The gap distance to the set of singular matrix pencils

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Abstract

We study matrix pencils $sE - A$ using the associated linear subspace $\ker[A, -E]$. The distance between subspaces is measured in terms of the gap metric. In particular, we investigate the gap distance of a regular matrix pencil to the set of singular pencils and provide upper and lower bounds for it. A relation to the distance to singularity in the Frobenius norm is provided.

Keywords: distance to singularity, matrix pencils, gap distance

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1 Introduction

We consider square matrix pencils $\mathcal{A}(s) = sE - A$ with $E, A \in \mathbb{C}^{n\times n}$, which are *regular*, i.e., $\det(sE - A)$ is not the zero polynomial. If $\det(sE - A) = 0$ for all $s \in \mathbb{C}$, then the matrix pencil $\mathcal{A}(s)$ is called *singular*. In the numerical treatment of matrix pencils it turns out that regular pencils which are close to a singular one are difficult to handle [16]. It is a hard task to compute canonical forms, because rank decisions seem to be impossible in general.

One way to characterize the *distance to singularity* $\delta(E, A)$ for a regular matrix pencil $sE - A$ is the Frobenius norm of the smallest perturbation which leads to a singular pencil

$$\delta(E, A) := \inf_{\Delta E, \Delta A \in \mathbb{C}^{n\times n}} \left\{ \|[\Delta E, \Delta A]\|_F \mid s(E + \Delta E) - (A + \Delta A) \text{ is singular} \right\}, \tag{1}$$

see [9]. Here $\|M\|_F := \sqrt{\operatorname{tr}(M^*M)}$ is the Frobenius norm of a matrix $M \in \mathbb{C}^{m\times n}$, and M^* is the adjoint of M.

Although in [9] several upper and lower bounds for $\delta(E, A)$ were obtained, they were all claimed to be insufficient. For current purposes we mention only (cf. [3, 9])

$$\frac{\sigma_{\min}(W_n(E, A))}{\sqrt{1 + \cos\left(\frac{\pi}{n+1}\right)}} \le \delta(E, A) \le \min\left\{ \sigma_{\min}\left(\begin{bmatrix} E \\ A \end{bmatrix}\right), \sigma_{\min}([E, A]) \right\}, \tag{2}$$

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where $W_n(E,A)$ is the block matrix

$$W_n(E,A) := \begin{bmatrix} E & 0 & \dots & 0 \\ A & E & & \vdots \\ 0 & \ddots & \ddots & 0 \\ \vdots & & A & E \\ 0 & \dots & 0 & A \end{bmatrix} \in \mathbb{C}^{(n+1)n\times n^2},$$

see also [18], and $\sigma_{\min}(M)$ is the smallest positive singular value of the matrix $M \in \mathbb{C}^{m\times k}$. Recently, in [21], new estimates were obtained in the case that the perturbation $s\Delta E - \Delta A$ in (1) has rank one and the pencil is Hermitian. In [16], the authors proposed a successful gradient based algorithm for finding a nearby singular pencil, however the cost function there is not the distance $\delta(E,A)$ itself.

Following [7], we associate with $\mathcal{A}(s) = sE - A$ the subspace $\mathcal{L}_{\mathcal{A}} = \ker[A, -E]$ of $\mathbb{C}^{2n}$, see also [2, 26]. Note that if E equals the identity, then $\mathcal{L}_{\mathcal{A}}$ coincides with the graph of A. For two pencils $\mathcal{A}(s) = sE - A$ and $\widetilde{\mathcal{A}}(s) = s\widetilde{E} - \widetilde{A}$ we define their *gap distance* as

$$\theta(\mathcal{A}, \widetilde{\mathcal{A}}) = \left\| P_{\mathcal{L}_{\mathcal{A}}} - P_{\mathcal{L}_{\widetilde{\mathcal{A}}}} \right\|,$$

where $P_{\mathcal{L}_{\mathcal{A}}}$ and $P_{\mathcal{L}_{\widetilde{\mathcal{A}}}}$ are the orthogonal projections onto $\mathcal{L}_{\mathcal{A}} = \ker[A, -E]$ and $\mathcal{L}_{\widetilde{\mathcal{A}}} = \ker[\widetilde{A}, -\widetilde{E}]$, respectively, and $\|M\| := \max_{\|x\|=1} \|Mx\|$ denotes the spectral norm. The central notion of the present paper is the *gap distance to singularity* $\theta_{\text{sing}}(E,A)$ of a pencil $\mathcal{A}(s)$, which is defined as the infimum of all $\theta(\mathcal{A}, \widetilde{\mathcal{A}})$ where $\widetilde{\mathcal{A}}(s)$ is a singular matrix pencil.

Let us mention here the basic property of $\theta_{\text{sing}}(E,A)$ that distinguishes it from $\delta(E,A)$. Namely, if the subspaces $\mathcal{L}_{\mathcal{A}}$ and $\mathcal{L}_{\widetilde{\mathcal{A}}}$ coincide (i.e., the pencils $\mathcal{A}(s)$ and $\widetilde{\mathcal{A}}(s)$ *generate the same linear relation*, see [7]) then $\theta_{\text{sing}}(E,A) = \theta_{\text{sing}}(\widetilde{E}, \widetilde{A})$. In other words, the distance $\theta_{\text{sing}}(E,A)$ depends on (the linear relation generated by) the subspace $\mathcal{L}_{\mathcal{A}}$ only. In particular, $\theta_{\text{sing}}(SE, SA) = \theta_{\text{sing}}(E,A)$ for any invertible S, while in contrast, $\delta(\tau E, \tau A) \to \infty$ for $\tau \to \infty$. Observe also that if $\theta(\mathcal{A}, \widetilde{\mathcal{A}}) < \theta_{\text{sing}}(E,A)$, than regularity of *any* matrix pencil generating the same linear relation as $\widetilde{\mathcal{A}}(s)$ is guaranteed. This fact allows to study large norm deviations of the matrices E and A, see Section 7.2.

Another important issue is the asymmetry of $\theta_{\text{sing}}(E,A)$ with respect to the Kronecker canonical form, see Section 3. This fact is particularly interesting when applied to classes with already restricted Kronecker canonical form. Applications to a recently studied class of pencils connected with port-Hamiltonian systems can be found in Section 7.3.

In the present paper, we give several bounds on $\theta_{\text{sing}}(E,A)$. For instance, in Theorem 4.3 we prove that

$$\theta_{\text{sing}}(E,A) \geq \frac{\sigma_{\min}(W_n(E,A))}{\sqrt{2}\,\|[E,A]\|}.$$

In Theorem 5.4 we obtain upper bounds in terms of the geometry of the underlying subspaces. A simplified version says that if $x, y \in \mathbb{C}^n \setminus \{0\}$ are such that $Ax = \lambda Ex$ and $Ay = \mu Ey$ for $\lambda, \mu \in \mathbb{C}$ with $\lambda \neq \mu$, then with

$$z := \begin{pmatrix} x - y \\ \mu y - \lambda y \end{pmatrix}, \quad \mathcal{J} := \operatorname{span}\left\{ \begin{pmatrix} x \\ \lambda x \end{pmatrix}, \begin{pmatrix} y \\ \mu y \end{pmatrix} \right\}$$

and $P_{\mathcal{J}^\perp}$ denoting the orthogonal projection onto the orthogonal complement $\mathcal{J}^\perp$ of $\mathcal{J}$, we have

$$\theta_{\text{sing}}(E,A) \leq \frac{\|P_{\mathcal{L}_{\mathcal{A}}^\perp} z\|}{\|P_{\mathcal{J}^\perp} z\|}. \tag{3}$$

Furthermore, in Theorem 6.2 we prove that $\theta_{\text{sing}}(E, A)$ and $\delta(E, A)$ are related by

$$\frac{\delta(E,A)}{\|[E,A]\|_F} \le \theta_{\text{sing}}(E,A) \le \frac{\delta(E,A)}{\sigma_{\min}([E,A]) - \delta(E,A)}. \tag{4}$$

Note that combining e.g. (3) and (4) yields a new upper bound for $\delta(E, A)$. Another bound, the proof of which is based on comparing the distances, is

$$\theta_{\text{sing}}(E,A) \ge \frac{1}{\sqrt{1 + \|A^{-1}E\|^2}},$$

where A is assumed to be invertible, see Theorem 6.5.

The paper is organized as follows: We recall the gap distance between subspaces in Section 2 together with some basic properties that are needed in due course. In order to define the gap distance between matrix pencils we associate with a pencil $\mathcal{A}(s) = sE - A$ the linear subspace $\mathcal{L}_{\mathcal{A}} = \ker[A, -E]$, which is discussed in Section 3 together with some of its properties. Then we introduce the gap distance between matrix pencils and the gap distance to singularity $\theta_{\text{sing}}(E, A)$. We derive upper and lower bounds for this number in Sections 4 and 5. A comparison of the gap distance to singularity with the distance to singularity $\delta(E, A)$ is discussed in Section 6. In Section 7 we discuss some examples and, in particular, we show that there are classes of matrix pencils for which regularity can be concluded using $\theta_{\text{sing}}(E, A)$, but not using $\delta(E, A)$.

Throughout this article, we use the following notation: For a subspace $\mathcal{L} \subseteq \mathbb{C}^n$ we denote by $S_{\mathcal{L}} := \{ x \in \mathcal{L} \mid \|x\| = 1 \}$ the unit sphere in $\mathcal{L}$ and by $d(x, \mathcal{L}) := \inf_{y \in \mathcal{L}} \|x - y\|$ the distance of a vector $x \in \mathbb{C}^n$ to $\mathcal{L}$. Furthermore, $P_{\mathcal{L}}$ denotes the orthogonal projection onto the subspace $\mathcal{L}$. By $\mathcal{L}_1 \dotplus \mathcal{L}_2$ we denote the direct sum of two subspaces $\mathcal{L}_1, \mathcal{L}_2 \subseteq \mathbb{C}^n$ with $\mathcal{L}_1 \cap \mathcal{L}_2 = \{0\}$, and by $\mathcal{L}_1 \oplus \mathcal{L}_2$ their orthogonal sum provided that $\mathcal{L}_1 \perp \mathcal{L}_2$. The singular value decomposition of a matrix $A \in \mathbb{C}^{n \times m}$ reads

$$A = U\Sigma V^*, \quad \Sigma = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma_r = \operatorname{diag}(\sigma_1, \ldots, \sigma_r), \tag{5}$$

with unitary matrices $U \in \mathbb{C}^{n \times n}$, $V \in \mathbb{C}^{m \times m}$ and singular values $\sigma_1 \ge \ldots \ge \sigma_r > 0$, $r = \operatorname{rk} A$. Note that $\|A\| = \sigma_1 = \sigma_{\max}(A)$, and for the *reduced minimum modulus* of A we have $\min_{\|x\|=1} \{ \|Ax\| \mid x \perp \ker A \} = \sigma_r = \sigma_{\min}(A)$. We denote by $\widetilde{\sigma}_{\min}(A)$ the *minimum modulus* of A, that is $\widetilde{\sigma}_{\min}(A) := \min_{\|x\|=1}\{\|Ax\|\}$. In other words, $\widetilde{\sigma}_{\min}(A) = \sigma_{\min}(A)$ if $\ker A = \{0\}$ and $\widetilde{\sigma}_{\min}(A) = 0$ otherwise.

2 The gap between subspaces

Recall from [13, 14, 19] that the *gap distance* between subspaces $\mathcal{L}, \mathcal{M} \subseteq \mathbb{C}^n$ with $\mathcal{L} \ne \{0\}$ or $\mathcal{M} \ne \{0\}$ is given by

$$\theta(\mathcal{L}, \mathcal{M}) := \max\left\{ \sup_{x \in S_{\mathcal{M}}} d(x, \mathcal{L}), \sup_{x \in S_{\mathcal{L}}} d(x, \mathcal{M}) \right\}. \tag{6}$$

The next proposition collects some well known properties of the gap distance, see [19, Corollary IV.2.6, Theorem IV.2.9], [14, Section S4.3].

Proposition 2.1. *For any two subspaces $\mathcal{L}, \mathcal{M} \subseteq \mathbb{C}^n$ the gap $\theta(\mathcal{L}, \mathcal{M})$ has the properties:*

(a) $\theta(\mathcal{L}, \mathcal{M}) = \|P_{\mathcal{M}} - P_{\mathcal{L}}\|$;

(b) $\theta(\mathcal{L}, \mathcal{M}) \le 1$, *and if* $\theta(\mathcal{L}, \mathcal{M}) < 1$ *then* $\dim \mathcal{L} = \dim \mathcal{M}$, $P_{\mathcal{M}}\mathcal{L} = \mathcal{M}$ *and* $P_{\mathcal{L}}\mathcal{M} = \mathcal{L}$;

(c) $\theta(\mathcal{L},\mathcal{M}) = \theta(\mathcal{L}^\perp,\mathcal{M}^\perp)$;

(d) $\theta(\mathcal{L},\mathcal{M}) = \max\{\|P_{\mathcal{M}^\perp}|_{\mathcal{L}}\|, \|P_{\mathcal{L}^\perp}|_{\mathcal{M}}\|\}$.

Every matrix $C \in \mathbb{C}^{n\times d}$ induces a subspace $\mathcal{L} \subseteq \mathbb{C}^n$ via $\mathcal{L} = \operatorname{ran} C$ and vice versa. For matrices $C \in \mathbb{C}^{n\times d}$ of full rank with $1 \le d \le n$, the following formula for the orthogonal projection onto the range of C holds.

$$P_{\operatorname{ran} C} = C(C^*C)^{-1}C^*. \tag{7}$$

If C has orthonormal columns then $C^*C = I_d$ and the equation (7) simplifies to $P_{\operatorname{ran} C} = CC^*$. Moreover, with Proposition 2.1 (d) we obtain the following corollary.

Corollary 2.2. *Let $\mathcal{L},\widetilde{\mathcal{L}} \subseteq \mathbb{C}^n$ be d-dimensional subspaces, $1 \le d \le n$, with $\mathcal{L} = \operatorname{ran} C$ and $\widetilde{\mathcal{L}} = \operatorname{ran}\widetilde{C}$ for some matrices $C,\widetilde{C} \in \mathbb{C}^{n\times d}$. Then*

$$\theta(\mathcal{L},\widetilde{\mathcal{L}}) = \max\{\|(I_n - C(C^*C)^{-1}C^*)|_{\operatorname{ran}\widetilde{C}}\|, \|(I_n - \widetilde{C}(\widetilde{C}^*\widetilde{C})^{-1}\widetilde{C}^*)|_{\operatorname{ran} C}\|\}.$$

For later use we record a formula for the gap between two d-dimensional subspaces of $\mathbb{C}^n$. In the case of $\mathbb{R}^n$, a proof using the CS-decomposition can be found in [15]; here we present a direct method.

Proposition 2.3. *Let $\mathcal{L},\widetilde{\mathcal{L}} \subseteq \mathbb{C}^n$ be d-dimensional subspaces, $1 \le d \le n$, such that $\mathcal{L} = \operatorname{ran} C$ and $\widetilde{\mathcal{L}} = \operatorname{ran}\widetilde{C}$ for some matrices $C,\widetilde{C} \in \mathbb{C}^{n\times d}$ of rank d with orthonormal columns. Then*

$$\theta(\mathcal{L},\widetilde{\mathcal{L}}) = \sqrt{1 - \widetilde{\sigma}_{\min}(C^*\widetilde{C})^2}. \tag{8}$$

Proof. Choose C_1 and $\widetilde{C}_1$ such that $U = [C, C_1]$ and $\widetilde{U} = [\widetilde{C}, \widetilde{C}_1]$ are unitary matrices. Then also $Q = U^*\widetilde{U}$ is unitary. Note that Q is of block form

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} = \begin{bmatrix} C^*\widetilde{C} & C^*\widetilde{C}_1 \\ C_1^*\widetilde{C} & C_1^*\widetilde{C}_1 \end{bmatrix}.$$

With $Q^*Q = QQ^* = I$ we find that

$$\begin{aligned} I &= Q_{11}^*Q_{11} + Q_{21}^*Q_{21}, \\ I &= Q_{11}Q_{11}^* + Q_{12}Q_{12}^*. \end{aligned}$$

The above relations imply that

$$\begin{aligned} 1 - \widetilde{\sigma}_{\min}(C^*\widetilde{C})^2 &= 1 - \min_{||x||=1} ||Q_{11}x||^2 = 1 - \min_{||x||=1} x^*(I - Q_{21}^*Q_{21})x \\ &= \max_{||x||=1} x^*Q_{21}^*Q_{21}x = ||Q_{21}||^2 \end{aligned} \tag{9}$$

Using that $\widetilde{\sigma}_{\min}(C^*\widetilde{C}) = \widetilde{\sigma}_{\min}(\widetilde{C}^*C)$ and $\|Q_{12}\| = \|Q_{12}^*\|$, similarly we obtain that

$$1 - \widetilde{\sigma}_{\min}(C^*\widetilde{C})^2 = ||Q_{12}||^2. \tag{10}$$

With $P_{\mathcal{L}} = CC^*$ and $P_{\widetilde{\mathcal{L}}} = \widetilde{C}\widetilde{C}^*$ we find that

$$\theta(\mathcal{L},\widetilde{\mathcal{L}}) = \|P_{\mathcal{L}} - P_{\widetilde{\mathcal{L}}}\| = \left\|CC^* - \widetilde{C}\widetilde{C}^*\right\| = \left\|U^*(CC^* - \widetilde{C}\widetilde{C}^*)\widetilde{U}\right\|, \tag{11}$$

and a straightforward calculation using $U^*U = \widetilde{U}^*\widetilde{U} = I$ gives

$$\begin{bmatrix} C^* \\ C_1^* \end{bmatrix} (CC^* - \widetilde{C}\widetilde{C}^*)[\widetilde{C}, \widetilde{C}_1] = \begin{bmatrix} 0 & C^*\widetilde{C}_1 \\ -C_1^*\widetilde{C} & 0 \end{bmatrix} = \begin{bmatrix} 0 & Q_{12} \\ -Q_{21} & 0 \end{bmatrix}.$$

Since

$$\left\| \begin{bmatrix} 0 & Q_{12} \\ -Q_{21} & 0 \end{bmatrix} \right\| = \max\{||Q_{21}||, ||Q_{12}||\},$$

the last relation together with (9), (10) and (11) implies the formula (8). $\square$

Remark 2.4. Note that also the following relations hold, cf. (9) and (10):

$$\theta(\mathcal{L}, \tilde{\mathcal{L}}) = \|C^* \widetilde{C}_1\| = \|C_1^* \widetilde{C}\|.$$

Furthermore, if $x_1, x_2 \in \mathbb{C}^n$ are non-zero vectors and $\mathcal{L}_i = \operatorname{span}\{x_i\}$, $i = 1, 2$, then Proposition 2.3 gives

$$\theta(\mathcal{L}_1, \mathcal{L}_2) = \sqrt{1 - \frac{|x_1^* x_2|^2}{\|x_1\|^2 \|x_2\|^2}}. \tag{12}$$

We stress that in complex vector spaces, the formula for the gap as in (12) is not the same as the sine of the angle between the two spanning vectors, which is defined by

$$\sin \sphericalangle(x_1, x_2) := \sqrt{1 - \cos^2 \sphericalangle(x_1, x_2)}, \quad \cos \sphericalangle(x_1, x_2) := \operatorname{Re} \frac{x_1^* x_2}{\|x_1\| \|x_2\|}.$$

For the convenience of the reader, some properties of the Frobenius norm $\|A\|_F$ of a matrix $A = (a_{ij}) \in \mathbb{C}^{n \times m}$ are mentioned:

$$\|A\|_F = \sqrt{\operatorname{tr}(A^* A)} = \sqrt{\operatorname{tr}(AA^*)} = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}|^2},$$

and for unitary matrices $U \in \mathbb{C}^{n \times n}$ and $V \in \mathbb{C}^{m \times m}$ we have

$$\|M\|_F = \|UMV\|_F.$$

A direct consequence of the above invariance is the following elementary lemma.

Lemma 2.5. *For $A \in \mathbb{C}^{n \times m}$ and $B \in \mathbb{C}^{m \times k}$ we have*

$$\|AB\|_F \le \|A\|_F \, \|B\| \quad \textit{and} \quad \|AB\|_F \le \|A\| \, \|B\|_F.$$

Proof. Consider the singular value decomposition $B = U \Sigma V^*$ with unitary $U \in \mathbb{C}^{m \times m}$, $V \in \mathbb{C}^{k \times k}$. Using the fact that $\|B\| = \sigma_{\max}(B)$ we find that

$$\|AB\|_F = \|AUU^* BV\|_F = \|AU\Sigma\|_F \le \sigma_{\max}(B) \|AU\|_F = \|A\|_F \, \|B\|.$$

The second inequality can be inferred in a similar way. $\square$

In the following we show that if the gap distance between the subspaces $\mathcal{L}, \widetilde{\mathcal{L}} \subseteq \mathbb{C}^n$ is small, then the representing matrices can be chosen in such way that the norm of their difference is small.

Proposition 2.6. *Let $\mathcal{L}, \widetilde{\mathcal{L}} \subseteq \mathbb{C}^n$ be subspaces with $\theta(\mathcal{L}, \widetilde{\mathcal{L}}) < 1$ and $\mathcal{L} = \operatorname{ran} C$ for some matrix $C \in \mathbb{C}^{n \times d}$ with linearly independent columns $c_1, \ldots, c_d \in \mathbb{C}^n$. Then $\widetilde{\mathcal{L}} = \operatorname{ran} P_{\widetilde{\mathcal{L}}} C$ and*

$$\|P_{\widetilde{\mathcal{L}}} C - C\| \le \theta(\mathcal{L}, \widetilde{\mathcal{L}}) \|C\|, \quad \|P_{\widetilde{\mathcal{L}}} C - C\|_F \le \theta(\mathcal{L}, \widetilde{\mathcal{L}}) \|C\|_F. \tag{13}$$

Proof. Since $\theta(\mathcal{L}, \widetilde{\mathcal{L}}) < 1$ we have from Proposition 2.1 (b) that $\widetilde{\mathcal{L}} = \operatorname{ran} P_{\widetilde{\mathcal{L}}} C$. The first inequality in (13) follows from

$$\|P_{\widetilde{\mathcal{L}}} C - C\| = \left\|P_{\widetilde{\mathcal{L}}} C - P_{\mathcal{L}} C\right\| \leq \left\|P_{\mathcal{L}} - P_{\widetilde{\mathcal{L}}}\right\| \|C\| \overset{\text{Prop. 2.1 (a)}}{=} \theta(\mathcal{L}, \widetilde{\mathcal{L}})\|C\|.$$

The second inequality in (13) can be inferred from

$$\left\|P_{\widetilde{\mathcal{L}}} C - C\right\|_F = \left\|P_{\mathcal{L}} C - P_{\widetilde{\mathcal{L}}} C\right\|_F \overset{\text{Lem. 2.5}}{\leq} \left\|P_{\mathcal{L}} - P_{\widetilde{\mathcal{L}}}\right\| \|C\|_F = \theta(\mathcal{L}, \widetilde{\mathcal{L}})\|C\|_F. \qquad \square$$

Next, we prove a converse to Proposition 2.6: a small distance of the representing matrices C and $\widetilde{C}$ implies a small gap distance. To this end, we need another elementary lemma.

Lemma 2.7. *Let $A \in \mathbb{C}^{n\times m}$ and $x \in \operatorname{ran} A$ such that $\|x\| = 1$. Then there exists $z \in \mathbb{C}^m$ such that $x = Az$ and $\|z\| \leq (\sigma_{\min}(A))^{-1}$.*

Proof. Consider a singular value decomposition $A = U\Sigma V^*$ of A as in (5) with unitary matrices $U \in \mathbb{C}^{n\times n}$, $V \in \mathbb{C}^{m\times m}$. Denote $\Sigma^+ = \begin{bmatrix} \Sigma_r^{-1} & 0 \\ 0 & 0 \end{bmatrix}$ and put $z = V\Sigma^+ U^* x$. Since $\operatorname{ran} A = \operatorname{ran} U\Sigma$, let $x = U\Sigma y$ for some $y \in \mathbb{C}^m$. It follows that $Az = U\Sigma V^* V \Sigma^+ U^* U \Sigma y = U\Sigma y = x$. Furthermore,

$$\|z\| \leq \|\Sigma^+\|\|U^* x\| \leq \sigma_r^{-1} = (\sigma_{\min}(A))^{-1}. \qquad \square$$

Proposition 2.8. *Let $\mathcal{L}, \widetilde{\mathcal{L}} \subseteq \mathbb{C}^n$ be subspaces with $\mathcal{L} = \operatorname{ran} C$ and $\widetilde{\mathcal{L}} = \operatorname{ran} \widetilde{C}$ for $C, \widetilde{C} \in \mathbb{C}^{n\times d} \setminus \{0\}$, $d \leq n$. Then we have*

$$\theta(\mathcal{L}, \widetilde{\mathcal{L}}) \leq \frac{\|\widetilde{C} - C\|}{\min\{\sigma_{\min}(C), \sigma_{\min}(\widetilde{C})\}}. \tag{14}$$

Proof. Let $x \in \operatorname{ran} C$ with $\|x\| = 1$ and choose, according to Lemma 2.7, $z \in \mathbb{C}^d$ such that $x = Cz$ and $\|z\| \leq (\sigma_{\min}(C))^{-1}$. Then we have

$$\begin{aligned} d(x, \operatorname{ran} \widetilde{C}) = \inf_{\widetilde{z}\in\mathbb{C}^d} \|Cz - \widetilde{C}\widetilde{z}\| &\leq \inf_{\widetilde{z}\in\mathbb{C}^d} \left(\|Cz - C\widetilde{z}\| + \|(\widetilde{C} - C)\widetilde{z}\|\right) \\ &\leq \min\{\|Cz\|, \|(\widetilde{C} - C)z\|\}, \end{aligned}$$

where the last inequality follows from setting $\widetilde{z} = z$ or $\widetilde{z} = 0$. Since $\|C\| \geq \sigma_{\min}(C)$ one finds that

$$d(x, \operatorname{ran} \widetilde{C}) \leq \frac{\|\widetilde{C} - C\|}{\sigma_{\min}(C)}, \tag{15}$$

and by symmetry it follows for all $x \in \operatorname{ran} \widetilde{C}$ with $\|x\| = 1$ that

$$d(x, \operatorname{ran} C) \leq \frac{\|\widetilde{C} - C\|}{\sigma_{\min}(\widetilde{C})}. \tag{16}$$

The inequalities (15) and (16) together with (6) imply (14). $\square$

Formula (14) is a slight improvement of a formula from [1, Proposition 1.1 (i)] for operators in a Hilbert space.

Remark 2.9. The upper bound (14) is sharp, but on the other hand it can be arbitrarily large. To see that (14) is sharp choose subspaces $\mathcal{L}, \tilde{\mathcal{L}} \subseteq \mathbb{C}^n$ and matrices $C, \widetilde{C} \in \mathbb{C}^{n\times d}$, $d \leq n$ with orthonormal columns such that $P_{\mathcal{L}} = CC^*$ and $P_{\widetilde{\mathcal{L}}} = \widetilde{C}\widetilde{C}^*$. We apply Proposition 2.8 to $P_{\mathcal{L}}$ and $P_{\widetilde{\mathcal{L}}}$. From $\sigma_{\min}(P_{\mathcal{L}}) = \sigma_{\min}(P_{\widetilde{\mathcal{L}}}) = 1$ and Proposition 2.1 (a) we see that (14) holds with equality. On the other hand, for $n \in \mathbb{N} \setminus \{0\}$ let

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} n & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $\mathcal{L} = \operatorname{ran} C = \operatorname{ran} \widetilde{C} = \widetilde{\mathcal{L}}$ and therefore $\theta(\mathcal{L}, \widetilde{\mathcal{L}}) = 0$ but $\sigma_{\min}(C) = \sigma_{\min}(\widetilde{C}) = 1$ and

$$\|\widetilde{C} - C\| = n - 1.$$

In the next lemma we compare the gap between two subspaces of a certain structure.

Lemma 2.10. *For subspaces $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L} \subseteq \mathbb{C}^n$ the following holds.*

(a) *If $\mathcal{L}_i \perp \mathcal{L}$, $i = 1, 2$, then $\theta(\mathcal{L}_1 \oplus \mathcal{L}, \mathcal{L}_2 \oplus \mathcal{L}) = \theta(\mathcal{L}_1, \mathcal{L}_2)$.*

(b) *If $\mathcal{L}_i \cap \mathcal{L} = \{0\}$, $i = 1, 2$, then $\theta(\mathcal{L}_1 \dotplus \mathcal{L}, \mathcal{L}_2 \dotplus \mathcal{L}) = \theta(P_{\mathcal{L}^\perp}\mathcal{L}_1, P_{\mathcal{L}^\perp}\mathcal{L}_2)$.*

(c) *If $\mathcal{L}_i \cap \mathcal{L} = \{0\}$ as well as $\mathcal{L}_i \neq \{0\}$, $\mathcal{L} \neq \{0\}$, $i = 1, 2$, then*

$$\theta(\mathcal{L}_1 \dotplus \mathcal{L}, \mathcal{L}_2 \dotplus \mathcal{L}) \leq \frac{\theta(\mathcal{L}_1, \mathcal{L}_2)}{\min\{\sigma_{\min}(P_{\mathcal{L}^\perp}P_{\mathcal{L}_1}), \sigma_{\min}(P_{\mathcal{L}^\perp}P_{\mathcal{L}_2})\}}.$$

Proof. Since $P_{\mathcal{L}_i \oplus \mathcal{L}} = P_{\mathcal{L}_i} + P_{\mathcal{L}}$, $i = 1, 2$, we have $P_{\mathcal{L}_1 \oplus \mathcal{L}} - P_{\mathcal{L}_2 \oplus \mathcal{L}} = P_{\mathcal{L}_1} - P_{\mathcal{L}_2}$ and then (a) follows from Proposition 2.1 (a).

To show (b), decompose $\mathbb{C}^n$ as $\mathbb{C}^n = \mathcal{L}_i \oplus \mathcal{L}_i^\perp = P_{\mathcal{L}}\mathcal{L}_i \oplus P_{\mathcal{L}^\perp}\mathcal{L}_i \oplus \mathcal{L}_i^\perp$, hence $(P_{\mathcal{L}^\perp}\mathcal{L}_i)^\perp = P_{\mathcal{L}}\mathcal{L}_i \oplus \mathcal{L}_i^\perp$. We conclude

$$(P_{\mathcal{L}^\perp}\mathcal{L}_i \oplus \mathcal{L})^\perp = (P_{\mathcal{L}^\perp}\mathcal{L}_i)^\perp \cap \mathcal{L}^\perp = \mathcal{L}_i^\perp \cap \mathcal{L}^\perp = (\mathcal{L}_i \dotplus \mathcal{L})^\perp .$$

By taking orthogonal complements we see $\mathcal{L}_i \dotplus \mathcal{L} = P_{\mathcal{L}^\perp}\mathcal{L}_i \oplus \mathcal{L}$ and (b) follows from (a) as

$$\theta(\mathcal{L}_1 \dotplus \mathcal{L}, \mathcal{L}_2 \dotplus \mathcal{L}) = \theta(P_{\mathcal{L}^\perp}\mathcal{L}_1 \oplus \mathcal{L}, P_{\mathcal{L}^\perp}\mathcal{L}_2 \oplus \mathcal{L}) = \theta(P_{\mathcal{L}^\perp}\mathcal{L}_1, P_{\mathcal{L}^\perp}\mathcal{L}_2).$$

Statement (c) follows from (b) and (14) by choosing $C := P_{\mathcal{L}^\perp}P_{\mathcal{L}_1}$ and $\widetilde{C} := P_{\mathcal{L}^\perp}P_{\mathcal{L}_2}$ since

$$\begin{aligned}\theta(P_{\mathcal{L}^\perp}\mathcal{L}_1, P_{\mathcal{L}^\perp}\mathcal{L}_2) &\leq \frac{\|P_{\mathcal{L}^\perp}P_{\mathcal{L}_1} - P_{\mathcal{L}^\perp}P_{\mathcal{L}_2}\|}{\min\{\sigma_{\min}(P_{\mathcal{L}^\perp}P_{\mathcal{L}_1}), \sigma_{\min}(P_{\mathcal{L}^\perp}P_{\mathcal{L}_2})\}} \\ &\leq \frac{\|P_{\mathcal{L}_1} - P_{\mathcal{L}_2}\|}{\min\{\sigma_{\min}(P_{\mathcal{L}^\perp}P_{\mathcal{L}_1}), \sigma_{\min}(P_{\mathcal{L}^\perp}P_{\mathcal{L}_2})\}}.\end{aligned}$$

□

3 Gap distance to singularity

In the following linear subspaces $\mathcal{L}$ of $\mathbb{C}^{2n}$ are considered, which are known under the name *linear relations*, see e.g. [10, 17, 24, 25]. To a matrix $A \in \mathbb{C}^{n\times n}$ the subspace $\operatorname{graph} A := \{(x, Ax) \in \mathbb{C}^{2n} \mid x \in \mathbb{C}^n\} \subseteq \mathbb{C}^{2n}$ is associated. Note that

$$\operatorname{graph} A = \ker[A, -I].$$

A similar correspondence can be obtained for matrix pencils. By [7, Theorem 3.3], to any subspace $\mathcal{L}$ of $\mathbb{C}^{2n}$ with $\dim \mathcal{L} = d$ there exist matrices $E, A \in \mathbb{C}^{n\times r}$ with $r = 2n - d$ such that

$$\mathcal{L} = \ker[A, -E],$$

which is called the *kernel representation* of $\mathcal{L}$. In what follows, to a matrix pencil $\mathcal{A}(s) = sE - A$ with $E, A \in \mathbb{C}^{n\times n}$ the subspace

$$\mathcal{L}_{\mathcal{A}} := \ker[A, -E]$$

is associated. These spaces are used to investigate the maximal gap distance between pencils $\mathcal{A}(s)$ and $\widetilde{\mathcal{A}}(s)$ that guarantees regularity of $\widetilde{\mathcal{A}}(s)$.

Definition 3.1. *For pencils $\mathcal{A}(s) = sE - A$ and $\widetilde{\mathcal{A}}(s) = s\widetilde{E} - \widetilde{A}$ with $E, A, \widetilde{E}, \widetilde{A} \in \mathbb{C}^{n\times n}$ the* gap distance *between two matrix pencils is defined as*

$$\theta(\mathcal{A}, \widetilde{\mathcal{A}}) := \theta(\mathcal{L}_{\mathcal{A}}, \mathcal{L}_{\widetilde{\mathcal{A}}}) = \left\| P_{\ker[A,-E]} - P_{\ker[\widetilde{A},-\widetilde{E}]} \right\|.$$

The gap distance to singularity *of a regular matrix pencil $\mathcal{A}(s) = sE - A$ is defined as*

$$\theta_{\text{sing}}(E, A) := \inf_{\widetilde{E},\widetilde{A}\in\mathbb{C}^{n\times n}} \left\{ \theta(\mathcal{A}, \widetilde{\mathcal{A}}) \;\middle|\; \widetilde{\mathcal{A}}(s) = s\widetilde{E} - \widetilde{A} \text{ is singular} \right\}.$$

Remark 3.2. Clearly $\theta_{\text{sing}}(E, A) \leq 1$ for any regular matrix pencil $sE - A$. It is also obvious that $\theta_{\text{sing}}(E, A) = 1$ for $E = A = [1]$. We leave it to the reader to show that $\theta_{\text{sing}}(A, A) = 1$ for any invertible matrix A.

Recall that every pencil $\mathcal{A}(s) = sE - A$ can be transformed into *Kronecker canonical form*, see e.g. [5, 6, 12]. To introduce this form the following notation is used: For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}^l$, $l \geq 1$, with absolute value $|\alpha| = \sum_{i=1}^{l} \alpha_i$ and $k \in \mathbb{N}$ we define the matrices

$$N_k := \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & \ddots \\ & & 1 \; 0 \end{bmatrix} \in \mathbb{C}^{k\times k}, \quad N_\alpha := \operatorname{diag}(N_{\alpha_1}, \ldots, N_{\alpha_l}) \in \mathbb{C}^{|\alpha|\times|\alpha|}.$$

If $k \geq 2$ rectangular matrices are defined as

$$K_k := \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \quad L_k := \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} \in \mathbb{C}^{(k-1)\times k},$$

and if $k = 1$

$$K_1 = L_1 := 0_{0\times 1}.$$

The expression $0_{0\times 1}$ means that there is a 0-column $(0, \ldots, 0)^\top \in \mathbb{C}^{n\times 1}$ in the matrix (17) below, and $0_{0\times 1}^\top$ means that there is a 0-row $(0, \ldots, 0) \in \mathbb{C}^{1\times n}$ in (17) at the corresponding block. The notation $0_{0\times 1}$ indicates that there is no contribution to the number of rows in (17), whereas $0_{0\times 1}^\top$ gives no contribution to the number of columns. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}^l$ we define

$$K_\alpha := \operatorname{diag}(K_{\alpha_1}, \ldots, K_{\alpha_l}), \; L_\alpha := \operatorname{diag}(L_{\alpha_1}, \ldots, L_{\alpha_l}) \in \mathbb{C}^{(|\alpha|-l)\times|\alpha|}.$$

According to a result of Kronecker [20], there exist invertible matrices $S, T \in \mathbb{C}^{n\times n}$ such that $S(sE - A)T$ has a block diagonal form

$$\begin{bmatrix} sI_{n_0} - J & 0 & 0 & 0 \\ 0 & sN_\alpha - I_{|\alpha|} & 0 & 0 \\ 0 & 0 & sK_\beta - L_\beta & 0 \\ 0 & 0 & 0 & sK_\gamma^\top - L_\gamma^\top \end{bmatrix} \tag{17}$$

for some $J \in \mathbb{C}^{n_0\times n_0}$ in Jordan canonical form, which is unique up to a permutation of its Jordan blocks, and multi-indices $\alpha \in \mathbb{N}^{n_\alpha}$, $\beta \in \mathbb{N}^{n_\beta}$, $\gamma \in \mathbb{N}^{n_\gamma}$ which are unique up to a permutation of their entries. If $k \in \mathbb{N}$ is one of the entries of the multi-indices β or γ in the Kronecker canonical form (17), we say that $\mathcal{A}(s)$ has a *singular block of size k*. Recall that $\mathcal{A}(s)$ is regular if and only if there are no singular blocks in the Kronecker canonical form. In the literature the numbers $\beta_1 - 1, \ldots, \beta_{n_\beta} - 1$ (respectively $\gamma_1 - 1, \ldots, \gamma_{n_\gamma} - 1$) are often called *right (left) minimal indices of $sE - A$*, see e.g. [11, 12].

Lemma 3.3. *For a matrix pencil $\mathcal{A}(s) = sE - A$ with $E, A \in \mathbb{C}^{n\times n}$ the following holds:*

(a) $\mathcal{L}_{\mathcal{A}}^{\perp} = \operatorname{ran} \begin{bmatrix} A^* \\ -E^* \end{bmatrix}$.

(b) $\dim \mathcal{L}_{\mathcal{A}} = 2n - \operatorname{rk}[A, -E] \geq n$.

(c) *If $\mathcal{A}(s)$ is regular, then* $\dim \mathcal{L}_{\mathcal{A}} = n$.

Proof. Property (a) follows from $\ker[A, -E] = \left(\operatorname{ran} \left[\begin{smallmatrix} A^* \\ -E^* \end{smallmatrix}\right]\right)^{\perp}$. Since $\mathcal{L}_{\mathcal{A}} = \ker[A, -E]$, we see that (b) is a consequence of the dimension formula. Obviously, $[A, -E] \in \mathbb{C}^{n\times 2n}$, hence its kernel has at least dimension n. To show (c) we use the Kronecker canonical form (17) without singular blocks since $\mathcal{A}(s)$ is regular, i.e., there exist invertible matrices $S, T \in \mathbb{C}^{n\times n}$ such that

$$\operatorname{rk}[A, -E] = \operatorname{rk}\left(S\ [A, -E] \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix}\right) = \operatorname{rk}\left[\begin{array}{cc|cc} J & 0 & -I_{n_0} & 0 \\ 0 & I_{|\alpha|} & 0 & -N_{\alpha} \end{array}\right] = n. \qquad \square$$

The following example shows that the converse of statement (c) in Lemma 3.3 is not true.

Example 3.4. Consider the matrix pencil given by

$$sE - A = s \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then $\mathcal{A}(s) = sE - A$ is singular but

$$\operatorname{rk}[A, -E] = \operatorname{rk} \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = 3$$

which implies by Lemma 3.3 (b) that $\dim \mathcal{L}_{\mathcal{A}} = 3$.

If the gap distance between a regular pencil and a singular pencil is smaller than one, then the singular pencil has a singular block of size at least two, as shown in the next proposition. Note that below we use the notation of (17) for the Kronecker canonical form of $\widetilde{\mathcal{A}}(s)$, not of $\mathcal{A}(s)$.

Proposition 3.5. *Let $E, A, \widetilde{E}, \widetilde{A} \in \mathbb{C}^{n\times n}$ be such that $\mathcal{A}(s) = sE - A$ is regular and $\widetilde{\mathcal{A}}(s) = s\widetilde{E} - \widetilde{A}$ is singular. If $\theta(\mathcal{A}, \widetilde{\mathcal{A}}) < 1$, then in the Kronecker canonical form* (17) *of $\widetilde{\mathcal{A}}(s)$ we have $n_\gamma > 0$ and $\gamma_i \geq 2$ for all $i = 1, \ldots, n_\gamma$, i.e., all left minimal indices of $\widetilde{\mathcal{A}}(s)$ are at least one.*

Proof. Since $\mathcal{A}(s) = sE - A$ is regular, $\dim \ker[A, -E] = n$ by Lemma 3.3, and the condition $\theta(\mathcal{A}, \widetilde{\mathcal{A}}) < 1$ implies by Proposition 2.1 (b) that

$$\dim \ker[\widetilde{A}, -\widetilde{E}] = \dim \ker[A, -E] = n. \tag{18}$$

Let $S, T \in \mathbb{C}^{n\times n}$ be invertible matrices such that $sS\widetilde{E}T - S\widetilde{A}T$ is in Kronecker canonical form (17). As $\widetilde{\mathcal{A}}(s)$ is a square singular pencil, the number n_γ of left minimal indices in (17) has to be nonzero, see e.g. [7]. Furthermore, if $\gamma_i = 1$ for some $i \in \{1, \ldots, n_\gamma\}$, then $sS\widetilde{E}T - S\widetilde{A}T$ contains a zero row. Hence, $\operatorname{rk}[S\widetilde{A}T, -S\widetilde{E}T] < n$, and consequently

$$\operatorname{rk}[\widetilde{A}, -\widetilde{E}] = \operatorname{rk}\left(S[\widetilde{A}, -\widetilde{E}] \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix}\right) = \operatorname{rk}[S\widetilde{A}T, -S\widetilde{E}T] < n,$$

which contradicts (18). $\square$

We show now that the assumptions of Proposition 3.5 do not restrict the right minimal indices of $\widetilde{\mathcal{A}}(s)$, i.e., we may have $\beta_i = 1$ for some $i \in \{1, \ldots, n_\beta\}$.

Example 3.6. Let

$$A = \begin{bmatrix} 0 & 0 \\ \varepsilon & 1 \end{bmatrix}, \quad \widetilde{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad E = \widetilde{E} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then clearly $\mathcal{A}(s) = sE - A$ is regular for $\varepsilon > 0$ and $\widetilde{\mathcal{A}}(s) = s\widetilde{E} - \widetilde{A}$ is in Kronecker canonical form (17) with one right minimal index 0 and one left minimal index 1, i.e., $\beta = (1)$ and $\gamma = (2)$. Furthermore,

$$\ker[A, -E] = \operatorname{span}\left\{\begin{pmatrix} 1 \\ -\varepsilon \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right\}, \quad \ker[\widetilde{A}, -\widetilde{E}] = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right\}.$$

By Lemma 2.10 (a) and equation (12) we see that $\theta(\mathcal{A}, \widetilde{\mathcal{A}})$ converges to zero for $\varepsilon \to 0$.

The above asymmetry of $\theta_{\text{sing}}(E, A)$ with respect to the Kronecker canonical form will be further discussed in Section 7.3.

4 Lower bounds for $\theta_{\text{sing}}(E, A)$

In this section we present lower bounds for $\theta_{\text{sing}}(E, A)$ of a regular matrix pencil $\mathcal{A}(s) = sE - A$ with $E, A \in \mathbb{C}^{n \times n}$. The main tool is the matrix

$$W_k(E, A) := \begin{bmatrix} E & 0 & \ldots & 0 \\ A & E & & \vdots \\ 0 & \ddots & \ddots & 0 \\ \vdots & & A & E \\ 0 & \ldots & 0 & A \end{bmatrix} \in \mathbb{C}^{(k+1)n \times kn}, \quad k \geq 1,$$

which has been studied e.g. in [12, 18]. The following characterization of regularity of $\mathcal{A}(s)$ is a consequence of [18, Theorem 3.1].

Theorem 4.1. *Let $E, A \in \mathbb{C}^{n \times n}$ and $\mathcal{A}(s) = sE - A$ be a matrix pencil with Kronecker canonical form (17). Then $\ker W_k(E, A) \neq \{0\}$, $k \geq 1$, if and only if there exists some entry β_i of the multi-index β in (17) with $\beta_i \leq k$. In particular, $\mathcal{A}(s)$ is regular if and only if $\ker W_n(E, A) = \{0\}$.*

The next lemma contains some properties of the matrices $W_k(E, A)$.

Lemma 4.2. *Let $E, A \in \mathbb{C}^{n \times n}$ and $k \in \mathbb{N}$. Then the following holds:*

(a) $\ker W_k(SE, SA) = \ker W_k(E, A)$ *for all invertible* $S \in \mathbb{C}^{n \times n}$*;*

(b) $\|W_k(E, A)\| \leq \sqrt{2}\|[E, A]\|$ *for all* $k \geq 1$.

Proof. The assertion (a) is an immediate consequence of

$$W_k(SE, SA) = \operatorname{diag}(S, \ldots, S)\, W_k(E, A).$$

To show (b) in case $k = 1$ note that

$$\left\| \begin{bmatrix} E \\ A \end{bmatrix} \right\|^2 = \max_{\|x\|=1} \left\{ \|Ex\|^2 + \|Ax\|^2 \right\} \leq \max_{\|(x_1,x_2)\|=\sqrt{2}} \left\{ \|Ex_1\|^2 + \|Ax_2\|^2 \right\} \leq 2\|[E, A]\|^2.$$

If $k \geq 2$ take $x = (x_1^\top, \ldots, x_k^\top)^\top \in \mathbb{C}^{kn}$ with $\|x\| = 1$, then

$$\begin{aligned} \|W_k(E, A)x\| &\leq \sqrt{\|E\|^2\|x_1\|^2 + \sum_{i=1}^{k-1} \|[E, A]\|^2(\|x_i\|^2 + \|x_{i+1}\|^2) + \|A\|^2\|x_k\|^2} \\ &\leq \sqrt{2}\|[E, A]\|. \end{aligned}$$

□

The following lower bounds for $\theta_{\text{sing}}(E, A)$ are one of our main results.

Theorem 4.3. *Let $\mathcal{A}(s) = sE - A$ and $\widetilde{\mathcal{A}}(s) = s\widetilde{E} - \widetilde{A}$ be two matrix pencils with $E, A, \widetilde{E}, \widetilde{A} \in \mathbb{C}^{n \times n}$ such that $\mathcal{A}(s)$ is regular. If for some $k \geq 1$ we have*

$$\theta(\mathcal{A}, \widetilde{\mathcal{A}}) < \frac{\sigma_{\min}(W_k(E, A))}{\sqrt{2}\,\|[E, A]\|}, \tag{19}$$

then $\beta_i \geq k + 1$ holds for all the entries β_i of the multi-index β in the Kronecker canonical form (17) of $\widetilde{\mathcal{A}}(s)$. Furthermore, we have

$$\theta_{\text{sing}}(E, A) \geq \sup \left\{ \frac{\sigma_{\min}(W_n(SE, SA))}{\sqrt{2}\,\|[SE, SA]\|} \;\middle|\; S \in \mathbb{C}^{n \times n}, S \text{ invertible} \right\}. \tag{20}$$

and, in particular,

$$\theta_{\text{sing}}(E, A) \geq \frac{\sigma_{\min}(W_n(E, A))}{\sqrt{2}\,\|[E, A]\|}. \tag{21}$$

If E (or A) is invertible, then

$$\theta_{\text{sing}}(E, A) \geq \frac{\sigma_{\min}(W_n(I_n, E^{-1}A))}{\sqrt{2}\sqrt{1 + \|E^{-1}A\|^2}} \quad \left(\text{resp. } \theta_{\text{sing}}(E, A) \geq \frac{\sigma_{\min}(W_n(I_n, A^{-1}E))}{\sqrt{2}\sqrt{1 + \|A^{-1}E\|^2}} \right). \tag{22}$$

Proof. Note that $\ker W_k(E, A) = \{0\}$ for all $k \geq 1$ by Theorem 4.1 since $\mathcal{A}(s)$ is regular. Furthermore, by regularity, the Lemma 3.3 yields that $\left[\begin{smallmatrix} A^* \\ -E^* \end{smallmatrix}\right]$ has full column rank. Now assume that (19) holds, then Lemma 4.2 (b) implies that $\theta(\mathcal{A}, \widetilde{\mathcal{A}}) < 1$. Define matrices $\widehat{E}, \widehat{A} \in \mathbb{C}^{n \times n}$ by

$$\begin{bmatrix} \widehat{A}^* \\ -\widehat{E}^* \end{bmatrix} = P_{\mathcal{L}_{\widetilde{\mathcal{A}}}^\perp} \begin{bmatrix} A^* \\ -E^* \end{bmatrix}.$$

Since $\mathcal{L}_{\widetilde{\mathcal{A}}}^\perp = \operatorname{ran} \left[\begin{smallmatrix} \widetilde{A}^* \\ -\widetilde{E}^* \end{smallmatrix}\right]$ and $\mathcal{L}_{\mathcal{A}}^\perp = \operatorname{ran} \left[\begin{smallmatrix} A^* \\ -E^* \end{smallmatrix}\right]$ according to Lemma 3.3 (a) and $\theta(\mathcal{L}_{\mathcal{A}}^\perp, \mathcal{L}_{\widetilde{\mathcal{A}}}^\perp) = \theta(\mathcal{A}, \widetilde{\mathcal{A}}) < 1$ by Proposition 2.1 (c), it follows with Proposition 2.6 that

$$\operatorname{ran} \begin{bmatrix} \widehat{A}^* \\ -\widehat{E}^* \end{bmatrix} = \operatorname{ran} \begin{bmatrix} \widetilde{A}^* \\ -\widetilde{E}^* \end{bmatrix} \tag{23}$$

and

$$\left\| \begin{bmatrix} A^* - \widehat{A}^* \\ -E^* + \widehat{E}^* \end{bmatrix} \right\| \leq \theta(\mathcal{L}_{\mathcal{A}}^\perp, \mathcal{L}_{\widetilde{\mathcal{A}}}^\perp) \left\| \begin{bmatrix} A^* \\ -E^* \end{bmatrix} \right\| = \theta(\mathcal{A}, \widetilde{\mathcal{A}})\|[E, A]\| < \frac{\sigma_{\min}(W_k(E, A))}{\sqrt{2}}.$$

The Lemma 4.2 (b) yields

$$\frac{\|W_k(E,A)-W_k(\widehat{E},\widehat{A})\|}{\sqrt{2}} = \frac{\|W_k(E-\widehat{E},A-\widehat{A})\|}{\sqrt{2}} \leq \|[E-\widehat{E},A-\widehat{A}]\| = \left\| \begin{bmatrix} A^*-\widehat{A}^* \\ -E^*+\widehat{E}^* \end{bmatrix} \right\|,$$

and a combination of the last two inequalities gives

$$\|W_k(E,A)-W_k(\widehat{E},\widehat{A})\| < \sigma_{\min}(W_k(E,A)). \tag{24}$$

Note that by [15, Theorem 2.5.3]

$$\sigma_{\min}(W_k(E,A)) = \min_{\operatorname{rk} B \leq r-1} \|W_k(E,A)-B\|, \quad \text{where } r = \operatorname{rk} W_k(E,A) = kn.$$

Therefore, it follows from the inequality (24) that $\operatorname{rk} W_k(\widehat{E},\widehat{A}) = \operatorname{rk} W_k(E,A)$, thus

$$\ker W_k(\widehat{E},\widehat{A}) = \ker W_k(E,A) = \{0\}. \tag{25}$$

In particular, the matrix $\left[\begin{smallmatrix} \widehat{A}^* \\ -\widehat{E}^* \end{smallmatrix}\right]$ has full column rank n. Moreover, the relation (23) implies that there is some invertible matrix $S \in \mathbb{C}^{n\times n}$ such that $\left[\begin{smallmatrix} \widehat{A}^* \\ -\widehat{E}^* \end{smallmatrix}\right] S^* = \left[\begin{smallmatrix} \widetilde{A}^* \\ -\widetilde{E}^* \end{smallmatrix}\right]$, which leads to $S\widehat{A} = \widetilde{A}$ and $S\widehat{E} = \widetilde{E}$. Now it follows from Lemma 4.2 (a) with the relation (25) that $\ker W_k(\widetilde{E},\widetilde{A}) = \{0\}$. By Theorem 4.1 this implies $\beta_i \geq k+1$ for all entries of β. For $k=n$, the Theorem 4.1 gives that $\widetilde{\mathcal{A}}(s)$ is regular and therefore (21) holds.

In the following we will use the fact that for $\mathcal{A}(s) = sE-A$ the pencil $S\mathcal{A}(s) = sSE-SA$ for any invertible $S \in \mathbb{C}^{n\times n}$ generates the same subspace, i.e., $\mathcal{L}_{\mathcal{A}} = \mathcal{L}_{S\mathcal{A}}$, hence we have

$$\theta_{\text{sing}}(SE,SA) = \theta_{\text{sing}}(E,A),$$

which shows (20). With $S = E^{-1}$ (or $S = A^{-1}$) in (20) and noting that $\|[I_n, E^{-1}A]\| = \sqrt{1+\|(E^{-1}A)\|^2}$ we immediately get (22). □

5 Upper bounds for $\theta_{\text{sing}}(E,A)$

In this section we show that Lemma 2.10 leads to an upper bound for $\theta_{\text{sing}}(E,A)$. If a priori a singular matrix pencil is known we obtain with Lemma 2.10 (c), with Lemma 2.10 (b) and (12) and with Lemma 2.10 (a) and (12) the following bounds.

Proposition 5.1. *Let $\mathcal{A}(s) = sE-A$ and $\widetilde{\mathcal{A}}(s) = s\widetilde{E}-\widetilde{A}$ be two matrix pencils with $E,A,\widetilde{E},\widetilde{A} \in \mathbb{C}^{n\times n}$ such that $\mathcal{A}(s)$ is regular and $\widetilde{\mathcal{A}}(s)$ is singular. Let $\mathcal{L},\mathcal{L}_1,\mathcal{L}_2 \subseteq \mathbb{C}^{2n}\setminus\{0\}$ be such that*

$$\mathcal{L}_{\mathcal{A}} = \ker[A,-E] = \mathcal{L}_1 \dotplus \mathcal{L} \quad \text{and} \quad \mathcal{L}_{\widetilde{\mathcal{A}}} = \ker[\widetilde{A},-\widetilde{E}] = \mathcal{L}_2 \dotplus \mathcal{L}.$$

Then

$$\theta_{\text{sing}}(E,A) \leq \theta(\mathcal{L}_{\mathcal{A}},\mathcal{L}_{\widetilde{\mathcal{A}}}) \leq \frac{\theta(\mathcal{L}_1,\mathcal{L}_2)}{\min\{\sigma_{\min}(P_{\mathcal{L}^\perp}P_{\mathcal{L}_1}),\sigma_{\min}(P_{\mathcal{L}^\perp}P_{\mathcal{L}_2})\}}.$$

If $\mathcal{L}_1 = \operatorname{span}\{x_1\}$ and $\mathcal{L}_2 = \operatorname{span}\{x_2\}$ for some non-zero vectors x_1 and x_2, then

$$\theta_{\text{sing}}(E,A) \leq \theta(\mathcal{L}_{\mathcal{A}},\mathcal{L}_{\widetilde{\mathcal{A}}}) = \theta(P_{\mathcal{L}^\perp}\mathcal{L}_1, P_{\mathcal{L}^\perp}\mathcal{L}_2) = \sqrt{1-\frac{|x_1^*P_{\mathcal{L}^\perp}x_2|^2}{\|P_{\mathcal{L}^\perp}x_1\|^2\|P_{\mathcal{L}^\perp}x_2\|^2}}.$$

If, in addition $x_1,x_2 \in \mathcal{L}^\perp$ then

$$\theta_{\text{sing}}(E,A) \leq \theta(\mathcal{L}_{\mathcal{A}},\mathcal{L}_{\widetilde{\mathcal{A}}}) = \sqrt{1-\frac{|x_1^*x_2|^2}{\|x_1\|^2\|x_2\|^2}}.$$

Proposition 5.1 is only applicable, if a singular pencil $\widetilde{\mathcal{A}}(s)$ is known. In the next theorem we construct a singular pencil in terms of the eigenvectors and eigenvalues of a given regular pencil and derive an upper bound for $\theta_{\text{sing}}(E, A)$.

For this let us introduce the following notions. Let $\mathcal{A}(s) = sE - A$ with $E, A \in \mathbb{C}^{n\times n}$ be a matrix pencil. We say that $x \in \mathbb{C}^n \setminus \{0\}$ is an *eigenvector corresponding to the eigenvalue* $\lambda \in \mathbb{C}$, if $\mathcal{A}(\lambda)x = 0$ or, equivalently, if $\left(\begin{smallmatrix} x \\ \lambda x \end{smallmatrix}\right) \in \mathcal{L}_{\mathcal{A}} = \ker[A, -E]$. Moreover, $x \in \mathbb{C}^n \setminus \{0\}$ is an *eigenvector corresponding to the eigenvalue* $\lambda = \infty$, if $Ex = 0$ or, equivalently, if $\left(\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right) \in \mathcal{L}_{\mathcal{A}}$. The set of all eigenvalues of $\mathcal{A}(s)$ is denoted by $\sigma(\mathcal{A})$. Observe that for a singular pencil we have $\sigma(\mathcal{A}) = \mathbb{C} \cup \{\infty\}$.

Recall from [4, 14] that a *Jordan chain of* $\mathcal{A}(s)$ *corresponding to the eigenvalue* λ is a sequence $(x_1, \ldots, x_k) \in \left(\mathbb{C}^n \setminus \{0\}\right)^k$ satisfying

$$(A - \lambda E)x_1 = 0,\ (A - \lambda E)x_2 = Ex_1,\ \ldots, (A - \lambda E)x_k = Ex_{k-1}, \quad \text{if } \lambda \in \mathbb{C}$$

and

$$Ex_1 = 0,\ Ex_2 = Ax_1,\ \ldots, Ex_k = Ax_{k-1}, \quad \text{if } \lambda = \infty.$$

Note that these conditions can be rewritten as

$$\left(\begin{smallmatrix} x_1 \\ \lambda x_1 \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} x_k \\ \lambda x_k + x_{k-1} \end{smallmatrix}\right) \in \mathcal{L}_{\mathcal{A}}, \quad \text{if } \lambda \in \mathbb{C} \tag{26}$$

and

$$\left(\begin{smallmatrix} 0 \\ x_1 \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} x_{k-1} \\ x_k \end{smallmatrix}\right) \in \mathcal{L}_{\mathcal{A}}, \quad \text{if } \lambda = \infty, \tag{27}$$

respectively. An entry in a Jordan chain corresponding to $\lambda \in \sigma(\mathcal{A})$ is called *a root vector of* λ. The linear span $\mathcal{R}_\lambda(\mathcal{A})$ of these vectors is called *the root subspace* of the matrix pencil $\mathcal{A}(s)$:

$$\mathcal{R}_\lambda(\mathcal{A}) := \text{span}\ \{\ x \in \mathbb{C}^n \ \mid x \text{ is a root vector of } \lambda\ \}.$$

The next result is a direct consequence of Theorem 3.7 in [8] for matrix pencils, it also follows from Corollary 3.3 in [24].

Proposition 5.2. *Let* $\mathcal{A}(s) = sE - A$ *with* $E, A \in \mathbb{C}^{n\times n}$ *be a matrix pencil. If there exist* $\lambda, \mu \in \mathbb{C} \cup \{\infty\}$ *with* $\lambda \neq \mu$ *such that*

$$\mathcal{R}_\lambda(\mathcal{A}) \cap \mathcal{R}_\mu(\mathcal{A}) \neq \{0\}, \tag{28}$$

then $\mathcal{A}(s)$ *is singular.*

Remark 5.3. Note that the above definitions of eigenvalues, eigenvectors and Jordan chains essentially differ from those used e.g. in [11, 21, 22, 23], where the eigenvalues are defined via the Kronecker form and the spectrum of any matrix pencil is a finite set. Another recent approach to matrix pencils are the *Wong sequences*, where the root subspaces can be defined via sequences of certain subspaces, see [4, 5, 6].

After these preparations we present our second main result: an upper bound for the gap distance to singularity $\theta_{\text{sing}}(E, A)$. Below, if $\mathcal{L}_2$ is a subspace of a linear space $\mathcal{L}_1 \subseteq \mathbb{C}^{2n}$, we use the symbol $\mathcal{L}_1 \ominus \mathcal{L}_2$ for $\mathcal{L}_1 \cap \mathcal{L}_2^\perp$ in the standard inner product on $\mathbb{C}^{2n}$.

Theorem 5.4. *Let* $E, A \in \mathbb{C}^{n\times n}$ *and* $\mathcal{A}(s) = sE - A$ *be a regular matrix pencil with a Jordan chain* $(x_1, \ldots, x_k)$ *corresponding to an eigenvalue* $\lambda \in \mathbb{C} \cup \{\infty\}$ *and a Jordan chain* $(y_1, \ldots, y_l)$ *corresponding to an eigenvalue* $\mu \in \mathbb{C}$ *with* $\lambda \neq \mu$. *Define*

$$\mathcal{J} := \begin{cases} \text{span}\left\{\left(\begin{smallmatrix} x_1 \\ \lambda x_1 \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} x_k \\ \lambda x_k + x_{k-1} \end{smallmatrix}\right), \left(\begin{smallmatrix} y_1 \\ \mu y_1 \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} y_l \\ \mu y_l + y_{l-1} \end{smallmatrix}\right)\right\}, & \textit{if } \lambda \in \mathbb{C}, \\ \text{span}\left\{\left(\begin{smallmatrix} 0 \\ x_1 \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} x_{k-1} \\ x_k \end{smallmatrix}\right), \left(\begin{smallmatrix} y_1 \\ \mu y_1 \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} y_l \\ \mu y_l + y_{l-1} \end{smallmatrix}\right)\right\}, & \textit{if } \lambda = \infty \end{cases}$$

and let

$$z := \begin{cases} \begin{pmatrix} x_k - y_l \\ \mu x_k - \lambda y_l \end{pmatrix}, & \text{if } \lambda \in \mathbb{C}, \\ \begin{pmatrix} x_k \\ \mu x_k + y_l \end{pmatrix}, & \text{if } \lambda = \infty. \end{cases}$$

Then we have

$$\theta_{\text{sing}}(E, A) \leq \frac{\|P_{\mathcal{L}_{\mathcal{A}}^{\perp}} z\|}{\|P_{\mathcal{J}^{\perp}} z\|}. \tag{29}$$

Furthermore, if $\theta_{\text{sing}}(E, A) = 1$*, then* $P_{\mathcal{L}_{\mathcal{A}} \ominus \mathcal{J}} z = 0$.

Proof. First note that $\mathcal{J} \subseteq \mathcal{L}_{\mathcal{A}}$, because of (26) and (27). Now we prove the following fact: If $\operatorname{span}\{z\} + \mathcal{J} \subseteq \mathcal{L}_{\widetilde{\mathcal{A}}}$ for some matrix pencil $\widetilde{\mathcal{A}}(s) = s\widetilde{E} - \widetilde{A}$ with $\widetilde{E}, \widetilde{A} \in \mathbb{C}^{n \times n}$, then the pencil $\widetilde{\mathcal{A}}(s)$ is singular. Indeed, for $\lambda \in \mathbb{C}$ we have

$$\begin{pmatrix} x_k - y_l \\ \mu x_k - \lambda y_l \end{pmatrix} = (\mu - \lambda) \begin{pmatrix} \eta \\ \lambda \eta + x_k \end{pmatrix} = (\mu - \lambda) \begin{pmatrix} \eta \\ \mu \eta + y_l \end{pmatrix}, \quad \eta := \frac{y_l - x_k}{\lambda - \mu}. \tag{30}$$

Note that $\eta \neq 0$, otherwise $y_l = x_k \in \mathcal{R}_\lambda(\mathcal{A}) \cap \mathcal{R}_\mu(\mathcal{A})$, which contradicts regularity of $\mathcal{A}(s)$ by Proposition 5.2. Hence, it follows from (30) that

$$\begin{pmatrix} \eta \\ \lambda \eta + x_k \end{pmatrix} = \begin{pmatrix} \eta \\ \mu \eta + y_l \end{pmatrix} \in \mathcal{L}_{\widetilde{\mathcal{A}}},$$

which implies that $(x_1, \ldots, x_k, \eta)$ is a Jordan chain of $\widetilde{\mathcal{A}}(s)$ corresponding to λ and $(y_1, \ldots, y_l, \eta)$ is a Jordan chain of $\widetilde{\mathcal{A}}(s)$ corresponding to μ. We thus obtain that $\eta \in \mathcal{R}_\lambda(\widetilde{\mathcal{A}}) \cap \mathcal{R}_\mu(\widetilde{\mathcal{A}})$ and $\widetilde{\mathcal{A}}(s)$ is singular by Proposition 5.2. For $\lambda = \infty$ we obtain similarly that $(y_1, \ldots, y_l, x_k)$ is a Jordan chain of $\widetilde{\mathcal{A}}(s)$ corresponding to μ, while $(x_1, \ldots, x_k)$ is a Jordan chain of $\widetilde{\mathcal{A}}(s)$ corresponding to ∞ and singularity of $\widetilde{\mathcal{A}}(s)$ follows, again by Proposition 5.2.

Now let $\tilde{z} := P_{\mathcal{L}_{\mathcal{A}} \ominus \mathcal{J}} z \neq 0$ and $\mathcal{L} := \mathcal{L}_{\mathcal{A}} \ominus \operatorname{span}\{\tilde{z}\}$. Note that $\mathcal{J} \subseteq \mathcal{L}$. Then, by what has been proved above, we have $z \notin \mathcal{L}$, otherwise the pencil $\mathcal{A}(s)$ would be singular. Consequently, $\dim(\operatorname{span}\{z\} + \mathcal{L}) = n$ and by [7, Theorem 3.3] there exist $\widetilde{E}, \widetilde{A} \in \mathbb{C}^{n \times n}$ such that $\widetilde{\mathcal{A}}(s) = s\widetilde{E} - \widetilde{A}$ satisfies $\mathcal{L}_{\widetilde{\mathcal{A}}} = \operatorname{span}\{z\} + \mathcal{L}$. By the first part of the proof, the pencil $\widetilde{\mathcal{A}}(s)$ is singular.

Assume that $z \neq 0$. Then Proposition 5.1 applied to $\mathcal{L}_1 = \operatorname{span}\{\tilde{z}\}$ and $\mathcal{L}_2 = \operatorname{span}\{z\}$ yields

$$\theta_{\text{sing}}(E, A) \leq \sqrt{1 - \frac{|z^* P_{\mathcal{L}^\perp} \tilde{z}|^2}{\|P_{\mathcal{L}^\perp} z\|^2 \|P_{\mathcal{L}^\perp} \tilde{z}\|^2}}. \tag{31}$$

The definition of $\mathcal{L}$ implies that $\mathcal{L}^\perp = \operatorname{span}\{\tilde{z}\} \oplus \mathcal{L}_{\mathcal{A}}^\perp$ and hence $P_{\mathcal{L}^\perp} \tilde{z} = \tilde{z} = P_{\mathcal{L}_{\mathcal{A}} \ominus \mathcal{J}} z$. Further, as $P_{\operatorname{span}\{\tilde{z}\}} = \frac{\tilde{z}\tilde{z}^*}{\|\tilde{z}\|^2}$ we find that

$$\|P_{\operatorname{span}\{\tilde{z}\}} z\| = \frac{|\tilde{z}^* z|}{\|\tilde{z}\|} = \frac{|z^* P_{\mathcal{L}_{\mathcal{A}} \ominus \mathcal{J}} z|}{\|P_{\mathcal{L}_{\mathcal{A}} \ominus \mathcal{J}} z\|} = \|P_{\mathcal{L}_{\mathcal{A}} \ominus \mathcal{J}} z\|,$$

thus

$$\|P_{\mathcal{L}^\perp} z\|^2 = \|P_{\mathcal{L}_{\mathcal{A}} \ominus \mathcal{J}} z\|^2 + \|P_{\mathcal{L}_{\mathcal{A}}^\perp} z\|^2. \tag{32}$$

A combination of (32) with (31) gives

$$\begin{aligned} \theta_{\text{sing}}(E, A) &\leq \sqrt{1 - \frac{|z^* P_{\mathcal{L}_{\mathcal{A}} \ominus \mathcal{J}} z|^2}{\|P_{\mathcal{L}^\perp} z\|^2 \|P_{\mathcal{L}_{\mathcal{A}} \ominus \mathcal{J}} z\|^2}} = \sqrt{\frac{\|P_{\mathcal{L}^\perp} z\|^2 - \|P_{\mathcal{L}_{\mathcal{A}} \ominus \mathcal{J}} z\|^2}{\|P_{\mathcal{L}^\perp} z\|^2}} \\ &= \frac{\|P_{\mathcal{L}_{\mathcal{A}}^\perp} z\|}{\sqrt{\|P_{\mathcal{L}_{\mathcal{A}} \ominus \mathcal{J}} z\|^2 + \|P_{\mathcal{L}_{\mathcal{A}}^\perp} z\|^2}}. \end{aligned} \tag{33}$$

Since $\mathcal{J}^\perp = (\mathcal{L}_\mathcal{A} \ominus \mathcal{J}) \oplus \mathcal{L}_\mathcal{A}^\perp$ holds, we have

$$\|P_{\mathcal{J}^\perp} z\|^2 = \|P_{\mathcal{L}_\mathcal{A} \ominus \mathcal{J}} z\|^2 + \|P_{\mathcal{L}_\mathcal{A}^\perp} z\|^2$$

and with this, (33) can be written as

$$\theta_{\text{sing}}(E, A) \le \frac{\|P_{\mathcal{L}_\mathcal{A}^\perp} z\|}{\|P_{\mathcal{J}^\perp} z\|}.$$

If $\tilde{z} = P_{\mathcal{L}_\mathcal{A} \ominus \mathcal{J}} z = 0$, then the upper bound in (33), and hence (29), is trivially satisfied, which finishes the proof of (29).

Assume now that $\theta_{\text{sing}}(E, A) = 1$. Then (33) immediately gives that $P_{\mathcal{L}_\mathcal{A} \ominus \mathcal{J}} z = 0$ as claimed. □

Remark 5.5. We stress that the Jordan chains $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_l)$ in Theorem 5.4 are not required to be maximal (cf. also Example 5.6). Manipulating with these chains may lead to different bounds on $\theta_{\text{sing}}(E, A)$.

Further, observe that the proof is based on the construction of an $(n-1)$-dimensional subspace $\mathcal{L}$ with $\mathcal{J} \subseteq \mathcal{L} \subseteq \mathcal{L}_\mathcal{A}$ and an element $\tilde{z} \in \mathcal{L}_\mathcal{A} \setminus \mathcal{L}$ in order to get

$$\mathcal{L}_\mathcal{A} = \mathcal{L} \dotplus \text{span}\{\tilde{z}\}, \quad \mathcal{L}_{\widetilde{\mathcal{A}}} = \mathcal{L} \dotplus \text{span}\{z\} \tag{34}$$

for some singular matrix pencil $\widetilde{\mathcal{A}}(s) = s\widetilde{E} - \widetilde{A}$. Note that for every such $\mathcal{L}$ and $\tilde{z}$ the inequality (31) holds. However, one may also easily see that the specific choice of $\mathcal{L}$ and $\tilde{z}$ constructed in the proof provide an optimal bound in (31) (for fixed z and $\mathcal{J}$).

Finally, we note that (34) essentially says that the singular pencil $\widetilde{\mathcal{A}}(s)$ is a rank one perturbation of the original regular pencil $\mathcal{A}(s)$. We refer the reader to [11, 21, 22] for other studies on low rank perturbations of singular pencils.

We illustrate Theorem 5.4 by the following example, where the right hand side of (29) can be made arbitrarily small.

Example 5.6. Consider the regular matrix pencil

$$\mathcal{A}(s) = sE - A = s\begin{bmatrix} 0 & 1 & 0 \\ -\varepsilon & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & \varepsilon \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \varepsilon \in (0, 1).$$

Then

$$\mathcal{L}_\mathcal{A} = \ker[A, -E] = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \varepsilon \\ 0 \end{pmatrix} \right\},$$

which implies $\sigma(\mathcal{A}) = \{0, \infty\}$ with eigenvectors $(0, 0, 1)^\top$ at ∞ and $(1, 0, 0)^\top$ at 0. The Kronecker canonical form (17) of $\mathcal{A}(s)$ is given by $\text{diag}(s, sN_2 - I_2)$. Then with

$$\mathcal{J} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad z = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \|z\| = \|P_{\mathcal{J}^\perp} z\| = \sqrt{2},$$

and

$$\|P_{\mathcal{L}_\mathcal{A}^\perp} z\| = \|z - P_{\mathcal{L}_\mathcal{A}} z\| = \frac{\sqrt{2}\varepsilon}{\sqrt{2 + \varepsilon^2}}$$

the bound (29) from Theorem 5.4 gives

$$\theta_{\text{sing}}(E, A) \le \frac{\varepsilon}{\sqrt{2 + \varepsilon^2}}.$$

6 Distance and gap distance to singularity

Here we derive some relations between the distance to singularity $\delta(E, A)$ from (1) and the gap distance to singularity $\theta_{\text{sing}}(E, A)$, which lead to new lower bounds for $\theta_{\text{sing}}(E, A)$. First note the following scaling property of $\delta(E, A)$.

Proposition 6.1. *Let $\mathcal{A}(s) = sE - A$ be a regular matrix pencil with $E, A \in \mathbb{C}^{n\times n}$, then for any invertible $S, T \in \mathbb{C}^{n\times n}$ we have*

$$\frac{\delta(SET, SAT)}{\|S\|\|T\|} \leq \delta(E, A) \leq \delta(SET, SAT)\|S^{-1}\|\|T^{-1}\|. \tag{35}$$

In particular, if $\tau \in \mathbb{C} \setminus \{0\}$, then $\delta(\tau E, \tau A) = |\tau|\delta(E, A)$.

Proof. As $\mathcal{A}(s)$ is regular, $\det(sSET - SAT) = \det S \det T \det(sE - A) \neq 0$ and therefore the pencil $sSET - SAT$ is regular. Let $\Delta E_k, \Delta A_k \in \mathbb{C}^{n\times n}$ and $s(E + \Delta E_k) - (A + \Delta A_k)$ be a sequence of singular matrix pencils with

$$\delta(E, A) = \lim_{k\to\infty} \|[\Delta E_k, \Delta A_k]\|_F.$$

Then

$$\delta(SET, SAT) \leq \lim_{k\to\infty} \|[S\Delta E_k T, S\Delta A_k T]\|_F \leq \|S\|\|T\|\delta(E, A),$$

where the last inequality follows from Lemma 2.5. This proves the lower bound for $\delta(E, A)$ in (35). The upper bound follows from the same inequality:

$$\delta\big(S^{-1}(SET)T^{-1}, S^{-1}(SAT)T^{-1}\big) \leq \|S^{-1}\|\|T^{-1}\|\delta(SET, SAT). \qquad \square$$

Next we show that the distance to singularity can be estimated by the gap distance to singularity.

Theorem 6.2. *Let $\mathcal{A}(s) = sE - A$ be a regular matrix pencil with $E, A \in \mathbb{C}^{n\times n}$, then*

$$\frac{\delta(E, A)}{\|[E, A]\|_F} \leq \theta_{\text{sing}}(E, A) \tag{36}$$

and

$$\big(\sigma_{\min}([E, A]) - \delta(E, A)\big) \cdot \theta_{\text{sing}}(E, A) \leq \delta(E, A). \tag{37}$$

Proof. Let $\widetilde{\mathcal{A}}(s) = s\widetilde{E} - \widetilde{A}$ be a matrix pencil with $\widetilde{E}, \widetilde{A} \in \mathbb{C}^{n\times n}$ such that

$$\theta(\mathcal{A}, \widetilde{\mathcal{A}}) < \frac{\delta(E, A)}{\|[E, A]\|_F}. \tag{38}$$

By regularity of $\mathcal{A}(s)$ and Lemma 3.3 we have that $\left[\begin{smallmatrix} A^* \\ -E^* \end{smallmatrix}\right]$ has full column rank. Also note that (38) together with (2) implies that $\theta(\mathcal{A}, \widetilde{\mathcal{A}}) < 1$. Hence, by Proposition 2.6, there exist $\widehat{E}, \widehat{A} \in \mathbb{C}^{n\times n}$ with

$$\operatorname{ran} \begin{bmatrix} \widehat{A}^* \\ -\widehat{E}^* \end{bmatrix} = \operatorname{ran} \begin{bmatrix} \widetilde{A}^* \\ -\widetilde{E}^* \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \widehat{A}^* \\ -\widehat{E}^* \end{bmatrix} = P_{\mathcal{L}_{\widetilde{\mathcal{A}}}^{\perp}} \begin{bmatrix} A^* \\ -E^* \end{bmatrix}, \tag{39}$$

where $\mathcal{L}_{\widetilde{\mathcal{A}}} = \ker[\widetilde{A}, -\widetilde{E}]$, and we have, invoking Lemma 3.3 (a),

$$\left\| \begin{bmatrix} A^* - \widehat{A}^* \\ -E^* + \widehat{E}^* \end{bmatrix} \right\|_F \leq \theta(\mathcal{L}_{\mathcal{A}}^{\perp}, \mathcal{L}_{\widetilde{\mathcal{A}}}^{\perp}) \left\| \begin{bmatrix} A^* \\ -E^* \end{bmatrix} \right\|_F = \theta(\mathcal{L}_{\mathcal{A}}, \mathcal{L}_{\widetilde{\mathcal{A}}}) \left\| \begin{bmatrix} A^* \\ -E^* \end{bmatrix} \right\|_F.$$

Then, from (38) we obtain

$$\|[E-\widehat{E}, A-\widehat{A}]\|_F = \left\| \begin{bmatrix} A^* - \widehat{A}^* \\ -E^* + \widehat{E}^* \end{bmatrix} \right\|_F \leq \theta(\mathcal{A}, \widetilde{\mathcal{A}}) \left\| \begin{bmatrix} A^* \\ -E^* \end{bmatrix} \right\|_F = \theta(\mathcal{A}, \widetilde{\mathcal{A}}) \, \|[E, A]\|_F < \delta(E, A).$$

Hence, by the definition of $\delta(E, A)$, the pencil $s\widehat{E} - \widehat{A}$ is regular and by Lemma 3.3 and (39) the pencil $\widetilde{\mathcal{A}}(s)$ is regular as well and (36) is shown.

We show (37). Note that by (2) we have $\delta(E, A) \leq \sigma_{\min}([E, A])$. If $\delta(E, A) = \sigma_{\min}([E, A])$, then (37) is true, so assume that $\delta(E, A) < \sigma_{\min}([E, A])$. Let $0 < \varepsilon < \sigma_{\min}([E, A]) - \delta(E, A)$ and let $\widetilde{\mathcal{A}}(s) = s\widetilde{E} - \widetilde{A}$ with $\widetilde{E}, \widetilde{A} \in \mathbb{C}^{n \times n}$ be a singular pencil such that $[\widetilde{E}, \widetilde{A}] \neq 0$ and $\left\| \left[\widetilde{E} - E, \widetilde{A} - A \right] \right\|_F \leq \delta(E, A) + \varepsilon$. Hence

$$\begin{aligned}
\theta_{\text{sing}}(E, A) &\leq \theta(\mathcal{A}, \widetilde{\mathcal{A}}) = \theta(\mathcal{L}_{\mathcal{A}}^{\perp}, \mathcal{L}_{\widetilde{\mathcal{A}}}^{\perp}) = \theta\left(\operatorname{ran} \begin{bmatrix} A^* \\ -E^* \end{bmatrix}, \operatorname{ran} \begin{bmatrix} \widetilde{A}^* \\ -\widetilde{E}^* \end{bmatrix} \right) \\
&\overset{(14)}{\leq} \frac{\left\| \begin{bmatrix} A^* - \widetilde{A}^* \\ -E^* + \widetilde{E}^* \end{bmatrix} \right\|}{\min\left\{ \sigma_{\min}\left(\begin{bmatrix} A^* \\ -E^* \end{bmatrix} \right), \sigma_{\min}\left(\begin{bmatrix} \widetilde{A}^* \\ -\widetilde{E}^* \end{bmatrix} \right) \right\}} \\
&= \frac{\left\| \left[\widetilde{E} - E, \widetilde{A} - A \right] \right\|}{\min\left\{ \sigma_{\min}([E, A]), \sigma_{\min}\left([\widetilde{E}, \widetilde{A}] \right) \right\}} \leq \frac{\delta(E, A) + \varepsilon}{\min\left\{ \sigma_{\min}([E, A]), \sigma_{\min}\left([\widetilde{E}, \widetilde{A}] \right) \right\}}.
\end{aligned}$$

According to Mirsky's Theorem [27, Theorem IV.4.11] the inequality

$$|\sigma_{\min}([E, A]) - \sigma_{\min}([\widetilde{E}, \widetilde{A}])| \leq \left\| \left[\widetilde{E} - E, \widetilde{A} - A \right] \right\|$$

holds, and by use of the matrix norm inequality $\| \cdot \| \leq \| \cdot \|_F$ one finds that

$$0 < \sigma_{\min}([E, A]) - \delta(E, A) - \varepsilon \leq \sigma_{\min}([\widetilde{E}, \widetilde{A}]).$$

As a consequence,

$$\theta_{\text{sing}}(E, A) \leq \frac{\delta(E, A) + \varepsilon}{\sigma_{\min}([E, A]) - \delta(E, A) - \varepsilon}$$

and for $\varepsilon \to 0$ we obtain (37). □

Using lower bounds for $\delta(E, A)$ from [9, Section 5.2] and from [3],

$$\delta(E, A) \geq \max_{(s,c) \in S^1} \sigma_{\min}(sE - cA) \quad \text{and} \quad \delta(E, A) \geq \frac{\sigma_{\min}(W_n(E, A))}{\sqrt{1 + \cos\left(\frac{\pi}{n+1}\right)}},$$

where S^1 is the unit circle in $\mathbb{C}^2$, we obtain the following corollary.

Corollary 6.3. *Let $\mathcal{A}(s) = sE - A$ be a regular matrix pencil with $E, A \in \mathbb{C}^{n \times n}$, then*

$$\theta_{\text{sing}}(E, A) \geq \frac{\max_{(s,c) \in S^1} \sigma_{\min}(sE - cA)}{\|[E, A]\|_F} \quad \text{and} \quad \theta_{\text{sing}}(E, A) \geq \frac{\sigma_{\min}(W_n(E, A))}{\sqrt{1 + \cos\left(\frac{\pi}{n+1}\right)} \|[E, A]\|_F}.$$

The inequalities (36) and (37) also yield the following.

Corollary 6.4. *For all $E, A \in \mathbb{C}^{n\times n}$ we have*

$$\frac{\theta_{\text{sing}}(E,A)}{1+\theta_{\text{sing}}(E,A)}\sigma_{\min}([E,A]) \le \delta(E,A) \le \theta_{\text{sing}}(E,A)\|[E,A]\|_F.$$

To conclude this section we improve the lower bound for $\theta_{\text{sing}}(E,A)$ for the case that E or A is invertible.

Theorem 6.5. *Let $\mathcal{A}(s) = sE - A$ be a regular pencil with $E, A \in \mathbb{C}^{n\times n}$. If E is invertible, then*

$$\theta_{\text{sing}}(E,A) \ge \frac{\max\{1, \widetilde{\sigma}_{\min}(E^{-1}A)\}}{\sqrt{1+\|E^{-1}A\|^2}}. \tag{40}$$

If A is invertible, then

$$\theta_{\text{sing}}(E,A) \ge \frac{\max\{1, \widetilde{\sigma}_{\min}(A^{-1}E)\}}{\sqrt{1+\|A^{-1}E\|^2}}. \tag{41}$$

Proof. We consider the case that E is invertible; the case of invertible A is analogous and omitted. Consider the distance to singularity in spectral norm,

$$\delta_2(E,A) = \inf_{\Delta E, \Delta A \in \mathbb{C}^{n\times n}} \left\{ \|[\Delta E, \Delta A]\| \;\middle|\; s(E+\Delta E) - (A+\Delta A) \text{ is singular } \right\}.$$

Note that $\delta_2(E,A) \le \delta(E,A)$ for all $E, A \in \mathbb{C}^{n\times n}$ as a consequence of the matrix norm inequality $\|\cdot\| \le \|\cdot\|_F$. Adapting the proof of Theorem 6.2 it is straightforward that

$$\frac{\delta_2(E,A)}{\|[E,A]\|} \le \theta_{\text{sing}}(E,A) \tag{42}$$

and

$$(\sigma_{\min}([E,A]) - \delta_2(E,A)) \cdot \theta_{\text{sing}}(E,A) \le \delta_2(E,A).$$

We prove that for all $M \in \mathbb{C}^{n\times n}$

$$\delta_2(I_n, M) \ge \max\{1, \widetilde{\sigma}_{\min}(M)\}. \tag{43}$$

Let $\Delta E, \Delta A \in \mathbb{C}^{n\times n}$ be such that $\|[\Delta E, \Delta A]\| < \max\{1, \widetilde{\sigma}_{\min}(M)\}$. We consider two cases.

Case 1: $\max\{1, \widetilde{\sigma}_{\min}(M)\} = 1$. Then $\|\Delta E\| \le \|[\Delta E, \Delta A]\| < 1$ and hence $I + \Delta E$ is invertible. Therefore, the pencil $s(I_n + \Delta E) - (M + \Delta A)$ is regular.

Case 2: $\max\{1, \widetilde{\sigma}_{\min}(M)\} = \sigma_{\min}(M)$. Then $\|\Delta A\| \le \|[\Delta E, \Delta A]\| < \sigma_{\min}(M)$ and hence

$$\|M^{-1}\Delta A\| \le \|M^{-1}\| \cdot \|\Delta A\| = \frac{\|\Delta A\|}{\sigma_{\min}(M)} < 1.$$

Therefore, $I + M^{-1}\Delta A$ is invertible, by which $M + \Delta A$ is invertible and the pencil $s(I_n + \Delta E) - (M + \Delta A)$ is regular.

This shows (43). With $S = E^{-1}$ in Theorem 4.3 we obtain

$$\theta_{\text{sing}}(E,A) = \theta_{\text{sing}}(I_n, E^{-1}A) \overset{(42)}{\ge} \frac{\delta_2(I_n, E^{-1}A)}{\|[I_n, E^{-1}A]\|} \overset{(43)}{\ge} \frac{\max\{1, \widetilde{\sigma}_{\min}(E^{-1}A)\}}{\sqrt{1+\|E^{-1}A\|^2}},$$

where for the last inequality we also used that $\left\|[I_n, E^{-1}A]\right\| = \sqrt{1+\|E^{-1}A\|^2}$. □

7 Applications

7.1 Example 4 from [9]

We consider the regular matrix pencil

$$sE - A = s \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad 0 \le \varepsilon < 1.$$

The singular values of $[E, A]$ are given by $\{1, \sqrt{1+\varepsilon^2}, \sqrt{2}\}$, hence $\sigma_{\min}([E,A]) = 1$ and $\|[E,A]\| = \sqrt{2}$. From [9] we have $\delta(E,A) = \varepsilon$ with

$$\Delta A := \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\varepsilon & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Delta E := 0$$

and a singular pencil $\widetilde{\mathcal{A}}(s) = s\widetilde{E} - \widetilde{A}$ is given by

$$s\widetilde{E} - \widetilde{A} = s \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have $\|[E - \widetilde{E}, A - \widetilde{A}]\| = \varepsilon$ and $\sigma_{\min}\left([\widetilde{E}, \widetilde{A}]\right) = 1$. Then (14) together with Theorem 6.2 implies

$$\frac{\varepsilon}{\sqrt{2}} = \frac{\delta(E,A)}{\|[E,A]\|} \le \theta_{\text{sing}}(E,A) \le \frac{\|[E - \widetilde{E}, A - \widetilde{A}]\|}{\min\left\{\sigma_{\min}([E,A]), \sigma_{\min}\left([\widetilde{E}, \widetilde{A}]\right)\right\}} = \varepsilon.$$

However, in this case, Proposition 5.1 and Theorem 6.5 yield better bounds. Since $\mathcal{A}(s)$ and $\widetilde{\mathcal{A}}(s)$ differ only by one row, we consider

$$\mathcal{L} = \operatorname{ran} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad x_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \varepsilon \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then $\mathcal{L}_{\mathcal{A}} = \mathcal{L} \oplus \operatorname{span}\{x_1\}$, $\mathcal{L}_{\widetilde{\mathcal{A}}} = \mathcal{L} \oplus \operatorname{span}\{x_2\}$ and hence Proposition 5.1 gives

$$\theta_{\text{sing}}(E,A) \le \sqrt{1 - \frac{|x_1^* x_2|^2}{\|x_1\|^2 \|x_2\|^2}} = \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}.$$

Applying (41) from Theorem 6.5 gives the improved lower bound

$$\theta_{\text{sing}}(E,A) \ge \frac{\max\{1, \widetilde{\sigma}_{\min}(A^{-1}E)^2\}}{\sqrt{1 + \|A^{-1}E\|^2}} = \frac{1}{\sqrt{1 + (1/\varepsilon)^2}} = \frac{\varepsilon}{\sqrt{1+\varepsilon^2}},$$

thus

$$\theta_{\text{sing}}(E,A) = \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}. \tag{44}$$

7.2 Example for regularity ensured by $\theta_{\text{sing}}(E,A)$ but not by $\delta(E,A)$

We show that there are classes of matrix pencils where for the investigation of regularity $\theta_{\text{sing}}(E,A)$ is more suitable than $\delta(E,A)$. Here we consider a family of matrix pencils that have a gap distance less than $\theta_{\text{sing}}(E,A)$, but the Frobenius norm of the coefficient matrices of the pencils gets arbitrarily large. Therefore, $\theta_{\text{sing}}(E,A)$ can be used to guarantee regularity of this family of matrix pencils, while $\delta(E,A)$ is not suitable for this. Consider the regular matrix pencil $\mathcal{A}(s) = sE - A$ from Section 7.1 and the pencils

$$\widetilde{\mathcal{A}}(s) = s\widetilde{E} - \widetilde{A} = s\begin{bmatrix} 0 & \tau_1 & 0 \\ \tau_2 a_1 & \tau_2 a_2 & \tau_2 a_3 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \tau_1 & 0 & 0 \\ \tau_2 a_2 & \tau_2 a_4 & 0 \\ 0 & 0 & \tau_3 \end{bmatrix}$$

with parameters $\tau_1, \tau_2, \tau_3 \in \mathbb{R}\backslash\{0\}$ and $a_1, a_2, a_3, a_4 \in \mathbb{R}$. We seek to investigate regularity of the pencils $\widetilde{\mathcal{A}}(s)$. To this end, we use that $\ker[A, -E]^\perp = \operatorname{ran}\left[\begin{smallmatrix} A^* \\ -E^* \end{smallmatrix}\right]$ and hence we compute the gap distance between

$$\operatorname{ran}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad \operatorname{ran}\begin{bmatrix} \tau_1 & \tau_2 a_2 & 0 \\ 0 & \tau_2 a_4 & 0 \\ 0 & 0 & \tau_3 \\ 0 & -\tau_2 a_1 & 0 \\ -\tau_1 & -a_2\tau_2 & 0 \\ 0 & -\tau_2 a_3 & 0 \end{bmatrix}.$$

Subtracting in the representing matrix the first column times $-\frac{\tau_2 a_2}{\tau_1}$ from the second column we can rewrite the second subspace as follows:

$$\operatorname{ran}\begin{bmatrix} \tau_1 & \tau_2 a_2 & 0 \\ 0 & \tau_2 a_4 & 0 \\ 0 & 0 & \tau_3 \\ 0 & -\tau_2 a_1 & 0 \\ -\tau_1 & -a_2\tau_2 & 0 \\ 0 & -\tau_2 a_3 & 0 \end{bmatrix} = \operatorname{ran}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \tau_2 a_4 & 0 \\ 0 & 0 & 1 \\ 0 & -\tau_2 a_1 & 0 \\ -1 & 0 & 0 \\ 0 & -\tau_2 a_3 & 0 \end{bmatrix} = \operatorname{ran}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix}}_{=:\mathcal{L}} \oplus \operatorname{span}\left\{\underbrace{\begin{pmatrix} 0 \\ \tau_2 a_4 \\ 0 \\ -\tau_2 a_1 \\ 0 \\ -\tau_2 a_3 \end{pmatrix}}_{=:x_2}\right\}.$$

With $x_1 := (0, \varepsilon, 0, 0, 0, -1)^\top$ we may observe that $\operatorname{ran}\left[\begin{smallmatrix} A^* \\ -E^* \end{smallmatrix}\right] = \mathcal{L} \oplus \operatorname{span}\{x_1\}$, hence an application of Proposition 5.1 yields

$$\theta(\mathcal{A}, \widetilde{\mathcal{A}}) = \theta\left(\mathcal{L}_{\mathcal{A}}^\perp, \mathcal{L}_{\widetilde{\mathcal{A}}}^\perp\right) = \sqrt{1 - \frac{|x_1^* x_2|^2}{\|x_1\|^2 \|x_2\|^2}} = \sqrt{1 - \frac{(a_3 + \varepsilon a_4)^2}{(1+\varepsilon^2)(a_1^2 + a_3^2 + a_4^2)}}. \tag{45}$$

Regularity of $\widetilde{\mathcal{A}}(s)$ is guaranteed if we choose a_1, a_2, a_3, a_4 such that (45) is less than $\theta_{\text{sing}}(E,A) = \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}$ (cf. (44)), which is equivalent to

$$\frac{(a_3 + \varepsilon a_4)^2}{a_1^2 + a_3^2 + a_4^2} > 1.$$

This condition is independent of the parameters τ_1, τ_2, τ_3. On the other hand, choosing these parameters large enough we see that the Frobenius norm $\|[\widetilde{E} - E, \widetilde{A} - A]\|_F$ becomes arbitrarily large, eventually exceeding $\delta(E,A) = \varepsilon$; in other words, for these parameters regularity of $\widetilde{\mathcal{A}}(s)$ cannot be concluded by investigating $\delta(E,A)$ only.

7.3 Pencils connected with linear systems

In this subsection we show how the properties of $\theta_{\text{sing}}(E, A)$ can be combined with structured assumptions on the matrix pencil. We investigate a recent class of pencils associated with linear time-invariant dissipative Hamiltonian descriptor systems, see [23]. Let $\mathcal{A}(s) = sE - A$ with $E, A \in \mathbb{C}^{n\times n}$ be such that there exist $Q, L \in \mathbb{C}^{n\times n}$ with

$$A = LQ, \quad E^*Q = Q^*E \geq 0, \quad L + L^* \leq 0, \quad sE - Q \text{ is regular.} \tag{46}$$

It was proved in [23] that if $\mathcal{A}(s)$ is singular then all left minimal indices of $\mathcal{A}(s)$ are zero, i.e., $\gamma_i = 1$ for all $i = 1, \ldots, n_\gamma$ in the Kronecker canonical form (17). Additionally, all right minimal indices of $\mathcal{A}(s)$ are at most one and there are several other constraints on the Kronecker canonical form, see [23]. Moreover, it was also shown in [23] that a singular pencil $\mathcal{A}(s) = sE - A$ satisfying

$$AE^* = EA^* \tag{47}$$

has only zero left minimal indices. Combining this with Proposition 3.5 we get the following result.

Corollary 7.1. *Let $E, A, \widetilde{E}, \widetilde{A} \in \mathbb{C}^{n\times n}$ and let $\mathcal{A}(s) = sE - A$ be a regular matrix pencil. Then for the pencil $\widetilde{\mathcal{A}}(s) = s\widetilde{E} - \widetilde{A}$ the following holds true:*

(a) *If $\widetilde{\mathcal{A}}(s)$ is singular and satisfies* (46) *or* (47)*, then $\theta(\mathcal{A}, \widetilde{\mathcal{A}}) = 1$.*

(b) *If $\theta_{\text{sing}}(E, A) < 1$, then there exists a singular pencil $\widetilde{\mathcal{A}}(s)$ with $\theta(\mathcal{A}, \widetilde{\mathcal{A}}) < 1$ which does neither satisfy* (46) *nor* (47)*.*

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