathcal{S}_{kq,l} := \{S_{i,j}^{kq,l} := [i/(kq), (i + 1)/(kq)] \times [j/l, (j + 1)/l], 0 \leq i < kq, 0 \leq j < l\} \quad (6.1)$$

which commutes with $\phi^{1/q}$ is a block slide type of map.

6.1 Description of the required combinatorics

We describe the underlying combinatorics of our construction. For this purpose, we recall our choice of the rotation number

$$\alpha_{n+1} = \alpha_n + \frac{1}{l_n \cdot k_n \cdot q_n^2}. \quad (6.2)$$

Additionally, we define the number

$$m_n = \frac{q_{n+1}}{2q_n^2} = \frac{l_n \cdot k_n}{2}. \quad (6.3)$$

By equations 6.2 and 2.22 we have $m_n \in \mathbb{N}$. In fact, m_n will be the height of the first tower and $m_n + 1$ the height of the second one. We put

$$r_n := m_n \cdot p_n \pmod{q_n}.$$

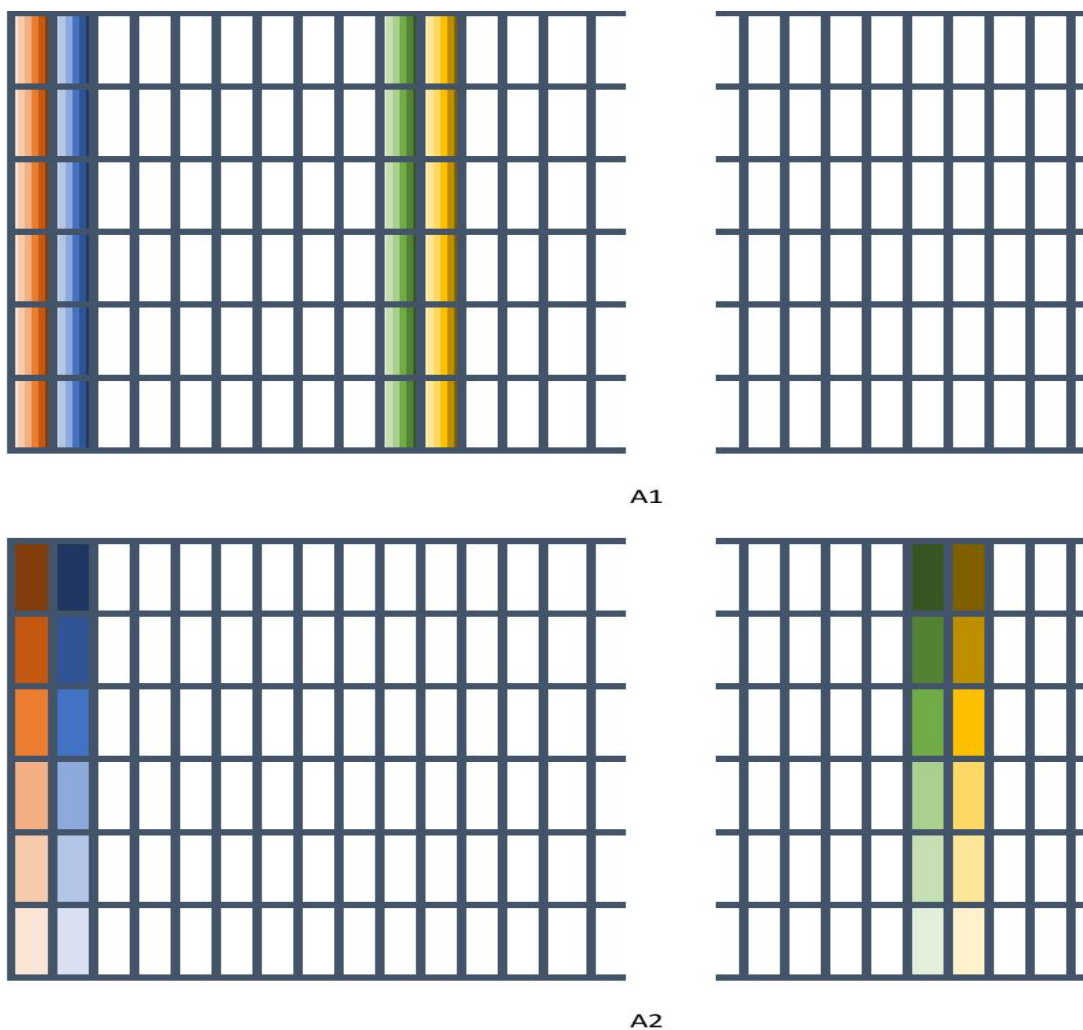


Figure 1: Illustration of the action of $\mathfrak{h}_{l,p,q,r}$ taking A1 to A2. It is drawn with $p = 1, q = 3, l = 6, k = 2$. Here \mathbb{T}^2 is partitioned into rectangles of horizontal length $1/(2l^2q^2)$ and vertical length $1/l$ and is colored with a shade of a certain color. For example the one on the left of A1 is colored with a light shade of red, and it progressively gets darker as we move to the right. Note that in A1, all the rectangles are colored by the same shade and color as we move vertically. As we move 6 blocks horizontally to the right, one for each $f, 0 \leq f < 6$, the color changes from red to a shade of blue and the same pattern repeats. Under the action of $\mathfrak{h}_{l,p,q,r}$, A1 gets mapped to A2 and, where the rectangles gets rearranged so the horizontal progression to the right becomes a vertical progression to top, and hence the shade changes likewise. Pattern repeats for blue. Note that the green and the yellow rectangles corresponds to $e \geq q$ and hence the behaviour changes, in fact they do not get mapped to the rectangle corresponding to the same a , rather to the rectangle corresponding to $a + e(r + p)$. However the pattern remains the same. Because of the larger size, we show this phenomenon with a break in the torus.

Altogether we calculate

$$m_n \cdot \alpha_{n+1} = \frac{r_n}{q_n} + \frac{1}{2q_n^2}, \quad (6.4)$$

$$(m_n + 1) \cdot \alpha_{n+1} = \frac{r_n + p_n}{q_n} + \frac{1}{2q_n^2} + \frac{1}{q_{n+1}}. \quad (6.5)$$

In the following, we consider the integers $0 \leq a < q$, $0 \leq b < 2q$, $0 \leq c < \frac{l}{2q}$ (we recall from equation 2.22 that $\frac{l}{2q} \in \mathbb{N}$ in case of our constructions), $0 \leq e < 2q$, $0 \leq f < l$ and $0 \leq t_i < l$ for $i = 1, \dots, d-2$. Under the map $\mathfrak{h}_{l,p,q,r}$ (see figure 1) the set

$$\begin{aligned} A_{a,b,c,t_1,\dots,t_{d-2},e,f,j} := & \left[\frac{a}{q} + \frac{b}{2q^2} + \frac{c}{lq} + \frac{t_1}{l^2q} + \dots + \frac{t_{d-2}}{l^{d-1}q} + \frac{e}{2l^{d-1}q^2} + \frac{f}{2l^d q^2}, \right. \\ & \left. \frac{a}{q} + \frac{b}{2q^2} + \frac{c}{lq} + \frac{t_1}{l^2q} + \dots + \frac{t_{d-2}}{l^{d-1}q} + \frac{e}{2l^{d-1}q^2} + \frac{f+1}{2l^d q^2} \right] \\ & \times \left[\frac{j}{l}, \frac{j+1}{l} \right] \end{aligned}$$

is mapped to

$$\begin{aligned} & \left[\frac{a+e \cdot r}{q} + \frac{e}{2q^2} + \frac{c}{lq} + \frac{t_1}{l^2q} + \dots + \frac{t_{d-2}}{l^{d-1}q} + \frac{b}{2l^{d-1}q^2} + \frac{j}{2l^d q^2}, \right. \\ & \left. \frac{a+e \cdot r}{q} + \frac{e}{2q^2} + \frac{c}{lq} + \frac{t_1}{l^2q} + \dots + \frac{t_{d-2}}{l^{d-1}q} + \frac{b}{2l^{d-1}q^2} + \frac{j+1}{2l^d q^2} \right] \times \left[\frac{f}{l}, \frac{f+1}{l} \right] \end{aligned}$$

in case of $0 \leq e < q$ and to

$$\begin{aligned} & \left[\frac{a+e \cdot (r+p)}{q} + \frac{e}{2q^2} + \frac{c}{lq} + \frac{t_1}{l^2q} + \dots + \frac{t_{d-2}}{l^{d-1}q} + \frac{b}{2l^{d-1}q^2} + \frac{j}{2l^d q^2}, \right. \\ & \left. \frac{a+e \cdot (r+p)}{q} + \frac{e}{2q^2} + \frac{c}{lq} + \frac{t_1}{l^2q} + \dots + \frac{t_{d-2}}{l^{d-1}q} + \frac{b}{2l^{d-1}q^2} + \frac{j+1}{2l^d q^2} \right] \times \left[\frac{f}{l}, \frac{f+1}{l} \right] \end{aligned}$$

in case of $q \leq e < 2q$.

Remark 6.6. We note several important properties of the map $\mathfrak{h}_{l,p,q,r}$:

1. $\mathfrak{h}_{l,p,q,r} \circ \phi^{\frac{1}{q}} = \phi^{\frac{1}{q}} \circ \mathfrak{h}_{l,p,q,r}$.
2. In case of $0 \leq e < q$ we have

$$\begin{aligned} & \mathfrak{h}_{l,p,q,r} \left(\bigcup_{f=0}^{l-1} A_{a,b,c,t_1,\dots,t_{d-2},e,f,j} \right) \\ & = \left[\frac{a+e \cdot r}{q} + \frac{e}{2q^2} + \frac{c}{lq} + \frac{t_1}{l^2q} + \dots + \frac{t_{d-2}}{l^{d-1}q} + \frac{b}{2l^{d-1}q^2} + \frac{j}{2l^d q^2}, \right. \\ & \quad \left. \frac{a+e \cdot r}{q} + \frac{e}{2q^2} + \frac{c}{lq} + \frac{t_1}{l^2q} + \dots + \frac{t_{d-2}}{l^{d-1}q} + \frac{b}{2l^{d-1}q^2} + \frac{j+1}{2l^d q^2} \right] \times [0, 1] \end{aligned}$$

and in case of $q \leq e < 2q$ we have

$$\begin{aligned} & \mathfrak{h}_{l,p,q,r} \left(\bigcup_{f=0}^{l-1} A_{a,b,c,t_1,\dots,t_{d-2},e,f,j} \right) \\ &= \left[\frac{a+e \cdot (r+p)}{q} + \frac{e}{2q^2} + \frac{c}{lq} + \frac{t_1}{l^2q} + \dots + \frac{t_{d-2}}{l^{d-1}q} + \frac{b}{2l^{d-1}q^2} + \frac{j}{2l^d q^2}, \right. \\ & \quad \left. \frac{a+e \cdot (r+p)}{q} + \frac{e}{2q^2} + \frac{c}{lq} + \frac{t_1}{l^2q} + \dots + \frac{t_{d-2}}{l^{d-1}q} + \frac{b}{2l^{d-1}q^2} + \frac{j+1}{2l^d q^2} \right] \times [0, 1] \end{aligned}$$

3. For all integers $0 \leq a < q$, $0 \leq c < \frac{l}{2q}$, $0 \leq e < 2q$, $0 \leq f < l$ and $0 \leq t_i < l$ for $i = 1, \dots, d-2$ we have in case of $0 \leq b < q-1$ and $0 \leq e < q-1$

$$\begin{aligned} \phi^{m_n \cdot \alpha_{n+1}} \circ \mathfrak{h}_{l_n, p_n, q_n, r_n} (A_{a,b,c,t_1,\dots,t_{d-2},e,f,j}) &= \phi^{m_n \cdot \alpha_{n+1}} (A_{a+e \cdot r_n, e, c, t_1, \dots, t_{d-2}, b, j, f}) \\ &= A_{a+(e+1) \cdot r_n, e+1, c, t_1, \dots, t_{d-2}, b, j, f} \\ &= \mathfrak{h}_{l_n, p_n, q_n, r_n} (A_{a,b,c,t_1,\dots,t_{d-2},e+1,f,j}) \end{aligned}$$

and in case of $q \leq b < 2q-1$ and $q \leq e < 2q-1$

$$\begin{aligned} & \phi^{(m_n+1) \cdot \alpha_{n+1} - \frac{1}{q_{n+1}}} \circ \mathfrak{h}_{l_n, p_n, q_n, r_n} (A_{a,b,c,t_1,\dots,t_{d-2},e,f,j}) \\ &= \phi^{(m_n+1) \cdot \alpha_{n+1} - \frac{1}{q_{n+1}}} (A_{a+e \cdot (r_n+p_n), e, c, t_1, \dots, t_{d-2}, b, j, f}) \\ &= A_{a+(e+1) \cdot (r_n+p_n), e+1, c, t_1, \dots, t_{d-2}, b, j, f} \\ &= \mathfrak{h}_{l_n, p_n, q_n, r_n} (A_{a,b,c,t_1,\dots,t_{d-2},e+1,f,j}) . \end{aligned}$$

6.2 Construction of the conjugation maps

In [BK, section 5.1] we proved the subsequent statement on the approximation of arbitrary permutations.

Proposition 6.7 ([BK], Theorem E). *Let $k, q, l \in \mathbb{N}$ and Π be any permutation of kql elements. We can naturally consider Π to be a permutation of the partition $\mathcal{S}_{kq,l}$ of the torus \mathbb{T}^2 . Assume that Π commutes with $\phi^{1/q}$. Then Π is a block-slide type of map.*

We point out that the result is proven by going to a finer partition $\mathcal{S}_{kq,2l}$. This will play a role in the definition of “good domains” of our conjugation map in Definition 6.14.

Since we know how to find real-analytic approximations of block-slide type of maps due to Proposition 2.17, this result will be very useful in order to find a conjugation map providing the combinatorics from the previous subsection. In fact, we find a real-analytic diffeomorphism $h_{2,n}$ $(\varepsilon_n, \delta_n)$ -close to $\mathfrak{h}_{l_n, p_n, q_n, r_n}$ involving the $x_1 - x_2$ -coordinates for our choices of $\delta_n = \frac{1}{n \cdot q_n}$ and $\varepsilon_n = \frac{\delta_n}{4d \cdot 2l_n^d q_n^2}$.

The higher dimensional case

In order to deal with higher dimensions, we consider the following step functions:

$$\begin{aligned}
\psi_{i,l,q}^{(1)} : [0, 1) \rightarrow \mathbb{R} & \quad \text{defined by} \quad \psi_{i,l,q}^{(1)}(x) = \sum_{j=1}^{l-1} \frac{l-j}{l^{d+2-i}q} \chi_{[\frac{j}{l}, \frac{j+1}{l}]}(x), \\
\psi_{i,l,q}^{(2)} : [0, 1) \rightarrow \mathbb{R} & \quad \text{defined by} \quad \psi_{i,l,q}^{(2)}(x) = \sum_{j=0}^{l^{d+2-i}q-1} \left(\frac{j}{l} - \left\lfloor \frac{j}{l} \right\rfloor \right) \chi_{[\frac{j}{l^{d+2-i}q}, \frac{j+1}{l^{d+2-i}q}]}(x), \\
\psi_{i,l,q}^{(3)} : [0, 1) \rightarrow \mathbb{R} & \quad \text{defined by} \quad \psi_{i,l,q}^{(3)}(x) = \sum_{j=0}^{l-1} \frac{j}{l^{d+2-i}q} \chi_{[\frac{j}{l}, \frac{j+1}{l}]}(x). \tag{6.8}
\end{aligned}$$

Then we define the following three types of block slide map:

$$\begin{aligned}
\mathfrak{g}_{i,l,q}^{(1)} : \mathbb{T}^d \rightarrow \mathbb{T}^d & \quad \text{defined by} \quad \mathfrak{g}_{i,l,q}^{(1)}(x_1, \dots, x_d) = (x_1 + \psi_{i,l,q}^{(1)}(x_i), x_2, \dots, x_d), \\
\mathfrak{g}_{i,l,q}^{(2)} : \mathbb{T}^d \rightarrow \mathbb{T}^d & \quad \text{defined by} \quad \mathfrak{g}_{i,l,q}^{(2)}(x_1, \dots, x_d) = (x_1, \dots, x_{i-1}, x_i + \psi_{i,l,q}^{(2)}(x_1), x_{i+1}, \dots, x_d), \\
\mathfrak{g}_{i,l,q}^{(3)} : \mathbb{T}^d \rightarrow \mathbb{T}^d & \quad \text{defined by} \quad \mathfrak{g}_{i,l,q}^{(3)}(x_1, \dots, x_d) = (x_1 - \psi_{i,l,q}^{(3)}(x_i), x_2, \dots, x_d).
\end{aligned}$$

Note that the composition

$$\mathfrak{g}_{i,l,q} : \mathbb{T}^d \rightarrow \mathbb{T}^d \quad \text{defined by} \quad \mathfrak{g}_{i,l,q} = \mathfrak{g}_{i,l,q}^{(3)} \circ \mathfrak{g}_{i,l,q}^{(2)} \circ \mathfrak{g}_{i,l,q}^{(1)} \tag{6.9}$$

maps the partition $\mathcal{G}_{i,l,q}$ to $\mathcal{G}_{i-1,l,q}$, where

$$\mathcal{G}_{j,l,q} := \left\{ \left[\frac{i_1}{l^{d+1-j}q}, \frac{i_1+1}{l^{d+1-j}q} \right) \times \left[\frac{i_2}{l}, \frac{i_2+1}{l} \right) \times \dots \times \left[\frac{i_j}{l}, \frac{i_j+1}{l} \right) \times \mathbb{T}^{d-j} : i_1 = 0, 1, \dots, l^{d+1-j}q - 1, \right. \\
\left. (i_2, \dots, i_j) \in \{0, 1, \dots, l-1\}^{j-1} \right\}. \tag{6.10}$$

So the composition

$$\mathfrak{g}_{l,q} : \mathbb{T}^d \rightarrow \mathbb{T}^d \quad \text{defined by} \quad \mathfrak{g}_{l,q} = \mathfrak{g}_{2,l,q} \circ \dots \circ \mathfrak{g}_{d,l,q} \tag{6.11}$$

maps the partition $\mathcal{G}_{l,q}$ to $\mathcal{T}_{l^d q}$.

In order to have conjugation maps that shift blocks of the same form, we modify the constructions slightly and define the block slide map $\mathfrak{g}_{i,l,q}^{(2)} : \mathbb{T}^d \rightarrow \mathbb{T}^d$ for $i = 3, \dots, d$ using the step function

$$\psi_{i,l,q}^{(2)} : [0, 1) \rightarrow \mathbb{R} \quad \text{defined by} \quad \psi_{i,l,q}^{(2)}(x) = \sum_{j=0}^{2l^d q^2 - 1} \left(\left\lfloor \frac{j}{2l^{i-2}q} \right\rfloor - \left\lfloor \frac{j}{2l^{i-1}q} \right\rfloor \right) \cdot \frac{1}{l} \cdot \chi_{[\frac{j}{2l^d q^2}, \frac{j+1}{2l^d q^2}]}(x). \tag{6.12}$$

Let $\varepsilon_n = \frac{\delta_n}{12d \cdot l_n^d \cdot q_n^2}$ and $\delta_n = \frac{1}{n \cdot q_n}$. With the aid of Proposition 2.17 we construct the real-analytic diffeomorphism $h_{i,l_n,q_n}^{(i)}$ (ε_n, δ_n)-approximating the maps $\mathfrak{g}_{i,l_n,q_n}^{(i)}$ from above. Finally, we put

$$h_{i,n} = h_{i,l_n,q_n}^{(3)} \circ h_{i,l_n,q_n}^{(2)} \circ h_{i,l_n,q_n}^{(1)}$$

for $i = 3, \dots, d$.

Piecing everything together

With the aid of the previous constructions we define our conjugation map

$$h_n = h_{2,n} \circ h_{3,n} \circ \cdots \circ h_{d,n}. \quad (6.13)$$

We point out that $\phi^{\alpha_n} \circ h_n = h_n \circ \phi^{\alpha_n}$.

Definition 6.14. The “good domain” G_n of h_n and h_n^{-1} respectively consists of all sets of the form

$$\left[\frac{i_1 + \delta_n}{2l_n^d q_n^2}, \frac{i_1 + 1 - \delta_n}{2l_n^d q_n^2} \right] \times \left[\frac{i_2 + \delta_n}{2l_n}, \frac{i_2 + 1 - \delta_n}{2l_n} \right] \times \prod_{j=3}^d \left[\frac{i_j + \delta_n - n}{l_n}, \frac{i_j + 1 - \delta_n}{l_n} \right],$$

where $i_t \in \mathbb{Z}$ for $t = 1, \dots, d$.

Remark 6.15. On the “good domain” the deviation of the composition h_n from the underlying block slide type of map is at most $\frac{\delta_n}{4l_n^d q_n^2}$ by our choice of ε_n in the construction of each map $h_{i,n}$.

7 Proof of the $(h, h + 1)$ -property

7.1 Construction of the towers

Let $\tilde{c}_{0,1}^{(n)}$ be the set

$$\begin{aligned} & \bigcup \left[\frac{i_1 \cdot r_n}{q_n} + \frac{i_1}{2q_n^2} + \frac{\delta_n}{2l_n^d q_n^2}, \frac{i_1 \cdot r_n}{q_n} + \frac{i_1}{2q_n^2} + \frac{\delta_n}{2l_n^d q_n^2} + \frac{q_n}{q_{n+1}} \right] \\ & \times \left[\frac{i_2 + \delta_n}{2l_n}, \frac{i_2 + 1 - \delta_n}{2l_n} \right] \times \prod_{j=3}^d \left[\frac{i_j + \delta_n}{l_n}, \frac{i_j + 1 - \delta_n}{l_n} \right], \end{aligned}$$

where the union is taken over $i_1 \in \mathbb{Z}$, $0 \leq i_1 < q_n - 1$, $i_2 \in \mathbb{Z}$, $0 \leq i_2 < 2l_n$, and $i_j \in \mathbb{Z}$, $0 \leq i_j < l_n$, for $j = 3, \dots, d$ (see figure 2). With this we define $c_{0,1}^{(n)} := H_n^{-1} \left(\tilde{c}_{0,1}^{(n)} \right)$. By Remark 6.6 we have

$$h_{2,n}^{-1} \left(\tilde{c}_{0,1}^{(n)} \right) \subset \left[0, \frac{1}{2l_n^{d-1} q_n} \right] \times \left[0, \frac{1}{l_n} \right] \times \prod_{j=3}^d [0, 1].$$

Altogether, we have

$$h_n^{-1} \left(\tilde{c}_{0,1}^{(n)} \right) \subset \left[0, \frac{1}{l_n q_n} \right] \times \left[0, \frac{1}{l_n} \right] \times \cdots \times \left[0, \frac{1}{l_n} \right].$$

Analogously we define the set $\tilde{c}_{0,2}^{(n)}$ by

$$\begin{aligned} & \bigcup \left[\frac{i_1 \cdot (r_n + p_n)}{q_n} + \frac{1}{2q_n} + \frac{i_1}{2q_n^2} + \frac{\delta_n}{2l_n^d q_n^2}, \frac{i_1 \cdot (r_n + p_n)}{q_n} + \frac{1}{2q_n} + \frac{i_1}{2q_n^2} + \frac{\delta_n}{2l_n^d q_n^2} + \frac{q_n}{q_{n+1}} \right] \\ & \times \left[\frac{i_2 + \delta_n}{2l_n}, \frac{i_2 + 1 - \delta_n}{2l_n} \right] \times \prod_{j=3}^d \left[\frac{i_j + \delta_n}{l_n}, \frac{i_j + 1 - \delta_n}{l_n} \right] \end{aligned}$$

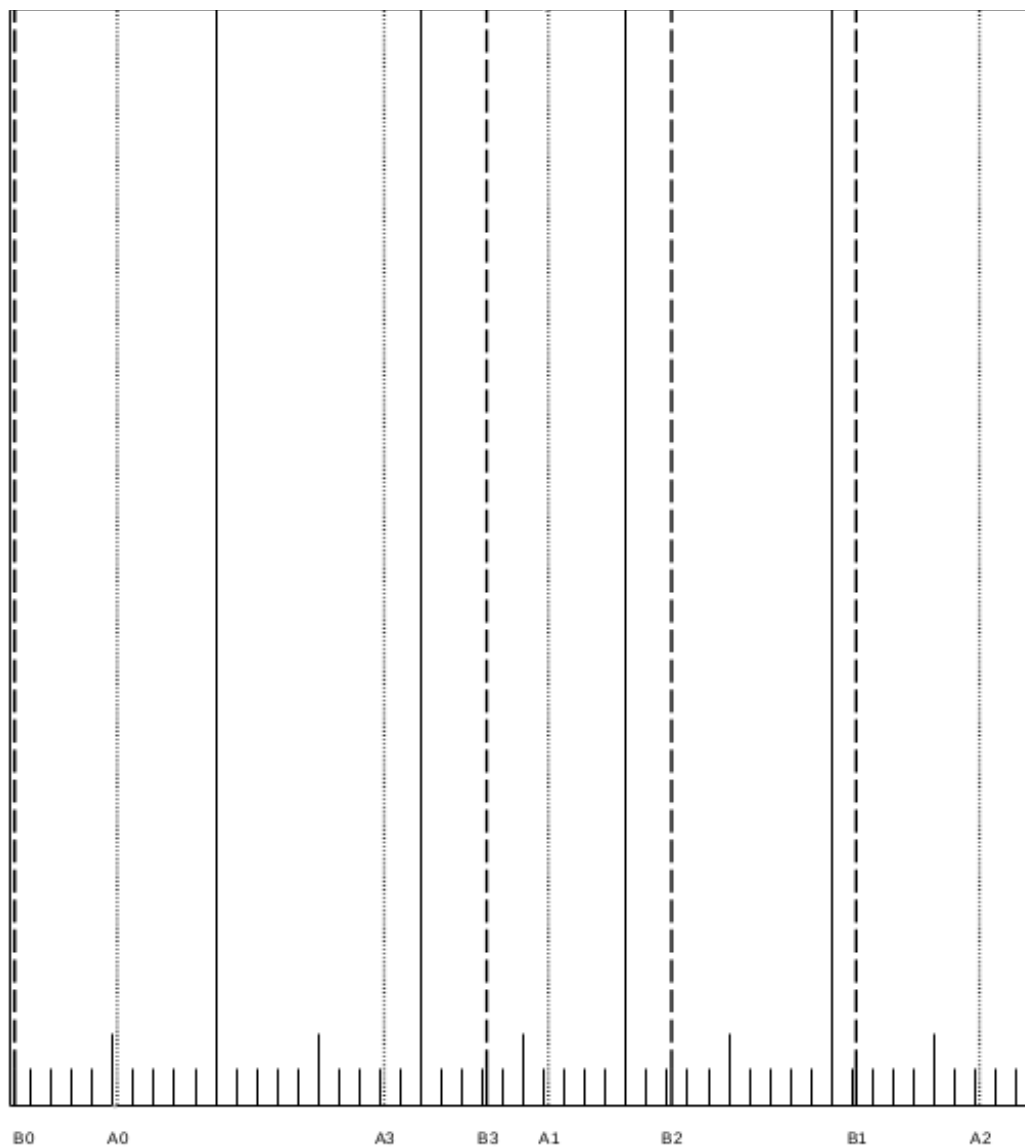


Figure 2: Sketch on \mathbb{T}^2 depicting the bases $\tilde{c}_{0,1}^{(n)} = A0 \cup A1 \cup A2 \cup A3$ and $\tilde{c}_{0,2}^{(n)} = B0 \cup B1 \cup B2 \cup B3$ of the two towers used in the approximation. This sketch is drawn with $p_n = 3, q_n = 5, m_n = 3$. So $r_n = m_n p_n = 4 \pmod{q_n}$. Our sketch is not fine enough to depict the $\delta_n / (2l_n^2 q_n^2)$ offset and the thickness of q_n / q_{n+1} . Note that the scale at the base is at a separation of $1 / (2q_n^2)$.

where the union is taken over $i_1 \in \mathbb{Z}$, $0 \leq i_1 < q_n - 1$, $i_2 \in \mathbb{Z}$, $0 \leq i_2 < 2l_n$, and $i_j \in \mathbb{Z}$, $0 \leq i_j < l_n$, for $j = 3, \dots, d$. With this we define $c_{0,1}^{(n)} := H_n^{-1} \left(\tilde{c}_{0,1}^{(n)} \right)$.

Remark 7.1. We note

$$\mu \left(c_{0,i}^{(n)} \right) = \mu \left(\tilde{c}_{0,i}^{(n)} \right) = (q_n - 1) \cdot l_n^{d-1} \cdot \frac{q_n}{q_{n+1}} \cdot \left(\frac{1 - 2\delta_n}{l_n} \right)^{d-1} = \left(1 - \frac{1}{q_n} \right) \cdot (1 - 2\delta_n)^{d-1} \cdot \frac{q_n^2}{q_{n+1}}.$$

With our number $m_n = \frac{q_{n+1}}{2q_n^2}$ we define the following sets:

$$\begin{aligned} c_{i,1}^{(n)} &:= T_n^i \left(c_{0,1}^{(n)} \right) \text{ for } i = 0, 1, \dots, m_n - 1, \\ c_{i,2}^{(n)} &:= T_n^i \left(c_{0,2}^{(n)} \right) \text{ for } i = 0, 1, \dots, m_n. \end{aligned}$$

It should be noted that these sets are disjoint by construction. Hence, we are able to define these two towers:

Definition 7.2. The first tower $B_1^{(n)}$ with base $c_{0,1}^{(n)}$ and height m_n consists of the sets $c_{i,1}^{(n)}$ for $i = 0, 1, \dots, m_n - 1$. The second tower $B_2^{(n)}$ with base $c_{0,2}^{(n)}$ and height $m_n + 1$ consists of the sets $c_{i,2}^{(n)}$ for $i = 0, 1, \dots, m_n$.

In the rest of this subsection we check that these towers satisfy the requirements of the definition of a $(h, h + 1)$ -approximation. First of all, we notice that both towers are substantial because we get by Remark 7.1:

$$\begin{aligned} \mu \left(B_1^{(n)} \right) &= m_n \cdot \mu \left(c_{0,1}^{(n)} \right) = \frac{q_{n+1}}{2q_n^2} \cdot \left(1 - \frac{1}{q_n} \right) \cdot (1 - 2\delta_n)^{d-1} \cdot \frac{q_n^2}{q_{n+1}} = \left(1 - \frac{1}{q_n} \right) \cdot (1 - 2\delta_n)^{d-1} \cdot \frac{1}{2}, \\ \mu \left(B_2^{(n)} \right) &= (m_n + 1) \cdot \mu \left(c_{0,2}^{(n)} \right) = \left(1 + \frac{2q_n^2}{q_{n+1}} \right) \cdot \left(1 - \frac{1}{q_n} \right) \cdot (1 - 2\delta_n)^{d-1} \cdot \frac{1}{2}. \end{aligned}$$

Using the notation from section 3 we consider the partial partition

$$\xi_n := \left\{ c_{i,1}^{(n)}, c_{k,2}^{(n)} : i = 0, 1, \dots, m_n - 1; k = 0, 1, \dots, m_n \right\}$$

and σ_n the associated permutation satisfying $\sigma_n \left(c_{i,1}^{(n)} \right) = c_{i+1,1}^{(n)}$ for $i = 0, 1, \dots, m_n - 2$, $\sigma_n \left(c_{m_n-1,1}^{(n)} \right) = c_{0,1}^{(n)}$ as well as $\sigma_n \left(c_{k,2}^{(n)} \right) = c_{k+1,2}^{(n)}$ for $k = 0, 1, \dots, m_n - 1$, $\sigma_n \left(c_{m_n,2}^{(n)} \right) = c_{0,2}^{(n)}$. We have to show: $\xi_n \rightarrow \varepsilon$ as $n \rightarrow \infty$. This property is fulfilled by the next lemma.

Lemma 7.3. *We have*

$$\xi_n \rightarrow \varepsilon \text{ as } n \rightarrow \infty.$$

Proof. The lemma is proven if we show that the partial partitions $\tilde{\xi}_n := \left\{ c \in \xi_n : \text{diam}(c) < \frac{1}{n} \right\}$ satisfy $\mu \left(\bigcup_{c \in \tilde{\xi}_n} c \right) \rightarrow 1$ as $n \rightarrow \infty$. For this purpose, we examine which tower elements satisfy the condition on their diameter. For this purpose, we note that

$$c_{i,j}^{(n)} = T_n^i \left(c_{0,j}^{(n)} \right) = H_{n-1}^{-1} \circ h_n^{-1} \circ \phi^{i \cdot \alpha_{n+1}} \circ H_n \left(H_n^{-1} \left(\tilde{c}_{0,j}^{(n)} \right) \right) = H_{n-1}^{-1} \circ h_n^{-1} \circ \phi^{i \cdot \alpha_{n+1}} \left(\tilde{c}_{0,j}^{(n)} \right)$$

for $j = 1, 2$ and the set $c_{i,j}^{(n)}$ is contained in one cuboid of x_1 -length $\frac{1}{l_n q_n}$ and length $\frac{1}{l_n}$ in the x_2, \dots, x_d -coordinates, if $\phi^{i \cdot \alpha_{n+1}}(\tilde{c}_{0,j}^{(n)})$ belongs to the “good domain” of h_n^{-1} (as introduced in Definition 6.14) and the deviation $i \cdot |\alpha_{n+1} - \alpha_n|$ is less than $\frac{1}{2q_n^2}$. By requirement 2.21 on the number l_n the diameter of the image of such a cuboid under H_{n-1}^{-1} is less than $\frac{1}{n^2}$. So we have to check for how many iterates i the set $\phi^{i \cdot \alpha_{n+1}}(\tilde{c}_{0,j}^{(n)})$ is contained in the “good domain” of h_n^{-1} . Note that the bases of both towers are positioned in this “good domain”. By the $\frac{1}{q_n}$ -equivariance of the map h_n and $i \cdot \alpha_{n+1} = \frac{i \cdot p_n}{q_n} + \frac{i}{q_{n+1}}$ we consider the displacement $\frac{i}{q_{n+1}}$, which is at most $\frac{1}{2q_n^2}$ because of $i \leq m_n \leq \frac{q_{n+1}}{2q_n^2}$. Since the projection on the x_1 -coordinate of the “good domain” from Definition 6.14 consists of all intervals of the form $\left[\frac{i_1 + \delta_n}{2l_n^d q_n^2}, \frac{i_1 + 1 - \delta_n}{2l_n^d q_n^2} \right]$, we can estimate the number of allowed iterates $i \in \{0, 1, \dots, m_n - 1\}$ by $(1 - 2\delta_n) \cdot m_n$. We conclude that there are at least $2 \cdot (1 - 2\delta_n) \cdot m_n$ partition elements $c_{i,j}^{(n)}$ in $\tilde{\xi}_n$ and this corresponds to a measure

$$\mu \left(\bigcup_{c \in \tilde{\xi}_n} c \right) \geq 2 \cdot (1 - 2\delta_n) \cdot m_n \cdot \mu(c_{i,j}^{(n)}) = \left(1 - \frac{1}{q_n}\right) \cdot (1 - 2\delta_n)^d,$$

which converges to 1 as $n \rightarrow \infty$. \square

Actually we have a linked approximation of type $(h, h + 1)$: Once again using the notation of section 3 we consider the partial partition

$$\eta_n := \left\{ c_{i,1}^{(n)} \cup c_{i,2}^{(n)}, c_{m_n,2}^{(n)} : 0 \leq i \leq m_n - 1 \right\}$$

and prove $\eta_n \rightarrow \varepsilon$ as $n \rightarrow \infty$.

Lemma 7.4. *We have*

$$\eta_n \rightarrow \varepsilon \text{ as } n \rightarrow \infty.$$

Proof. To see this we examine the partition

$$\tilde{\eta}_n := \left\{ \tilde{c}_i := c_{i,1}^{(n)} \cup c_{i,2}^{(n)} : 0 \leq i \leq m_n - 1, \text{diam}(\tilde{c}_i) < \frac{1}{n} \right\}$$

and show $\lim_{n \rightarrow \infty} \mu \left(\bigcup_{\tilde{c} \in \tilde{\eta}_n} \tilde{c} \right) = 1$. For this purpose, we note that $c_{i,1}^{(n)} \cup c_{i,2}^{(n)}$ is contained in one cuboid of x_1 -length $\frac{1}{l_n q_n}$ and length $\frac{1}{l_n}$ in the x_2, \dots, x_d -coordinates, if $\phi^{i \cdot \alpha_{n+1}}(\tilde{c}_{0,1}^{(n)} \cup \tilde{c}_{0,2}^{(n)})$ belongs to the “good domain” of h_n^{-1} and the deviation $i \cdot |\alpha_{n+1} - \alpha_n|$ is less than $\frac{1}{2q_n^2}$. Hence, the calculations from the previous lemma apply. \square

7.2 Speed of approximation

In this section we want we prove that T admits a good linked approximation of type $(h, h + 1)$:

Proposition 7.5. *The constructed diffeomorphism $T \in \text{Diff}_\rho^{\omega}(\mathbb{T}^d, \mu)$ admits a good linked approximation of type $(h, h + 1)$.*

Since T admits a linked approximation by Lemma 7.4, we have to compute the speed of the approximation in order to prove this statement. First of all, we observe

$$\sum_{c \in \xi_n} \mu(T(c) \Delta \sigma_n(c)) \leq \sum_{c \in \xi_n} (\mu(T(c) \Delta T_n(c)) + \mu(T_n(c) \Delta \sigma_n(c))) \quad (7.6)$$

recalling that σ_n is the associated permutation satisfying $\sigma_n(c_{i,1}^{(n)}) = c_{i+1,1}^{(n)}$ for $i = 0, 1, \dots, m_n - 2$, $\sigma_n(c_{m_n-1,1}^{(n)}) = c_{0,1}^{(n)}$ as well as $\sigma_n(c_{k,2}^{(n)}) = c_{k+1,2}^{(n)}$ for $k = 0, 1, \dots, m_n - 1$, $\sigma_n(c_{m_n,2}^{(n)}) = c_{0,2}^{(n)}$. In the subsequent lemmas we examine each summand.

Lemma 7.7. *We have*

$$\sum_{c \in \xi_n} \mu(T_n(c) \Delta \sigma_n(c)) \leq \frac{3q_n}{q_{n+1}}.$$

Proof. We note $\sigma_n|_{c_{i,1}^{(n)}} = f_n|_{c_{i,1}^{(n)}}$ for $i = 0, \dots, m_n - 2$ and $\sigma_n(c_{m_n-1,1}^{(n)}) = c_{0,1}^{(n)}$ as well as $\sigma_n|_{c_{i,2}^{(n)}} = T_n|_{c_{i,2}^{(n)}}$ for $i = 0, \dots, m_n - 1$, $\sigma_n(c_{m_n,2}^{(n)}) = c_{0,2}^{(n)}$.

To estimate the expression $\sum_{c \in \xi_n} \mu(T_n(c) \Delta \sigma_n(c))$ we recall equation 6.4 and Remark 6.6, 3. Hereby, we observe that

$$T_n(c_{m_n-1,1}^{(n)}) = H_n^{-1} \circ \phi^{\alpha_{n+1}} \circ H_n \circ H_n^{-1} \circ \phi^{(m_n-1) \cdot \alpha_{n+1}} \circ H_n \left(H_n^{-1} \left(\tilde{c}_{0,1}^{(n)} \right) \right) = H_n^{-1} \circ \phi^{m_n \cdot \alpha_{n+1}} \left(\tilde{c}_{0,1}^{(n)} \right)$$

and $c_{0,1}^{(n)} = H_n^{-1} \left(\tilde{c}_{0,1}^{(n)} \right)$ differ in that part of $c_{0,1}^{(n)}$ corresponding to the value $i_1 = q_n - 2$ which is not mapped back to $c_{0,1}^{(n)}$ under $T_n^{m_n}$. This discrepancy yields a measure difference of $\frac{q_n}{q_{n+1}} \cdot (1 - 2\delta_n)^{d-1}$. Analogously we recall equation 6.5 stating

$$(m_n + 1) \cdot \alpha_{n+1} = \frac{r_n + p_n}{q_n} + \frac{1}{2q_n^2} + \frac{1}{q_{n+1}}$$

and observe that $T_n(c_{m_n,2}^{(n)})$ and $c_{0,2}^{(n)}$ differ in the part of $c_{0,2}^{(n)}$ corresponding to $i_1 = q_n - 2$ which is not mapped back to $c_{0,2}^{(n)}$. Additionally, for each other allowed value of i_1 there is the discrepancy of $(m_n + 1) \cdot (\alpha_{n+1} - \alpha_n)$ from $\frac{1}{2q_n^2}$ with value $\frac{1}{q_{n+1}}$. Thus, we get

$$\mu \left(T_n^{m_n+1} \left(c_{0,2}^{(n)} \right) \Delta c_{0,2}^{(n)} \right) \leq \frac{2q_n}{q_{n+1}}.$$

This yields

$$\sum_{c \in \xi_n} \mu(T_n(c) \Delta \sigma_n(c)) \leq \frac{3q_n}{q_{n+1}}.$$

□

In the next step we consider $\sum_{c \in \xi_n} \mu(T(c) \Delta T_n(c))$:

Lemma 7.8. *We have*

$$\sum_{c \in \xi_n} \mu(T(c) \Delta T_n(c)) \leq 40d \cdot \delta_{n+1}.$$

Proof. First of all we examine $\sum_{c \in \xi_n} \mu(T_{n+1}(c) \Delta T_n(c))$. We compute for every $c \in \xi_n$ using the notation $\bar{c} := H_n(c) = \phi^{i \cdot \alpha_{n+1}}(\tilde{c}_{0,j}^{(n)})$:

$$\begin{aligned} \mu(T_{n+1}(c) \Delta T_n(c)) &= \mu(H_{n+1}^{-1} \circ \phi^{\alpha_{n+2}} \circ h_{n+1}(\bar{c}) \Delta H_n^{-1} \circ \phi^{\alpha_{n+1}}(\bar{c})) \\ &= \mu(H_{n+1}^{-1} \circ \phi^{\alpha_{n+2}} \circ h_{n+1}(\bar{c}) \Delta H_n^{-1} \circ h_{n+1}^{-1} \circ \phi^{\alpha_{n+1}} \circ h_{n+1}(\bar{c})) \\ &= \mu(\phi^{\alpha_{n+2}} \circ h_{n+1}(\bar{c}) \Delta \phi^{\alpha_{n+1}} \circ h_{n+1}(\bar{c})). \end{aligned}$$

Since we have no control on $h_{n+1}(\bar{c})$ for these areas of \bar{c} that do not belong to the “good domain” of the map h_{n+1} , they will be part of the measure difference in our estimates. On the other hand, for the part of \bar{c} belonging to the “good domain” of the map h_{n+1} the difference is caused by the deviation $|\alpha_{n+2} - \alpha_{n+1}|$ and by the approximation error between the block slide type of map and its real-analytic approximation. By Definition 6.14 the “good domain” of the map h_{n+1} on a block of the form

$$\Delta_{i_1, \dots, i_d} = \left[\frac{i_1}{2l_{n+1}^d q_{n+1}^2}, \frac{i_1 + 1}{2l_{n+1}^d q_{n+1}^2} \right] \times \left[\frac{i_2}{2l_{n+1}}, \frac{i_2 + 1}{2l_{n+1}} \right] \times \left[\frac{i_3}{l_{n+1}}, \frac{i_3 + 1}{l_{n+1}} \right] \times \dots \times \left[\frac{i_d}{l_{n+1}}, \frac{i_d + 1}{l_{n+1}} \right]$$

is

$$\left[\frac{i_1 + \delta_{n+1}}{2l_{n+1}^d q_{n+1}^2}, \frac{i_1 + 1 - \delta_{n+1}}{2l_{n+1}^d q_{n+1}^2} \right] \times \left[\frac{i_2 + \delta_{n+1}}{2l_{n+1}}, \frac{i_2 + 1 - \delta_{n+1}}{2l_{n+1}} \right] \times \prod_{j=3}^d \left[\frac{i_j + \delta_{n+1}}{l_{n+1}}, \frac{i_j + 1 - \delta_{n+1}}{l_{n+1}} \right].$$

So its measure is

$$\frac{(1 - 2\delta_{n+1})^d}{4 \cdot l_{n+1}^{2d-1} \cdot q_{n+1}^2} \geq \frac{(1 - 2d \cdot \delta_{n+1})}{4 \cdot l_{n+1}^{2d-1} \cdot q_{n+1}^2}.$$

By our choice of ε_{n+1} the approximation error is at most $\frac{\delta_{n+1}}{4l_{n+1}^d \cdot q_{n+1}^2}$ in each coordinate x_2, \dots, x_d . It follows that the measure difference of $\phi^{\alpha_{n+2}} \circ h_{n+1}(\bar{c})$ and $\phi^{\alpha_{n+1}} \circ h_{n+1}(\bar{c})$ on a set of the form Δ_{i_1, \dots, i_d} is at most

$$2 \cdot \left(\frac{2d \cdot \delta_{n+1}}{4 \cdot l_{n+1}^{2d-1} \cdot q_{n+1}^2} + |\alpha_{n+2} - \alpha_{n+1}| \cdot \frac{(1 - 2d \cdot \delta_{n+1})}{4 \cdot l_{n+1}^{2d-1} \cdot q_{n+1}^2} + \frac{(d-1) \cdot \delta_{n+1}}{4l_{n+1}^{3d-2} \cdot q_{n+1}^4} \right) \leq \frac{3d \cdot \delta_{n+1}}{l_{n+1}^{2d-1} \cdot q_{n+1}^2}.$$

Moreover, we recall that $\tilde{c}_{0,j}^{(n)}$ consists of at most $4 \cdot q_n^2 \cdot l_{n+1}^{2d-1} \cdot q_{n+1}$ such blocks Δ_{i_1, \dots, i_d} . Then we conclude

$$\mu(T_{n+1}(c) \Delta T_n(c)) \leq 4 \cdot q_n^2 \cdot l_{n+1}^{2d-1} \cdot q_{n+1} \cdot \frac{3d \cdot \delta_{n+1}}{l_{n+1}^{2d-1} \cdot q_{n+1}^2} = q_n^2 \cdot \frac{12d \cdot \delta_{n+1}}{q_{n+1}}$$

Each of the $(2m_n + 1)$ elements $c \in \xi_n$ contributes and so we obtain

$$\sum_{c \in \xi_n} \mu(f_{n+2}(c) \Delta f_{n+1}(c)) \leq 20d \cdot \delta_{n+1}.$$

Analogously estimating the other summands we observe

$$\sum_{c \in \xi_n} \mu(T(c) \Delta T_n(c))$$

$$\begin{aligned}
&\leq \sum_{k=0}^{\infty} \sum_{c \in \xi_n} \mu \left(T_{n+k+1} \left(c \cap \bigcap_{j=0}^{k-1} H_{n+j}^{-1} (G_{n+j+1}) \right) \Delta T_{n+k} \left(c \cap \bigcap_{j=0}^{k-1} H_{n+j}^{-1} (G_{n+j+1}) \right) \right) \\
&\leq \sum_{k=n+1}^{\infty} 20d\delta_k \leq 40d \cdot \delta_{n+1}.
\end{aligned}$$

□

Proof of Proposition 7.5. Using equation 7.6 and the precedent three lemmas we conclude

$$\sum_{c \in \xi_n} \mu(T(c) \Delta \sigma_n(c)) \leq \frac{3 \cdot q_n}{q_{n+1}} + 40d \cdot \delta_{n+1}.$$

In order to prove that this speed of approximation is of order $o\left(\frac{1}{m_n}\right)$ we compute

$$\frac{\frac{3 \cdot q_n}{q_{n+1}} + 40d \cdot \delta_{n+1}}{\frac{1}{m_n}} = \frac{q_{n+1}}{2q_n^2} \cdot \left(\frac{3 \cdot q_n}{q_{n+1}} + \frac{40d}{(n+1) \cdot q_{n+1}} \right) = \frac{3}{2 \cdot q_n} + \frac{20d}{(n+1) \cdot q_n^2}$$

by our choice of $\delta_{n+1} = \frac{1}{(n+1) \cdot q_{n+1}}$. Since this converges to 0 as $n \rightarrow \infty$, we have a good linked approximation of type $(h, h+1)$. □

8 Proof of good cyclic approximation

8.1 Tower for good cyclic approximation

Let $\tilde{d}_0^{(n)}$ be the set

$$\bigcup \left[\frac{\delta_n}{2l_n^d q_n^2}, \frac{\delta_n}{2l_n^d q_n^2} + \frac{1}{q_{n+1}} \right] \times \left[\frac{i_2 + \delta_n}{2l_n}, \frac{i_2 + 1 - \delta_n}{2l_n} \right] \times \prod_{j=3}^d \left[\frac{i_j + \delta_n}{l_n}, \frac{i_j + 1 - \delta_n}{l_n} \right],$$

where the union is taken over $i_2 \in \mathbb{N}$, $0 \leq i_2 < 2l_n$, and $i_j \in \mathbb{N}$, $0 \leq i_j < l_n$, for $j = 3, \dots, d$. With this we define $d_0^{(n)} := H_n^{-1}(\tilde{d}_0^{(n)})$ and the tower levels by

$$d_i^{(n)} := T_n^i(d_0^{(n)}) \quad \text{for } i = 0, \dots, q_{n+1} - 1.$$

Recall $\alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}}$, where p_{n+1} and q_{n+1} are relatively prime. Hence, the tower levels are disjoint sets of equal measure not less than $\frac{(1-2\delta_n)^{d-1}}{q_{n+1}}$. Moreover, the associated cyclic permutation $\tilde{\sigma}_n$ is given by the description $\tilde{\sigma}_n|_{d_i^{(n)}} = T_n|_{d_i^{(n)}}$ for $i = 0, \dots, q_{n+1} - 2$ and $\tilde{\sigma}_n(d_{q_{n+1}-1}^{(n)}) = d_0^{(n)}$. Since $T_n^{q_{n+1}} = \text{id}$ we also have $\tilde{\sigma}_n|_{d_{q_{n+1}-1}^{(n)}} = T_n|_{d_{q_{n+1}-1}^{(n)}}$.

In order to see that this provides a cyclic approximation of the constructed map T , we show that the partial partition $\Gamma_n := \{d_i^{(n)} : i = 0, \dots, q_{n+1} - 1\}$ converges to the decomposition into points.

Lemma 8.1. *We have $\Gamma_n \rightarrow \varepsilon$ as $n \rightarrow \infty$.*

Proof. It suffices to show that the partial partition $\tilde{\Gamma}_n := \left\{ d_i^{(n)} \in \Gamma_n : \text{diam} \left(d_i^{(n)} \right) < \frac{1}{n} \right\}$ satisfies $\lim_{n \rightarrow \infty} \mu \left(\bigcup_{d \in \tilde{\Gamma}_n} d \right) = 1$. As in the previous section we have to check for how many iterates $i \in \{0, 1, \dots, q_{n+1} - 1\}$ the set $\phi^{i \cdot \alpha_{n+1}} \left(\tilde{d}_0^{(n)} \right)$ is contained in the “good domain” of the map h_n^{-1} whose corresponding length on the x_1 -axis is at least $1 - 2\delta_n$ by Definition 6.14. Since $\phi^{i \cdot \alpha_{n+1}}$ is equally distributed on \mathbb{S}^1 , there are at least $(1 - 2\delta_n) \cdot q_{n+1}$ such iterates i . Then we conclude

$$\mu \left(\bigcup_{d \in \tilde{\Gamma}_n} d \right) \geq (1 - 2\delta_n) \cdot q_{n+1} \cdot \mu \left(d_0^{(n)} \right) \geq (1 - 2\delta_n)^d,$$

which converges to 1 as $n \rightarrow \infty$. \square

8.2 Speed of approximation

In this subsection we show that T admits a good cyclic approximation.

Proposition 8.2. *The constructed diffeomorphism $T \in \text{Diff}_\rho^w(\mathbb{T}^d, \mu)$ admits a good cyclic approximation.*

As observed in the previous subsection $\tilde{\sigma}_n|_d = T_n|_d$ for every $d \in \Gamma_n$. Thus, for the speed of approximation it holds:

$$\sum_{d \in \Gamma_n} \mu(T(d) \Delta \tilde{\sigma}_n(d)) = \sum_{d \in \Gamma_n} \mu(T(d) \Delta T_n(d)). \quad (8.3)$$

Lemma 8.4. *We have*

$$\sum_{d \in \Gamma_n} \mu(T(d) \Delta T_n(d)) \leq 20d \cdot \delta_{n+1}.$$

Proof. In order to estimate $\mu(T_{n+1}(d) \Delta T_n(d))$ we denote $\bar{d} = H_n(d)$ and argue as in the proof of Lemma 7.8 that we have no control on $h_{n+1}(\bar{d})$ for these areas of \bar{d} that do not belong to the “good domain” of the map h_{n+1} . Hence, these will be part of the measure difference in our estimates. On the other hand, for the part of \bar{d} belonging to the “good domain” of the map h_{n+1} the difference is caused by the deviations $|\alpha_{n+2} - \alpha_{n+1}|$ as well as the approximation error between the block slide type of map and its real-analytic approximation.

Using the estimate on the “good domain” of h_{n+1} on sets of the form Δ_{i_1, \dots, i_d} in Lemma 7.8 and that $H_n(d)$ consists of $4 \cdot l_{n+1}^{2d-1} \cdot q_{n+1}$ many of those sets Δ_{i_1, \dots, i_d} , we obtain

$$\mu(T_{n+1}(d) \Delta T_n(d)) \leq 4 \cdot l_{n+1}^{2d-1} \cdot q_{n+1} \cdot \frac{3d \cdot \delta_{n+1}}{l_{n+1}^{2d-1} \cdot q_{n+1}^2} \leq \frac{12d \cdot \delta_{n+1}}{q_{n+1}}.$$

Each of the q_{n+1} elements $d \in \Gamma_n$ contributes and so we obtain

$$\sum_{d \in \Gamma_n} \mu(T_{n+1}(d) \Delta T_n(d)) \leq 12d \cdot \delta_{n+1}.$$

Analogously we estimate the other summands and obtain

$$\sum_{d \in \Gamma_n} \mu(T(d) \Delta T_n(d))$$

$$\begin{aligned}
&\leq \sum_{k=0}^{\infty} \sum_{d \in \Gamma_n} \mu \left(T_{n+k+1} \left(d \cap \bigcap_{j=0}^{k-1} H_{n+j}^{-1}(G_{n+j+1}) \right) \Delta T_{n+k} \left(d \cap \bigcap_{j=0}^{k-1} H_{n+j}^{-1}(G_{n+j+1}) \right) \right) \\
&\leq \sum_{k=n}^{\infty} 12d\delta_{k+1} \leq 20d \cdot \delta_{n+1}.
\end{aligned}$$

□

Proof of Proposition 8.2. Using equation 8.3 as well as Lemma 8.4 we conclude

$$\sum_{d \in \Gamma_n} \mu(T(d) \Delta \tilde{\sigma}_n(d)) \leq 20d \cdot \delta_{n+1}.$$

In order to prove that this speed of approximation is of order $o\left(\frac{1}{q_{n+1}}\right)$ we compute

$$\frac{20d \cdot \delta_{n+1}}{\frac{1}{q_{n+1}}} = q_{n+1} \cdot \frac{20d}{(n+1) \cdot q_{n+1}} = \frac{20d}{n+1}$$

by our choice of δ_{n+1} . Since this converges to 0 as $n \rightarrow \infty$, we have a good cyclic approximation. □

9 References

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