An optimal shape design problem for plates

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Abstract

We consider an optimal shape design problem for the plate equation, where the variable thickness of the plate is the design function. This problem can be formulated as a control in the coefficient PDE-constrained optimal control problem with additional control and state constraints. The state constraints are treated with a Moreau-Yosida regularization of a dual problem. Variational discretization is employed for discrete approximation of the optimal control problem. For discretization of the state in the mixed formulation we compare the standard continuous piecewise linear ansatz with a piecewise constant one based on the lowest-order Raviart-Thomas mixed finite element. We derive bounds for the discretization and regularization errors and also address the coupling of the regularization parameter and finite element grid size. The numerical solution of the optimal control problem is realized with a semismooth Newton algorithm. Numerical examples show the performance of the method.

Key words. elliptic optimal control problem, optimal shape design, pointwise state constraints, Moreau-Yosida regularization, error estimates.

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1 Introduction

This work is devoted to the numerical analysis and solution of an optimal control problem for a plate with variable thickness. The state equation

$$\Delta(u^3 \Delta y) = f \qquad \text{in } \Omega$$

can be used to model the relation between the (small) deflection y and the thickness u of a (thin) plate under the force of a transverse load f. The domain $\Omega \subset \mathbb{R}^2$ represents the unloaded plate's midplane and we assume its boundary to be simply supported, i.e.,

$$y = \Delta y = 0 \qquad \text{on } \partial\Omega.$$

Invoking a pointwise lower bound on the state y and pointwise almost everywhere box constraints on the control u, and minimizing the volume of the plate given by the cost functional

$$\int_\Omega u(x)\,\mathrm{d}x$$

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lead to a control in coefficients problem, which can also be viewed as an optimal shape design problem. In [Sprekels and Tiba 1999] this optimization problem is analyzed using a transformation to a dual problem.

Building upon this duality, it is our aim to solve the control problem with a finite element approximation that is suitable with regard to the necessary optimality conditions. To this end we compare variational discretization of the control problem (cf. [Hinze 2005]) based on either the lowest-order Raviart-Thomas mixed finite element or piecewise linear continuous finite elements for the discretization of the Poisson equation. The pointwise state constraints, which are responsible for the low regularity of the Lagrange multiplier, are treated with the help of Moreau-Yosida regularization (cf. [Hintermüller and Kunisch 2006]). The numerical solution to the control problem is computed via a path-following algorithm that simultaneously refines the mesh and follows the homotopy generated by the regularization parameter. The resulting subproblems are solved by a semismooth Newton method.

To the best of the authors' knowledge this is the first contribution to numerical analysis of a "control in the coefficients" problem for biharmonic equations including state constraints. The mathematical techniques applied in the numerical analysis of the regularized control problem are related to the relaxation of state constraints as proposed in [Hintermüller and Hinze 2009] and to [Deckelnick, Günther and Hinze 2009], where the Raviart-Thomas mixed finite element was employed in the context of gradient constraints.

The present work is organized as follows. In Section 2 the optimal control problem and its dual problem are introduced. The regularization of the dual problem is investigated in Section 3. Section 4 deals with the discretization of the regularized problems and with the related error bounds. Finally, in Section 5 the original control problem is solved with a Newton-type path-following method. Numerical examples are presented which validate our analytical findings.

2 The optimization problems

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded domain with smooth boundary $\partial\Omega$. The Dirichlet problem for the Poisson equation

$$\begin{aligned} -\Delta y &= g \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.1}$$

admits for every $g \in L^2(\Omega)$ a unique solution $y := T(g) \in V := H^2(\Omega) \cap H_0^1(\Omega)$ satisfying

$$\|y\|_{H^2(\Omega)} \leq C \|g\|_{L^2(\Omega)} . \tag{2.2}$$

In order to define the control problems considered in this paper we introduce the admissible sets for controls and states according to

$$\begin{aligned} U_{\mathrm{ad}} &:= \left\{ u \in L^\infty(\Omega) \,\middle|\, m \leq u \leq M \quad \text{a.e. in } \Omega \right\}, \\ L_{\mathrm{ad}} &:= \left\{ l \in L^\infty(\Omega) \,\middle|\, M^{-3} \leq l \leq m^{-3} \quad \text{a.e. in } \Omega \right\}, \\ Y_{\mathrm{ad}} &:= \left\{ y \in C(\bar{\Omega}) \,\middle|\, y \geq -\tau \quad \text{in } \Omega \right\}, \end{aligned}$$

where $\tau > 0$ and $0 < m < M$ are positive real constants. For a given $f \in L^2(\Omega)$ we consider the following optimal control problems (cf. [Sprekels and Tiba 1999], problems $\mathbf{P}_1$ and $\mathbf{D}_1$):

$$\min_{u \in L^\infty(\Omega)} \tilde{J}(u) := \int_\Omega u \,\mathrm{d}x \tag{$\mathbb{P}$}$$

subject to

$$
\begin{aligned}
\Delta(u^3 \Delta y) &= f && \text{in } \Omega, && (2.3)\\
y &= 0 && \text{on } \partial\Omega, && (2.4)\\
-\Delta y &= 0 && \text{on } \partial\Omega, && (2.5)\\
u &\in U_{\text{ad}}, && &&\\
y &\in Y_{\text{ad}}, && &&
\end{aligned}
$$

denoted as the primal problem ($\mathbb{P}$), representing the physical control problem motivated in the introduction, and secondly, with the datum $z \in V$ induced by f

$$
\begin{aligned}
-\Delta z &= f \quad \text{in } \Omega,\\
y &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
$$

the dual problem ($\mathbb{D}$)

$$\min_{l \in L^\infty(\Omega)} J(l) := \int_\Omega l^{-1/3} \,\mathrm{d}x \qquad (\mathbb{D})$$

subject to

$$
\begin{aligned}
-\Delta y &= z\, l && \text{in } \Omega, && (2.6)\\
y &= 0 && \text{on } \partial\Omega, && (2.7)\\
l &\in L_{\text{ad}}, && &&\\
y &\in Y_{\text{ad}}, && &&
\end{aligned}
$$

which is analytically and numerically advantageous, in that it is convex and contains two coupled second order equations instead of the fourth order equation (2.3). It will therefore serve as a basis for our analysis in the remaining sections. For every $u \in U_{\text{ad}}$ the system (2.3)–(2.5) has a unique weak solution $y = y(u) \in V$ with $u^3 \Delta y \in V$. Due to $z \in L^2(\Omega)$ we have $z\, l \in L^2(\Omega)$, and there is a strong solution $y = y(l) \in V$ to the dual system (2.6)–(2.7).

We impose the following Slater condition:

$$\exists u_s \in U_{\text{ad}}, \varepsilon_s > 0\colon \quad y_s = y(u_s) > -\tau + \varepsilon_s, \qquad (2.8)$$

and recall from [Sprekels and Tiba 1999] that for each pair (y, u) admissible for ($\mathbb{P}$), the pair $(y, l = u^{-3})$ is admissible for ($\mathbb{D}$) with the same cost and vice versa. Moreover, there exists a unique solution u_{opt} of problem ($\mathbb{P}$), and $l_{\text{opt}} = u_{\text{opt}}^{-3}$ is the unique solution of ($\mathbb{D}$). We denote the associated state by y_{opt}.

Next we derive optimality conditions characterizing l_{opt}. For this purpose we introduce $\mathscr{M}(\bar{\Omega})$, the space of regular Borel measures, which equipped with norm

$$\|\cdot\|_{\mathscr{M}(\bar{\Omega})} = \sup_{f \in C(\bar{\Omega}), |f| \le 1} \int_{\bar{\Omega}} f \,\mathrm{d}(\cdot)$$

is the dual space of $C(\bar{\Omega})$. Arguments similar to those used in [Casas 1986, Theorem 2] yield the

Theorem 2.1. *Let the assumption* (2.8) *hold. A control* $l \in L^\infty(\Omega)$ *with associated state* $y = y(l)$ *is optimal for the dual problem* ($\mathbb{D}$) *if and only if there exist* $q \in L^2(\Omega)$ *and* $\nu \in \mathscr{M}(\bar\Omega)$, *such that*

$$\begin{aligned} -\int_\Omega q\,\Delta w \,\mathrm{d}x &= \int_\Omega w \,\mathrm{d}\nu(x) && \forall\, w \in V, \\ \int_\Omega \left(q\,z - \frac{1}{3}\,l^{-4/3}\right)(k-l)\,\mathrm{d}x &\geq 0 && \forall\, k \in L_{\mathrm{ad}}, \quad (2.9) \\ \int_{\bar\Omega} (w-y)\,\mathrm{d}\nu(x) &\leq 0 && \forall\, w \in Y_{\mathrm{ad}}, \quad (2.10) \\ l \in L_{\mathrm{ad}}, &\qquad y \in Y_{\mathrm{ad}} \end{aligned}$$

are satisfied.

The variational inequality (2.9) can be written as a projection formula

$$l = \left(P_{[m^4,M^4]}\,(3\,q\,z)\right)^{-3/4} \quad \text{a.e. in } \Omega, \tag{2.11}$$

where $P_{[a,b]}$ is the orthogonal projection onto the real interval $[a,b]$. For later use, we note that (2.10) is equivalent to

$$\nu \leq 0, \qquad \int_{\bar\Omega} (y+\tau)\,\mathrm{d}\nu(x) = 0. \tag{2.12}$$

It follows from (2.12) that the support of the measure ν is concentrated in the state-active set $\{\, x \in \Omega \mid y(x) = -\tau \,\}$. In particular $\nu_{|\partial\Omega} \equiv 0$, see [Casas 1986]. Furthermore, from Theorems 4 and 5 in [Casas 1986] we deduce $q \in W_0^{1,s}(\Omega)$ for all $1 \leq s < d/(d-1)$ and $q \in H^2_{\mathrm{loc}}(\Omega \setminus \{y = -\tau\})$.

3 Moreau-Yosida regularized problem

To relax the state constraints, we introduce the Moreau-Yosida regularization ($\mathbb{D}^\gamma$) of problem ($\mathbb{D}$) for a parameter $\gamma > 0$. It reads

$$\min_{l \in L_{\mathrm{ad}}} J^\gamma(l) := \int_\Omega l^{-1/3}\,\mathrm{d}x + \frac{\gamma}{2}\int_\Omega \left((y+\tau)^-\right)^2 \mathrm{d}x \tag{$\mathbb{D}^\gamma$}$$

subject to

$$y = T(z\,l).$$

Here we set $(\cdot)^- := \min\{0, (\cdot)\}$. It admits a unique solution l^γ of ($\mathbb{D}^\gamma$) with associated state denoted by $y^\gamma = T(z\,l^\gamma)$. Furthermore, there exists a unique $q^\gamma \in L^2(\Omega)$ which together with y^γ and l^γ satisfies

$$-\int_\Omega q^\gamma\,\Delta w\,\mathrm{d}x = \int_\Omega \gamma(y^\gamma+\tau)^-\,w\,\mathrm{d}x \quad \forall\, w \in V, \tag{3.1}$$

$$\int_\Omega \left(q^\gamma\,z - \frac{1}{3}\,(l^\gamma)^{-4/3}\right)(k-l^\gamma)\,\mathrm{d}x \geq 0 \qquad \forall\, k \in L_{\mathrm{ad}}, \tag{3.2}$$

$$l^\gamma \in L_{\mathrm{ad}}.$$

The term $\nu^\gamma := \gamma(y^\gamma + \tau)^-$ can be regarded as a regularized version of the Lagrange multiplier $\nu \in \mathscr{M}(\bar\Omega)$ in Theorem 2.1. There holds $q^\gamma = T(\nu^\gamma) \in V$ and

$$\langle y^\gamma - w \,,\, \nu^\gamma \rangle_{C(\bar\Omega),\mathscr{M}(\bar\Omega)} \geq 0 \qquad \forall\, w \in Y_{\mathrm{ad}}.$$

This inequality for $w \in Y_{\mathrm{ad}}$ can be argued as follows:

$$\begin{aligned}\langle y^\gamma - w \,,\, \nu^\gamma \rangle_{C(\bar\Omega),M(\bar\Omega)} &= (y^\gamma - w \,,\, \nu^\gamma)_{L^2(\Omega)} = (y^\gamma + \tau - \tau - w \,,\, \nu^\gamma)_{L^2(\Omega)} \\ &= \big(y^\gamma + \tau \,,\, \gamma(y^\gamma + \tau)^-\big)_{L^2(\Omega)} + (-(w+\tau) \,,\, \nu^\gamma)_{L^2(\Omega)} \\ &= \gamma \left\|(y^\gamma + \tau)^-\right\|^2_{L^2(\Omega)} + (-(w+\tau) \,,\, \nu^\gamma)_{L^2(\Omega)} \geq 0.\end{aligned}$$

Now we want to show convergence of the parameterized subproblems ($\mathbb{D}^\gamma$) towards the unregularized problem ($\mathbb{D}$). We begin with uniform boundedness of primal and dual variables with respect to γ. While the former is obtained immediately through the control constraints and (2.2), the latter can be shown as follows.

Lemma 3.1. *Let $\gamma > 0$ and $l^\gamma \in L_{\mathrm{ad}}$ be the solution to the problem ($\mathbb{D}^\gamma$) with associated state y^γ and multipliers q^γ, ν^γ according to the optimality conditions. Then there exists a constant $C > 0$ independent of γ such that*

$$\|\nu^\gamma\|_{L^1(\Omega)}\,, \quad \|q^\gamma\|_{L^2(\Omega)} \quad \leq \quad C.$$

Proof. To uniformly bound ν^γ in $L^1(\Omega)$ we test (3.2) with the Slater element l_s. With the help of the adjoint equation (3.1) we get

$$C \geq \left(\frac{1}{3}(l^\gamma)^{-4/3} \,,\, l^\gamma - l_s\right)_{L^2(\Omega)} \geq (q^\gamma \,,\, z(l^\gamma - l_s)_{L^2(\Omega)} = (\nu^\gamma \,,\, y^\gamma - y_s)_{L^2(\Omega)}\,,$$

with a constant C independent of γ. The desired estimate now follows from

$$\begin{aligned}(\nu^\gamma \,,\, y^\gamma - y_s)_{L^2(\Omega)} &= (\nu^\gamma \,,\, y^\gamma + \tau - \tau - y_s)_{L^2(\Omega)} \\ &= \gamma \left\|(y^\gamma + \tau)^-\right\|^2_{L^2(\Omega)} + (-\nu^\gamma \,,\, y_s + \tau)_{L^2(\Omega)} \\ &\geq 0 + (-\nu^\gamma \,,\, \varepsilon_s)_{L^2(\Omega)} \\ &= \varepsilon_s \left\|\nu^\gamma\right\|_{L^1(\Omega)}.\end{aligned}$$

With this bound we want to prove the one on the dual state q^γ. Let $w \in V$ solve

$$\begin{aligned}-\Delta w &= q^\gamma \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega,\end{aligned}$$

Using (3.1), the embedding of $C(\bar\Omega)$ into $H^2(\Omega)$ and the continuous dependence of w on q^γ, we have that

$$\begin{aligned}\|q^\gamma\|^2_{L^2(\Omega)} &= \int_\Omega q^\gamma(-\Delta w)\,\mathrm{d}x = \int_\Omega \nu^\gamma w\,\mathrm{d}x \leq C\left\|\nu^\gamma\right\|_{L^1(\Omega)} \|w\|_{L^\infty(\Omega)} \\ &\leq C\left\|\nu^\gamma\right\|_{L^1(\Omega)} \|w\|_{H^2(\Omega)} \leq C \left\|q^\gamma\right\|_{L^2(\Omega)}\,,\end{aligned}$$

hence $\|q^\gamma\|_{L^2(\Omega)} \leq C$. □

Next we need to estimate the violation of the state constraint measured in the maximum norm with the help of techniques developed in [Hintermüller, Schiela and Wollner 2014].

Lemma 3.2. *Let l^γ be the solution of problem $(\mathbb{D}^\gamma)$, y^γ the corresponding state. Then for $d \in \{2,3\}$ we have for every $\varepsilon > 0$ a constant $C_\varepsilon > 0$, independent of γ, such that*

$$\left\|(y^\gamma + \tau)^-\right\|_{L^\infty(\Omega)} \leq C_\varepsilon \left(\gamma^{d/4-1+\varepsilon}\right).$$

Proof. We show that Corollary 2.6 of [Hintermüller, Schiela and Wollner 2014] is applicable. To begin with, we note that $l^\gamma = P(3q^\gamma z)^{-3/4}$ is uniformly bounded in $W^{1,s}$ for every $s \in [1, d/(d-1))$, since ν^γ is uniformly bounded in $L^1(\Omega)$. This implies that $zl^\gamma \in W^{1,s}(\Omega)$ is uniformly bounded in γ and that $y^\gamma \in W^{3,s}(\Omega)$ is uniformly bounded w.r.t. γ, which by Sobolev imbedding theorems holds also in $C^\beta(\bar\Omega)$, for $\beta = 4 - d - \varepsilon$ and all $\varepsilon > 0$. Thus Corollary 2.6 in [Hintermüller, Schiela and Wollner 2014] is applicable and delivers our desired bound. □

We are now in position to estimate the regularization error.

Theorem 3.3. *Let l and l^γ be the solutions to $(\mathbb{D})$ and $(\mathbb{D}^\gamma)$, resp., with corresponding states y and y^γ. Then for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$, independent of γ, for which it holds that*

$$\|l - l^\gamma\|_{L^2(\Omega)} + \|y - y^\gamma\|_{H^2(\Omega)} + \|y - y^\gamma\|_{L^\infty(\Omega)} \leq C_\varepsilon \gamma^{-\frac{1}{2}(1-\frac{d}{4})+\varepsilon}.$$

Proof. Using l as test function in (3.2) and l^γ as test function in (2.9) we obtain

$$\begin{aligned}
&C\,\|l - l^\gamma\|^2_{L^2(\Omega)}\\
&\leq (q^\gamma - q\,,\, z\,l - z\,l^\gamma)_{L^2(\Omega)} = -\,(q^\gamma - q\,,\, \Delta(y - y^\gamma))_{L^2(\Omega)}\\
&= \left(y - y^\gamma\,,\, \gamma(y^\gamma + \tau)^-\right)_{L^2(\Omega)} - \langle y - y^\gamma\,,\, \nu\rangle_{C(\bar\Omega),\mathscr{M}(\bar\Omega)}\\
&= \left(y + \tau\,,\, \gamma(y^\gamma + \tau)^-\right)_{L^2(\Omega)} - \left(y^\gamma + \tau\,,\, \gamma(y^\gamma + \tau)^-\right)_{L^2(\Omega)} + \langle y^\gamma - y\,,\, \nu\rangle_{C(\bar\Omega),\mathscr{M}(\bar\Omega)}\\
&=: (I) + (II) + (III).
\end{aligned}$$

Since $y \in Y_{\text{ad}}$ we have $(I) \leq 0$. Moreover $(II) = -\gamma\,\|(y^\gamma + \tau)^-\|^2_{L^2(\Omega)} \leq 0$. The third addend is treated with the complementarity condition (2.12) for the multiplier ν:

$$\begin{aligned}
(III) &= \langle y^\gamma + \tau\,,\, \nu\rangle_{C(\bar\Omega),\mathscr{M}(\bar\Omega)} + \underbrace{\langle -y - \tau\,,\, \nu\rangle_{C(\bar\Omega),\mathscr{M}(\bar\Omega)}}_{=0}\\
&= \langle y^\gamma + \tau\,,\, \nu\rangle_{C(\bar\Omega),\mathscr{M}(\bar\Omega)}\\
&\leq \left\langle (y^\gamma + \tau)^-\,,\, \nu\right\rangle_{C(\bar\Omega),\mathscr{M}(\bar\Omega)}.
\end{aligned}$$

With the help of Lemma 3.2 we arrive at

$$\begin{aligned}
C\,\|l - l^\gamma\|^2_{L^2(\Omega)} &\leq \left\langle (y^\gamma + \tau)^-\,,\, \nu\right\rangle_{C(\bar\Omega),\mathscr{M}(\bar\Omega)} \leq \left\|(y^\gamma + \tau)^-\right\|_{L^\infty(\Omega)} \|\nu\|_{\mathscr{M}(\bar\Omega)}\\
&\leq C_\varepsilon \left(\gamma^{d/4-1+\varepsilon}\right),
\end{aligned}$$

with γ-independent constants C, C_ϵ. The continuous dependence of the states on the controls and the continuous embedding $H^2(\Omega) \hookrightarrow C(\bar\Omega)$ allow to extend this estimate to $\|y - y^\gamma\|_{H^2(\Omega)}$ and $\|y - y^\gamma\|_{L^\infty(\Omega)}$. □

4 Finite element discretization

4.1 Mixed piecewise constant versus piecewise linear approximation

In this section we turn to the variational discretization of the regularized control problems, taking into account the structure imposed by the optimality systems, especially the projection formula (2.11) and its discrete counterparts (4.7) and (4.15). The function $(P_{[m^4,M^4]}(\cdot))^{-3/4}$ applied to the product of two state variables is evaluated with little effort if those variables are approximated piecewise constant and yields an implicit piecewise constant discretization of the optimal control. Further, the finite element system of the semismooth Newton method in Section 5, in particular the parts (A.2) and (A.3) involving the projection formula and its generalized derivative, in this situation is easily assembled exactly.

Approximating the states with piecewise linear, continuous finite elements delivers a more involved variational discretization of the controls, since the projection formula then no longer implies a piecewise polynomial discretization of the control variable, but rather the negative power of the pointwise projection of a piecewise quadratic function. Moreover, the approximate computation of the terms (A.4) and (A.5) introduces an additional error. On the other hand, a piecewise linear ansatz delivers the higher approximation order two for the states, as opposed to an order of at most one for a piecewise constant ansatz. This is supported by the convergence rates w.r.t. the grid size h in the error plots in Section 5, and also allows for a better resolution of the control active sets.

In the remainder of this section we give estimates for the overall error in both discrete approaches.

4.2 Variational discretization of $(\mathbb{D}^\gamma)$ with mixed finite elements

Following the above remarks, we use a mixed finite element method based on the lowest-order Raviart-Thomas element. To begin with we recall the mixed formulation of the Dirichlet-problem for the Poisson equation, i.e., for $g \in L^2(\Omega)$, $y = T(g)$ and $\mathbf{v} = \nabla y$ there holds

$$\begin{aligned} \int_\Omega \mathbf{v} \cdot \mathbf{w} \, \mathrm{d}x + \int_\Omega y \, \operatorname{div} \mathbf{w} \, \mathrm{d}x = 0 \qquad & \forall\, \mathbf{w} \in H(\operatorname{div}, \Omega), \\ \int_\Omega \phi \, \operatorname{div} \mathbf{v} \, \mathrm{d}x + \int_\Omega g \, \phi \, \mathrm{d}x = 0 \qquad & \forall\, \phi \in L^2(\Omega), \end{aligned} \tag{4.1}$$

where $H(\operatorname{div}, \Omega) := \left\{ \mathbf{w} \in L^2(\Omega)^d \mid \operatorname{div} \mathbf{w} \in L^2(\Omega) \right\}$. For a given right-hand side $g \in L^2(\Omega)$ we represent the solution of this mixed problem by $G(g) := (y, \mathbf{v})$. In particular, with $\mathbf{v}_z := \nabla z$ this means $G(f) = (z, \mathbf{v}_z)$.

Let a triangulation $\mathcal{T}_h$ of Ω be given, where $h := \max_{T \in \mathcal{T}_h} \operatorname{diam}(T)$ and $\bar{\Omega}$ be the union of the elements of $\mathcal{T}_h$, with boundary elements allowed to have one curved face. We additionally assume that the triangulation is quasi-uniform, i.e., there exists a constant $\rho > 0$, independent of h, such that each $T \in \mathcal{T}_h$ is contained in a ball of radius $\rho^{-1} h$ and contains a ball of radius ρh. To define the discrete version of (4.1) let us introduce the spaces

$$\begin{aligned} \mathbf{V}_h &:= RT_0(\Omega, \mathcal{T}_h) := \left\{ \mathbf{w}_h \in H(\operatorname{div}, \Omega) \mid \mathbf{w}_{h|T} \in RT_0(T) \quad \forall\, T \in \mathcal{T}_h \right\}, \\ RT_0(T) &:= \left\{ \mathbf{w} \colon T \to \mathbb{R}^d \;\middle|\; \exists\, a \in \mathbb{R}^d, \exists\, \beta \in \mathbb{R} : \mathbf{w}(x) = a + \beta x \quad \forall\, x \in \mathbb{R}^d \right\}, \\ Y_h &:= \left\{ \phi_h \in L^2(\Omega) \mid \forall\, T \in \mathcal{T}_h \ \exists\, \beta_T \in \mathbb{R} : \phi_{h|T} \equiv \beta_T \right\}. \end{aligned}$$

For a given $g \in L^2(\Omega)$ we set $G_h(g) := (y_h, \mathbf{v}_h) \in Y_h \times \mathbf{V}_h$ to be the solution of

$$\begin{aligned} \int_\Omega \mathbf{v}_h \cdot \mathbf{w}_h \,\mathrm{d}x + \int_\Omega y_h \operatorname{div} \mathbf{w}_h \,\mathrm{d}x = 0 \qquad \forall\, \mathbf{w}_h \in \mathbf{V}_h, \\ \int_\Omega \phi_h \operatorname{div} \mathbf{v}_h \,\mathrm{d}x + \int_\Omega g\, \phi_h \,\mathrm{d}x = 0 \qquad \forall\, \phi_h \in Y_h. \end{aligned}$$

The resulting error satisfies (see [Brezzi and Fortin 1991])

$$\begin{aligned} \|y - y_h\|_{L^2(\Omega)} + \|\mathbf{v} - \mathbf{v}_h\|_{L^2(\Omega)^d} &\le Ch \left(\|y\|_{H^1(\Omega)} + \|\mathbf{v}\|_{H^1(\Omega)^d} \right) \\ &\le Ch \,\|y\|_{H^2(\Omega)} \le Ch \,\|g\|_{L^2(\Omega)}, \end{aligned} \tag{4.2}$$

as well as, if $g \in L^\infty(\Omega)$, the pointwise estimate (see [Gastaldi and Nochetto 1989, Cor. 5.5])

$$\|y - y_h\|_{L^\infty(\Omega)} + \|\mathbf{v} - \mathbf{v}_h\|_{L^\infty(\Omega)^d} \le Ch \,|\log h|\, \|g\|_{L^\infty(\Omega)}. \tag{4.3}$$

The load f induces a discrete datum $z_h \in Y_h$ via $(z_h, \mathbf{v}_{z,h}) = G_h(f)$.

Remark 4.1. For our error analysis we require that $\|z_h\|_{L^\infty(\Omega)}$ is bounded uniformly in h in both of the considered discretization approaches.

This is satisfied, e.g., if $f \in L^\infty(\Omega)$, since then $\|z - z_h\|_{L^\infty(\Omega)} \to 0$ by (4.3) and (4.11), resp. Note that this regularity restriction is not essentially necessary, cf. [Deckelnick and Hinze 2014, Lemma 3.4], which holds analogously for the scalar states in both discrete approaches and allows for $f \in L^p(\Omega)$, $p > 2$, with smaller powers of h in (4.3) and (4.11).

Let $Y_{\mathrm{ad},h} := \left\{ \phi_h \in Y_h \;\middle|\; \phi_{h|T} \ge -\tau \quad \forall T \in \mathcal{T}_h \right\}$. The variational discretization ($\mathbb{D}_h^\gamma$) of the regularized control problems ($\mathbb{D}^\gamma$) reads

$$\min_{l \in L^\infty(\Omega)} J_h^\gamma(l) := \int_\Omega l^{-1/3} \,\mathrm{d}x + \frac{\gamma}{2} \int_\Omega \left((y_h + \tau)^- \right)^2 \mathrm{d}x \tag{$\mathbb{D}_h^\gamma$}$$

subject to

$$\begin{aligned} (y_h, \mathbf{v}_h) &= G_h(z_h\, l), \\ l &\in L_{\mathrm{ad}}. \end{aligned}$$

We note that ($\mathbb{D}_h^\gamma$) is still an infinite-dimensional optimization problem similar to ($\mathbb{D}^\gamma$) since the control l is not discretized. It admits a unique solution l_h^γ, which is characterized by the optimality system

$$(y_h^\gamma, \mathbf{v}_h^\gamma) = G_h(z_h\, l_h^\gamma), \tag{4.4}$$

$$(q_h^\gamma, \mathbf{v}_{q,h}^\gamma) = G_h\left(\gamma (y_h^\gamma + \tau)^- \right), \tag{4.5}$$

$$\int_\Omega \left(q_h^\gamma z_h - \frac{1}{3} (l_h^\gamma)^{-4/3} \right) (k - l_h^\gamma) \,\mathrm{d}x \ge 0 \qquad \forall\, k \in L_{\mathrm{ad}}, \tag{4.6}$$

$$l_h^\gamma \in L_{\mathrm{ad}}.$$

Condition (4.6) is equivalent to the projection formula

$$l_h^\gamma = \left(P_{[m^4, M^4]} \left(3\, q_h^\gamma z_h \right) \right)^{-3/4}. \tag{4.7}$$

We denote $\nu_h^\gamma := \gamma (y_h^\gamma + \tau)^-$ and similarly to the proof of Lemma 3.1 one obtains boundedness uniformly in h and γ:

Lemma 4.2. *Let $\gamma > 0$ and $l_h^\gamma \in L_{\mathrm{ad}}$ be the solution to the problem $(\mathbb{D}_h^\gamma)$ with state $(y_h^\gamma, \mathbf{v}_h^\gamma) = G_h(z_h\, l_h^\gamma)$. Then there exists an $h_0 > 0$ and a constant $C > 0$ independent of γ and of h such that*

$$\left\|\nu_h^\gamma\right\|_{L^1(\Omega)} \leq C \qquad \forall\, 0 < h < h_0, \quad \forall\, \gamma > 0.$$

Proof. Let $(q_h^\gamma, \mathbf{v}_{q,h}^\gamma)$ be the adjoint state and l_s the Slater element with corresponding discrete state $(y_{s,h}, \mathbf{v}_{s,h}) = G_h(z_h l_s)$, which is a discrete Slater state for all $0 < h < h_0$ with some $h_0 > 0$ small enough. In fact, with y_s and ε_s from (2.8) we obtain from $\|y_s - y_{s,h}\|_{L^\infty(\Omega)} \to 0$ that $y_{s,h} > -\tau + \varepsilon_s/2$ for $0 < h < h_0$.

We test (4.6) with l_s and with the help of the adjoint equation (4.5) and the definition of G_h we get

$$C \geq \left(\frac{1}{3}(l_h^\gamma)^{-4/3}\,,\, l_h^\gamma - l_s\right)_{L^2(\Omega)} \geq \left(q_h^\gamma\,,\, z_h(l_h^\gamma - l_s)\right)_{L^2(\Omega)} = \left(\nu_h^\gamma\,,\, y_h^\gamma - y_{s,h}\right)_{L^2(\Omega)},$$

with a constant C independent of γ and h. We then have

$$\begin{aligned}
\left(\nu_h^\gamma\,,\, y_h^\gamma - y_{s,h}\right)_{L^2(\Omega)} &= \left(\nu_h^\gamma\,,\, y_h^\gamma + \tau - \tau - y_{s,h}\right)_{L^2(\Omega)} \\
&= \gamma \left\|(y_h^\gamma + \tau)^-\right\|_{L^2(\Omega)}^2 + \left(-\nu_h^\gamma\,,\, y_{s,h} + \tau\right)_{L^2(\Omega)} \\
&\geq 0 + \left(-\nu_h^\gamma\,,\, \frac{\varepsilon_s}{2}\right)_{L^2(\Omega)} \qquad \forall\, 0 < h < h_0 \\
&= \frac{\varepsilon_s}{2} \left\|\nu_h^\gamma\right\|_{L^1(\Omega)} \qquad \forall\, 0 < h < h_0.
\end{aligned}$$

□

Aiming for an estimate of the overall error induced by regularization and discretization we apply the approach from [Hintermüller and Hinze 2009] to our problem setting and derive a similar asymptotic h,γ-dependent bound on $(l - l_h^\gamma)$ in $L^2(\Omega)$, which further allows to couple the regularization parameter efficiently to the grid size parameter. To this end we need to estimate the discretization error for the regularized problems.

Theorem 4.3. *Let l^γ and l_h^γ be the solutions of $(\mathbb{D}^\gamma)$ and $(\mathbb{D}_h^\gamma)$, resp., with corresponding states y^γ and $(y_h^\gamma, \mathbf{v}_h^\gamma)$. Then there is an $h_0 > 0$ and a γ- and h-independent constant C, such that for all $h \in (0, h_0)$ and all $\gamma > 0$*

$$\left\|l^\gamma - l_h^\gamma\right\|_{L^2(\Omega)} + \left\|y^\gamma - y_h^\gamma\right\|_{L^\infty(\Omega)} \leq C h^{1/2} \left|\log h\right|^{1/2}.$$

Proof. We define the auxiliary variable $(q_h^\nu, \mathbf{v}_h^\nu) = G_h(\nu^\gamma)$ and test the problems' variational inequalities with the respective solutions to obtain

$$\begin{aligned}
&C \left\|l^\gamma - l_h^\gamma\right\|_{L^2(\Omega)}^2 \leq \left(q^\gamma z - q_h^\gamma z_h\,,\, l_h^\gamma - l^\gamma\right)_{L^2(\Omega)} \\
&= \left(q^\gamma z - q^\gamma z_h\,,\, l_h^\gamma - l^\gamma\right)_{L^2(\Omega)} + \left(q^\gamma z_h - q_h^\nu z_h\,,\, l_h^\gamma - l^\gamma\right)_{L^2(\Omega)} + \left(q_h^\nu z_h - q_h^\gamma z_h\,,\, l_h^\gamma - l^\gamma\right)_{L^2(\Omega)} \\
&=: (I) + (II) + (III).
\end{aligned}$$

In view of Lemma 3.1 we have for sufficiently small $0 < h < h_0$ that

$$(I) \leq \|q^\gamma\|_{L^2(\Omega)} \|z - z_h\|_{L^2(\Omega)} \left\|l_h^\gamma - l^\gamma\right\|_{L^\infty(\Omega)} \leq Ch \left\|l_h^\gamma - l^\gamma\right\|_{L^\infty(\Omega)} \leq Ch.$$

For the second addend we find

$$(II) \leq \|z_h\|_{L^\infty(\Omega)} \|q^\gamma - q_h^\nu\|_{L^1(\Omega)} \left\|l_h^\gamma - l^\gamma\right\|_{L^\infty(\Omega)} \leq C \|q^\gamma - q_h^\nu\|_{L^1(\Omega)}\,.$$

To estimate the finite element error $\|q^\gamma - q_h^\nu\|_{L^1(\Omega)}$ we set $p = \operatorname{sgn}(q^\gamma - q_h^\nu) \in L^2(\Omega)$ (cf. the proof of Lemma 8.3.11 on p. 228 in [Brenner and Scott 2008]) and consider

$$(q^\gamma - q_h^\nu\,,\, p)_{L^2(\Omega)} = (q^\gamma\,,\, p)_{L^2(\Omega)} - (q_h^\nu\,,\, p)_{L^2(\Omega)}\,. \tag{4.8}$$

Defining $(y^p, \mathbf{v}^p) = G(p)$ we have (cf. [Casas 1985], proof of Theorem 3 with piecewise linear finite elements)

$$(q^\gamma\,,\, p)_{L^2(\Omega)} = (q^\gamma\,,\, -\Delta y^p)_{L^2(\Omega)} = (-\Delta q^\gamma\,,\, y^p)_{L^2(\Omega)} = (y^p\,,\, \nu^\gamma)_{L^2(\Omega)}$$

and furthermore, setting $(y_h^p, \mathbf{v}_h^p) = G_h(p)$ and using the definitions of q_h^ν and $\mathbf{v}_h^\nu$,

$$\begin{aligned}(q_h^\nu\,,\, p)_{L^2(\Omega)} &= -\left(q_h^\nu\,,\, \operatorname{div} \mathbf{v}_h^p\right)_{L^2(\Omega)} = \left(\mathbf{v}_h^\nu \cdot \mathbf{v}_h^p\,,\, 1\right)_{L^2(\Omega)} = -\left(y_h^p\,,\, \operatorname{div} \mathbf{v}_h^\nu\right)_{L^2(\Omega)} \\ &= \left(y_h^p\,,\, \nu^\gamma\right)_{L^2(\Omega)}\,.\end{aligned}$$

Inserting these into (4.8), we conclude with $\|p\|_{L^\infty(\Omega)} \leq 1$ that

$$\begin{aligned}\|q^\gamma - q_h^\nu\|_{L^1(\Omega)} &= (q^\gamma - q_h^\nu\,,\, p)_{L^2(\Omega)} = (q^\gamma - q_h^\nu\,,\, \operatorname{sgn}(q^\gamma - q_h^\nu))_{L^2(\Omega)} = \left(y^p - y_h^p\,,\, \nu^\gamma\right)_{L^2(\Omega)} \\ &\leq \left\|y^p - y_h^p\right\|_{L^\infty(\Omega)} \|\nu^\gamma\|_{L^1(\Omega)} \leq Ch\,|\log h|\, \|p\|_{L^\infty(\Omega)} \leq Ch\,|\log h|\,.\end{aligned}$$

Finally, we set $(\bar{y}_h, \bar{\mathbf{v}}_h) = G_h(z_h\, l^\gamma)$ and rewrite the third addend (III) using the definitions of y_h^γ, $\bar{y}_h$, q_h^ν and q_h^γ together with the monotonicity of the min-function

$$\begin{aligned}&(III) = \left(\gamma(y^\gamma + \tau)^- - \gamma(y_h^\gamma + \tau)^-\,,\, y_h^\gamma - \bar{y}_h\right)_{L^2(\Omega)} \\ &= \underbrace{\left(\gamma(y^\gamma + \tau)^- - \gamma(y_h^\gamma + \tau)^-\,,\, y_h^\gamma - y^\gamma\right)_{L^2(\Omega)}}_{\leq 0} + \left(\gamma(y^\gamma + \tau)^- - \gamma(y_h^\gamma + \tau)^-\,,\, y^\gamma - \bar{y}_h\right)_{L^2(\Omega)} \\ &\leq \left(\gamma(y^\gamma + \tau)^- - \gamma(y_h^\gamma + \tau)^-\,,\, y^\gamma - \bar{y}_h\right)_{L^2(\Omega)}\,.\end{aligned}$$

Now let $(\bar{y}, \bar{\mathbf{v}}) = G(z_h\, l^\gamma)$ and similar to the proof of [Hintermüller and Hinze 2009, Theorem 3.5] one has for $0 < h < h_0$

$$\begin{aligned}&\left(\gamma(y^\gamma + \tau)^- - \gamma(y_h^\gamma + \tau)^-\,,\, y^\gamma - \bar{y}_h\right)_{L^2(\Omega)} \leq \max\left\{\|\nu^\gamma\|_{L^1(\Omega)}\,,\, \left\|\nu_h^\gamma\right\|_{L^1(\Omega)}\right\} \|y^\gamma - \bar{y}_h\|_{L^\infty(\Omega)} \\ &\leq C\,\|y^\gamma - \bar{y}\|_{L^\infty(\Omega)} + C\,\|\bar{y} - \bar{y}_h\|_{L^\infty(\Omega)} \leq C\,\|y^\gamma - \bar{y}\|_{H^2(\Omega)} + Ch\,|\log h|\, \|z_h\|_{L^\infty(\Omega)} \|l^\gamma\|_{L^\infty(\Omega)} \\ &\leq C\,\|l^\gamma\|_{L^\infty(\Omega)} \|z - z_h\|_{L^2(\Omega)} + Ch\,|\log h| \leq Ch + Ch\,|\log h| \leq Ch\,|\log h|\,.\end{aligned}$$

Altogether, we obtain the proposed error bound for the controls. The estimate for the states follows similarly to the proof of Theorem 3.3 and with (4.2) and (4.3). With $(\bar{y}^h, \bar{\mathbf{v}}^h) = G(z_h\, l_h^\gamma)$ there holds for sufficiently small $h > 0$ that

$$\begin{aligned}\left\|y^\gamma - y_h^\gamma\right\|_{L^\infty(\Omega)} &\leq \|y^\gamma - \bar{y}\|_{L^\infty(\Omega)} + \left\|\bar{y} - \bar{y}^h\right\|_{L^\infty(\Omega)} + \left\|\bar{y}^h - y_h^\gamma\right\|_{L^\infty(\Omega)} \\ &\leq C\,\|y^\gamma - \bar{y}\|_{H^2(\Omega)} + C\left\|\bar{y} - \bar{y}^h\right\|_{H^2(\Omega)} + Ch\,|\log h|\, \|z_h\|_{L^\infty(\Omega)} \left\|l_h^\gamma\right\|_{L^\infty(\Omega)} \\ &\leq C\,\|l^\gamma\|_{L^\infty(\Omega)} \|z - z_h\|_{L^2(\Omega)} + C\,\|z_h\|_{L^\infty(\Omega)} \left\|l^\gamma - l_h^\gamma\right\|_{L^2(\Omega)} + Ch\,|\log h| \\ &\leq Ch\,\|f\|_{L^2(\Omega)} + Ch^{1/2}\,|\log h|^{1/2} + Ch\,|\log h| \\ &\leq Ch^{1/2}\,|\log h|^{1/2}\,.\end{aligned}$$

□

Combination of the estimates for the two error components immediately gives a bound for the overall error.

Theorem 4.4. *Let l and l_h^γ denote the solutions of* $(\mathbb{D})$ *and* $(\mathbb{D}_h^\gamma)$*, resp., with corresponding states* $(y, \mathbf{v})$ *and* $(y_h^\gamma, \mathbf{v}_h^\gamma)$*. Then there exist an $h_0 > 0$ and for every $\varepsilon > 0$ a γ- and h-independent constant C_ε, such that for all $0 < h < h_0$ and $\gamma > 0$*

$$\left\|l - l_h^\gamma\right\|_{L^2(\Omega)} + \left\|y - y_h^\gamma\right\|_{L^\infty(\Omega)} \leq C_\varepsilon \left(\gamma^{-\frac{1}{2}(1-\frac{d}{4})+\varepsilon} + h^{\frac{1}{2}} \left|\log h\right|^{\frac{1}{2}}\right). \tag{4.9}$$

Proof. We split the overall control error into the sum of $\|l - l^\gamma\|_{L^2(\Omega)}$ and $\left\|l^\gamma - l_h^\gamma\right\|_{L^2(\Omega)}$ and apply Theorems 3.3 and 4.3 to estimate the regularization and discretization errors. For the states we proceed analogously. □

Estimate (4.9) now suggests a coupling of γ and the grid size h of the form $\gamma = O(h^{-\kappa})$. Equilibrating the errors depending on the dimension d, we obtain

$$\left\|l - l_h^\gamma\right\|_{L^2(\Omega)} + \left\|y - y_h^\gamma\right\|_{L^\infty(\Omega)} \leq C_\varepsilon h^{\frac{1}{2}-\varepsilon} \begin{cases} \text{for } \kappa = 2, \text{ if } d = 2, \\ \text{for } \kappa = 4, \text{ if } d = 3. \end{cases}$$

4.3 Variational discretization of $(\mathbb{D}^\gamma)$ with continuous piecewise linear finite elements

Next we consider piecewise linear and continuous finite element approximations of the state in problem $(\mathbb{D}^\gamma)$. For $g \in L^2(\Omega)$ given we denote by $y_h = G_h(g) \in Y_h$ the solution to

$$(\nabla y_h\,,\, \nabla w_h)_{L^2(\Omega)^d} = (g\,,\, w_h)_{L^2(\Omega)} \quad \forall w_h \in Y_h,$$

where

$$Y_h := \{\, w_h \in C(\bar{\Omega}) \mid w_{h|T} \in P^1(T) \quad \forall T \in \mathcal{T}_h \,\} \cap H_0^1(\Omega).$$

Then, with $y = T(g)$ we have the well known error estimate

$$\|y - y_h\|_{L^2(\Omega)} + h\,\|\nabla(y - y_h)\|_{L^2(\Omega)^d} \leq Ch^2\,\|y\|_{H^2(\Omega)} \leq Ch^2\,\|g\|_{L^2(\Omega)}\,. \tag{4.10}$$

Provided that $g \in L^\infty(\Omega)$, one can use the $L^p(\Omega)$-estimate from [Schatz 1998, Theorem 2.2 and the subsequent Remark] together with careful control of the constant in the a-priori $L^p(\Omega)$-estimate for (2.1) to prove the following bound on the error in the maximum norm

$$\|y - y_h\|_{L^\infty(\Omega)} \leq Ch^2\,|\log h|^2\,\|g\|_{L^\infty(\Omega)}\,. \tag{4.11}$$

We refer to [Deckelnick and Hinze 2008, Lemma 1] for the complete argument.

Enforcing the state constraints in the (inner) nodes $\{x_i\}_{i=1,\ldots,N}$ of the grid, i.e., using the space $Y_{\mathrm{ad},h} := \{\, \phi_h \in Y_h \mid \phi_h(x_i) \geq -\tau, \quad 1 \leq i \leq N \,\}$ of admissible states, the variational discretization of the regularized optimization problem $(\mathbb{D}^\gamma)$ is now straightforward. In problem $(\mathbb{D}_h^\gamma)$ we only have to replace the discrete solution operator by its counterpart of the present solution. With this and the implicit datum $z_h = G_h(f)$ we obtain

$$\min_{l \in L^\infty(\Omega)} J_h^\gamma(l) := \int_\Omega l^{-1/3}\,\mathrm{d}x + \frac{\gamma}{2}\int_\Omega \left((y_h + \tau)^-\right)^2 \mathrm{d}x \tag{$\mathbb{D}_{h,1}^\gamma$}$$

subject to

$$y_h = G_h(z_h\, l),$$
$$l \in L_{\text{ad}}.$$

Problem $(\mathbb{D}^\gamma_{h,1})$ admits a unique solution l^γ_h, which together with the discrete adjoint state q^γ_h satisfies

$$y^\gamma_h = G_h(z_h\, l^\gamma_h), \tag{4.12}$$
$$q^\gamma_h = G_h\left(\gamma(y^\gamma_h + \tau)^-\right), \tag{4.13}$$
$$\int_\Omega \left(q^\gamma_h\, z_h - \frac{1}{3}\,(l^\gamma_h)^{-4/3}\right)(k - l^\gamma_h)\,\mathrm{d}x \geq 0 \qquad \forall\, k \in L_{\text{ad}}, \tag{4.14}$$
$$l^\gamma_h \in L_{\text{ad}}.$$

The variational inequality (4.14) can again be rewritten as a projection formula

$$l^\gamma_h = \left(P_{[m^4, M^4]}\left(3\, q^\gamma_h\, z_h\right)\right)^{-3/4}. \tag{4.15}$$

We note that $(q_h z_h)_{|T}$ is a quadratic function, whose projection in general can not be represented by a polynomial over T. As in Lemma 4.2 we infer $\left\|\nu^\gamma_h := \gamma(y^\gamma_h + \tau)^-\right\|_{L^1(\Omega)} \leq C$ independently of γ and h. Moreover, Theorem 4.3 holds accordingly.

Theorem 4.5. *Let l^γ and l^γ_h be the solutions of $(\mathbb{D}^\gamma)$ and $(\mathbb{D}^\gamma_{h,1})$, resp., with corresponding states y^γ and y^γ_h. Then there exists an $h_0 > 0$ and a γ- and h-independent positive constant C, such that for all $h \in (0, h_0)$ and all $\gamma > 0$ we have*

$$\left\|l^\gamma - l^\gamma_h\right\|_{L^2(\Omega)} + \left\|y^\gamma - y^\gamma_h\right\|_{H^1(\Omega)} + \left\|y^\gamma - y^\gamma_h\right\|_{L^\infty(\Omega)} \leq Ch\,|\log h|\,.$$

Proof. With the adjoint states q^γ and q^γ_h from (3.1) and (4.13), resp., and $q^\nu_h = G_h(\nu^\gamma)$, we obtain as in the proof of Theorem 4.3

$$\begin{aligned}
&C\left\|l^\gamma - l^\gamma_h\right\|^2_{L^2(\Omega)} \leq \left(q^\gamma z - q^\gamma_h z_h\,,\, l^\gamma_h - l^\gamma\right)_{L^2(\Omega)} \\
&= \left(q^\gamma z - q^\gamma z_h\,,\, l^\gamma_h - l^\gamma\right)_{L^2(\Omega)} + \left(q^\gamma z_h - q^\nu_h\, z_h\,,\, l^\gamma_h - l^\gamma\right)_{L^2(\Omega)} + \left(q^\nu_h z_h - q^\gamma_h\, z_h\,,\, l^\gamma_h - l^\gamma\right)_{L^2(\Omega)} \\
&=: (I) + (II) + (III),
\end{aligned}$$

where, using Lemma 3.1, (4.10), (4.11) and $\bar{y} = T(z_h l^\gamma)$, $\bar{y}_h = T_h(z_h l^\gamma)$ (cf. proof of Theorem 4.3), we now can estimate the three addends as follows

$$\begin{aligned}
(I) &\leq \|q^\gamma\|_{L^2(\Omega)}\,\|z - z_h\|_{L^2(\Omega)}\left\|l^\gamma - l^\gamma_h\right\|_{L^\infty(\Omega)} \leq Ch^2, \\
(II) &\leq \|z_h\|_{L^\infty(\Omega)}\,\|q^\gamma - q^\nu_h\|_{L^1(\Omega)}\left\|l^\gamma - l^\gamma_h\right\|_{L^\infty(\Omega)} \leq Ch^2\,|\log h|^2\,, \\
(III) &\leq C\,\|z - z_h\|_{L^2(\Omega)} + C\,\|\bar{y} - \bar{y}_h\|_{L^\infty(\Omega)} \leq C\left(h^2\,\|f\|_{L^2(\Omega)} + h^2\,|\log h|^2\,\|z_h l^\gamma\|_{L^2(\Omega)}\right) \\
&\leq Ch^2\,|\log h|^2\,,
\end{aligned}$$

with (III) and the estimate of $\|q^\gamma - q^\nu_h\|_{L^1(\Omega)}$ deduced similarly as in the proof of Theorem 4.3. Combining the above estimates completes the proof for the control error. The bounds on the state errors can be obtained using similar arguments as for Theorem 4.3. □

The estimate for the overall error follows from combining Theorems 3.3 and 4.5.

Theorem 4.6. *Let l and l_h^γ be the solutions of* $(\mathbb{D})$ *and* $(\mathbb{D}_{h,1}^\gamma)$*, resp., with corresponding states y and y_h^γ. Then there exist an $h_0 > 0$ and for every $\varepsilon > 0$ a γ- and h-independent constant C_ε, such that for all $h \in (0, h_0)$ and all $\gamma > 0$ we have*

$$\left\|l - l_h^\gamma\right\|_{L^2(\Omega)} + \left\|y - y_h^\gamma\right\|_{H^1(\Omega)} + \left\|y - y_h^\gamma\right\|_{L^\infty(\Omega)} \le C_\varepsilon \left(\gamma^{-\frac{1}{2}(1-\frac{d}{4})+\varepsilon} + h\left|\log h\right|\right).$$

Coupling γ and h again in the form $\gamma = O(h^{-\kappa})$ to balance the error contributions, we obtain

$$\left\|l - l_h^\gamma\right\|_{L^2(\Omega)} + \left\|y - y_h^\gamma\right\|_{H^1(\Omega)} + \left\|y - y_h^\gamma\right\|_{L^\infty(\Omega)} \le C_\varepsilon h^{1-\varepsilon} \begin{cases} \text{for } \kappa = 4, & \text{if } d = 2, \\ \text{for } \kappa = 8, & \text{if } d = 3. \end{cases}$$

It turns out that variational discretization with piecewise linear, continuous finite elements yields a better approximation order for our optimal control problem than variational discretization with lowest-order Raviart-Thomas mixed finite elements. In order to achieve this, however, a more progressive coupling of regularization parameter and grid size is necessary in the case of piecewise linear, continuous finite elements. While this would mean a better approximation of the unregularized solution, it can be a drawback if at the same time the problems become harder to solve, cf. the numerical examples in the following section.

5 Numerical examples

To approximate the solution of problem $(\mathbb{D})$ we apply a path-following algorithm in the parameter γ, which is sent to ∞ (cf., e.g., [Hintermüller and Kunisch 2006]). The subproblems $(\mathbb{D}_h^\gamma)$ and $(\mathbb{D}_{h,1}^\gamma)$, resp., are solved with a semismooth Newton method, where the parameter γ is coupled to h according to Theorems 4.4 and 4.6, respectively. The semismooth Newton method is described in the Appendix. We conclude this work with two example problems to supplement our numerical analysis.

Example 1: In order to construct an example problem with known solution (cf. Section 2.9 in [Tröltzsch 2010]), we add the term

$$(\alpha/2)\left\|T(z\,l) - y_\Omega\right\|_{L^2(\Omega)}^2$$

to the cost functional, where $\alpha > 0$ and $y_\Omega \in L^2(\Omega)$, and change the state equation to

$$-\Delta y = z\,l + e_\Omega,$$

with a suitable function $e_\Omega \in L^2(\Omega)$, see below. To construct the exact solution we set $\Omega = (0,1)^2$ and define $r(x) := |x - \bar{x}|$, where $\bar{x} = (1/2, 1/2)$. We choose $\tau = 0.1$, $m = 0.35$, $M = 0.45$ and $\alpha = 1$, and define the (optimal) state

$$y(r) = \begin{cases} -0.1, & r \le \frac{1}{8}, \\ 614.4r^5 - 768r^4 + 352r^3 - 72r^2 + \frac{27}{4}r - \frac{27}{80}, & r \in (\frac{1}{8}, \frac{3}{8}), \\ 0, & r \ge \frac{3}{8} \end{cases}$$

and the adjoint state

$$q(r) = \begin{cases} -r^2 + \frac{1}{64}, & r < \frac{1}{8}, \\ 0, & r \ge \frac{1}{8}. \end{cases}$$

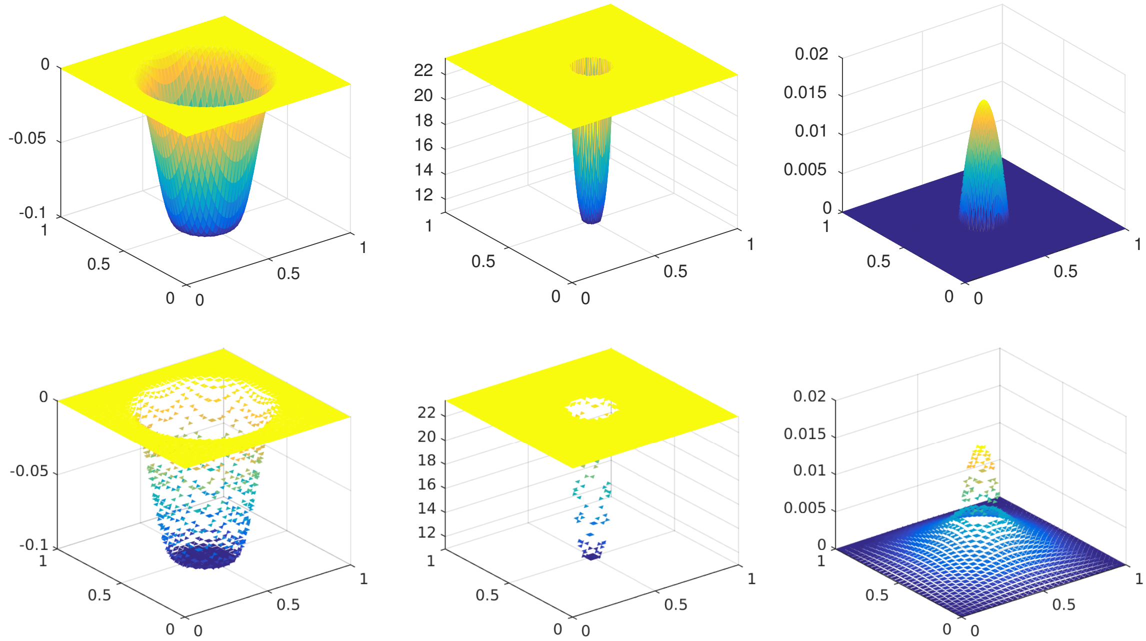

Figure 1: Optimal (top) and optimal discrete piecewise constant (bottom) states y, y_h^γ (left), controls l, l_h^γ (middle) and adjoint states q, q_h^γ (right) with $\gamma = 10^8$ and $h = h_6$ in Example 1.

The multiplier ν is composed of a regular part concentrated in $\Omega_1 := B(\bar{x}, 1/8)$, and a part concentrated on the boundary $\partial\Omega_1$. Taking

$$y_\Omega(r) = \begin{cases} -5.1, & r < \frac{1}{8}, \\ y(r), & r \geq \frac{1}{8}, \end{cases}$$

we obtain as action of $\nu \in \mathscr{M}(\bar{\Omega})$ applied to an element $g \in C(\bar{\Omega})$

$$\int_{\bar{\Omega}} g \,\mathrm{d}\nu = -\int_{\Omega_1} g \,\mathrm{d}x - \frac{1}{4}\int_{\partial\Omega_1} g \,\mathrm{d}s.$$

The auxiliary state z and the corresponding load f are set to

$$\begin{aligned} z(x) &= \sin(\pi x_1)\sin(\pi x_2), \quad \text{and} \\ f(x) &= -\Delta z(x) = 2\pi^2 \sin(\pi x_1)\sin(\pi x_2). \end{aligned}$$

The optimal control l is given by

$$l(x) = \Big(P_{[m^4, M^4]}\big(3(q \circ r)(x) z(x)\big)\Big)^{-3/4},$$

and

$$e_\Omega(x) := -\Delta(y \circ r)(x) - z(x) l(x).$$

The control and state variables are depicted in Figure 1, where we set

$$h_k = \sqrt{2} \cdot 2^{1-k}, \quad k > 1.$$

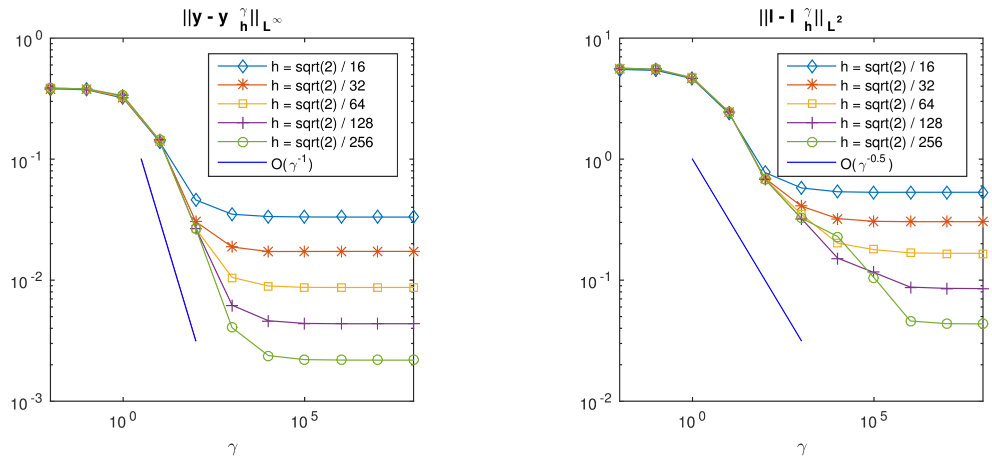

Figure 2: Errors in the states y_h^γ (left) and controls l_h^γ (right) plotted against γ for different values of h in Example 1 with piecewise constant state approximation.

In all our numerical tests we approximate integrals using a 3-point Gauß quadrature rule. In the case of piecewise linear elements, however, we distinguish between those parts of the triangles on which the involved projections are active and those parts where they are inactive.

Let us begin with the mixed state approximation. Table 1 contains $L^\infty(\Omega)$-errors of the state and $L^2(\Omega)$-errors of the control variables for a run of the path-following Algorithm 1, using the coupling $\gamma = O(h^{-2})$, starting with $\gamma_4 = 400$ at $h = h_4$. We observe errors of the size $O(h)$, or equivalently $O(\gamma^{-1/2})$, which is twice the rate predicted by Theorem 4.6. Figure 2 shows the development of the errors in the state and the control variables, respectively, over a large range of regularization parameters. It can be seen from the graphs that the discretization error for both variables is approximately of order one, which explains the above convergence rate of size $O(h)$, and is the square of the expected error bound derived in the previous section. Furthermore, the regularization error appears to be of order $O(\gamma^{-1})$ for the states and between $O(\gamma^{-0.3})$ and $O(\gamma^{-0.5})$ for the controls, while Theorem 3.3 predicts an order of $O(\gamma^{-0.25})$. Computation on a series of random unstructured meshes with grid sizes in the range of the considered uniform meshes results in the same convergence rates, and thus rules out superconvergence effects. The observed convergence rates might be explained by the high regularity of the constructed solution.

With the path-following Algorithm 1 in the appendix 3 to 4 Newton steps are needed to compute the numerical solution, where we use the tolerance 10^{-3}. This result is achieved independent of the grid size of the underlying mesh and thus indicates mesh-independence of the algorithm.

Using piecewise linear, continuous state approximations, however, we in this example are not able to sufficiently progress in the regularization parameter. Even for small values of γ and with damped Newton steps the semismooth Newton iteration failed to converge. This may be due to large slopes contributed by the term e_Ω and by the jump in y_Ω. Similar observations are reported in [Günther and Hinze 2011], where an interior point solver is used to treat gradient constraints in elliptic optimal control. There a jump in the exact control leads to oscillations in the discrete approximation with piecewise linear, continuous finite elements, which results in convergence problems for the solver.

Table 1: Errors and corresponding experimental orders of convergence EOC for the piecewise constant states y_h^γ and controls l_h^γ in Example 1. Parameters are coupled via $\gamma = O(h^{-2})$, $\gamma_4 = 400$.

h_k	$\|y - y_h^\gamma\|_{L^\infty}$	EOC_y	$\|l - l_h^\gamma\|_{L^2}$	EOC_l
h_5	3.45e−2	–	5.66e−1	–
h_6	1.74e−2	0.99	3.32e−1	0.77
h_7	8.80e−3	0.98	1.82e−1	0.87
h_8	4.39e−3	1.00	9.47e−2	0.94
h_9	2.19e−3	1.00	4.83e−2	0.97

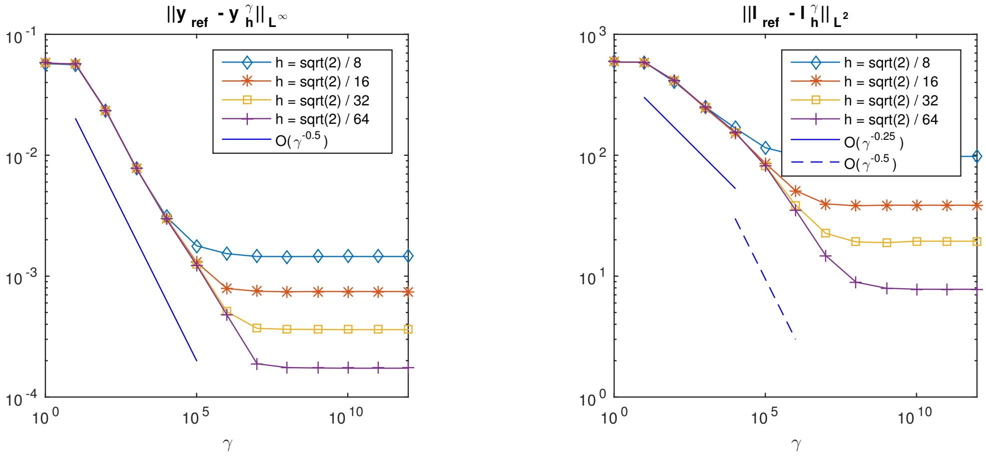

Figure 3: Errors in the states y_h^γ (left) and controls l_h^γ (right) plotted against γ for different values of h in Example 2 with piecewise constant state approximation.

Example 2: Here we consider problem ($\mathbb{P}$) with parameters $\tau = 0.01$, $m = 0.1$ and $M = 0.2$, so that the physical assumptions of a thin plate are satisfied. We again set $\Omega = (0,1)^2$ and define the load

$$f(x_1, x_2) := \begin{cases} -0.04, & x_1 \leq \frac{1}{2} \\ 0.01, & x_1 > \frac{1}{2}. \end{cases}$$

We consider the numerical solutions for $h = h_9$ as reference solutions.

Figure 3 shows the behaviors of the state and control errors in the piecewise constant case. The large magnitudes of the errors in the dual control $l := u^{-3}$ stem from the small bounds on the primary control u. In this example the regularization error in the control exhibits two consecutive convergence behaviors: Up to $\gamma = 10^5$ we observe the theoretically derived order $O(\gamma^{-0.25})$, which for larger values of γ improves to $O(\gamma^{-0.5})$. In this example the piecewise linear, continuous state approximation works well with our path-following Algorithm 1. In Table 2 we report our numerical findings for the state and control variables with piecewise constant and piecewise linear and continuous state approximation, resp. The parameters are coupled according to Theorems 4.4 and 4.6, i.e., $\gamma = O(h^{-2})$, starting with $\gamma = 400$ on h_4,

Table 2: Errors relative to the reference solution $(y_{\text{ref}}, l_{\text{ref}})$ on grid h_9 and corresponding experimental orders of convergence for the states y_h^γ and controls l_h^γ in Example 2. Parameters are coupled via $\gamma = O(h^{-2})$, $\gamma_4 = 400$, and $\gamma = O(h^{-4})$, $\gamma_4 = 16$, in the mixed and the piecewise linear, continuous case, resp.

	p.w. constant ansatz		p.w. linear ansatz		
h_k	$\lVert y_{\text{ref}} - y_h^\gamma \rVert_{L^\infty}$	$\lVert l_{\text{ref}} - l_h^\gamma \rVert_{L^2}$	$\lVert y_{\text{ref}} - y_h^\gamma \rVert_{L^\infty}$	$\lVert y_{\text{ref}} - y_h^\gamma \rVert_{H^1}$	$\lVert l_{\text{ref}} - l_h^\gamma \rVert_{L^2}$
h_4	1.22e−2	3.04e+2	3.82e−2	9.95e−2	5.81e+2
h_5	6.30e−3	2.24e+2	1.42e−2	3.42e−2	3.61e+2
h_6	3.54e−3	1.71e+2	3.80e−3	1.21e−2	2.01e+2
h_7	2.09e−3	1.23e+2	1.29e−3	4.58e−3	9.81e+1
	0.96	0.44	1.43	1.54	0.69
	0.83	0.39	1.90	1.50	0.85
	0.76	0.48	1.56	1.40	1.03

and $\gamma = O(h^{-4})$, starting with $\gamma = 16$ on h_4. With this coupling, the errors in the controls are roughly of the predicted orders, whereas the state errors seem to converge at a faster rate. Large slopes now occur in the dual control l, due to the small control constraints on u and the asymptotically singular behaviour of the adjoint state near the state active set, which reduces to a point in the limit $\gamma \to \infty$.

Figure 4 displays the resulting approximations to the optimal state y and control u on the grid with $h = h_7$. Similar to the experiment in [Arnautu et al. 2000, Figure 4-1] with changing sign of the load, we observe a bang-bang-like control with minimal thickness along the boundary.

Let us comment on the active set shapes of the variationally discretized control variable l, compare the discussion in Subsection 4.1. Since the approximation of the state with RT_0-elements yields piecewise constant control approximations the boundary of the active set in this case follows the finite element mesh. For piecewise linear, continuous states the control active set is generally bounded by piecewise, non-degenerate hyperbolas. In Figure 5 this is depicted for the numerical solutions of Example 2 with $\gamma = 500$ on the mesh with $h = h_5$. As expected, in the piecewise linear, continuous case the boundary of the upper active set is already well approximated on this coarse mesh, given the rather small regularization parameter.

Appendix A: Semismooth Newton method

In order to solve the subproblems $(\mathbb{D}_h^\gamma)$ and $(\mathbb{D}_{h,1}^\gamma)$ via a semismooth Newton method (cf., e.g., [Hintermüller, Ito and Kunisch 2003]), we rewrite the system of primal and adjoint equations of the associated optimality systems, (4.4)–(4.5) and (4.12)–(4.13), respectively, in the matrix form

$$F^\gamma(\mathbf{x}^\gamma) := \begin{pmatrix} A & k_1 \\ k_2^\gamma & A \end{pmatrix} (\mathbf{x}^\gamma) = 0, \tag{A.1}$$

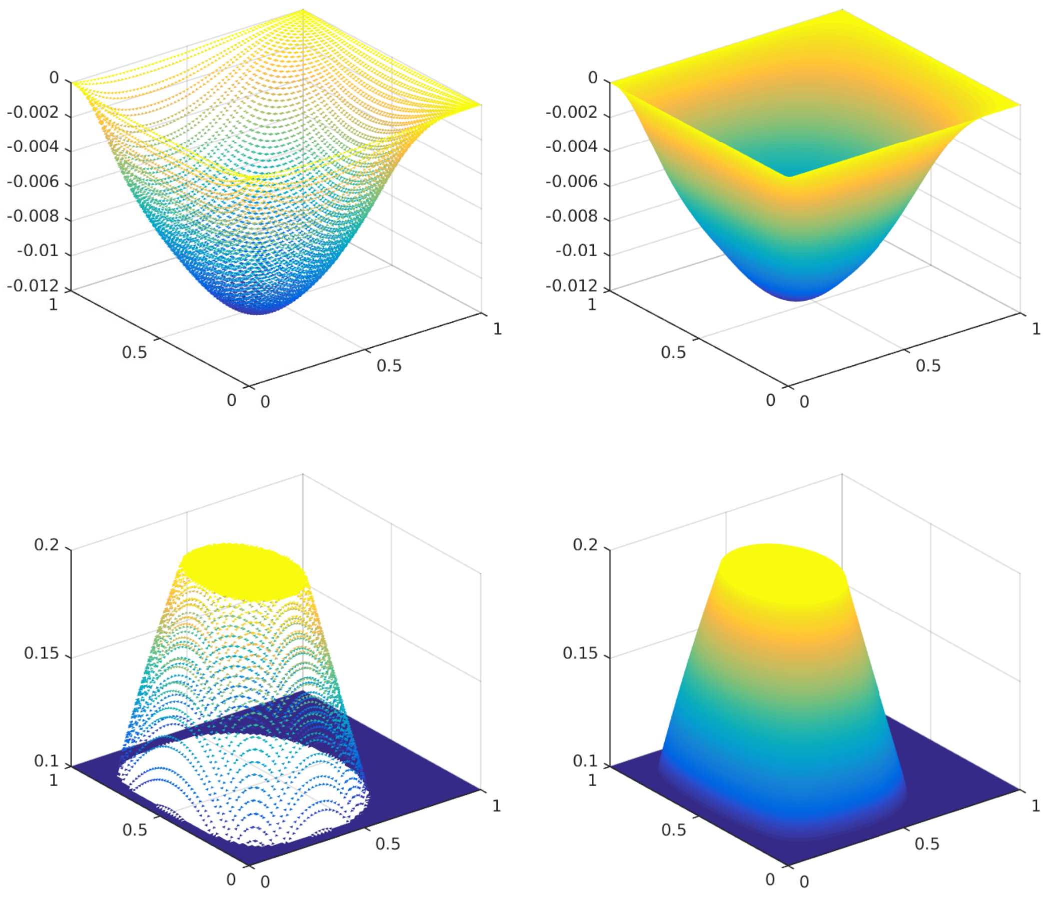

Figure 4: Final states y_h^γ (top row) and corresponding controls u_h^γ (bottom row) for $h = h_7$ in the path-following runs of Example 2. Piecewise constant (left) and piecewise linear and continuous (right) state approximation for $\gamma = 2.5600\text{e}{+}4$ and $\gamma = 6.5536\text{e}{+}4$, resp.

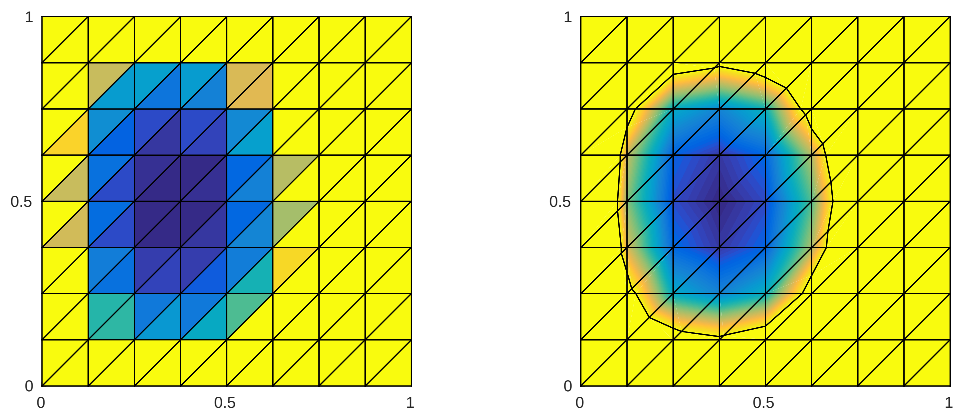

Figure 5: Comparison of the upper active set boundaries of the control l in Example 2 with $h = h_5$ and $\gamma = 500$ in the Raviart-Thomas (left) and the piecewise linear, continuous case (right).

where the variational inequality, in the equivalent form of the projection formula, has been substituted for the control in the primal equation. The matrix A denotes the according finite element system matrix of the respective discretization ansatz, whereas the operators k_1 and k_2^γ result from the right-hand sides of the primal and adjoint equations, respectively. Due to the underlying projections these operators will not be Fréchet-differentiable. Similarly to [Günther and Tber 2009] we define the following generalized derivative of F^γ as

$$DF^\gamma(\mathbf{x}) := \begin{pmatrix} A & Dk_1(\mathbf{x}) \\ Dk_2^\gamma(\mathbf{x}) & A \end{pmatrix},$$

with D denoting the generalized derivative of Clarke (cf. [Clarke 1983]).

In the Raviart-Thomas case we use from [Bahriawati and Carstensen 2005] the Matlab code *EBmfem* in order to discretize the Poisson equations and refer the reader to this reference for further details. With m and n denoting the numbers of edges and elements of the triangulation, let then $\mathbf{x}^\gamma = (\mathbf{v}_h^\gamma, y_h^\gamma, \mathbf{v}_{q,h}^\gamma, q_h^\gamma)^T \in \mathbb{R}^{2mn}$ be the combined primal and dual state vector, and $\mathbf{x} = (\mathbf{v}_h, y_h, \mathbf{v}_{q,h}, q_h)^T \in \mathbb{R}^{2mn}$ another vector. We have

$$\begin{aligned} k_1(\mathbf{v}_{q,h}^\gamma, q_h^\gamma) &:= \left(k_1(q_h^\gamma, T_j)\right)_{j=1,\ldots,n}, \\ k_2^\gamma(\mathbf{v}_h^\gamma, y_h^\gamma) &:= \left(k_2^\gamma(y_h^\gamma, T_j)\right)_{j=1,\ldots,n}, \end{aligned}$$

where

$$k_1(q_h^\gamma, T) := \begin{cases} z_{h|T}(3\, q_{h|T}^\gamma\, z_{h|T})^{-3/4}\, |T|\,, & T \in i(q_h^\gamma), \\ M^{-3}\, |T|\, z_{h|T}, & T \in l(q_h^\gamma), \\ m^{-3}\, |T|\, z_{h|T}, & T \in u(q_h^\gamma), \end{cases} \tag{A.2}$$

$$k_2^\gamma(y_h^\gamma, T) := \begin{cases} \gamma(y_{h|T}^\gamma + \tau)\, |T|\,, & T \in a(y_h^\gamma), \\ 0, & \text{else}, \end{cases}$$

with the control and state active and inactive sets

$$\begin{aligned} i(q_h^\gamma) &:= \left\{ T \in \mathcal{T}_h \;\middle|\; m^4 < 3q_h^\gamma z_h < M^4 \right\}, \\ u(q_h^\gamma) &:= \left\{ T \in \mathcal{T}_h \;\middle|\; 3q_h^\gamma z_h \le m^4 \right\}, \\ l(q_h^\gamma) &:= \left\{ T \in \mathcal{T}_h \;\middle|\; M^4 \le 3q_h^\gamma z_h \right\}, \end{aligned}$$

and

$$a(y_h^\gamma) := \left\{ T \in \mathcal{T}_h \;\middle|\; y_h^\gamma + \tau \le 0 \right\}.$$

The generalized derivatives of k_1 and k_2^γ are given by the diagonal matrices

$$\begin{aligned} Dk_1(\mathbf{x}) &:= \operatorname{diag}\left(Dk_1(q_h, T_j)\right)_{j=1,\ldots,n}, \\ Dk_2^\gamma(\mathbf{x}) &:= \operatorname{diag}\left(Dk_2^\gamma(y_h, T_j)\right)_{j=1,\ldots,n}, \end{aligned}$$

where

$$Dk_1(q_h, T) := \begin{cases} -\frac{9}{4} z_{h|T}^2\, |T|\, (3\, q_{h|T}\, z_{h|T})^{-7/4}, & T \in i(q_h), \\ 0, & \text{else}, \end{cases} \tag{A.3}$$

$$Dk_2^\gamma(y_h, T) := \begin{cases} \gamma\, |T|\,, & T \in a(y_h), \\ 0, & \text{else}. \end{cases}$$

Algorithm 1 Sketch of the path-following method for solving problem ($\mathbb{D}$).

$h_0, \gamma_0 \leftarrow$ positive initial grid size and regularization parameter
$\mathbf{x}_0 \leftarrow$ given initial state vector of appropriate dimension, with subvectors y_0, q_0
$n \leftarrow 0$
loop
  $z_n \leftarrow$ (scalar component of) $G_{h_n}(f)$
  while $\|F^{\gamma_n}(\mathbf{x}_n)\| >$ given tolerance **do**
    $\mathbf{x}_n \leftarrow \mathbf{x}_n - DF^{\gamma_n}(\mathbf{x}_n)^{-1} F^{\gamma_n}(\mathbf{x}_n)$
  end while
  $l_n \leftarrow \big(P_{[m^4,M^4]}(3\, q_n\, z_n)\big)^{-3/4}$
  if $J^{\gamma_n}(y_n, l_n) = \|l_n^{-1/3}\|_{L^1(\Omega)} + \gamma_n/2 \|(y_n+\tau)^-\|^2_{L^2(\Omega)}$ is sufficiently small **then**
    Stop and return the last iterate
  else
    $h_{n+1} \leftarrow h_n/2$
    $\gamma_{n+1} \leftarrow \gamma_n \cdot 2^\kappa$ (with κ chosen as suggested by Theorems 4.4 and 4.6)
    $\mathbf{x}_{n+1} \leftarrow$ interpolation of $\mathbf{x}_n$ on the refined mesh
    $n \leftarrow n+1$
  end if
end loop

In the case of piecewise linear elements, let N be the number of inner nodes in the triangulation and $(\phi_j)_{j=1}^N$ a canonical basis of Y_h. With $\mathbf{x}^\gamma = (y_h^\gamma, q_h^\gamma)^T \in \mathbb{R}^{2N}$ and an arbitrary $\mathbf{x} = (y_h, q_h)^T \in \mathbb{R}^{2N}$ we have that

$$\begin{aligned} k_1(q_h^\gamma) &:= \left(- \int_\Omega z_h \left(P_{[m^4,M^4]}(3\, q_h^\gamma\, z_h)\right)^{-3/4} \phi_j \,\mathrm{d}x \right)_{j=1,\dots,N}, \\ k_2^\gamma(y_h^\gamma) &:= \left(- \int_\Omega \gamma\, P_{(-\infty,0]}(y_h^\gamma + \tau)\, \phi_j \,\mathrm{d}x \right)_{j=1,\dots,N}. \end{aligned} \tag{A.4}$$

Denoting by $g_1 \in \partial P_{[m^4,M^4]}$ and $g_2 \in \partial P_{(-\infty,0]}$ subgradients of the projection operators, we further obtain

$$\begin{aligned} Dk_1(\mathbf{x})_{j,k} &:= \int_\Omega \frac{9}{4} z_h^2 \left(P_{[m^4,M^4]}(3\, q_h\, z_h)\right)^{-7/4} g_1(3\, q_h\, z_h)\, \phi_k\, \phi_j \,\mathrm{d}x, \\ Dk_2^\gamma(\mathbf{x})_{j,k} &:= - \int_\Omega \gamma\, g_2(y_h + \tau)\, \phi_k\, \phi_j \,\mathrm{d}x. \end{aligned} \tag{A.5}$$

The nonlinear equation (A.1) can now be solved with a semismooth Newton method. For this purpose, and an initial iterate $\mathbf{x}_0$, we generate a sequence of semismooth Newton steps via

$$\mathbf{x}_{n+1} := \mathbf{x}_n - DF^\gamma(\mathbf{x}_n)^{-1} F^\gamma(\mathbf{x}_n), \qquad n \in \mathbb{N},$$

which, very similar to Proposition 4.1 of [Günther and Tber 2009], can be shown to be well-defined and locally convergent.

In order to approximate the solution of ($\mathbb{D}$) one can perform a path-following method, as outlined in Algorithm 1. To this end, the above iteration can be continued until the Euclidean norm of the vector $F^\gamma(\mathbf{x}_n)$, which measures how much $\mathbf{x}_n$ violates the optimality system, is below a given tolerance, and use the obtained approximate solution to $(\mathbb{D}^\gamma)$ as the starting iterate on a refined mesh with increased γ. This nested iteration approach helps to stay within the convergence radius of the Moreau-Yosida-penalized Newton method. Overall, in our computations we observed a marked sensitivity of the Newton-type method with respect to the regularization parameter. Should the Newton solver diverge, however, we were able to return into the domain of convergence by choosing a more conservative increase in γ. As mentioned before, the associated controls can then be extracted from the state variables $\mathbf{x}_n$ by means of the projection formula (4.7).

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