Simple heteroclinic networks in \mathbb{R}^4

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Abstract

We classify simple heteroclinic networks for a Γ -equivariant system in \mathbb{R}^4 with finite $\Gamma \subset O(4)$, proceeding as follows: we define a graph associated with a given $\Gamma \subset O(n)$ and identify all so-called simple graphs associated with subgroups of O(4). Then, knowing the graph associated with a given Γ , we determine the types of heteroclinic networks that the group admits. Our study is restricted to networks that are maximal in the sense that they have the highest possible number of connections – any non-maximal network can then be derived by deleting one or more connections. Finally, for networks of type A, i.e., admitted by $\Gamma \subset SO(4)$, we give necessary and sufficient conditions for fragmentary and essential asymptotic stability. (For other simple hete-roclinic networks the conditions for stability are known.) The results are illustrated by a numerical example of a simple heteroclinic network that involves two subcycles that can be essentially asymptotically stable simultaneously.

Keywords: equivariant dynamics, heteroclinic cycle, heteroclinic network, stability *Mathematics Subject Classification:* 34C14, 34C37, 37C29, 37C75, 37C80

1 Introduction

Heteroclinic cycles and networks are flow-invariant sets in a dynamical system that are associated with stop-and-go dynamics encountered in a variety of applications. As such they have been studied from various angles over the last decades. The present paper is a systematic investigation of simple robust heteroclinic networks in \mathbb{R}^4 . Our interest lies in systems of the form

$$\dot{x} = f(x)$$
, with $f(\gamma x) = \gamma f(x)$ for all $x \in \mathbb{R}^n$, $\gamma \in \Gamma$. (1)

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map and $\Gamma \subset O(n)$ is a finite group. While some constructions we present are general we mainly focus on the case n = 4. In a system (1) a heteroclinic cycle is a set of equilibria ξ_1, \ldots, ξ_M and connections $\kappa_j \subset W^u(\xi_j) \cap W^s(\xi_{j+1})$ in the intersection of the respective unstable and stable manifolds of subsequent equilibria, with the convention M + 1 = 1. It is well known that if all connections are of saddle-sink type in fixed-point subspaces P_j of the system, then the cycle persists under Γ -equivariant perturbations and is therefore said to be robust. A cycle in \mathbb{R}^4 is called simple if dim $P_j = 2$ for all j, see section 2 for details. A (simple) heteroclinic network is a connected union of (simple) heteroclinic cycles.

Simple cycles have previously been studied by many authors, see e.g. [6, 9] for a general overview. Conditions for their asymptotic stability were derived in [10, 11]. In order to improve our systematic understanding of heteroclinic dynamics it is desirable to classify lowdimensional heteroclinic networks, which can possibly be combined to form more complex structures in higher dimensions. A full classification of simple cycles in \mathbb{R}^4 has been achieved step by step in [11, 17, 18, 19], while pseudo-simple cycles (that differ from simple ones in the isotypic decomposition of \mathbb{R}^4 w.r.t. certain subgroups) were addressed in [2]. In this paper we take the next step by deriving a complete list of simple networks in \mathbb{R}^4 . To do so, we introduce an intuitive way of associating a subgroup $\Gamma \subset O(n)$ with a graph: by drawing isotropy semiaxes as points and isotropy planes as lines, such that a line contains a point if and only if the respective semiaxis is contained in the respective plane. We say that such a graph is *simple* if for any isotropy semiaxis its isotropy subgroup decomposes \mathbb{R}^n into onedimensional isotypic components. There is a close relation between simple graphs associated with $\Gamma \subset O(4)$ and the structure of heteroclinic cycles/networks that a Γ -equivariant vector field may exhibit.

In a heteroclinic network no individual cycle can be asymptotically stable, so the situation becomes more subtle than in [10, 11]: in the context of networks it is of particular interest to identify subcycles with the strongest possible attraction properties. There are several notions of non-asymptotic stability available in the literature, ranging from the strongest, essential asymptotic stability (e.a.s) [13], to the weakest, fragmentary asymptotic stability (f.a.s.) [14]. We characterise the stability configurations in networks of type A which corresponds to the case $\Gamma \subset SO(4)$.

Thus, the two main contributions of this work are (i) a complete description of simple graphs for $\Gamma \subset O(4)$, the groups associated with them and the heteroclinic networks they admit, and (ii) a stability study for heteroclinic networks of type A in \mathbb{R}^4 . The former is achieved by building on results about simple heteroclinic cycles in [17], where the subgroups of O(4) have been studied in detail, using the quaternionic representation from [8] in order to decide which groups admit heteroclinic cycles. The latter closes a gap in the study of stability in heteroclinic networks in \mathbb{R}^4 : networks of type Z (as opposed to type A) have been thoroughly investigated before and they give rise to complex stability configurations, [3, 15]. For type A networks we show that the situation is comparatively simple in the sense that fragmentary and essential asymptotic stability of the network are directly linked to the existence of one or more subcycles with the same property. Moreover, we numerically investigate a heteroclinic network in \mathbb{R}^4 consisting of four equilibria and six connections forming two cycles with two equilibria and one cycle with four equilibria. As expected from our theoretic results it is possible to have each cycle e.a.s., but no two cycles sharing a connections can be f.a.s. simultaneously.

This paper is organized as follows. In section 2 we recall necessary background information and terminology regarding heteroclinic cycles and their stability properties as well as the quaternionic approach for representing subgroups of O(4). Section 3 explains how we associate a given subgroup $\Gamma \subset O(n)$ with a (simple) graph. In Theorems 1 and 2 we identify all types of simple graphs that can occur for $\Gamma \subset O(4)$ and list the corresponding groups. This enables us to classify maximal simple heteroclinic networks in \mathbb{R}^4 in Theorem 3. In section 4 we give necessary and sufficient conditions for fragmentary and essential asymptotic stability of type A simple heteroclinic networks in \mathbb{R}^4 . Finally, in section 5 we study numerically an example of a heteroclinic network identified in section 4 that may exhibit one attracting cycle with four equilibria or two attracting cycles with two equilibria each. Section 6 concludes.

2 Background

2.1 Heteroclinic cycles and notions of stability

In this subsection we recall basic terminology in the context of (robust) heteroclinic cycles and their stability properties. Given a Γ -equivariant dynamical system (1) recall that for $x \in \mathbb{R}^n$ the *isotropy subgroup of* x is the subgroup of all elements in Γ that fix x. On the other hand, for a subgroup $\Sigma \subset \Gamma$ we denote by Fix (Σ) its *fixed point space*, i.e. the space of points in \mathbb{R}^n that are fixed by all elements of Σ .

Let ξ_1, \ldots, ξ_M be hyperbolic equilibria of a system (1) with stable and unstable manifolds $W^s(\xi_j)$ and $W^u(\xi_j)$, respectively. Also, let $\kappa_j \subset W^u(\xi_j) \cap W^s(\xi_{j+1}) \neq \emptyset$ for $j = 1, \ldots, M$ be connections between them, where we set $\xi_{M+1} = \xi_1$. Then the collection of equilibria $\{\xi_1, \ldots, \xi_M\}$ together with the connecting trajectories $\{\kappa_1, \ldots, \kappa_M\}$ is called a *heteroclinic cycle*. A connected union of heteroclinic cycles is a *heteroclinic network*.

A heteroclinic cycle is structurally stable or robust if for all j there are subgroups $\Sigma_j \subset \Gamma$ such that ξ_{j+1} is a sink in $P_j := \operatorname{Fix}(\Sigma_j)$ and κ_j is contained in P_j , see [10]. As usual we divide the eigenvalues of the Jacobian $df(\xi_j)$ into radial (eigenspace belonging to $P_{j-1} \cap P_j$), contracting (belonging to $P_{j-1} \oplus (P_{j-1} \cap P_j)$), expanding (belonging to $P_j \oplus (P_{j-1} \cap P_j)$) and transverse (all others), where we write $X \oplus Y$ for a complementary subspace of Y in X.

We are interested in cycles where (i) dim $P_j = 2$ for all j, and (ii) the heteroclinic cycle intersects each connected component of $P_{j-1} \cap P_j = \text{Fix}(\Delta_j)$, where $\Delta_j \subset \Gamma$ is some subgroup, at most once. Note that then dim $(P_{j-1} \cap P_j) = 1$, and so we refer to P_j as an *isotropy plane* and to $P_{j-1} \cap P_j$ as an *isotropy axis*. The latter is often denoted by L_j , we differ slightly from this notation by using L_j for a connected component of $P_{j-1} \cap P_j \setminus \{0\}$, i.e. for an *isotropy semiaxis*. Note that different connected components may or may not be related by symmetry. In \mathbb{R}^4 , there is then one eigenvalue of each type and we denote the corresponding contracting, expanding and transverse eigenspaces of $df(\xi_j)$ by V_j , W_j and T_j , respectively. In [17] it is shown that under these conditions there are three possibilities for the unique Δ_j -isotypic decomposition of \mathbb{R}^4 :

(1)
$$\mathbb{R}^4 = \operatorname{Fix}(\Delta_j) \oplus V_j \oplus W_j \oplus T_j$$

(2) $\mathbb{R}^4 = \operatorname{Fix}(\Delta_j) \oplus V_j \oplus \widetilde{W}_j$, where $\widetilde{W}_j = W_j \oplus T_j$ is two-dimensional

(3) $\mathbb{R}^4 = \operatorname{Fix}(\Delta_j) \oplus W_j \oplus \widetilde{V}_j$, where $\widetilde{V}_j = V_j \oplus T_j$ is two-dimensional

Here \oplus denotes the orthogonal direct sum. This inspires the following definition.

Definition 1 ([17]) We call a heteroclinic cycle satisfying conditions (i) and (ii) above simple if case 1 holds true for all j, and pseudo-simple otherwise. A heteroclinic network is called simple if it consists only of simple cycles, and pseudo-simple if at least one of its cycles is pseudo-simple.

We aim to identify all subgroups of O(4) that admit simple heteroclinic networks in the following sense.

Definition 2 ([17]) We say that a subgroup Γ of O(n) admits robust heteroclinic cycles (network) if there exists an open subset of the set of smooth Γ -equivariant vector fields in \mathbb{R}^n , such that vector fields in this subset possess a robust heteroclinic cycle (network).

In order to discuss stability properties a further distinction of simple cycles into different types has proved useful. There are several established ways to do this, we reproduce here only the types that are relevant for our results in section 4.

Definition 3 ([11, 14]) A simple heteroclinic cycle (network) in \mathbb{R}^4 is said to be of

- (i) type A if $\Sigma_j \cong \mathbb{Z}_2$ for all j,
- (ii) type Z if Σ_j decomposes P_j^{\perp} into one-dimensional isotypic components for all j.

Note that $\Gamma \subset SO(4)$ admits heteroclinic cycles only of type A and that type Z cycles are admitted by $\Gamma \not\subset SO(4)$ only. Type A homoclinic and heteroclinic cycles can exist in Γ -equivariant systems with $\Gamma \not\subset SO(4)$.

A heteroclinic cycle that belongs to a network cannot be asymptotically stable because it does not contain the entire unstable manifolds of all its equilibria. Moreover, it has been proved recently that a compact robust heteroclinic network comprised of equilibria and a finite number of connecting trajectories is never asymptotically stable [16]. Various weaker notions of stability have been introduced over the last decades. The strongest one is essential asymptotic stability which goes back to [13]. In the following $\ell(.)$ denotes the Lebesgue measure of a set in \mathbb{R}^n and for $\varepsilon > 0$ we denote by $B_{\varepsilon}(X)$ an ε -neighbourhood of X. For an ε -neighbourhood of 0 we simply write B_{ε} . **Definition 4** A set X is essentially asymptotically stable (e.a.s) if

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \frac{\ell(B_{\varepsilon}(X) \setminus \mathcal{B}_{\delta}(X))}{\ell(B_{\varepsilon}(X))} = 0,$$

where

$$\mathcal{B}_{\delta}(X) = \left\{ x \in \mathbb{R}^n : d(\Phi(x,t),X) < \delta \text{ for all } t > 0 \text{ and } \lim_{t \to +\infty} d(\Phi(x,t),X) = 0 \right\}.$$

is the δ -basin of attraction of X and $\Phi(x,t)$ denotes the flow of the system.

Heteroclinic cycles that are not e.a.s. may still attract a set of positive Lebesgue measure within their neighbourhood. This is captured in the following term from [14].

Definition 5 A heteroclinic cycle X is fragmentarily asymptotically stable (f.a.s.) if $\ell(\mathcal{B}_{\delta}(X)) > 0$ for any $\delta > 0$.

We also refer to the (local) stability index along a connection as a way to characterise stability and attraction of heteroclinic cycles and networks. It is a quantity which can be computed with respect to any flow-invariant set and is constant along solution trajectories, first defined in [15] and studied further e.g. in [12]. In particular, a heteroclinic cycle/network is e.a.s. if and only if the local stability indices along all of its connections are positive. It is f.a.s. as soon as one of them is greater than $-\infty$.

Finally, we introduce the notions of thin and thick cusps, which we use in our proofs in section 4. For $\alpha > 1$ we define the following subset of \mathbb{R}^2 :

$$V(a_1, a_2, \alpha) := \left\{ (x_1, x_2) \in \mathbb{R}^2 : |a_1 x_1 + a_2 x_2| < \max(|x_1|, |x_2|)^{\alpha} \right\}$$

Definition 6 We say that $U \subset \mathbb{R}^2$ is a thin cusp, if

- $\ell(B_{\delta} \cap U) > 0$ for all $\delta > 0$;
- there exist $a_1, a_2, \alpha > 1$ and $\delta > 0$ such that $U \cap B_{\delta} \subset V(a_1, a_2, \alpha)$.

Definition 7 We say that $U \subset \mathbb{R}^2$ is a thick cusp, if its complement in \mathbb{R}^2 is a union of a finite number of thin cusps.

- **Remark 1** 1. Let U_1, U_2 be thin cusps. Then, generically, for sufficiently small $\delta > 0$, we have $U_1 \cap U_2 \cap B_{\delta} = \emptyset$.
 - 2. Let U_1 be a thin cusp and U_2 be a thick cusp. Then, generically, for sufficiently small $\delta > 0$, we have $U_1 \cap B_{\delta} \subset U_2$.

2.2 Quaternions and subgroups of O(4)

In this section we briefly describe the presentation of finite subgroups of O(4) with quaternions, for more on this topic see [7, 8]. A real quaternion is a set of four real numbers, $\mathbf{q} = (q_1, q_2, q_3, q_4)$. Multiplication of quaternions is defined as

$$\mathbf{qw} = (q_1w_1 - q_2w_2 - q_3w_3 - q_4w_4, q_1w_2 + q_2w_1 + q_3w_4 - q_4w_3, q_1w_3 - q_2w_4 + q_3w_1 + q_4w_2, q_1w_4 + q_2w_3 - q_3w_2 + q_4w_1).$$
(2)

The conjugate of **q** is defined as $\tilde{\mathbf{q}} = (q_1, -q_2, -q_3, -q_4)$. For a unit quaternion we have $\tilde{\mathbf{q}} = \mathbf{q}^{-1}$. We denote by \mathcal{Q} the multiplicative group of unit quaternions; obviously, its identity element is (1, 0, 0, 0).

Due to the existence of a 2-to-1 homomorphism of \mathcal{Q} onto SO(3), finite subgroups of \mathcal{Q} are labelled after the respective subgroups of SO(3). They are:

$$\mathbb{Z}_{n} = \bigoplus_{r=0}^{n-1} (\cos 2r\pi/n, 0, 0, \sin 2r\pi/n)
\mathbb{D}_{n} = \mathbb{Z}_{2n} \oplus \bigoplus_{r=0}^{2n-1} (0, \cos r\pi/n, \sin r\pi/n, 0)
\mathbb{V} = ((\pm 1, 0, 0, 0))
\mathbb{T} = \mathbb{V} \oplus (\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2})
\mathbb{O} = \mathbb{T} \oplus \sqrt{\frac{1}{2}} ((\pm 1, \pm 1, 0, 0))
\mathbb{I} = \mathbb{T} \oplus \frac{1}{2} ((\pm \tau, \pm 1, \pm \tau^{-1}, 0)),$$
(3)

where $\tau = (\sqrt{5}+1)/2$. Double parenthesis denote all even permutations of quantities within the parenthesis. Any other finite subgroup of Q is conjugate to one of these under an inner automorphism of Q.

For (q_1, q_2, q_3, q_4) regarded as Euclidean coordinates of a point in \mathbb{R}^4 , a pair of unit quaternions $(\mathbf{l}; \mathbf{r})$ defines the transformation $\mathbf{q} \to \mathbf{lqr}^{-1}$, which is a rotation in \mathbb{R}^4 , i.e. an element of the group SO(4). The mapping $\Phi : \mathcal{Q} \times \mathcal{Q} \to \text{SO}(4)$ that relates the pair $(\mathbf{l}; \mathbf{r})$ with the rotation $\mathbf{q} \to \mathbf{lqr}^{-1}$ is a 2-to-1 homomorphism, the kernel of which consists of (1; 1) and (-1; -1).

Therefore, a finite subgroup of SO(4) is a subgroup of a product of two finite subgroups of \mathcal{Q} . Denote by \mathbf{L} and \mathbf{R} the finite subgroups of \mathcal{Q} comprised of \mathbf{l}_j and \mathbf{r}_j , $1 \leq j \leq J_{\mathbf{l}}, J_{\mathbf{r}}$, respectively. To any element $\mathbf{l} \in \mathbf{L}$ there are several corresponding elements \mathbf{r}_i , such that $(\mathbf{l}; \mathbf{r}_i) \in \mathcal{Q}$, and similarly for any $\mathbf{r} \in \mathbf{R}$. This establishes a correspondence between \mathbf{L} and \mathbf{R} . Following [8], we denote by \mathbf{L}_K and \mathbf{R}_K the subgroups of \mathbf{L} and \mathbf{R} corresponding to the unit elements in \mathbf{R} and \mathbf{L} , respectively, and write $(\mathbf{L} \mid \mathbf{L}_K; \mathbf{R} \mid \mathbf{R}_K)$ for the group Γ . The isomorphism between \mathbf{L}/\mathbf{L}_K and \mathbf{R}/\mathbf{R}_K may not be unique and different isomorphisms may give rise to different subgroups of SO(4). If this is the case, such subgroups are indicated by additional subscripts or superscripts.

A reflection in \mathbb{R}^4 can be expressed in the quaternionic presentation as $\mathbf{q} \to \mathbf{a} \tilde{\mathbf{q}} \mathbf{b}$, where **a** and **b** are unit quaternions. We write this reflection as $(\mathbf{a}; \mathbf{b})^*$. A group $\Gamma^* \subset O(4)$,

 $\Gamma^* \not\subset \mathrm{SO}(4)$, can be decomposed as

 $\Gamma^* = \Gamma \oplus \sigma \Gamma$, where $\Gamma \subset SO(4)$ and $\sigma = (\mathbf{a}; \mathbf{b})^* \notin SO(4)$.

3 Graphs

3.1 Groups and graphs

With a given $\Gamma \subset O(n)$ we associate a graph by the following rules:

- (A) A group orbit of isotropy semiaxes is drawn by a point.
- (B) A group orbit of isotropy planes is drawn by a line.
- (C) If for some representatives of the group orbits, an isotropy semiaxis belongs to an isotropy plane, then the respective point is drawn on the respective line.

Definition 8 We say that an isotropy semiaxis of $\Gamma \subset O(n)$ is simple if its isotropy group decomposes \mathbb{R}^n into one-dimensional isotypic components. We call the graph associated with $\Gamma \subset O(n)$ simple if all isotropy semiaxes of Γ are simple.

In this subsection we identify all simple graphs associated with finite subgroups of O(4).

An isotropy semiaxis is a connected component of $L \setminus \{0\}$, where L is an isotropy axis, i.e. $\tilde{L} = \operatorname{Fix} \Delta$ for some $\Delta \subset \Gamma$. There are two possibilities for the normalizer $N_{\Gamma}(\Delta)$: either $N_{\Gamma}(\Delta) = \Delta$ or $N_{\Gamma}(\Delta) = \Delta \times \mathbb{Z}_2$. In the latter case the two semiaxes comprising $\tilde{L} \setminus \{0\}$ belong to the same group orbit, while in the former case they do not. We use the notion of a semiaxis instead of an axis because in the case $N_{\Gamma}(\Delta) = \Delta$ an isotropy axis can contain two distinct (i.e., not related by a symmetry) steady states, one on each isotropy semiaxis.

The following lemma proves properties of finite groups with simple graphs which will be used further on to identify graphs associated with subgroups of O(4).

Lemma 1 Consider a finite $\Gamma \subset O(n)$, such that

- (a) the graph associated with Γ is simple;
- (b) Γ has at least one isotropy semiaxis;
- (c) any isotropy semiaxis of Γ belongs to an intersection of at least two isotropy planes.

Then

- (i) Any isotropy plane contains isotropy semiaxes belonging to at most two group orbits.
- (ii) Γ admits simple heteroclinic cycles.
- (iii) If in addition any isotropy semiaxis is an intersection of at least three isotropy planes and any isotropy plane contains isotropy semiaxes belonging to two group orbits, then the group admits simple heteroclinic networks.

(iv) Consider a connected component of the graph associated with Γ . If $N_{\Gamma}(\Delta_0) = \Delta_0$ for one isotropy semiaxis of this component, then $N_{\Gamma}(\Delta_j) = \Delta_j$ for all isotropy semiaxes, which belong to this component. The component involves at most two group orbits of semiaxes and each isotropy plane contains representatives of both orbits.

Proof: (i) Let P_j be an isotropy plane of Γ . Denote by Σ_j its isotropy subgroup and by $N_{\Gamma}(\Sigma_j)$ the normalizer of Σ_j in Γ . The plane P_j is invariant under $N_{\Gamma}(\Sigma_j)$ where $N_{\Gamma}(\Sigma_j)/\Sigma_j \cong \mathbb{D}_{K_j}$ for some $K_j \ge 0$ (we assume $\mathbb{D}_1 \equiv \mathbb{Z}_2$ and $\mathbb{D}_0 \equiv I$). The definition of simple cycles and condition (c) imply that any isotropy semiaxis that belongs to P_j is an isotropy semiaxis of $N_{\Gamma}(\Sigma_j)/\Sigma_j$. The group \mathbb{D}_{K_j} splits semiaxes in P_j into two group orbits. If isotropy semiaxes from different \mathbb{D}_{K_j} -group orbits are related by some $\gamma \in \Gamma$ such that $\gamma \notin N_{\Gamma}(\Sigma_j)$, then P_j contains semiaxes from just one group orbit.

(ii) To begin with, we prove that the group Γ has two sequences of isotropy semiaxes L_j and isotropy planes P_j , $1 \le j \le m$, such that:

• $L_i \neq \gamma L_j$ and $P_i \neq \gamma P_j$ for any $i \neq j, 1 \leq i, j \leq m$ and $\gamma \in \Gamma$.

•
$$L_j = P_{j-1} \cap P_j$$
 for $j = 1, ..., m$, and $L_1 = P_m \cap \gamma P_1$ for some $\gamma \in \Gamma$.

(We do not exclude m = 1, in which case the constructed cycle is homoclinic.)

The existence of such sequences follows from conditions (a) and (b): we take L'_1 to be any isotropy semiaxis of Γ and P'_1 a plane that L'_1 belongs to. We denote by P'_2 another isotropy plane that contains L'_1 and by L'_2 the other isotropy axis in P'_2 . Repeating this procedure, we finally obtain a sequence L'_k, \ldots, L'_{k+m+2} , such that $L'_k = \gamma L'_{k+m+2}$ for some $\gamma \in \Gamma$, but $L'_i \neq \sigma L'_j$ for any $\sigma \in \Gamma$ and $i \neq j$, $k \leq i, j \leq k+m+1$. Denoting $L_j = L'_{k+j-1}$ and $P_j = P'_{k+j-1}$ we obtain the desired sequences. The proof of the existence of a Γ -equivariant system with $\xi_j \in L_j$ and $\kappa_j \subset P_j$ follows the same ideas as the proof of lemma 1 in [2] and is omitted.

(iii) Decompose the set of isotropy semiaxes into disjoint sets $\mathcal{L}_j = \{\Gamma L_{j1}, \ldots, \Gamma L_{js_j}\},\$ where for any L_{jk} and L_{jl} there exist sequences $(\{P_1, \ldots, P_m\}; \{L_1, \ldots, L_m\})$ as constructed in part (ii), where $L_{jk} = L_i$ and $L_{jl} = L_{i'}$ for some $1 \le i, i' \le m$. Here ΓL_{jk} is the group orbit of the semiaxis L_{jk} . The set \mathcal{L}_j can be comprised of just one group orbit. We assume that the sets \mathcal{L}_j are maximal, namely that for any L_{jk} and $L_{j'l}$, where $j \neq j'$, such sequences do not exist. For a set \mathcal{L}_j denote by $\mathcal{P}_j = \{\Gamma P_{j1}, \ldots, \Gamma P_{jt_i}\}$ the set of isotropy planes, such that P_{ij} contains isotropy semiaxes from two group orbits, one of which belongs to \mathcal{L}_j and the other one does not. The assumption that the sets \mathcal{L}_i are maximal implies the existence of at least one \mathcal{L}_i such that $t_i \leq 1$, let it be \mathcal{L}_1 . By the conditions in part (iii) above, \mathcal{L}_1 has at least two group orbits of isotropy semiaxes and at least one of them is comprised of axes that do not belong to P_{11} . Denote by L_1 an isotropy semiaxis that does not belong to P_{11} . Since L_1 is an intersection of at least three isotropy planes, different from P_{11} , there exist sequences $(\{P_1, ..., P_m\}; \{L_1, ..., L_m\})$ and $(\{P'_1, ..., P'_{m'}\}; \{L_1, ..., L_{m'}\})$ obtained by the same procedure as in part (ii) with $L_j \in \mathcal{L}_1$ and $L'_j \in \mathcal{L}_1$, such that $L_i \neq L_j$ for any $2 \leq i \leq s$ and $2 \leq j \leq s'$, and $L_{s+k} = L'_{s'+k}$ for any $1 \leq k \leq m-s = m'-s'$. Similar to [2] and part (ii), we construct a dynamical system with $\xi_j \in L_j$ and $\xi'_i \in L'_i$ (where $\xi_j = \xi'_i$ for j > s) and $\kappa_j \in P_j$ and $\kappa'_i \in P'_i$ (where $\kappa_j = \kappa'_i$ for j > s).

(iv) If L_1 and L_m belong to the same connected component of the graph, then there exist sequences $(\{P_1, \ldots, P_{m-1}\}; \{L_1, \ldots, L_m\})$ such that $L_j \subset P_j$ and $L_{j+1} \subset P_j$. If $N_{\Gamma}(\Delta_1) = \Delta_1$ then K_j is odd and $-I \notin \mathbb{D}_{K_1}$, which implies that $N_{\Gamma}(\Delta_2) = \Delta_2$. Hence, $-I \notin \mathbb{D}_{K_2}$, which implies that $N_{\Gamma}(\Delta_3) = \Delta_3$. Repeating this procedure m-1 times, we obtain that $N_{\Gamma}(\Delta_m) = \Delta_m$.

Since all K_j are odd, any P_j in the above sequence contains just one isotropy type of axis, \tilde{L}_j , and two connected components of $\tilde{L}_j \setminus \{0\}$ are not related by symmetries of Γ . Hence, all \tilde{L}_j , $1 \leq j \leq m$, are of the same isotropy type and the connected components of $\tilde{L}_j \setminus \{0\}$ for all $1 \leq j \leq m$ belong to two distinct group orbits. **QED**

Theorem 1 Let Γ be a finite subgroup of SO(4), such that Γ has at least one isotropy axis and all its isotropy axes are simple. Then the graph associated with Γ is non-empty and simple and it is one of those shown in figure 1. The subgroups of SO(4) with associated simple graphs and the types of these graphs are listed in (4) below.

Group	Graph
$(\mathbb{D}_{2K_1} \mathbb{D}_{2K_1}; \mathbb{D}_{2K_2} \mathbb{D}_{2K_2}), K_1 \wedge K_2 = 1, K_1 + K_2 \text{ even}$	V
$(\mathbb{D}_{2K_1} \mathbb{D}_{2K_1}; \mathbb{D}_{2K_2} \mathbb{D}_{2K_2}), K_1 \wedge K_2 = 1, K_1 + K_2 \ odd$	VI
$(\mathbb{D}_{2K_1r} \mathbb{Z}_{4K_1}; \mathbb{D}_{2K_2r} \mathbb{Z}_{4K_2})_s, K_1, K_2, r, s \text{ satisfy } (5)$	IV
$(\mathbb{D}_{2K_1r} \mathbb{Z}_{2K_1}; \mathbb{D}_{2K_2r} \mathbb{Z}_{2K_2})_s, K_1, K_2 \text{ odd, } K_1, K_2, r, s \text{ satisfy } (6)$	III
$(\mathbb{D}_{2K_1} \mid \mathbb{D}_{K_1}; \mathbb{D}_{2K_2} \mid \mathbb{D}_{K_2}), K_1 \wedge K_2 = 1$	IV
$(\mathbb{D}_{2K_1} \mathbb{D}_{K_1}; \mathbb{D}_{2K_2} \mathbb{Z}_{4K_2}), K_1 \text{ even, } K_1/2 \wedge K_2 = 1$	II
$(\mathbb{D}_{2K_1} \mathbb{D}_{K_1}; \mathbb{D}_{2K_2} \mathbb{Z}_{4K_2}), K_1 \text{ odd}, K_1 \wedge K_2 = 1$	III
$(\mathbb{D}_{2K} \mathbb{D}_{2K}; \mathbb{T} \mathbb{T}), K even$	Ι
$(\mathbb{D}_{2K} \mid \mathbb{D}_{2K}; \mathbb{T} \mid \mathbb{T}), K odd$	II
$(\mathbb{D}_{2K} \mathbb{D}_{2K}; \mathbb{O} \mathbb{O}), K \text{ odd}, K \neq 3k$	III
$(\mathbb{D}_{2K} \mathbb{Z}_{4K}; \mathbb{O} \mathbb{T}), K \neq 3k, K even$	Ι
$(\mathbb{D}_{2K} \mathbb{Z}_{4K}; \mathbb{O} \mathbb{T}), K \neq 3k, K \text{ odd}$	II
$(\mathbb{D}_{2K} \mathbb{D}_K; \mathbb{O} \mathbb{T}), \ K \neq 2(2k+1), \ K \neq 3k, \ K \ even$	Ι
$(\mathbb{D}_{2K} \mid \mathbb{D}_K; \mathbb{O} \mid \mathbb{T}), \ K \neq 2(2k+1), \ K \neq 3k, \ K \ odd$	II
$(\mathbb{D}_{2K} \mathbb{D}_{2K}; \mathbb{I} \mathbb{I}), \ K \neq 5k, \ K \ even$	Ι
$(\mathbb{D}_{2K} \mid \mathbb{D}_{2K}; \mathbb{I} \mid \mathbb{I}), \ K \neq 5k, \ K \ odd$	II
$(\mathbb{D}_{2K_1r} \mathbb{Z}_{K_1}; \mathbb{D}_{2K_2r} \mathbb{Z}_{K_2})_s, K_1, K_2 \text{ odd}, K_1, K_2, r, s \text{ satisfy } (7)$	II

(4)

The conditions mentioned are

$$K_1 \wedge K_2 = 1, \quad r \wedge K_2 - sK_1 = 1,$$
(5)

$$K_1 \wedge K_2 = 1, \quad r \wedge (K_2 \pm sK_1)/2 = 1,$$
(6)

$$K_1 \wedge K_2 = 1, \quad r \wedge (K_2 \pm sK_1)/2 = 1, \quad r \wedge (K_2 \pm sK_1)/4 = 1,$$
(7)

where plus or minus are taken so that the ratios are integer and $K_1 \wedge K_2$ denotes the greatest common divisor of K_1 and K_2 .



Figure 1: Simple graphs associated with subgroups of O(4).

Proof: The graph associated with Γ is simple and non-empty, because all isotropy axes are simple and it has at least one isotropy axis. Since any plane has isotropy semiaxes belonging to at most two group orbits, we draw the points representing the semiaxes as endpoints of the line representing the plane. So, instead of points we can use the term *vertices*. A line that corresponds to an isotropy plane either connects two vertices, or it begins and ends at the same vertex. Since Γ is a subgroup of SO(4), a simple isotropy axis belongs to an intersection of three isotropy planes. Hence, the number of isotropy types of semiaxes is 2/3 of the number of isotropy types of planes and the number of isotropy semiaxes is even. Our proof uses results of [17], where simple heteroclinic cycles in \mathbb{R}^4 were studied. The table in appendix C *ibid* lists conjugacy classes of isotropy subgroups of finite groups $\Gamma \subset SO(4)$ satisfying dim Fix $(\Sigma) = 2$ and dim Fix $(\Delta) = 1$. Such groups Σ always satisfy $\Sigma \cong \mathbb{Z}_2$. In the table only selected Δ such that $\Delta \cong (\mathbb{Z}_2)^2$ are listed.

Since $P = \text{Fix}(\Sigma)$ implies that $\gamma P = \text{Fix}(\gamma \Sigma \gamma^{-1})$, the number of conjugacy classes of Σ equals the number of group orbits of isotropy planes. From the table we see that the number of group orbits of intersecting isotropy planes for various $\Gamma \subset \text{SO}(4)$ can be 3, 6 or 9. Hence the number of group orbits of semiaxes is 2, 4 or 6. (If an isotropy plane does not intersect with another isotropy plane, then it does not contain a heteroclinic connection and, therefore, can be ignored.) If the number of group orbits of semiaxes is 2, an isotropy plane can be homoclinic (i.e., all its isotropy semiaxes belong to the same group orbit). In such a case there exists another (not related by a symmetry of Γ) homoclinic plane and the graph is of type I. Another possibility for a graph with two vertices is type II where two vertices are connected by three lines.

For a connected graph with four vertices and without homoclinic isotropy planes (connectedness and absence of homoclinic planes follow from appendix C in [17]) there are two possibilities: a graph without two-vertices connected subgraphs (type III) and the one that involves such subgraphs (type IV). There can possibly exist several types of graphs with six vertices, however, as we conclude from table C, the only groups that have six isotropy types of axes are $(\mathbb{D}_{2K_1} | \mathbb{D}_{2K_1}; \mathbb{D}_{2K_2} | \mathbb{D}_{2K_2})$. Explicitly constructing graphs associated with these groups using data from tables in appendices B and C in [17], we obtain that the graph is of type V when $K_1 + K_2$ is even and of type VI when it is odd.

$\left(\mathbb{D}_{2K_1} \mathbb{D}_{2K_1}; \mathbb{D}_{2K_2} \mathbb{D}_{2K_2} ight)$	$(\mathbb{D}_{2K} \mid \mathbb{Z}_{4K}; \mathbb{O} \mid \mathbb{T}), \ K \neq 3k$
$\left[\left(\mathbb{D}_{2K_{1}r} \mathbb{Z}_{4K_{1}}; \mathbb{D}_{2K_{2}r} \mathbb{Z}_{4K_{2}} \right)_{s}, K_{1}, K_{2}, r, s \text{ satisfy } (5) \right]$	$(\mathbb{D}_{2K} \mathbb{D}_K; \mathbb{O} \mathbb{T}), \ K \neq 2(2k+1)$
$(\mathbb{D}_{2K_1r} \mathbb{Z}_{2K_1}; \mathbb{D}_{2K_2r} \mathbb{Z}_{2K_2})_s, \ K_1 \wedge K_2 = 1,$	$(\mathbb{D}_{2K} \mid \mathbb{D}_{2K}; \mathbb{I} \mid \mathbb{I}), \ K \neq 5k$
K_1, K_2, r, s satisfy (6)	$(\mathbb{T} \mathbb{Z}_2; \mathbb{T} \mathbb{Z}_2)$
$\left(\mathbb{D}_{2K_1} \mathbb{D}_{K_1};\mathbb{D}_{2K_2} \mathbb{D}_{K_2} ight)$	$(\mathbb{T} \mid \mathbb{T}; \mathbb{O} \mid \mathbb{O})$
$(\mathbb{D}_{2K_1} \mathbb{D}_{K_1}; \mathbb{D}_{2K_2} \mathbb{Z}_{4K_2}), K_1 \text{ even}, K_1/2 \wedge K_2 = 1$	$(\mathbb{O} \mid \mathbb{O}; \mathbb{I} \mid \mathbb{I})$
$(\mathbb{D}_{2K_1} \mathbb{D}_{K_1}; \mathbb{D}_{2K_2} \mathbb{Z}_{4K_2}), K_1 \text{ odd}$	$(\mathbb{D}_{2K_1r} \mathbb{Z}_{K_1}; \mathbb{D}_{2K_2r} \mathbb{Z}_{K_2})_s, K_1, K_2 \text{ odd},$
$ig(\mathbb{D}_{2K} \mathbb{D}_{2K};\mathbb{T} \mathbb{T}ig)$	K_1, K_2, r, s satisfy (7)
$(\mathbb{D}_{2K} \mathbb{D}_{2K}; \overline{\mathbb{O}} \mathbb{O}), K \text{ odd}$	

Subgroups of SO(4) admitting simple heteroclinic cycles are listed in theorem 2 in [17]. For consistency of presentation we reproduce this list below:

In this list some subgroups (e.g., $(\mathbb{T} | \mathbb{T}; \mathbb{O} | \mathbb{O})$ or $(\mathbb{D}_{2K_1} | \mathbb{D}_{2K_2} | \mathbb{D}_{2K_2})$ where K_1 and K_2 are not co-prime) have non-simple isotropy axes. For $\Gamma \subset SO(4)$ any non-simple axis is an intersection of two planes, that are not orthogonal. The existence of isotropy planes, intersecting non-orthogonally, can be identified using appendices B and C and lemma 3

in [17]. Excluding from the above table subgroups that have isotropy planes with nonorthogonal intersections we obtain the list (4) of subgroups that have associated simple graphs.

We use the data from appendices B-D in [17] to identify the graph for each group. The groups with three group orbits of isotropy planes we split into types I and II depending on whether the group admits homoclinic cycles or not. The groups with six group orbits of isotropy planes split into types of III and IV, depending on whether they have two-vertices subgraphs or not. For the groups $(\mathbb{D}_{2K_1} | \mathbb{D}_{2K_1}; \mathbb{D}_{2K_2} | \mathbb{D}_{2K_2})$, the only ones with nine isotropy types of planes, we construct graphs from the data in the tables, as stated in the beginning of the proof of this lemma. **QED**

Theorem 2 The graph associated with a group $\Gamma^* \subset O(4)$,

$$\Gamma^* = \Gamma \oplus \sigma \Gamma$$
, where $\Gamma \subset SO(4)$ and $\sigma \notin SO(4)$,

is non-empty and simple if and only if Γ and σ are listed in (8).

Γ	σ	
$(\mathbb{D}_2 \mathbb{Z}_2; \mathbb{D}_2 \mathbb{Z}_2)$	$((0, 1, 0, 0), (0, 1, 0, 0))^*$	(8)
$(\mathbb{D}_2 \mathbb{Z}_1; \mathbb{D}_2 \mathbb{Z}_1)$	$((1,0,0,0),(1,0,0,0))^*$	

Proof: If the graph associated with a group Γ^* is non-empty and simple, then its subgroup Γ admits simple heteroclinic cycles. According to theorem 3 in [17], such Γ is one of the following:

$$(\mathbb{D}_{4} | \mathbb{Z}_{2}; \mathbb{D}_{4} | \mathbb{Z}_{2}), \ (\mathbb{D}_{4} | \mathbb{Z}_{1}; \mathbb{D}_{4} | \mathbb{Z}_{1})_{3}, \ (\mathbb{D}_{2} | \mathbb{Z}_{2}; \mathbb{D}_{2} | \mathbb{Z}_{2}), (\mathbb{T} | \mathbb{Z}_{2}; \mathbb{T} | \mathbb{Z}_{2}), \ (\mathbb{D}_{2} | \mathbb{Z}_{1}; \mathbb{D}_{2} | \mathbb{Z}_{1}), \ (\mathbb{D}_{2K} | \mathbb{D}_{K}; \mathbb{D}_{2K} | \mathbb{D}_{K}).$$

In the above list, the only groups where all isotropy axes are simple are the two given in (8). The respective elements σ are also given in [17]. The graphs, associated with Γ^* , are the same as the ones associated with Γ , namely type III for $(\mathbb{D}_2 \mid \mathbb{Z}_2; \mathbb{D}_2 \mid \mathbb{Z}_2)$ and type II for $(\mathbb{D}_2 \mid \mathbb{Z}_1; \mathbb{D}_2 \mid \mathbb{Z}_1)$. **QED**

Note that not every simple heteroclinic cycle is admitted by a group Γ that is associated with a simple graph. This is because Γ may admit simple heteroclinic cycles, but have additional non-simple isotropy axes. However, it can be deduced from tables B-D in [17] that all such groups admit at most one simple heteroclinic cycle, and thus no simple heteroclinic network. Therefore, our method allows us to find all simple heteroclinic networks in \mathbb{R}^4 .

3.2 Graphs and networks

In this subsection we address the following question: For a given $\Gamma \subset O(4)$, what kinds of heteroclinic networks can a Γ -equivariant dynamical systems possess? For the group $\Gamma^* = \Gamma \oplus \sigma \Gamma \cong \mathbb{Z}_2^4$, with $\Gamma = (\mathbb{D}_2 | \mathbb{Z}_2; \mathbb{D}_2 | \mathbb{Z}_2)$ and the corresponding σ from theorem 2, this question was addressed in [5], where it was shown that the group supports three different types of networks associated with type III: a network of two cycles with three equilibria each (and one common connection), a network of one cycle with three and one with four equilibria (and two common connections), and a network of two cycles with three and one with four equilibria. In the latter network every available isotropy plane contains a heteroclinic connection, while in the former two there is always a plane that is not required for the construction.

Heteroclinic cycles and networks can be represented by diagrams similar to the graphs in figure 1. In these diagrams a steady state is drawn as a point, a heteroclinic connection a a line and arrows indicate the directions of the heteroclinic connections. Although our graphs are formally different from the diagrams (a diagram is drawn for a dynamical system that possesses a heteroclinic cycle or network, while a graph is drawn for a subgroup of O(n)), the graphs can be related to the heteroclinic diagrams using arguments similar to the ones employed to prove lemma 1 in [2]. Namely, we can explicitly construct a Γ -equivariant dynamical system such that each isotropy semiaxis contains a steady state and each isotropy plane contains a heteroclinic connection, which can go in either direction. Hence, we can relate heteroclinic cycles to the graphs shown in figure 1. Note that the geometric structure of the graph, associated with a group, determines the geometric structure of the networks admitted by this group. In this subsection we consider maximal networks in the sense that C1:

- any isotropy semiaxis contains a steady state,
- any isotropy plane contains a heteroclinic connection,
- all such steady states and connections belong to the network.

The group Γ can admit non-maximal networks, see e.g., [3] or [5], where the graph of type III was considered, however all other networks can be constructed by removing one or more connections from a maximal network.

Theorem 3 Suppose that a Γ -equivariant system, where Γ is a finite subgroup of SO(4), possesses a heteroclinic network that is simple and maximal. Then

- (i) If Γ has exactly two group orbits of isotropy semiaxes, then the network is of type II in figure 2.
- (ii) If the graph associated with Γ is of type III, then the network is of type III in figure 2.
- (iii) If the graph associated with Γ is of type IV, then the network is one of types IVa-IVc in figure 3.

- (iv) If the graph associated with Γ is of type V, then the network is one of types Va-Vc in figure 4.
- (v) If the graph associated with Γ is of type VI, then the network is one of types VIa-VIg in figure 5.

Proof: (i) If Γ has two group orbits of isotropy semiaxes, then the associated graph is of type I or II. The groups with associated graphs of type I admit two distinct (i.e., not related by a symmetry of Γ) homoclinic cycles, however they do not admit heteroclinic networks. The graph of type II involves two group orbits of semiaxes, and three group orbits of planes, hence, the heteroclinic networks admitted by the respective groups involve two (group orbits of) equilibria, ξ_1 and ξ_2 (one for each of isotropy semiaxes), and three (group orbits of) heteroclinic connections (one for each of isotropy planes), say, two from ξ_1 to ξ_2 and one from ξ_2 to ξ_1 .

(ii) If the graph associated with Γ is of type III, then the network is a union of graphs involving three or four equilibria. Evidently, it has at least one cycle with three equilibria, let it be $\xi_1 \to \xi_2 \to \xi_3 \to \xi'_1$. There are four possibilities for the remaining connections:

(a) $\xi_3 \to \xi_4, \ \xi'_4 \to \xi_1, \ \xi''_4 \to \xi_2;$ (b) $\xi_3 \to \xi_4, \ \xi'_4 \to \xi_1, \ \xi_2 \to \xi''_4;$ (c) $\xi_3 \to \xi_4, \ \xi_1 \to \xi'_4, \ \xi''_4 \to \xi_2;$ (d) $\xi_3 \to \xi_4, \ \xi'_4 \to \xi_1, \ \xi_2 \to \xi''_4.$

The network (b) is mapped to the network (a) by the permutation of equilibria $(\xi_1, \xi_2, \xi_3, \xi_4) \mapsto (\xi_4, \xi_1, \xi_2, \xi_3)$; the network (c) is mapped to the network (a) by the permutation of equilibria $(\xi_1, \xi_2, \xi_3, \xi_4) \mapsto (\xi_4, \xi_2, \xi_3, \xi_1)$; the network (d) is mapped to the network (a) by the permutation of equilibria $(\xi_1, \xi_2, \xi_3, \xi_4) \mapsto (\xi_3, \xi_1, \xi_2, \xi_4)$. Hence, the only possibility is the network (a) that is shown in figure 2.

(iii) If the graph associated with Γ is of type IV, then the network can involve zero, one or two cycles that are comprised of two equilibria. Hence, there are three different heteroclinic networks, shown in figure 3.

(iv) If the graph is of type V, the network either does or does not involve a cycle with six equilibria. In case there is such a cycle, say $\xi_1 \to \xi_2 \to \ldots \to \xi_6 \to \xi'_1$, there are two possibilities (up to cyclic permutation of equilibria) for the remaining connections: the equilibria unstable in the transverse directions are (ξ_1, ξ_2, ξ_3) or (ξ_1, ξ_3, ξ_5) . If the network does not involve a cycle with six equilibria, there is only one such network up to a permutation of ξ_j . The proof is similar to that of part (ii) and we omit it. The latter network can be thought of as a union of four cycles, each connecting four equilibria. These networks are shown in figure 4.

(v) The graph of type VI can be considered as a union of two subgraphs (upper and lower) with three vertices each, and three vertical lines connecting these subgraphs. Let the equilibria of the upper subgraph be (ξ_1, ξ_2, ξ_3) and those of the lower (ξ_4, ξ_5, ξ_6) such that vertical lines connect ξ_j and ξ_{j+3} . According to the definition of a network, two of the vertical connections must go in the same direction and the third one in the opposite. We

assume there are connections $\xi_1 \to \xi_4$, $\xi_2 \to \xi_5$ and $\xi_6 \to \xi_3$. Then there are four possibilities for the connections between ξ_1 , ξ_2 and ξ_3 :

(a)
$$\xi_1 \to \xi_2, \ \xi_2 \to \xi_3, \ \xi_3 \to \xi_1';$$
 (b) $\xi_1 \to \xi_2, \ \xi_3 \to \xi_2, \ \xi_3 \to \xi_1';$
(c) $\xi_2 \to \xi_1, \ \xi_3 \to \xi_2, \ \xi_3 \to \xi_1';$ (d) $\xi_2 \to \xi_1, \ \xi_3 \to \xi_2, \ \xi_1' \to \xi_3.$

Similarly, there are four possibilities for the connections on the lower subgraph. The only permutation of ξ_j that preserves the directions of connections on the vertical lines is $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) \mapsto (\xi_2, \xi_1, \xi_3, \xi_5, \xi_4, \xi_6)$, hence the number of different networks is eight. They are shown in figure 5. QED



Figure 2: Heteroclinic networks of types II and III.



Figure 3: Heteroclinic networks of type IV.

4 Stability of type A heteroclinic networks

In this section we consider an equivariant dynamical system (1) in \mathbb{R}^4 that possesses a simple heteroclinic network of type A. We give necessary and sufficient conditions for its fragmentary and essential asymptotic stability below. In accordance with the terminology in [1] and [10], we use the following definitions.



Figure 4: Heteroclinic networks of type V.

Definition 9 Let Y be a simple heteroclinic network in \mathbb{R}^n and $\xi_j \in Y$ an equilibrium in Y. The eigenvalue of $df(\xi_j)$ with maximal real part is called the principal eigenvalue at ξ_j . The corresponding eigenspace is the principal eigenspace, and the branch of $W^u(\xi_j)$ that is tangent to it is the principal unstable manifold, denoted by $W^{pu}(\xi_j)$. A cross section to $W^{pu}(\xi_j)$ near ξ_j is called principal outgoing cross section. If $W^{pu}(\xi_j)$ connects to another equilibrium in Y, we speak of a principal connection. Finally, a cycle comprised only of principal connections is called a principal cycle.

Definition 10 Let Y be a simple heteroclinic network in \mathbb{R}^n and $\xi_j \in Y$ an equilibrium in Y. The eigenvalue of $df(\xi_j)$ with minimal real part, excluding the radial ones, is called the minus-principal eigenvalue at ξ_j . As in the previous definition we extend this terminology to minus-principal eigenspaces, cross sections and cycles.

Remark 2 With this terminology, recall the following two facts from [15].

- (i) If a type A heteroclinic cycle in \mathbb{R}^4 is f.a.s., then it is e.a.s.
- (ii) If a type A heteroclinic cycle in \mathbb{R}^4 is f.a.s., then it is principal.

In the following, let $Y = (\bigcup_{1 \leq j \leq J} \xi_j) \cup (\bigcup_{(ij) \in Q} \kappa_{ij}) \subset \mathbb{R}^4$ be a simple heteroclinic network, consisting of equilibria ξ_j and connections κ_{ij} . We use the standard notation for cross sections H_{ij}^{in} and H_{jk}^{out} to incoming and outgoing connections κ_{ij} and κ_{jk} near ξ_j . These can be considered as two-dimensional since the radial directions are irrelevant in the study of stability properties. We denote the principal outgoing cross section near ξ_j by H_j^p .

For equilibria $\xi_i, \xi_j, \xi_k \in Y$ with connections $\kappa_{ij}, \kappa_{jk} \subset Y$ we employ the usual notation for local maps $\phi_{ijk} : H_{ij}^{\text{in}} \to H_{jk}^{\text{out}}$, global maps $\psi_{jk} : H_{jk}^{\text{out}} \to H_{jk}^{\text{in}}$ and their compositions $g_{ijk} := \psi_{jk} \circ \phi_{ijk} : H_{ij}^{\text{in}} \to H_{jk}^{\text{in}}$. Note that this is a slight abuse of notation because the domain of ϕ_{ijk} is only a subset of H_{ij}^{in} if there is more than one positive eigenvalue at ξ_j . If there is no ambiguity we drop subscripts and write ϕ_j, ψ_j and g_j or even simply ϕ, ψ and g.

Lemma 2 Consider a local map $\phi : H_{ij}^{\text{in}} \to H_j^p$. Then, generically, there exists a thick cusp $V \subset H_{ij}^{\text{in}}$ such that

(i) $\phi(V)$ is a thin cusp;



Figure 5: Heteroclinic networks of type VI.

(ii) for any $\delta > 0$ we can find $\varepsilon > 0$ such that $\phi(B_{\varepsilon} \cap V) \subset B_{\delta}$.

Proof: The local map is given by $(y_1, y_2) = \phi(x_1, x_2) = (Ax_1^{\alpha}, x_2|x_1|^{\beta})$, where $A \neq 0, \alpha > 0$ and $\beta > -1$, see e.g. [15]. Suppose first that $\alpha < 1 + \beta$. Set

$$V := \{ (x_1, x_2) \in \mathbb{R}^2 : |x_2| < |x_1|^h \text{ and } |x_2| < |x_1|^{-\beta+s} \},\$$

where $h < 1, s > 0, \alpha < h + \beta$ and $-\beta + s < 1$. Then V is a thick cusp and for $(y_1, y_2) \in \phi(V)$ we obtain

$$|y_2| < |x_1|^h |x_1|^\beta = A' |y_1|^{\frac{h+\beta}{\alpha}}$$

for suitable A' > 0, which shows that $\phi(V)$ is a thin cusp.

Now suppose that $\alpha > 1 + \beta$. In this case we set

$$V := \{ (x_1, x_2) \in \mathbb{R}^2 : |x_2| > |x_1|^h \text{ and } |x_2| < |x_1|^{-\beta+s} \},\$$

where $h > 1, s > 0, \alpha > h + \beta$ and $-\beta + s < 1$. Again V is a thick cusp and for $(y_1, y_2) \in \phi(V)$ we obtain

$$|y_2| > |x_1|^h |x_1|^\beta = A' |y_1|^{\frac{h+\beta}{\alpha}}$$

for suitable A' > 0, which as before means that $\phi(V)$ is a thin cusp. Hence, (i) is proven. Note that s > 0 is chosen such that V is contained in the domain of ϕ if $\beta < 0$. To prove (ii), we note that for $(x_1, x_2) \in V$ we have $|\phi(x_1, x_2)| < k(\max(|x_1|, |x_2|))^{\min(\alpha, s)}$ with a constant k > 0. QED **Corollary 1** Suppose that a set $U \subset H_{ij}^{\text{in}}$ is a thin cusp and κ_{jk} is a principal connection.

- (i) Then, generically, for any small $\delta > 0$ there exists $\varepsilon > 0$ such that for all points $x \in B_{\varepsilon} \cap U \setminus \{0\}$ the trajectories $\Phi(x,t)$ leave the δ -neighbourhood of ξ_j through $H_j^p = H_{jk}^{\text{out}}$.
- (ii) The set $g_j(B_{\varepsilon} \cap U) \subset H_{jk}^{\text{in}}$ is generically a thin cusp. We also express this as "thin cusps generically follow principal connections".

Proof: (i) Since κ_{jk} is principal, the domain of ϕ_j is a thick cusp in H_{ij}^{in} . Then since U is a thin cusp, by remark 1 the set $U \cap B_{\varepsilon}$ is generically contained in it for sufficiently small $\varepsilon > 0$.

(ii) By lemma 2 and (i) above the set $\phi_j(B_{\varepsilon} \cap U) \subset H_{jk}^{\text{out}}$ is a thin cusp. The global map ψ_j is a generic linear map, so it maps a thin cusp in H_{ik}^{out} into a thin cusp in H_{ik}^{in} . **QED**

Corollary 2 Let X be an f.a.s. type A heteroclinic cycle in \mathbb{R}^4 with equilibria ξ_k , $1 \le k \le K$. Then, generically, for all k and $\delta > 0$ the set $\mathcal{B}_{\delta}(X) \cap H_{k-1,k}^{\text{in}}$ is a thick cusp.

Proof: By lemma 2 there is a thick cusp $V \subset H_{k-1,k}^{\text{in}}$ such that its image $\phi_k(V) \subset H_{k,k+1}^{\text{out}}$ is a thin cusp. Then by corollary 1 (ii) $\phi_k(V)$ generically follows principal connections. Since X is f.a.s., the trajectories passing through V stay near X for all positive times. Therefore, $V \cap B_{\varepsilon} \subset \mathcal{B}_{\delta}(X) \cap H_{k-1,k}^{\text{in}}$ for sufficiently small $\varepsilon > 0$. QED

We are now in a position to prove necessary and sufficient conditions for essential asymptotic stability of Y. Recall that there are many ways in which type Z networks can be e.a.s., see [3, 15]. For type A networks the situation is simpler as the following theorem shows.

Theorem 4 Generically, a type A heteroclinic network Y in \mathbb{R}^4 is e.a.s. if and only if the following holds.

- (i) All principal subcycles of Y are f.a.s.
- (ii) For any equilibrium $\xi_j \in Y$, the network contains a principal connection $\kappa_{jk} \subset Y$.

Proof: Assume first that (i) and (ii) are satisfied. Then by remark 2 every principal subcycle of Y is e.a.s. For any connection κ_{12} there is a unique sequence of equilibria $\xi_3, \ldots, \xi_k \in Y$ with principal connections $\kappa_{j,j+1}$ for $j = 2, \ldots, k - 1$, such that ξ_l belongs to a principal subcycle $X \subset Y$ if and only if l = k. By lemma 2 there is a thick cusp $V \subset H_{12}^{\text{in}}$ such that $\phi_2(V) \subset H_{23}^{\text{out}} \equiv H_2^p$ is a thin cusp. By corollary 1, thin cusps generically follow principal connections, so $g_k \circ \ldots \circ g_2(V) \subset H_k^{\text{out}}$ is also a thin cusp. By remark 1, for sufficiently small $\varepsilon > 0$, the set $g_k \circ \ldots \circ g_2(V) \cap B_{\varepsilon}$ is generically contained in the thick cusp that is $\mathcal{B}_{\delta}(X) \cap H_k^{\text{out}}$ (corollary 2). Applying these arguments to all connections in Y, we obtain that Y is e.a.s.

We now prove the other implication. Let Y be e.a.s, $X = \xi_1 \to \dots \to \xi_K \subset Y$ be a principal cycle and $V \subset H_1^{\text{in}}$ be the set considered in lemma 2. Since V is a thick cusp

and Y is e.a.s., the intersection $\mathcal{B}_{\delta}(Y) \cap V$ has positive measure. By corollary 1 almost all trajectories starting there stay near X for all t > 0. Hence, X is f.a.s., which proves (i). If (ii) is not satisfied, then there exists $\xi_j \in Y$, such that its principal unstable manifold does not belong to the network. Then by lemma 2 there is a thick cusp in H_{ij}^{in} that is mapped to the principal outgoing cross section and thus does not stay near Y, which contradicts Y being e.a.s. QED

We now turn to fragmentary asymptotic stability. Given $\delta > 0$ we denote by \mathfrak{W} the set of positive measure subsets $U \subset \mathcal{B}_{\delta}(Y) \cap H_{ij}^{\text{in}}$. In what follows we drop the superscript "in" to simplify notation. By \mathfrak{g} we define the map acting on subsets $\mathfrak{U} \subset \mathfrak{W}$, $\mathfrak{U} = \{U_1, ..., U_K\}$, as follows:

• If $\mathfrak{U} = \{U\}$, where $U \subset H_{ij}$, and ξ_j has M outgoing connections, $\kappa_{jj_1}, ..., \kappa_{jj_M}$, then $\mathfrak{gU} = \{U_1, ..., U_K\}$ and

$$U_k = \{ x \in H_{jj_m} : \Phi(y,0) \in U, \quad \Phi(y,t_0) = x \text{ for some } t_0 > 0 \\ \text{and } \Phi(y,t) \notin H_{i'j'} \text{ for any } 0 < t < t_0 \text{ and } (i'j') \in Q \}$$

$$(9)$$

Since the sets U_k have positive measures, we have $0 < K \leq M$.

•
$$\mathfrak{g}(\mathfrak{U}_1 \cup \mathfrak{U}_2) = (\mathfrak{gU}_1) \cup (\mathfrak{gU}_2)$$

Lemma 3 Let $U \subset H_{ij} \cap \mathcal{B}_{\delta}(Y)$ be a thin cusp. Then generically for $\mathfrak{U} = \{B_{\varepsilon} \cap U\}$, any l > 0 and sufficiently small $\varepsilon > 0$ we have $\mathfrak{g}^{l}\mathfrak{U} = \{U_{l}\}$, where U_{l} is a thin cusp.

Proof: We start with the case l = 1. The equilibrium ξ_j can have one or two outgoing connections. First, we assume that there is one outgoing connection, $\kappa_{j,j+1}$. The map $g: H_{ij} \to H_{j,j+1}$ that maps $x \in H_{ij}$ to $\Phi(x,t) \in H_{j,j+1}$ is known – see, e.g., [15] and also our comments before lemma 2. It is

$$g(x_1, x_2) = (A_1 x_1^{\alpha} + A_2 x_2 |x_1|^{\beta}, A_3 x_1^{\alpha} + A_4 x_2 |x_1|^{\beta}),$$

where $\alpha > 0$, α and β depend on eigenvalues of $df(\xi_j)$ and generically $\alpha \neq \beta + 1$, $A_j \neq 0$ and $A_1/A_2 \neq A_3/A_4$.

Assume first that $\alpha < \beta + 1$. By definition of a thin cusp, for any c > 0 and small $\varepsilon = \varepsilon(c) > 0$ the set U satisfies $B_{\varepsilon} \cap U \subset W_c$, where

$$W_c = \{ (x_1, x_2) \in \mathbb{R}^2 : |x_1 + \frac{a_2}{a_1} x_2| < c|x_2| \}.$$

We have

$$g(qx_2, x_2) = (A_1 q^{\alpha} x_2^{\alpha} + A_2 q^{\beta} x_2^{1+\beta}, A_3 q^{\alpha} x_2^{\alpha} + A_4 q^{\beta} x_2^{1+\beta})$$

Therefore, for any $0 < s < (1+\beta)/\alpha - 1$ and sufficiently small x_2 , any $(y_1, y_2) \in gW_c \subset H_{j,j+1}$ satisfies

$$|y_1 - \frac{A_1}{A_3}y_2| < |y_1|^{(1+\beta)/\alpha-s},$$

which implies that gW_c is a thin cusp. Hence, $U_1 = g(B_{\varepsilon} \cap U)$ is a thin cusp. The arguments in the case $\alpha > \beta + 1$ are similar.

In the case of two outgoing connections from ξ_j , for small δ the trajectories from U (except for a set of zero measure) follow the principal connection of ξ_j . Hence, $\mathfrak{g}U = \{g(B_{\varepsilon} \cap U)\}$, where $g(B_{\varepsilon} \cap U)$ is a thin cusp as it is shown above and g is the map along the principal connection.

Applying these arguments l times we prove the lemma.

Lemma 4 Suppose that $\xi_j \in X$ has two incoming connections, κ_{j_1j} and κ_{j_2j} . Let $U^1 \subset H_{j_1j}$ and $U^2 \subset H_{j_2j}$ be such that $\ell(B_{\varepsilon} \cap U^k) > 0$ for any $\varepsilon > 0$ and k = 1, 2. Then generically at least one of the sets W^1 and W^2 , where $\{W^k\} = \mathfrak{g}\{U^k\}$, is a thin cusp.

Proof: The maps $g_k : H_{j_k j} \to H_{ji}$ are

$$g_k(x_1, x_2) = (A_{k1}x_1^{\alpha_k} + A_{k2}x_2|x_1|^{\beta_k}, A_{k3}x_1^{\alpha_k} + A_{k4}x_2|x_1|^{\beta_k}),$$

where $\beta_k > 0$, $\beta_1 = \alpha_2$, $\beta_2 = \alpha_1$ and generically $\alpha_1 \neq \beta_1$ and $A_j \neq 0$. Hence, we have either $\alpha_1 < \beta_1$ or $\alpha_2 < \beta_2$. Suppose that $\alpha_1 < \beta_1$. Then for s > 0 such that $\alpha_1 < \beta_1 - s$ and sufficiently small ε the points $(y_1, y_2) \in g_1(U^1 \cap B_{\varepsilon})$ satisfy

$$|y_1 - \frac{A_{11}}{A_{13}}y_2| < |y_2|^{(\beta_1 - s)/\alpha_1}$$

which implies that $W^1 = g_1 U^1$ is a thin cusp.

Lemma 5 Let $X_1, ..., X_L$ be minus-principal cycles that are subsets of Y. (If there are no such cycles, L = 0 is assumed.) Consider a heteroclinic connection $\kappa_{ij} \subset Y$ such that $\kappa_{ij} \not\subset X_l$ for any $1 \leq l \leq L$. Then there exists $\delta > 0$ and M > 0 such that for $\mathfrak{U} = \{U\}$, where $U = \mathcal{B}_{\delta}(Y) \cap H_{ij}$ we have

$$\mathfrak{g}^M\mathfrak{U} = \{U_{M1}, ..., U_{MK}\},\$$

where all U_{Mk} , $1 \leq k \leq K$, are thin cusps.

Proof: Suppose the statement of the theorem does not hold true. We take M to be the number of all connection comprising Y plus one and consider

$$\mathfrak{g}^M\mathfrak{U} = \{U_{M1}, ..., U_{MK}\}.$$

For $\delta \to 0$ the set remains non-empty, hence there exists some U_{Mk} that is not a thin cusp for any small δ . The set U_{Mk} is the image of a subset of U under $g = g_M \dots g_1$, where $g_m : H_{j_{m-1}j_m} \to H_{j_m j_{m+1}}$, with $j_0 = i$ and $j_1 = j$, is the map discussed in lemma 3. Since the number of connections $\kappa_{j_m j_{m+1}}$ involved in the map g is larger than the total number of connections of Y, at least one connection occurs more than once.

QED

QED

Here we have two possibilities: either g overlaps with a cyclic map and we have $g_s = g_{d+s}$ for all d < s < M or at least one of the equilibria involved in g has two incoming connections, say $\kappa_{j_m j_{m+1}}$ and $\kappa_{j_s j_{s+1}}$. In the former case, the connection κ_{ij} belongs to a minus-principal cycle, which contradicts the conditions in the statement of this lemma. In the latter case due to lemmas 3 and 4 the set U_{Mk} is a thin cusp. **QED**

Theorem 5 Suppose that Y is f.a.s. Generically, for sufficiently small $\delta > 0$ almost all $x \in \mathcal{B}_{\delta}(Y)$ satisfy $\omega(x) = X$, where $X \subset Y$ is a principal cycle.

Proof: To prove the theorem, it is sufficient to consider $x \in H_{ij}$. If κ_{ij} does not belong to a minus-principal cycle, by lemma 5 for almost all $x \in \mathcal{B}_{\delta}(Y) \cap H_{ij}$ the trajectory $\Phi(x,t)$ for some t > 0 (depending on x) belongs to one of the sets $\{U_{M1}, ..., U_{MK}\}$ that are thin cusps. Hence, by corollary 1 the trajectory is attracted by a principal cycle as $t \to \infty$.

If κ_{ij} belongs to a minus-principal cycle which is not f.a.s., almost all trajectories $\Phi(x, t)$ with $x \in \mathcal{B}_{\delta}(Y) \cap H_{ij}$ for some t > 0 escape from the δ -neighbourhood of this cycle. They escape along connections that do not belong to other minus-principal cycles (two minusprincipal cycles do not have common equilibria), hence the arguments given above imply that they are attracted by principal cycles.

If a minus-principal cycle is f.a.s. (and thus principal), then the above arguments apply to those trajectories that escape from it. **QED**

Corollary 3 If Y is f.a.s., then its principal cycles that are e.a.s. are Milnor attractors.

Corollary 4 Consider a type A network Y in \mathbb{R}^4 . The network is f.a.s. if and only if it contains at least one subcycle that is f.a.s.

Note that a corresponding result for networks of type Z in \mathbb{R}^4 follows from [4, Theorem 3.4]: two cycles in a network of type Z always have a connection in common, but it is not possible for a trajectory to switch from a neighbourhood of one cycle to the other and back. Therefore, whenever such a network is f.a.s., one of its subcycles must be f.a.s.

5 Numerical examples: attracting heteroclinic cycles in $(\mathbb{D}_2 | \mathbb{Z}_4; \mathbb{D}_2 | \mathbb{Z}_4)$ -equivariant systems

In this section we construct two $(\mathbb{D}_2 | \mathbb{Z}_4; \mathbb{D}_2 | \mathbb{Z}_4)$ -equivariant systems with attracting heteroclinic cycles. According to theorem 1, the graph associated with this group is of type IV. We aim at constructing systems with type IVc networks, where in one system the four-equilibria subcycle is stable, while in the other both two-equilibria subcycles are stable. To achieve this we employ ideas from lemma 1 in [2] that we recall in subsection 5.1. The construction itself is discussed in subsection 5.2.

5.1 Construction of a Γ -equivariant system with a desirable network

In [2] we proved that a group Γ with a set of invariant planes satisfying certain conditions admits pseudo-simple heteroclinic cycles. In the proof we explicitly built a Γ -equivariant dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ possessing the desirable cycle. As noted *ibid*, the proof can be generalised for the construction of systems with other types of cycles and networks. Below we describe such a construction for a given $\Gamma \subset SO(4)$ admitting a simple heteroclinic network. Moreover, by varying constants employed in the construction, we can change the eigenvalues of $df(\xi_j)$, giving us control over the asymptotic stability of the cycles in the network.

We assume that the network under construction is given as a directed graph, comprised of heteroclinic connections $\kappa_j : \xi_j \to \xi'_j$, $1 \leq j \leq m$. By $P_j = \operatorname{Fix} \Sigma_j$ we denote the isotropy plane the connection belongs to, by L_j and L'_j the isotropy semiaxes in P_j (since the network is simple, there are exactly two isotropy types of semiaxes for each P_j), by $N_{\Gamma}(\Sigma_j), N_{\Gamma}(\Sigma_j)/\Sigma_j \cong \mathbb{D}_{K_j}$, the normalizer of Σ_j in Γ . If P_i and P_j intersect, then one of L_i or L'_i coincides with one of L_j or L'_j . The construction is done in three steps:

As a first step, for each P_j we define a two-dimensional vector field \mathbf{h}_j , which in polar coordinates (r, θ) is:

$$\mathbf{h}_{j}(r,\theta) = (r(1-r), \ \sin(K_{j}\theta)(A_{j1} + A_{j2}\cos(K_{j}\theta))).$$
(10)

We choose the angle of L_j to be $\theta = 0$, hence the angles of the nearest L'_j are $\pm \pi/K_j$. We assume that the sign of $A_{j1} + A_{j2}$ is positive and the sign of $-A_{j1} + A_{j2}$ is negative. In [2] we assumed that $A_{j2} = 0$. Here non-vanishing values of A_{j2} are taken, because by varying this parameter we change the eigenvalues of the linearisation df at ξ_j and ξ'_j , hence, change the stability properties of the cycles. For the flow of $(\dot{r}, \dot{\theta}) = \mathbf{h}_j(r, \theta)$ each of the axes $\theta = 2k\pi/K_j$ (L_j) and $\theta = (2k+1)\pi/K_j$ (L'_j) is invariant and has an equilibrium at r = 1which is attracting along the direction of r. Moreover, there are heteroclinic connections between equilibria on neighbouring axes, going from $\xi_j \in L_j$ to $\xi'_j \in L'_j$.

As a second step we extend the vector fields \mathbf{h}_j to $\mathbf{g}_j : \mathbb{R}^4 \to \mathbb{R}^4$ as follows: Denote by π_j and π_j^{\perp} the projections onto the plane P_j and its orthogonal complement in \mathbb{R}^4 , respectively. We set

$$\pi_j \mathbf{g}_j(\mathbf{x}) = \frac{\mathbf{h}_j(\pi_j \mathbf{x})}{1 + B |\pi_j^{\perp} \mathbf{x}|^2}, \quad \pi_j^{\perp} \mathbf{g}_j(\mathbf{x}) = 0,$$
(11)

with a positive constant B, which should be taken sufficiently large.

Finally, we define the vector field $\mathbf{f} : \mathbb{R}^4 \to \mathbb{R}^4$ as

$$\mathbf{f}(\mathbf{x}) = \sum_{j=1}^{m} \sum_{\gamma_{ij} \in \mathcal{G}_j} \gamma_{ij} \mathbf{g}_j(\gamma_{ij}^{-1} \mathbf{x}), \qquad (12)$$

where $\mathcal{G}_j = \Gamma / N_{\Gamma}(\Sigma_j)$.

One can prove then that the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{13}$$

possesses steady states $\xi_j \in L_j$, for all L_j involved in the network, and heteroclinic connections $\xi_j \to \xi'_j \subset P_j$. By construction the system (13) is Γ -equivariant, which implies the invariance of all axes L_j and planes P_j . The eigenvalues of $df(\xi_j)$ and $df(\xi'_j)$ associated with eigenvectors in P_j for large values of B are approximated by $A_{j1} + A_{j2} > 0$ and $-A_{j1} + A_{j2} < 0$, respectively.

5.2 $(\mathbb{D}_2 \,|\, \mathbb{Z}_4; \mathbb{D}_2 \,|\, \mathbb{Z}_4)$ -equivariant systems with attracting heteroclinic cycles

Using the results of the previous subsection, we construct two $(\mathbb{D}_2 | \mathbb{Z}_4; \mathbb{D}_2 | \mathbb{Z}_4)$ -equivariant dynamical systems possessing a type IVc network, such that in the first system the four-equilibria cycle is attracting and in the second both two-equilibria cycles are attracting.

The group $\Gamma = (\mathbb{D}_2 | \mathbb{Z}_4; \mathbb{D}_2 | \mathbb{Z}_4)$ is comprised of the following elements:

$$\begin{aligned} \kappa_1(\pm) &= \pm ((1,0,0,0); (1,0,0,0)) \quad , \quad \kappa_2(\pm) = \pm ((1,0,0,0); (0,0,0,1)), \\ \kappa_3(\pm) &= \pm ((0,0,0,1); (1,0,0,0)) \quad , \quad \kappa_4(\pm) = \pm ((0,0,0,1); (0,0,0,1)), \\ \kappa_5(\pm) &= \pm ((0,1,0,0); (0,1,0,0)) \quad , \quad \kappa_6(\pm) = \pm ((0,0,1,0); (0,0,0,1)), \\ \kappa_7(\pm) &= \pm ((0,1,0,0); (0,0,1,0)) \quad , \quad \kappa_8(\pm) = \pm ((0,0,1,0); (0,1,0,0)). \end{aligned}$$

The group has six isotropy types of subgroups Σ satisfying dim Fix $\Sigma = 2$, the respective fixed-point subspaces are:

Plane	Fixed by	Coordinates	
P_1	$\kappa_4(+)$	$(x_1, 0, 0, x_4)$	
P_2	$\kappa_4(-)$	$(0, x_2, x_3, 0)$	
P_3	$\kappa_5(\pm)$	$(x_1, x_2, 0, 0), (0, 0, x_3, x_4)$	(14)
P_4	$\kappa_6(\pm)$	$(x_1, 0, x_3, 0), (0, x_2, 0, x_4)$	
P_5	$\kappa_7(\pm)$	$(x_1, x_2, x_2, -x_1), (x_1, x_2, -x_2, x_1)$	
P_6	$\kappa_8(\pm)$	$(x_1, x_2, x_2, x_1), (x_1, x_2, -x_2, -x_1).$	

The invariant axes are:

Axis	Intersection of	Coordinates	
L_1	P_1, P_3, P_4	$(x_1, 0, 0, 0), (0, 0, 0, x_4)$	
L_2	P_2, P_3, P_4	$(0, x_2, 0, 0), (0, 0, x_3, 0)$	(15)
L_3	P_1, P_5, P_6	$(x_1, 0, 0, x_1), (x_1, 0, 0, -x_1)$	
L_4	P_2, P_5, P_6	$(0, x_2, x_2, 0), (0, x_2, -x_2, 0)$	

Since $-I \in \Gamma$, two semi-axes of any axis are of the same isotropy type.

We have $K_1 = K_2 = 4$ and $K_j = 2$ for $3 \le j \le 6$, therefore in agreement with (10) set

$$\mathbf{h}_j(r,\theta) = (r(1-r), \sin 4\theta (A_{j1} + A_{j2}\cos 4\theta)) \text{ for } j = 1,2, \mathbf{h}_j(r,\theta) = (r(1-r), \sin 2\theta (A_{j1} + A_{j2}\cos 2\theta)) \text{ for } 3 \le j \le 6$$

For the plots shown in figure 6 the coefficients A_{j1} and A_{j2} are:

(a):
$$A_{11} = A_{21} = 25$$
, $A_{12} = A_{22} = -5$, $A_{j1} = 15$, $A_{j2} = -5$, for $3 \le j \le 6$,

(b):
$$A_{11} = A_{21} = 2$$
, $A_{12} = A_{22} = -1$, $A_{j1} = 25$, $A_{j2} = -5$, for $3 \le j \le 6$,

and in both cases B = 100. Recall that a type A heteroclinic cycle is e.a.s. if

$$\prod_{1 \le j \le J} \frac{|c_j|}{e_j} > 1 \text{ and } e_j > t_j \text{ for all } 1 \le j \le J,$$
(16)

where we use the usual convention of writing $-c_j, e_j$ and t_j for the contracting, expanding and transverse eigenvalues, respectively. In case (a) the eigenvalues of $df(\xi_j)$ for the fourequilibria cycle are:

$$e_1 = e_3 \approx 10, \ c_1 = c_3 \approx -30, \ t_1 = t_3 \approx -20, \ e_2 = e_4 \approx 20, \ c_2 = c_4 \approx -20, \ t_2 = t_4 \approx 10,$$

therefore (16) implies that the cycle is e.a.s. In case (b) for both two-equilibria cycles the eigenvalues are

$$e_1 = e_2 \approx 20, \ c_1 = c_2 \approx -30, \ t_1 = -3, \ t_2 = 1,$$

hence (16) implies that these cycles are e.a.s. For both cases trajectories approaching attracting the heteroclinic cycles are shown on figure 6.

6 Conclusion

We have contributed to the systematic study of heteroclinic dynamics in low dimensions in two ways: (i) we derived a complete list of maximal simple heteroclinic networks in \mathbb{R}^4 ; and (ii) we proved conditions for fragmentary and essential asymptotic stability of type A heteroclinic networks in \mathbb{R}^4 , i.e. admitted by a group $\Gamma \subset SO(4)$. Along the way we introduced the concept of a graph associated with a group, presented a list of simple graphs for subgroups of O(4) and for each graph the subgroups that the graph is associated with.

Possible continuations of this work include a similar study for pseudo-simple heteroclinic networks in \mathbb{R}^4 , extending the work on pseudo-simple cycles in [2]. The employed approach is likely to be applicable for the classification of graphs and eventually heteroclinic cycles/networks associated with groups $\Gamma \subset O(n)$ for n > 4. By contrast, the proofs of the stability results rely on properties of thin and thick cusps, for which the generalisation to higher dimensions is not obvious.

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Figure 6: Projection of the heteroclinic connections in the planes P_1 and P_2 (solid lines), P_3 and P_4 (dashed lines), P_5 and P_6 (dotted lines) and trajectories (bold lines) approaching the four-equilibria cycle (a) and the two-equilibria cycles (b) into the plane $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$, where $\mathbf{v}_1 = (4, 2, 4, 1.5)$ and $\mathbf{v}_2 = (2, 4, -1.5, 4)$. The steady states ξ_1 and ξ_2 are denoted by filled circles and stars, the steady states ξ_3 and ξ_4 by hollow circles and stars, respectively.

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