MATRIX COEFFICIENT IDENTIFICATION IN AN ELLIPTIC EQUATION WITH THE CONVEX ENERGY FUNCTIONAL METHOD

Michael Hinze and Tran Nhan Tam Quyen

University of Hamburg, Bundesstrasse 55, 20146 Hamburg, Germany
Email: michael.hinze@uni-hamburg.de and quyen.tran@uni-hamburg.de

Abstract: In this paper we study the inverse problem of identifying the *diffusion matrix* in an elliptic PDE from measurements. The convex energy functional method with Tikhonov regularization is applied to tackle this problem. For the discretization we use the variational discretization concept, where the PDE is discretized with piecewise linear, continuous finite elements. We show the convergence of approximations. Using a suitable source condition, we prove an error bound for discrete solutions. For the numerical solution we propose a gradient-projection algorithm and prove the strong convergence of its iterates to a solution of the identification problem. Finally, we present a numerical experiment which illustrates our theoretical results.

Key words and phrases: Coefficient identification, diffusion matrix, Tikhonov regularization, convex energy function, source condition, convergence rates, finite element method, gradient-projection algorithm, Dirichlet problem, ill-posed problems.

1 Introduction

Let Ω be an open bounded connected domain of R^d, $d \leq 3$ with boundary $\partial\Omega$. We investigate the problem of identifying the spatially varying diffusion matrix Q in the Dirichlet problem for the elliptic equation

$$-\mathrm{div}(Q\nabla u) = f \text{ in } \Omega, \tag{1.1}$$

$$u = 0 \text{ on } \partial\Omega \tag{1.2}$$

from the observation z^δ of the solution u in the domain Ω. Here, the function $f \in L^2(\Omega)$ is given.

In this paper we assume that $z^\delta \in H_0^1(\Omega)$. For related research we refer the reader to [5, 6, 8, 21, 24, 31, 35, 47].

Our identification problem can be considered as a generalization of identifying the scalar diffusion coefficient q in the elliptic equation

$$-\mathrm{div}(q\nabla u) = f \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega. \tag{1.3}$$

The problem has been studied extensively in the last 30 years or so. The identification results can be found in [9, 34, 41, 46]. Error estimates for finite element approximation solutions have been obtained, for example, in [18, 25, 35, 47]. A survey of numerical methods for the identification problem can be found in [7, 32, 38].

Compared to the identification q in (1.3), the problem of identifying the matrix Q in (1.1) has received less attention. However, there are some contributions treating this problem. Hoffmann and Sprekels in [27] proposed a dynamical system approach to reconstruct the matrix Q in equation

(1.1). In [40] Rannacher and Vexler employed the finite element method and showed error estimates for a matrix identification problem from pointwise measurements of the state variable, provided that the sought matrix is constant and the exact data is smooth enough.

In the present paper we adopt the convex energy functional approach of Kohn and Vogelius in [36, 37] to the matrix case. In fact, for estimating the matrix Q in (1.1)–(1.2) from the observation z^δ of the solution u, we use the *non-negative convex functional* (see §2.4)

$$\mathcal{J}^\delta(Q) := \int_\Omega Q\nabla\big(\mathcal{U}(Q) - z^\delta\big) \cdot \nabla\big(\mathcal{U}(Q) - z^\delta\big)dx$$

together with Tikhonov regularization and consider the *strictly convex* minimization problem

$$\min_{Q\in\mathcal{Q}_{ad}} \mathcal{J}^\delta(Q) + \rho\|Q\|^2_{L^2(\Omega)^{d\times d}}$$

over the admissible set $\mathcal{Q}_{ad}$ (see §2.2), and consider its *unique global* solution $Q^{\rho,\delta}$ as reconstruction. Here $\rho > 0$ is the regularization parameter and $\mathcal{U}$ the non-linear coefficient-to-solution operator.

For the discretization we use the variational discretization method introduced in [26] and show the convergence of approximations. Under a source condition, which is weaker than that of the existing theories in [14, 15], we prove an error bound for discrete regularized solutions. Finally, we employ a gradient-projection algorithm for the numerical solution of the regularized problems. The strong convergence of iterates to a solution of the identification problem is ensured without smoothness requirements on the sought matrix. Numerical results show an efficiency of our theoretical findings.

In [14, 15] the authors investigated the convergence of Tikhonov regularized solutions via the standard output least squares method for the general non-linear ill-posed equation in Hilbert spaces. They proved some rates of convergence for this approach under a source condition and the so-called *small enough condition* on source elements. In the present paper, by working with a convex energy functional for our concrete identification problem, we in the proof of Theorem 5.1 are not faced with a smallness condition. Furthermore, our source condition does not require additional smoothness assumption of the sought matrix and the exact data (see §5). We also remark that such a source condition without the smallness condition was proposed in [21, 22, 23, 24] for the *scalar* coefficient identification problem in elliptic PDEs and in some concrete cases the source condition was proved to satisfy if sought coefficients belong to certain smooth function spaces.

We mention that in [16], by utilizing a modified kind of adjoint, the authors for the inverse heat conduction problem introduced a source condition in the form of a variational identity without the smallness condition on source elements. The advantage of this source condition is that it does not involve the Fréchet derivative of the coefficient-to-solution operator. However, the source condition requires some smoothness assumptions on the sought coefficient.

Starting with [28], the authors in [20, 30, 48] have proposed new source conditions in the form of variational inequalities. They proved some convergence rates for Tikhonov-type regularized solutions via the misfit functional method of the discrepancy for the general non-linear ill-posed equation in Banach spaces. The novelty of this theory is that the source conditions do not involve the Fréchet derivative of forward operators and so avoid differentiability assumptions. Furthermore, the theory is applied to inverse problems with PDEs (see, for example, [29]).

Recently, by using several sets of observations and a suitable projected source condition motivated by [19] as well as certain smoothness requirements on the sought coefficient and the exact solution, the authors of [13] derived an error bound for the finite element solutions of a standard output least squares approach to identify the diffusion matrix in (1.1). Due to the non-linearity of the identification problem the method presented in [13] solves a non-convex minimization problem. Our

approach in the present paper is different. We utilize a convex cost functional and a different source condition without smoothness assumptions. Therefore, the theory in [13] and its proof techniques are not directly comparable with our approach. Furthermore, taking the advantage of the convexity to account, we here are able to prove that iterates via a gradient-projection algorithm converge to the identified diffusion matrix.

The remaining part of this paper is organized as follows. In Section 2 and Section 3 we describe the direct and inverse problems and the finite element method which is applied to the identification problem, respectively. Convergence analysis of the finite element method is presented in Section 4. In Section 5 we show convergence rates obtained with this technique. Section 6 is devoted to a gradient-projection algorithm. Finally, in Section 7 we present a numerical experiment which illustrates our theoretical results.

Throughout the paper we use the standard notion of Sobolev spaces $H^1(\Omega)$, $H_0^1(\Omega)$, $W^{k,p}(\Omega)$, etc from, for example, [45]. If not stated otherwise we write $\int_\Omega \cdots$ instead of $\int_\Omega \cdots dx$.

2 Problem setting and preliminaries

2.1 Notations

Let $\mathcal{S}_d$ denote the set of all symmetric $d \times d$-matrices equipped with the inner product $M \cdot N := \text{trace}(MN)$ and the norm

$$\|M\|_{\mathcal{S}_d} = (M \cdot M)^{1/2} = \left(\sum_{i,j=1}^d m_{ij}^2 \right)^{1/2},$$

where $M = (m_{ij})_{1 \le i,j \le d}$. Let M and N be in $\mathcal{S}_d$, then

$$M \preceq N$$

if and only if

$$M\xi \cdot \xi \le N\xi \cdot \xi \text{ for all } \xi \in R^d.$$

We note that if $0 \preceq M \in \mathcal{S}_d$ the root $M^{1/2}$ is well defined.

In $\mathcal{S}_d$ we introduce the convex subset

$$\mathcal{K} := \{M \in \mathcal{S}_d \mid \underline{q} I_d \preceq M \preceq \overline{q} I_d\},$$

where $\underline{q}$ and $\overline{q}$ are given positive constants and I_d is the unit $d \times d$-matrix. Furthermore, let $\xi := (\xi_1, \cdots, \xi_d)$ and $\eta := (\eta_1, \cdots, \eta_d)$ be two arbitrary vectors in R^d, we use the notation

$$(\xi \otimes \eta)_{1 \le i,j \le d} \in \mathcal{S}_d \text{ with } (\xi \otimes \eta)_{ij} := \frac{1}{2}(\xi_i \eta_j + \xi_j \eta_i) \text{ for all } i,j = 1, \cdots, d.$$

Finally, in the space $L^\infty(\Omega)^{d \times d}$ we use the norm

$$\|H\|_{L^\infty(\Omega)^{d \times d}} := \max_{1 \le i,j \le d} \|h_{ij}\|_{L^\infty(\Omega)},$$

where $H = (h_{ij})_{1 \le i,j \le d} \in L^\infty(\Omega)^{d \times d}$.

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2.2 Direct and inverse problems

We recall that a function u in $H_0^1(\Omega)$ is said to be a weak solution of the Dirichlet problem (1.1)–(1.2) if the identity

$$\int_\Omega Q\nabla u \cdot \nabla v = \int_\Omega fv \tag{2.1}$$

holds for all $v \in H_0^1(\Omega)$. Assume that the matrix Q belongs to the set

$$\mathcal{Q}_{ad} := \left\{ Q \in L^\infty(\Omega)^{d\times d} \mid Q(x) \in \mathcal{K} \text{ a.e. in } \Omega \right\}. \tag{2.2}$$

Then, by the aid of the Poincaré-Friedrichs inequality in $H_0^1(\Omega)$, there exists a positive constant κ depending only on $\underline{q}$ and the domain Ω such that the coercivity condition

$$\int_\Omega Q\nabla u \cdot \nabla u \geq \kappa\|u\|_{H^1(\Omega)}^2 \tag{2.3}$$

holds for all u in $H_0^1(\Omega)$ and $Q \in \mathcal{Q}_{ad}$. Hence, by the Lax-Milgram lemma, we conclude that there exists a unique solution u of (1.1)–(1.2) satisfying the following estimate

$$\|u\|_{H^1(\Omega)} \leq \frac{1}{\kappa}\|f\|_{L^2(\Omega)}. \tag{2.4}$$

Therefore, we can define the non-linear coefficient-to-solution operator

$$\mathcal{U} : \mathcal{Q}_{ad} \subset L^\infty(\Omega)^{d\times d} \to H_0^1(\Omega)$$

which maps the matrix $Q \in \mathcal{Q}_{ad}$ to the unique solution $\mathcal{U}(Q) := u$ of the problem (1.1)–(1.2). Then, the inverse problem is stated as follows:

Given $\overline{u} := \mathcal{U}(\overline{Q}) \in H_0^1(\Omega)$, find a matrix $\overline{Q} \in \mathcal{Q}_{ad}$ such that (2.1) is satisfied with $\overline{u}$ and $\overline{Q}$.

2.3 Tikhonov regularization

According to our problem setting $\overline{u}$ is the exact solution of (1.1)–(1.2), so there exists some $\overline{Q} \in \mathcal{Q}_{ad}$ such that $\overline{u} = \mathcal{U}(\overline{Q})$. We assume that instead of the exact $\overline{u}$ we have only measurements $z^\delta \in H_0^1(\Omega)$ with

$$\|z^\delta - \overline{u}\|_{H^1(\Omega)} \leq \delta \text{ for some } \delta > 0. \tag{2.5}$$

Our problem is to reconstruct the matrix $\overline{Q}$ from z^δ. For solving this problem we consider the non-negative *convex* functional (see §2.4)

$$\mathcal{J}^\delta(Q) := \int_\Omega Q\nabla\big(\mathcal{U}(Q) - z^\delta\big) \cdot \nabla\big(\mathcal{U}(Q) - z^\delta\big). \tag{2.6}$$

Furthermore, since the problem is ill-posed, in this paper we shall use Tikhonov regularization to solve it in a stable way. Namely, we consider

$$\min_{Q\in\mathcal{Q}_{ad}} \Upsilon^{\rho,\delta}, \tag{$\mathcal{P}^{\rho,\delta}$}$$

where

$$\Upsilon^{\rho,\delta} := \mathcal{J}^\delta(Q) + \rho\|Q\|_{L^2(\Omega)^{d\times d}}^2$$

and $\rho > 0$ is the regularization parameter.

In the present paper we assume that the gradient-type observation is available. Concerning this assumption, we refer the reader to [24, 8, 31, 6, 5, 35] and the references therein, where discussions about the interpolation of discrete measurements of the solution $\overline{u}$ which results the data z^δ satisfying (2.5) are given.

2.4 Auxiliary results

Now we summarize some properties of the coefficient-to-solution operator. The proofs of the following results are based on standard arguments and therefore omitted.

Lemma 2.1. *The coefficient-to-solution operator $\mathcal{U} : \mathcal{Q}_{ad} \subset L^\infty(\Omega)^{d \times d} \to H_0^1(\Omega)$ is infinitely Fréchet differentiable on $\mathcal{Q}_{ad}$. For each $Q \in \mathcal{Q}_{ad}$ and $m \geq 1$ the action of the Fréchet derivative $\mathcal{U}^{(m)}(Q)$ in direction $(H_1, H_2, \cdots, H_m) \in \left(L^\infty(\Omega)^{d \times d} \right)^m$ denoted by $\eta := \mathcal{U}^{(m)}(Q)(H_1, H_2, \cdots, H_m)$ is the unique weak solution in $H_0^1(\Omega)$ to the equation*

$$\int_\Omega Q \nabla \eta \cdot \nabla v = -\sum_{i=1}^m \int_\Omega H_i \nabla \mathcal{U}^{(m-1)}(Q) \xi_i \cdot \nabla v \tag{2.7}$$

for all $v \in H_0^1(\Omega)$ with $\xi_i := (H_1, \cdots, H_{i-1}, H_{i+1}, \cdots, H_m)$. Furthermore, the following estimate is fulfilled

$$\|\eta\|_{H^1(\Omega)} \leq \frac{md}{\kappa^{m+1}} \|f\|_{L^2(\Omega)} \prod_{i=1}^m \|H_i\|_{L^\infty(\Omega)^{d \times d}}.$$

Now we prove the following useful results.

Lemma 2.2. *The functional $\mathcal{J}^\delta$ defined by (2.6) is convex on the convex set $\mathcal{Q}_{ad}$.*

Proof. From Lemma 2.1 we have that $\mathcal{J}^\delta$ is infinitely differentiable. A short calculation with $\eta := \mathcal{U}'(Q)H$ gives

$$\mathcal{J}^{\delta''}(Q)(H, H) = 2 \int_\Omega Q \nabla \eta \cdot \nabla \eta \geq 0$$

for all $Q \in \mathcal{Q}_{ad}$ and $H \in L^\infty(\Omega)^{d \times d}$, which proves the lemma. □

Lemma 2.3 ([39, 44])**.** *Let $(Q_n)_n$ be a sequence in $\mathcal{Q}_{ad}$. Then, there exists a subsequence, again denoted $(Q_n)_n$, and an element $Q \in \mathcal{Q}_{ad}$ such that*

> $\mathcal{U}(Q_n)$ *weakly converges to* $\mathcal{U}(Q)$ *in* $H_0^1(\Omega)$ *and*
> $Q_n \nabla \mathcal{U}(Q_n)$ *weakly converges to* $Q \nabla \mathcal{U}(Q)$ *in* $L^2(\Omega)^d$.

The sequence $(Q_n)_n$ is then said to be H-convergent to Q.

The concept of H-convergence generalizes that of G-convergence introduced by Spagnolo in [43]. Furthermore, the H-limit of a sequence is unique.

A relationship between the H-convergence and the weak* convergence in $L^\infty(\Omega)^{d \times d}$ is given by the following lemma.

Lemma 2.4 ([39]). *Let $(Q_n)_n$ be a sequence in $\mathcal{Q}_{ad}$. Assume that $(Q_n)_n$ is H-convergent to Q and $(Q_n)_n$ weak* converges to $\widehat{Q}$ in $L^\infty(\Omega)^{d\times d}$. Then, $Q(x) \preceq \widehat{Q}(x)$ a.e. in Ω and*

$$\|Q\|^2_{L^2(\Omega)^{d\times d}} \le \|\widehat{Q}\|^2_{L^2(\Omega)^{d\times d}} \le \liminf_n \|Q_n\|^2_{L^2(\Omega)^{d\times d}}.$$

Theorem 2.5. *There exists a unique minimizer $Q^{\rho,\delta}$ of the problem $\left(\mathcal{P}^{\rho,\delta}\right)$, which is called the regularized solution of the identification problem.*

Proof. Let $(Q_n)_n$ be a minimizing sequence of the problem $(\mathcal{P}^{\rho,\delta})$, i.e.,

$$\lim_n \Upsilon^{\rho,\delta}(Q_n) = \inf_{Q\in\mathcal{Q}_{ad}} \Upsilon^{\rho,\delta}(Q).$$

By Lemma 2.3 and Lemma 2.4, it follows that there exists a subsequence which is not relabelled and elements $Q \in \mathcal{Q}_{ad}$ and $\widehat{Q} \in L^\infty(\Omega)^{d\times d}$ such that

$(Q_n)_n$ is H-convergent to Q,
$(Q_n)_n$ weak* converges to $\widehat{Q}$ in $L^\infty(\Omega)^{d\times d}$,
$Q(x) \preceq \widehat{Q}(x)$ a.e. in Ω and
$\|Q\|^2_{L^2(\Omega)^{d\times d}} \le \|\widehat{Q}\|^2_{L^2(\Omega)^{d\times d}} \le \liminf_n \|Q_n\|^2_{L^2(\Omega)^{d\times d}}.$

We have that

$$\mathcal{J}^\delta(Q_n) = \int_\Omega Q_n \nabla\mathcal{U}(Q_n)\cdot\nabla(\mathcal{U}(Q_n) - z^\delta) - \int_\Omega Q_n\nabla(\mathcal{U}(Q_n) - z^\delta)\cdot\nabla z^\delta$$

$$= \int_\Omega f(\mathcal{U}(Q_n) - z^\delta) - \int_\Omega Q_n\nabla\mathcal{U}(Q_n)\cdot\nabla z^\delta + \int_\Omega Q_n\nabla z^\delta\cdot\nabla z^\delta.$$

And so that

$$\lim_n \mathcal{J}^\delta(Q_n) = \int_\Omega f(\mathcal{U}(Q) - z^\delta) - \int_\Omega Q\nabla\mathcal{U}(Q)\cdot\nabla z^\delta + \int_\Omega \widehat{Q}\nabla z^\delta\cdot\nabla z^\delta$$

$$= \mathcal{J}^\delta(Q) + \int_\Omega (\widehat{Q} - Q)\nabla z^\delta\cdot\nabla z^\delta$$

$$\ge \mathcal{J}^\delta(Q).$$

We therefore get

$$\Upsilon^{\rho,\delta}(Q) \le \lim_n \mathcal{J}^\delta(Q_n) + \liminf_n \rho\|Q_n\|^2_{L^2(\Omega)^{d\times d}}$$

$$= \liminf_n \left(\mathcal{J}^\delta(Q_n) + \rho\|Q_n\|^2_{L^2(\Omega)^{d\times d}}\right)$$

$$= \inf_{Q\in\mathcal{Q}_{ad}} \mathcal{J}^\delta(Q) + \rho\|Q\|^2_{L^2(\Omega)^{d\times d}}.$$

Since $\Upsilon^{\rho,\delta}$ is strictly convex, the minimizer is unique. $\qquad\square$

3 Discretization

Let $(\mathcal{T}_h)_{0<h<1}$ be a family of regular and quasi-uniform triangulations of the domain $\overline{\Omega}$ with the mesh size h. For the definition of the discretization space of the state functions let us denote

$$\mathcal{V}_h^1 := \left\{v_h \in C(\overline{\Omega}) \cap H_0^1(\Omega) \mid v_{h|T} \in \mathcal{P}_1(T),\ \forall T \in \mathcal{T}_h\right\}$$

with $\mathcal{P}_1$ consisting all polynomial functions of degree less than or equal to 1. Similar to the continuous case we have the following result.

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Proposition 3.1. *Let Q be in $\mathcal{Q}_{ad}$. Then the variational equation*

$$\int_\Omega Q\nabla u_h \cdot \nabla v_h = \int_\Omega f v_h, \quad \forall v_h \in \mathcal{V}_h^1 \tag{3.1}$$

admits a unique solution $u_h = u_h(Q) \in \mathcal{V}_h^1$. Further, the prior estimate

$$\|u_h\|_{H^1(\Omega)} \le \frac{1}{\kappa} \|f\|_{L^2(\Omega)}, \tag{3.2}$$

is satisfied.

Definition 3.2. The map $\mathcal{U}_h : \mathcal{Q}_{ad} \to \mathcal{V}_h^1$ from each $Q \in \mathcal{Q}_{ad}$ to the unique solution u_h of variational equation (3.1) is called *the discrete coefficient-to-solution operator.*

We note that the operator $\mathcal{U}_h$ is Fréchet differentiable on the set $\mathcal{Q}_{ad}$. For each $Q \in \mathcal{Q}_{ad}$ and $H \in L^\infty(\Omega)^{d\times d}$ the Fréchet differential $\eta_h := \mathcal{U}_h'(Q)H$ is an element of $\mathcal{V}_h^1$ and satisfies the equation

$$\int_\Omega Q\nabla\eta_h \cdot \nabla v_h = -\int_\Omega H\nabla\mathcal{U}_h(Q) \cdot \nabla v_h \tag{3.3}$$

for all v_h in $\mathcal{V}_h^1$.

Before presenting our results we need some facts on data interpolation.

3.1 Data interpolation

It is well known that there is a usual nodal value interpolation operator

$$I_h^1 : C(\overline{\Omega}) \to \left\{ v_h \in C(\overline{\Omega}) \mid v_{h|T} \in \mathcal{P}_1(T), \ \forall T \in \mathcal{T}_h \right\}$$

such that

$$I_h^1\left(H_0^1(\Omega) \cap C(\overline{\Omega}) \right) \subset \mathcal{V}_h^1.$$

Since $H^2(\Omega)$ is continuously embedded in $C(\overline{\Omega})$ as $d \le 3$ (see, for example, [1]), the following result is standard in the theory of the finite element method, the proof of which can be found, for example, in [4, 10].

Lemma 3.3. *Let ψ be in $H_0^1(\Omega) \cap H^2(\Omega)$. Then, we have*

$$\left\|\psi - I_h^1\psi\right\|_{H^k(\Omega)} \le Ch^{m-k} \|\psi\|_{H^m(\Omega)},$$

where $0 \le k < m \le 2$.

3.2 Data mollification

Since the data z^δ is not smooth enough, in general we cannot define I_h^1 for them, when $d \ge 2$. Instead, we use Clément's interpolation operator

$$\Pi_h : L^2(\Omega) \to \left\{ v_h \in C(\overline{\Omega}) \mid v_{h|T} \in \mathcal{P}_1(T), \ \forall T \in \mathcal{T}_h \right\}$$

with

$$\Pi_h\left(H_0^1(\Omega) \right) \subset \mathcal{V}_h^1$$

and satisfying the following convergence properties and estimates

$$\lim_{h\to 0} \|\vartheta - \Pi_h\vartheta\|_{H^k(\Omega)} = 0 \ \text{ for all } \ k \in \{0,1\} \tag{3.4}$$

and

$$\|\vartheta - \Pi_h\vartheta\|_{H^k(\Omega)} \le Ch^{m-k}\|\vartheta\|_{H^m(\Omega)} \tag{3.5}$$

for $0 \le k < m \le 2$ (see [11], and some generalizations of which [2, 3, 42]).

3.3 Finite element method

Using the operator Π_h in §3.2, we introduce the discrete cost functional

$$\mathcal{J}_h^\delta(Q) := \int_\Omega Q\nabla\big(\mathcal{U}_h(Q) - \Pi_h z^\delta\big) \cdot \nabla\big(\mathcal{U}_h(Q) - \Pi_h z^\delta\big) \tag{3.6}$$

with $Q \in \mathcal{Q}_{ad}$.

We note that the cost functional $\mathcal{J}_h^\delta$ contains the interpolation $\Pi_h z^\delta$ of the measurement z^δ. This is different from the approaches in [13, 18, 35, 47]. However, it is unavoidable for a numerical experiment since in general we cannot define the pointwise values of z^δ at the nodes of $\mathcal{T}_h$.

The following results are exactly obtained as in the continuous case.

Lemma 3.4. *For each $h > 0$ the functional $\mathcal{J}_h^\delta$ defined by (3.6) is convex on the convex set $\mathcal{Q}_{ad}$.*

We adapt a finite element version of Lemma 2.3 and Lemma 2.4.

Lemma 3.5 ([12])**.** *Let $(\mathcal{T}_{h_n})_n$ be a sequence of triangulations with $\lim_n h_n = 0$ and $(Q_n)_n$ be a sequence in $\mathcal{Q}_{ad}$. Then there exists a subsequence which is not relabelled and an element $Q \in \mathcal{Q}_{ad}$ such that*

$\mathcal{U}_{h_n}(Q_n)$ *weakly converges to* $\mathcal{U}(Q)$ *in* $H_0^1(\Omega)$ *and*
$Q_n\nabla\mathcal{U}_{h_n}(Q_n)$ *weakly converges to* $Q\nabla\mathcal{U}(Q)$ *in* $L^2(\Omega)^d$.

The sequence $(Q_n)_n$ is then said to be Hd-convergent to Q.

Lemma 3.6 ([12])**.** *Let $(Q_n)_n$ be a sequence in $\mathcal{Q}_{ad}$. Assume that $(Q_n)_n$ is Hd-convergent to Q and $(Q_n)_n$ weak* converges to $\widehat{Q}$ in $L^\infty(\Omega)^{d\times d}$. Then, $Q(x) \preceq \widehat{Q}(x)$ a.e. in Ω and*

$$\|Q\|_{L^2(\Omega)^{d\times d}}^2 \le \|\widehat{Q}\|_{L^2(\Omega)^{d\times d}}^2 \le \liminf_n \|Q_n\|_{L^2(\Omega)^{d\times d}}^2.$$

Lemma 3.7. *Let*

$$\Upsilon_h^{\rho,\delta}(Q) := \mathcal{J}_h^\delta(Q) + \rho\, \|Q\|_{L^2(\Omega)^{d\times d}}^2 .$$

There exists a unique minimizer $Q_h^{\rho,\delta}$ of the strictly convex minimization problem

$$\min_{Q\in\mathcal{Q}_{ad}} \Upsilon_h^{\rho,\delta}(Q). \tag{$\mathcal{P}_h^{\rho,\delta}$}$$

Now we consider the orthogonal projection $P_\mathcal{K} : \mathcal{S}_d \to \mathcal{K}$ characterised by

$$(A - P_\mathcal{K}(A))(B - P_\mathcal{K}(A)) \le 0$$

for all $A \in \mathcal{S}_d$ and $B \in \mathcal{K}$.

Lemma 3.8. *Let $Q_h^{\rho,\delta} \in \mathcal{Q}_{ad}$. Then $Q_h^{\rho,\delta}$ is the unique solution of the problem $\left(\mathcal{P}_h^{\rho,\delta}\right)$ if and only if the equation*

$$Q_h^{\rho,\delta}(x) = P_\mathcal{K}\left(\frac{1}{2\rho}\left(\nabla\mathcal{U}_h(Q_h^{\rho,\delta})(x) \otimes \nabla\mathcal{U}_h(Q_h^{\rho,\delta})(x) - \nabla\Pi_h z^\delta(x) \otimes \nabla\Pi_h z^\delta(x)\right)\right)$$

holds for a.e. in Ω.

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Proof. Since the problem $\left(\mathcal{P}_h^{\rho,\delta}\right)$ is strictly convex, an element $Q_h^{\rho,\delta} \in \mathcal{Q}_{ad}$ is the unique solution of $\left(\mathcal{P}_h^{\rho,\delta}\right)$ if and only if the inequality

$$\mathcal{J}_h^{\delta'}(Q_h^{\rho,\delta})(Q - Q_h^{\rho,\delta}) + 2\rho \int_\Omega Q_h^{\rho,\delta} \cdot (Q - Q_h^{\rho,\delta}) \geq 0 \tag{3.7}$$

is satisfied for all $Q \in \mathcal{Q}_{ad}$.

By (3.6) and (3.3), we have that

$$
\begin{aligned}
\mathcal{J}_h^{\delta'}(Q_h^{\rho,\delta})(Q - Q_h^{\rho,\delta}) &= \int_\Omega (Q - Q_h^{\rho,\delta}) \nabla(\mathcal{U}_h(Q_h^{\rho,\delta}) - \Pi_h z^\delta) \cdot \nabla(\mathcal{U}_h(Q_h^{\rho,\delta}) - \Pi_h z^\delta) \\
&\quad + 2\int_\Omega Q_h^{\rho,\delta} \nabla \mathcal{U}_h{}'(Q_h^{\rho,\delta})(Q - Q_h^{\rho,\delta}) \cdot \nabla(\mathcal{U}_h(Q_h^{\rho,\delta}) - \Pi_h z^\delta) \\
&= \int_\Omega (Q - Q_h^{\rho,\delta}) \nabla(\mathcal{U}_h(Q_h^{\rho,\delta}) - \Pi_h z^\delta) \cdot \nabla(\mathcal{U}_h(Q_h^{\rho,\delta}) - \Pi_h z^\delta) \\
&\quad - 2\int_\Omega (Q - Q_h^{\rho,\delta}) \nabla \mathcal{U}_h(Q_h^{\rho,\delta}) \cdot \nabla(\mathcal{U}_h(Q_h^{\rho,\delta}) - \Pi_h z^\delta) \\
&= -\int_\Omega (Q - Q_h^{\rho,\delta}) \left(\nabla \mathcal{U}_h(Q_h^{\rho,\delta}) \cdot \nabla \mathcal{U}_h(Q_h^{\rho,\delta}) - \nabla \Pi_h z^\delta \cdot \nabla \Pi_h z^\delta \right) \\
&= -\int_\Omega \left(\nabla \mathcal{U}_h(Q_h^{\rho,\delta}) \otimes \nabla \mathcal{U}_h(Q_h^{\rho,\delta}) - \nabla \Pi_h z^\delta \otimes \nabla \Pi_h z^\delta \right) \cdot (Q - Q_h^{\rho,\delta}). \quad (3.8)
\end{aligned}
$$

It follows from (3.7) and (3.8) that

$$\int_\Omega \left(\frac{1}{2\rho} \left(\nabla \mathcal{U}_h(Q_h^{\rho,\delta}) \otimes \nabla \mathcal{U}_h(Q_h^{\rho,\delta}) - \nabla \Pi_h z^\delta \otimes \nabla \Pi_h z^\delta \right) - Q_h^{\rho,\delta} \right) \cdot (Q - Q_h^{\rho,\delta}) \leq 0$$

for all $Q \in \mathcal{Q}_{ad}$. Then a localization argument infers

$$
\begin{aligned}
\left(\frac{1}{2\rho} \left(\nabla \mathcal{U}_h(Q_h^{\rho,\delta})(x) \otimes \nabla \mathcal{U}_h(Q_h^{\rho,\delta})(x) - \nabla \Pi_h z^\delta(x) \otimes \nabla \Pi_h z^\delta(x) \right) - Q_h^{\rho,\delta}(x) \right) \cdot & \\
\cdot \left(M - Q_h^{\rho,\delta}(x) \right) \leq 0 &
\end{aligned}
$$

for all $M \in \mathcal{K}$. The proof is completed. $\qquad \square$

Remark. Since $\mathcal{U}_h(Q_h^{\rho,\delta})$ and $\Pi_h z^\delta$ are both in $\mathcal{V}_h^1$, the assertion of Lemma 3.8 shows that the solution of $\left(\mathcal{P}_h^{\rho,\delta}\right)$ is a piecewise constant matrix over $\mathcal{T}_h$, so that it belongs to the set $\mathcal{Q}_{ad} \cap \mathcal{V}_h$, where

$$
\begin{aligned}
\mathcal{V}_h := \Big\{ M := (m_{ij})_{1 \leq i,j \leq d} \in L^\infty(\Omega)^{d \times d} \ \Big| \ & M(x) \in \mathcal{S}_d \text{ a.e. in } \Omega \text{ and} \\
& m_{ij|T} = \text{const for all } i,j \text{ with } 1 \leq i,j \leq d \text{ and } T \in \mathcal{T}_h \Big\}. \quad (3.9)
\end{aligned}
$$

Taking this into account, a discretization of the admissible set $\mathcal{Q}_{ad}$ can be avoided. Furthermore, we note that $\mathcal{Q}_{ad} \cap \mathcal{V}_h$ is a non-empty, convex, bounded and closed set in the $L^2(\Omega)^{d \times d}$-norm in the finite dimensional space $\mathcal{V}_h$.

In what follows C is a generic positive constant which is independent of the mesh size h of $\mathcal{T}_h$, the noise level δ and the regularization parameter ρ.

4 Convergence

In this section we analyze the convergence of Tikhonov regularization. To this end, we introduce

$$\sigma_h(Q) := \|\mathcal{U}(Q) - \mathcal{U}_h(Q)\|_{H^1(\Omega)} \tag{4.1}$$

and

$$\gamma_h(\varphi) := \|\varphi - \Pi_h\varphi\|_{H^1(\Omega)}, \tag{4.2}$$

where $Q \in \mathcal{Q}_{ad}$ and $\varphi \in H_0^1(\Omega)$. We note that

$$\lim_{h \to 0} \sigma_h(Q) = 0 \text{ and } \lim_{h \to 0} \gamma_h(\varphi) = 0.$$

Theorem 4.1. *Let $(\mathcal{T}_{h_n})_n$ be a sequence of triangulations with $\lim_n h_n = 0$. Assume that $(Q_{h_n}^{\rho,\delta})_n$ is the sequence of unique minimizers of $(\mathcal{P}_{h_n}^{\rho,\delta})$. Then $(Q_{h_n}^{\rho,\delta})_n$ converges to minimizer $Q^{\rho,\delta}$ of $(\mathcal{P}^{\rho,\delta})$ in the $L^2(\Omega)^{d \times d}$-norm.*

Proof. For the sake of the notation we denote by

$$Q_n := Q_{h_n}^{\rho,\delta}.$$

In view of Lemma 3.5 and Lemma 3.6 there exists a subsequence, again denoted $(Q_n)_n$ and elements $Q^{\rho,\delta} \in \mathcal{Q}_{ad}$, $\widehat{Q} \in L^\infty(\Omega)^{d \times d}$ such that $(Q_n)_n$ is Hd-convergent to $Q^{\rho,\delta}$, $(Q_n)_n$ weak* converges to $\widehat{Q}$ in $L^\infty(\Omega)^{d \times d}$, $Q^{\rho,\delta}(x) \preceq \widehat{Q}(x)$ a.e. in Ω and $\|Q^{\rho,\delta}\|_{L^2(\Omega)^{d \times d}}^2 \leq \|\widehat{Q}\|_{L^2(\Omega)^{d \times d}}^2 \leq \liminf_n \|Q_n\|_{L^2(\Omega)^{d \times d}}^2$.

First we show that

$$\lim_n \mathcal{J}_{h_n}^\delta(Q_n) \geq \mathcal{J}^\delta(Q^{\rho,\delta}). \tag{4.3}$$

Indeed, we write

$$\mathcal{J}_{h_n}^\delta(Q_n) = \int_\Omega Q_n \nabla \mathcal{U}_{h_n}(Q_n) \cdot \nabla \mathcal{U}_{h_n}(Q_n) - 2 \int_\Omega Q_n \nabla \mathcal{U}_{h_n}(Q_n) \cdot \nabla \Pi_{h_n} z^\delta + \int_\Omega Q_n \nabla \Pi_{h_n} z^\delta \cdot \nabla \Pi_{h_n} z^\delta. \tag{4.4}$$

We have that

$$\int_\Omega Q_n \nabla \mathcal{U}_{h_n}(Q_n) \cdot \nabla \mathcal{U}_{h_n}(Q_n) = \int_\Omega f \mathcal{U}_{h_n}(Q_n)$$
$$\to \int_\Omega f \mathcal{U}(Q^{\rho,\delta}) \tag{4.5}$$

and, by (3.4),

$$\int_\Omega Q_n \nabla \mathcal{U}_{h_n}(Q_n) \cdot \nabla \Pi_{h_n} z^\delta = \int_\Omega Q_n \nabla \mathcal{U}_{h_n}(Q_n) \cdot \nabla z^\delta + \int_\Omega Q_n \nabla \mathcal{U}_{h_n}(Q_n) \cdot \nabla(\Pi_{h_n} z^\delta - z^\delta)$$
$$\to \int_\Omega Q^{\rho,\delta} \nabla \mathcal{U}(Q^{\rho,\delta}) \cdot \nabla z^\delta \tag{4.6}$$

and

$$\int_\Omega Q_n \nabla \Pi_{h_n} z^\delta \cdot \nabla \Pi_{h_n} z^\delta = \int_\Omega Q_n \nabla z^\delta \cdot \nabla z^\delta + \int_\Omega Q_n \nabla (\Pi_{h_n} z^\delta - z^\delta) \cdot \nabla \Pi_{h_n} z^\delta$$

$$+ \int_\Omega Q_n \nabla (\Pi_{h_n} z^\delta - z^\delta) \cdot \nabla z^\delta$$

$$\to \int_\Omega \widehat{Q} \nabla z^\delta \cdot \nabla z^\delta$$

$$= \int_\Omega Q^{\rho,\delta} \nabla z^\delta \cdot \nabla z^\delta + \int_\Omega (\widehat{Q} - Q^{\rho,\delta}) \nabla z^\delta \cdot \nabla z^\delta$$

$$\geq \int_\Omega Q^{\rho,\delta} \nabla z^\delta \cdot \nabla z^\delta. \tag{4.7}$$

By (4.4)–(4.7), we arrive at (4.3). Furthermore, in view of (4.1) and (3.4), for all $Q \in \mathcal{Q}_{ad}$ we also get

$$\lim_n \mathcal{J}_{h_n}^\delta(Q) = \mathcal{J}^\delta(Q).$$

Hence it follows that for all $Q \in \mathcal{Q}_{ad}$

$$\mathcal{J}^\delta(Q^{\rho,\delta}) + \rho \|Q^{\rho,\delta}\|_{L^2(\Omega)^{d\times d}}^2 \leq \lim_n \mathcal{J}_{h_n}^\delta(Q_n) + \liminf_n \rho \|Q_n\|_{L^2(\Omega)^{d\times d}}^2$$

$$= \liminf_n \left(\mathcal{J}_{h_n}^\delta(Q_n) + \rho \|Q_n\|_{L^2(\Omega)^{d\times d}}^2 \right)$$

$$\leq \limsup_n \left(\mathcal{J}_{h_n}^\delta(Q_n) + \rho \|Q_n\|_{L^2(\Omega)^{d\times d}}^2 \right)$$

$$\leq \limsup_n \left(\mathcal{J}_{h_n}^\delta(Q) + \rho \|Q\|_{L^2(\Omega)^{d\times d}}^2 \right)$$

$$= \lim_n \left(\mathcal{J}_{h_n}^\delta(Q) + \rho \|Q\|_{L^2(\Omega)^{d\times d}}^2 \right)$$

$$= \mathcal{J}^\delta(Q) + \rho \|Q\|_{L^2(\Omega)^{d\times d}}^2. \tag{4.8}$$

Thus, $Q^{\rho,\delta}$ is a unique solution to $(\mathcal{P}^{\rho,\delta})$. It remains to show that $(Q_n)_n$ converges to $Q^{\rho,\delta}$ in the $L^2(\Omega)^{d\times d}$-norm. To this end, we rewrite

$$\rho \|Q^{\rho,\delta} - Q_n\|_{L^2(\Omega)^{d\times d}}^2 = \rho \|Q^{\rho,\delta}\|_{L^2(\Omega)^{d\times d}}^2 + \rho \|Q_n\|_{L^2(\Omega)^{d\times d}}^2 - 2\rho \langle Q^{\rho,\delta}, Q_n \rangle_{L^2(\Omega)^{d\times d}}$$

$$= \rho \|Q^{\rho,\delta}\|_{L^2(\Omega)^{d\times d}}^2 + \left(\mathcal{J}_{h_n}^\delta(Q_n) + \rho \|Q_n\|_{L^2(\Omega)^{d\times d}}^2 \right)$$

$$- 2\rho \langle Q^{\rho,\delta}, Q_n \rangle_{L^2(\Omega)^{d\times d}} - \mathcal{J}_{h_n}^\delta(Q_n).$$

By (4.8), we have that

$$\limsup_n \left(\mathcal{J}_{h_n}^\delta(Q_n) + \rho \|Q_n\|_{L^2(\Omega)^{d\times d}}^2 \right) = \mathcal{J}^\delta(Q^{\rho,\delta}) + \rho \|Q^{\rho,\delta}\|_{L^2(\Omega)^{d\times d}}^2.$$

Therefore, by (4.3) and the fact that $(Q_n)_n$ weak* converges to $\widehat{Q}$ in $L^\infty(\Omega)^{d\times d}$ with $Q^{\rho,\delta}(x) \preceq \widehat{Q}(x)$ a.e. in Ω, we deduce that

$$\rho \lim_n \|Q^{\rho,\delta} - Q_n\|_{L^2(\Omega)^{d\times d}}^2 \leq \rho \|Q^{\rho,\delta}\|_{L^2(\Omega)^{d\times d}}^2 + \left(\mathcal{J}^\delta(Q^{\rho,\delta}) + \rho \|Q^{\rho,\delta}\|_{L^2(\Omega)^{d\times d}}^2 \right)$$

$$- 2\rho \langle Q^{\rho,\delta}, \widehat{Q} \rangle_{L^2(\Omega)^{d\times d}} - \mathcal{J}^\delta(Q^{\rho,\delta})$$

$$\leq 2\rho \|Q^{\rho,\delta}\|_{L^2(\Omega)^{d\times d}}^2 - 2\rho \langle Q^{\rho,\delta}, Q^{\rho,\delta} \rangle_{L^2(\Omega)^{d\times d}}$$

$$= 0.$$

The proof is completed. $\qquad\square$

Next we show convergence of discrete regularized solutions to identification problem. Before presenting our result we introduce the notion of the minimum norm solution of the identification problem.

Lemma 4.2. *The set*

$$\mathcal{I}_{\mathcal{Q}_{ad}}(\overline{u}) := \{Q \in \mathcal{Q}_{ad} \mid \mathcal{U}(Q) = \overline{u}\}$$

is non-empty, convex, bounded and closed in the $L^2(\Omega)^{d \times d}$-norm. Hence there is a unique minimizer $Q^\dagger$ of the problem

$$\min_{Q \in \mathcal{I}_{\mathcal{Q}_{ad}}(\overline{u})} \|Q\|^2_{L^2(\Omega)^{d \times d}}$$

which is called by the minimum norm solution of the identification problem.

Theorem 4.3. *Let $(\mathcal{T}_{h_n})_n$ be a sequence of triangulations with mesh sizes $(h_n)_n$. Let $(\delta_n)_n$ and $(\rho_n)_n$ be any positive sequences such that*

$$\rho_n \to 0, \ \frac{\delta_n^2}{\rho_n} \to 0, \ \frac{\sigma_{h_n}^2(Q^\dagger)}{\rho_n} \to 0 \ and \ \frac{\gamma_{h_n}^2(\mathcal{U}(Q^\dagger))}{\rho_n} \to 0$$

as $n \to \infty$. Moreover, assume that $(z^{\delta_n})_n$ is a sequence satisfying $\|\mathcal{U}(Q^\dagger) - z^{\delta_n}\|_{H_0^1(\Omega)} \le \delta_n$ and $(Q_{h_n}^{\rho_n,\delta_n})_n$ is the sequence of unique minimizers of $(\mathcal{P}_{h_n}^{\rho_n,\delta_n})$. Then $(Q_{h_n}^{\rho_n,\delta_n})_n$ converges to $Q^\dagger$ in the $L^2(\Omega)^{d \times d}$-norm as $n \to \infty$.

Proof. Denoting

$$Q_n := Q_{h_n}^{\rho_n,\delta_n}$$

and due to the definition of Q_n, we get

$$\mathcal{J}_{h_n}^{\delta_n}(Q_n) + \rho_n \|Q_n\|^2_{L^2(\Omega)^{d \times d}} \le \mathcal{J}_{h_n}^{\delta_n}(Q^\dagger) + \rho_n \|Q^\dagger\|^2_{L^2(\Omega)^{d \times d}}.$$

We have that

$$\begin{aligned}
\mathcal{J}_{h_n}^{\delta_n}&(Q^\dagger) \\
&= \int_\Omega Q^\dagger \nabla \big(\mathcal{U}_{h_n}(Q^\dagger) - \Pi_{h_n} z^{\delta_n}\big) \cdot \big(\mathcal{U}_{h_n}(Q^\dagger) - \Pi_{h_n} z^{\delta_n}\big) \\
&\le \overline{q} \big\|\mathcal{U}_{h_n}(Q^\dagger) - \Pi_{h_n} z^{\delta_n}\big\|^2_{H^1(\Omega)} \\
&= \overline{q} \big\|\mathcal{U}_{h_n}(Q^\dagger) - \mathcal{U}(Q^\dagger) + \Pi_{h_n}\big(\mathcal{U}(Q^\dagger) - z^{\delta_n}\big) + \mathcal{U}(Q^\dagger) - \Pi_{h_n} \mathcal{U}(Q^\dagger)\big\|^2_{H^1(\Omega)} \\
&\le 3\overline{q} \left(\big\|\mathcal{U}_{h_n}(Q^\dagger) - \mathcal{U}(Q^\dagger)\big\|^2_{H^1(\Omega)} + \big\|\Pi_{h_n}\big(\mathcal{U}(Q^\dagger) - z^{\delta_n}\big)\big\|^2_{H^1(\Omega)} + \big\|\mathcal{U}(Q^\dagger) - \Pi_{h_n}\mathcal{U}(Q^\dagger)\big\|^2_{H^1(\Omega)} \right) \\
&\le 3C\overline{q} \left(\sigma_{h_n}^2(Q^\dagger) + \delta_n^2 + \gamma_{h_n}^2(\mathcal{U}(Q^\dagger)) \right),
\end{aligned}$$

where C is the positive constant defined by

$$C := \max \big(1, \|\Pi_{h_n}\|_{\mathcal{L}(H^1(\Omega), H^1(\Omega))}\big).$$

So that

$$\mathcal{J}_{h_n}^{\delta_n}(Q_n) + \rho_n \|Q_n\|^2_{L^2(\Omega)^{d \times d}} \le 3C\overline{q}\big(\sigma_{h_n}^2(Q^\dagger) + \delta_n^2 + \gamma_{h_n}^2(\mathcal{U}(Q^\dagger))\big) + \rho_n \|Q^\dagger\|^2_{L^2(\Omega)^{d \times d}}. \tag{4.9}$$

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This implies

$$\limsup_n \|Q_n\|^2_{L^2(\Omega)^{d\times d}} \le \limsup_n \left(3C\overline{q} \frac{\sigma^2_{h_n}(Q^\dagger) + \delta^2_n + \gamma^2_{h_n}(\mathcal{U}(Q^\dagger))}{\rho_n} + \|Q^\dagger\|^2_{L^2(\Omega)^{d\times d}} \right)$$

$$= \|Q^\dagger\|^2_{L^2(\Omega)^{d\times d}}. \tag{4.10}$$

By Lemma 3.5 and Lemma 3.6 there exists a subsequence which is not relabelled and elements $\Theta \in \mathcal{Q}_{ad}$, $\widehat{Q} \in L^\infty(\Omega)^{d\times d}$ such that $(Q_n)_n$ is Hd-convergent to Θ, $(Q_n)_n$ weak* converges to $\widehat{Q}$ in $L^\infty(\Omega)^{d\times d}$, $\Theta(x) \preceq \widehat{Q}(x)$ a.e. in Ω and

$$\|\Theta\|^2_{L^2(\Omega)^{d\times d}} \le \|\widehat{Q}\|^2_{L^2(\Omega)^{d\times d}} \le \liminf_n \|Q_n\|^2_{L^2(\Omega)^{d\times d}}. \tag{4.11}$$

Moreover, the proof of Theorem 4.1 includes an argument which can be used to show that

$$\lim_n \mathcal{J}^{\delta_n}_{h_n}(Q_n) \ge \int_\Omega \Theta \nabla \big(\mathcal{U}(\Theta) - \mathcal{U}(Q^\dagger) \big) \cdot \nabla \big(\mathcal{U}(\Theta) - \mathcal{U}(Q^\dagger) \big).$$

Then by (2.3) and (4.9), we arrive at

$$\kappa \big\| \mathcal{U}(\Theta) - \mathcal{U}(Q^\dagger) \big\|^2_{H^1(\Omega)} \le \lim_n \mathcal{J}^{\delta_n}_{h_n}(Q_n) = 0.$$

Therefore, $\Theta \in \mathcal{I}_{\mathcal{Q}_{ad}}(\overline{u})$. Furthermore, by (4.11) and (4.10) and the uniqueness of the minimum norm solution, we obtain that $\Theta = Q^\dagger$. Finally, by (4.10), the fact that $(Q_n)_n$ weak* converges to $\widehat{Q}$ in $L^\infty(\Omega)^{d\times d}$ and $Q^\dagger(x) \preceq \widehat{Q}(x)$ a.e. in Ω, we infer that

$$\limsup_n \|Q_n - Q^\dagger\|^2_{L^2(\Omega)^{d\times d}} = \limsup_n \left(\|Q_n\|^2_{L^2(\Omega)^{d\times d}} + \|Q^\dagger\|^2_{L^2(\Omega)^{d\times d}} - 2\langle Q_n, Q^\dagger \rangle_{L^2(\Omega)^{d\times d}} \right)$$

$$\le 2\|Q^\dagger\|^2_{L^2(\Omega)^{d\times d}} - 2\langle \widehat{Q}, Q^\dagger \rangle_{L^2(\Omega)^{d\times d}}$$

$$\le 2\|Q^\dagger\|^2_{L^2(\Omega)^{d\times d}} - 2\langle Q^\dagger, Q^\dagger \rangle_{L^2(\Omega)^{d\times d}}$$

$$= 0.$$

The proof is completed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

5 Convergence rates

Now we state the result on convergence rates for Tikhonov regularization of our identification problem. Before presenting the result we recall some notions.

Any $\Psi \in L^\infty(\Omega)$ can be considered as an element in $L^\infty(\Omega)^*$ by

$$\langle \Psi, \psi \rangle_{(L^\infty(\Omega)^*, L^\infty(\Omega))} := \int_\Omega \Psi\psi \tag{5.1}$$

for all ψ in $L^\infty(\Omega)$ with $\|\Psi\|_{L^\infty(\Omega)^*} \le |\Omega| \|\Psi\|_{L^\infty(\Omega)}$.

According to Lemma 2.1, for each $Q \in \mathcal{Q}_{ad}$ the mapping

$$\mathcal{U}'(Q) : L^\infty(\Omega)^{d\times d} \to H^1_0(\Omega)$$

is a continuous operator with the dual

$$\mathcal{U}'(Q)^* : H^1_0(\Omega)^* \to \left(L^\infty(\Omega)^{d\times d} \right)^*.$$

Let $w^* \in H_0^1(\Omega)^*$ be arbitrary but fixed. We consider the Dirichlet problem

$$-\mathrm{div}(Q^\dagger \nabla w) = w^* \text{ in } \Omega \text{ and } w = 0 \text{ on } \partial\Omega \tag{5.2}$$

which has a unique weak solution $w \in H_0^1(\Omega)$. Then for all $H \in L^\infty(\Omega)^{d\times d}$ we have

$$\langle \mathcal{U}'(Q^\dagger)^* w^*, H \rangle_{\left(L^\infty(\Omega)^{d\times d*}, L^\infty(\Omega)^{d\times d}\right)} = \langle w^*, \mathcal{U}'(Q^\dagger)H \rangle_{\left(H_0^1(\Omega)^*, H_0^1(\Omega)\right)}$$
$$= \int_\Omega Q^\dagger \nabla w \cdot \nabla \mathcal{U}'(Q^\dagger)H. \tag{5.3}$$

Theorem 5.1. *Assume that there is a functional $w^* \in H_0^1(\Omega)^*$ such that*

$$\mathcal{U}'(Q^\dagger)^* w^* = Q^\dagger. \tag{5.4}$$

Then

$$\frac{\kappa}{4}\left\|\mathcal{U}_h(Q_h) - \mathcal{U}(Q^\dagger)\right\|_{H^1(\Omega)}^2 + \rho\left\|Q_h - Q^\dagger\right\|_{L^2(\Omega)^{d\times d}}^2 = \mathcal{O}\left(\delta^2 + \sigma_h^2(Q^\dagger) + \gamma_h^2\big(\mathcal{U}(Q^\dagger)\big) + \gamma_h^2(w) + \rho^2\right), \tag{5.5}$$

where $Q_h := Q_h^{\rho,\delta}$ is the unique solution of $\left(\mathcal{P}_h^{\rho,\delta}\right)$.

We remark that in case $\overline{u}, w \in H^2(\Omega)$ with w from (5.2), by the Céa's lemma and (3.5), we infer that $\sigma_h(Q^\dagger) \leq Ch$, $\gamma_h\big(\mathcal{U}(Q^\dagger)\big) \leq Ch$ and $\gamma_h(w) \leq Ch$. Therefore, the convergence rate

$$\frac{\kappa}{4}\left\|\mathcal{U}_h(Q_h) - \mathcal{U}(Q^\dagger)\right\|_{H^1(\Omega)}^2 + \rho\left\|Q_h - Q^\dagger\right\|_{L^2(\Omega)^{d\times d}}^2 = \mathcal{O}\left(\delta^2 + h^2 + \rho^2\right)$$

is obtained.

By (2.7), (5.1) and (5.3), the source condition (5.4) is satisfied if there exists a functional $w \in H_0^1(\Omega)$ such that for all $H \in L^\infty(\Omega)^{d\times d}$ the equation

$$\int_\Omega H \cdot Q^\dagger = -\int_\Omega H\nabla\mathcal{U}(Q^\dagger) \cdot \nabla w$$

holds. However, as we can see in (5.10) below, the convergence rate (5.5) is obtained under the weaker condition that there exists a functional $w \in H_0^1(\Omega)$ such that

$$\int_\Omega (Q^\dagger - Q_h) \cdot Q^\dagger \leq \left|\int_\Omega (Q^\dagger - Q_h)\nabla\mathcal{U}(Q^\dagger) \cdot \nabla w\right|. \tag{5.6}$$

Lemma 5.2. *If there exists a functional $w \in H_0^1(\Omega)$ such that*

$$Q^\dagger(x) = P_\mathcal{K}\left(\nabla\mathcal{U}(Q^\dagger)(x) \otimes \nabla w(x)\right) \quad a.e \text{ in } \Omega, \tag{5.7}$$

then the condition (5.6) holds. Thus the convergence rate (5.5) is obtained.

We note that (5.7) is the projected source condition introduced in [13]. However, we here do not require any of the smoothness of the sought matrix and the exact data.

Proof. We have

$$
\begin{aligned}
\int_\Omega (Q^\dagger - Q_h) \cdot Q^\dagger &= - \int_\Omega P_{\mathcal{K}} \left(\nabla \mathcal{U}(Q^\dagger) \otimes \nabla w \right) \cdot \left(Q_h - P_{\mathcal{K}} \left(\nabla \mathcal{U}(Q^\dagger) \otimes \nabla w \right) \right) \\
&= \int_\Omega \left(\nabla \mathcal{U}(Q^\dagger) \otimes \nabla w - P_{\mathcal{K}} \left(\nabla \mathcal{U}(Q^\dagger) \otimes \nabla w \right) \right) \cdot \left(Q_h - P_{\mathcal{K}} \left(\nabla \mathcal{U}(Q^\dagger) \otimes \nabla w \right) \right) \\
&\quad - \int_\Omega \nabla \mathcal{U}(Q^\dagger) \otimes \nabla w \cdot \left(Q_h - P_{\mathcal{K}} \left(\nabla \mathcal{U}(Q^\dagger) \otimes \nabla w \right) \right) \\
&\leq - \int_\Omega \nabla \mathcal{U}(Q^\dagger) \otimes \nabla w \cdot \left(Q_h - P_{\mathcal{K}} \left(\nabla \mathcal{U}(Q^\dagger) \otimes \nabla w \right) \right) \\
&\leq \left| \int_\Omega \nabla \mathcal{U}(Q^\dagger) \otimes \nabla w \cdot (Q^\dagger - Q_h) \right| \\
&= \left| \int_\Omega (Q^\dagger - Q_h) \nabla \mathcal{U}(Q^\dagger) \cdot \nabla w \right|,
\end{aligned}
$$

which finishes the proof. $\qquad\square$

To prove Theorem 5.1 we need the following auxiliary result.

Lemma 5.3. *The estimate*

$$
\mathcal{J}_h^\delta(Q^\dagger) \leq C \left(\delta^2 + \sigma_h^2(Q^\dagger) + \gamma_h^2 \big(\mathcal{U}(Q^\dagger) \big) \right)
$$

holds.

Proof. The stated inequality follows from an argument which has included in the proof in Theorem 4.3 and therefore omitted. $\qquad\square$

Proof of Theorem 5.1. Since Q_h is the solution of the problem $\left(\mathcal{P}_h^{\rho,\delta} \right)$, we have that

$$
\begin{aligned}
\mathcal{J}_h^\delta(Q_h) + \rho \|Q_h\|_{L^2(\Omega)^{d\times d}}^2 &\leq \mathcal{J}_h^\delta(Q^\dagger) + \rho \|Q^\dagger\|_{L^2(\Omega)^{d\times d}}^2 \\
&\leq C \left(\delta^2 + \sigma_h^2(Q^\dagger) + \gamma_h^2 \big(\mathcal{U}(Q^\dagger) \big) \right) + \rho \|Q^\dagger\|_{L^2(\Omega)^{d\times d}}^2, \quad (5.8)
\end{aligned}
$$

by Lemma 5.3. Thus, we get

$$
\begin{aligned}
&\mathcal{J}_h^\delta(Q_h) + \rho \|Q_h - Q^\dagger\|_{L^2(\Omega)^{d\times d}}^2 \\
&\quad \leq C \left(\delta^2 + \sigma_h^2(Q^\dagger) + \gamma_h^2 \big(\mathcal{U}(Q^\dagger) \big) \right) \\
&\qquad + \rho \left(\|Q^\dagger\|_{L^2(\Omega)^{d\times d}}^2 - \|Q_h\|_{L^2(\Omega)^{d\times d}}^2 + \|Q_h - Q^\dagger\|_{L^2(\Omega)^{d\times d}}^2 \right) \\
&\quad = C \left(\delta^2 + \sigma_h^2(Q^\dagger) + \gamma_h^2 \big(\mathcal{U}(Q^\dagger) \big) \right) + 2\rho \langle Q^\dagger, Q^\dagger - Q_h \rangle_{L^2(\Omega)^{d\times d}}. \quad (5.9)
\end{aligned}
$$

By (5.1), (5.4) and (5.3), we have with w from (5.2)

$$
\begin{aligned}
\langle Q^\dagger, Q^\dagger - Q_h \rangle_{L^2(\Omega)^{d\times d}} &= \langle Q^\dagger, Q^\dagger - Q_h \rangle_{\left(L^\infty(\Omega)^{d\times d*}, L^\infty(\Omega)^{d\times d} \right)} \\
&= \langle \mathcal{U}'(Q^\dagger)^* w^*, Q^\dagger - Q_h \rangle_{\left(L^\infty(\Omega)^{d\times d*}, L^\infty(\Omega)^{d\times d} \right)} \\
&= \langle w^*, \mathcal{U}'(Q^\dagger)(Q^\dagger - Q_h) \rangle_{\left(H_0^1(\Omega)^*, H_0^1(\Omega) \right)} \\
&= \int_\Omega Q^\dagger \nabla \mathcal{U}'(Q^\dagger)(Q^\dagger - Q_h) \cdot \nabla w \\
&= - \int_\Omega (Q^\dagger - Q_h) \nabla \mathcal{U}(Q^\dagger) \cdot \nabla w, \quad (5.10)
\end{aligned}
$$

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here we used the equation (2.7). Hence by (2.1), we get

$$\langle Q^\dagger, Q^\dagger - Q_h \rangle_{L^2(\Omega)^{d \times d}} = \int_\Omega Q_h \nabla \mathcal{U}(Q^\dagger) \cdot \nabla w - \int_\Omega fw$$

$$= \int_\Omega Q_h \nabla \mathcal{U}(Q^\dagger) \cdot \nabla w - \int_\Omega Q_h \nabla \mathcal{U}(Q_h) \cdot \nabla w$$

$$= \int_\Omega Q_h \nabla (\mathcal{U}(Q^\dagger) - \mathcal{U}(Q_h)) \cdot \nabla w$$

$$= \int_\Omega Q_h \nabla (\mathcal{U}(Q^\dagger) - \Pi_h z^\delta) \cdot \nabla w$$

$$+ \int_\Omega Q_h \nabla (\mathcal{U}_h(Q_h) - \mathcal{U}(Q_h)) \cdot \nabla w$$

$$+ \int_\Omega Q_h \nabla (\Pi_h z^\delta - \mathcal{U}_h(Q_h)) \cdot \nabla w$$

$$:= S_1 + S_2 + S_3.$$

We have that

$$\|\mathcal{U}(Q^\dagger) - \Pi_h z^\delta\|_{H^1(\Omega)} \le \|\Pi_h(\mathcal{U}(Q^\dagger) - z^\delta)\|_{H^1(\Omega)} + \|\mathcal{U}(Q^\dagger) - \Pi_h \mathcal{U}(Q^\dagger)\|_{H^1(\Omega)}$$

$$\le \|\Pi_h\|_{\mathcal{L}(H^1(\Omega), H^1(\Omega))} \|\mathcal{U}(Q^\dagger) - z^\delta\|_{H^1(\Omega)} + \|\mathcal{U}(Q^\dagger) - \Pi_h \mathcal{U}(Q^\dagger)\|_{H^1(\Omega)}$$

$$\le \max\left(1, \|\Pi_h\|_{\mathcal{L}(H^1(\Omega), H^1(\Omega))}\right) \left(\delta + \gamma_h(\mathcal{U}(Q^\dagger))\right)$$

$$= C\left(\delta + \gamma_h(\mathcal{U}(Q^\dagger))\right). \tag{5.11}$$

Thus we obtain

$$S_1 := \int_\Omega Q_h \nabla (\mathcal{U}(Q^\dagger) - \Pi_h z^\delta) \cdot \nabla w$$

$$\le C\|\mathcal{U}(Q^\dagger) - \Pi_h z^\delta\|_{H^1(\Omega)}$$

$$\le C\left(\delta + \gamma_h(\mathcal{U}(Q^\dagger))\right).$$

We deduce from (2.1) and (3.1) that

$$\int_\Omega Q_h \nabla (\mathcal{U}_h(Q_h) - \mathcal{U}(Q_h)) \cdot \nabla v_h = 0$$

for all $v_h \in \mathcal{V}_h^1$. Therefore, we obtain

$$S_2 := \int_\Omega Q_h \nabla (\mathcal{U}_h(Q_h) - \mathcal{U}(Q_h)) \cdot \nabla w$$

$$= \int_\Omega Q_h \nabla (\mathcal{U}_h(Q_h) - \mathcal{U}(Q_h)) \cdot \nabla (w - \Pi_h w)$$

$$\le \|Q_h \nabla (\mathcal{U}_h(Q_h) - \mathcal{U}(Q_h))\|_{L^2(\Omega)} \|\nabla(w - \Pi_h w)\|_{L^2(\Omega)}$$

$$\le C\|w - \Pi_h w\|_{H^1(\Omega)}$$

$$\le C\gamma_h(w).$$

Since $0 \preceq Q_h(x) \in \mathcal{S}_d$ for a.e. $x \in \Omega$, the root $Q_h(x)^{1/2}$ is well defined. Furthermore,

$$Q_h(x) \nabla (\Pi_h z^\delta(x) - \mathcal{U}_h(Q_h)(x)) \cdot \nabla w(x)$$

$$= Q_h^{1/2}(x) \nabla (\Pi_h z^\delta(x) - \mathcal{U}_h(Q_h)(x)) \cdot Q_h^{1/2}(x) \nabla w(x).$$

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Then applying the Cauchy-Schwarz inequality, we have

$$S_3 := \int_\Omega Q_h \nabla \big(\Pi_h z^\delta - \mathcal{U}_h(Q_h)\big) \cdot \nabla w$$

$$\leq \left(\int_\Omega Q_h \nabla \big(\mathcal{U}_h(Q_h) - \Pi_h z^\delta\big) \cdot \nabla \big(\mathcal{U}_h(Q_h) - \Pi_h z^\delta\big) \right)^{1/2} \left(\int_\Omega Q_h \nabla w \cdot \nabla w \right)^{1/2}$$

$$\leq C \left(\int_\Omega Q_h \nabla \big(\mathcal{U}_h(Q_h) - \Pi_h z^\delta\big) \cdot \nabla \big(\mathcal{U}_h(Q_h) - \Pi_h z^\delta\big) \right)^{1/2}.$$

Using Young's inequality, we obtain

$$S_3 \leq C\rho + \frac{1}{4\rho} \int_\Omega Q_h \nabla \big(\mathcal{U}_h(Q_h) - \Pi_h z^\delta\big) \cdot \nabla \big(\mathcal{U}_h(Q_h) - \Pi_h z^\delta\big).$$

Therefore, we arrive at

$$2\rho \langle Q^\dagger, Q^\dagger - Q_h \rangle_{L^2(\Omega)^{d\times d}} \leq C \left(\delta^2 + \gamma_h^2(\mathcal{U}(Q^\dagger)) + \gamma_h^2(w) + \rho^2 \right) + \frac{1}{2} \mathcal{J}_h^\delta(Q_h).$$

Combining this with (5.9), we infer that

$$\frac{1}{2} \mathcal{J}_h^\delta(Q_h) + \rho \| Q_h - Q^\dagger \|_{L^2(\Omega)^{d\times d}}^2 \leq C \left(\delta^2 + \sigma_h^2(Q^\dagger) + \gamma_h^2(\mathcal{U}(Q^\dagger)) + \gamma_h^2(w) + \rho^2 \right).$$

Now we have

$$\frac{\kappa}{4} \big\| \mathcal{U}_h(Q_h) - \mathcal{U}(Q^\dagger) \big\|_{H^1(\Omega)}^2 \leq \frac{\kappa}{2} \big\| \mathcal{U}_h(Q_h) - \Pi_h z^\delta \big\|_{H^1(\Omega)}^2 + \frac{\kappa}{2} \big\| \Pi_h z^\delta - \mathcal{U}(Q^\dagger) \big\|_{H^1(\Omega)}^2$$

$$\leq \frac{1}{2} \mathcal{J}_h^\delta(Q_h) + \kappa C^2 \left(\delta^2 + \gamma_h^2(\mathcal{U}(Q^\dagger)) \right),$$

by (2.3) and (5.11). Thus, we arrive at (5.5), which finishes the proof. $\qquad\square$

6 Gradient-projection algorithm

For the numerical solution we here use the gradient-projection algorithm of [17]. We note that many other efficient solution methods are available, see for example [33].

We consider the finite dimensional space $\mathcal{V}_h$ defined by (3.9). Let $\underline{C}_h$ and $\overline{C}_h$ be positive constants such that

$$\underline{C}_h \| H \|_{L^2(\Omega)^{d\times d}} \leq \| H \|_{L^\infty(\Omega)^{d\times d}} \leq \overline{C}_h \| H \|_{L^2(\Omega)^{d\times d}} \tag{6.1}$$

for all $H \in \mathcal{V}_h$.

The following results are useful.

Lemma 6.1. *The discrete coefficient-to-solution operator $\mathcal{U}_h$ is Lipschitz continuous on $\mathcal{Q}_{ad} \cap \mathcal{V}_h$ in the $L^2(\Omega)^{d\times d}$-norm with a Lipschitz constant*

$$\frac{\overline{C}_h d}{\kappa^2} \| f \|_{L^2(\Omega)}.$$

Proof. For all $M, N \in \mathcal{Q}_{ad} \cap \mathcal{V}_h$ it follows from (3.1) that

$$\int_\Omega M \nabla \mathcal{U}_h(M) \cdot \nabla v_h = \int_\Omega f v_h$$
$$= \int_\Omega N \nabla \mathcal{U}_h(N) \cdot \nabla v_h$$

for all $v_h \in \mathcal{V}_h^1$. Thus

$$\int_\Omega M \nabla (\mathcal{U}_h(M) - \mathcal{U}_h(N)) \cdot \nabla v_h = \int_\Omega (N - M) \nabla \mathcal{U}_h(N) \cdot \nabla v_h.$$

Choosing $v_h = \mathcal{U}_h(M) - \mathcal{U}_h(N)$, by (2.3), we have

$$\kappa \|\mathcal{U}_h(M) - \mathcal{U}_h(N)\|_{H^1(\Omega)}^2 \le d \|M - N\|_{L^\infty(\Omega)^{d \times d}} \|\mathcal{U}_h(N)\|_{H^1(\Omega)} \|\mathcal{U}_h(M) - \mathcal{U}_h(N)\|_{H^1(\Omega)}.$$

Therefore, from (3.2) and (6.1) we arrive at

$$\|\mathcal{U}_h(M) - \mathcal{U}_h(N)\|_{H^1(\Omega)} \le \frac{\overline{C}_h d}{\kappa^2} \|f\|_{L^2(\Omega)} \|M - N\|_{L^2(\Omega)^{d \times d}}. \tag{6.2}$$

This finishes the proof. $\qquad\square$

Lemma 6.2. *The objective functional $\Upsilon_h^{\rho,\delta}$ of $\left(\mathcal{P}_h^{\rho,\delta}\right)$ has the property that the gradient is Lipschitz continuous on $\mathcal{Q}_{ad} \cap \mathcal{V}_h$ in the $L^2(\Omega)^{d \times d}$-norm with a Lipschitz constant*

$$L_h := 2\overline{C}_h d \left(\frac{\overline{C}_h d}{\kappa^3} \|f\|_{L^2(\Omega)}^2 + \rho |\Omega|^{1/2} \right).$$

In other word, the estimate

$$\|\Upsilon_h^{\rho,\delta'}(M) - \Upsilon_h^{\rho,\delta'}(N)\|_{\mathcal{L}(L^2(\Omega)^{d \times d}, R)} \le L_h \|M - N\|_{L^2(\Omega)^{d \times d}}$$

is satisfied for all $M, N \in \mathcal{Q}_{ad} \cap \mathcal{V}_h$.

Proof. Since any norm on $\mathcal{V}_h$ is equivalent, $\mathcal{U}_h$ is Fréchet differentiable on the set $\mathcal{Q}_{ad} \cap \mathcal{V}_h$ in the $L^\infty(\Omega)^{d \times d}$-norm and thus in the $L^2(\Omega)^{d \times d}$-norm. For all $M, N \in \mathcal{Q}_{ad} \cap \mathcal{V}_h$ and $H \in \mathcal{V}_h$, in view of (3.8), we get

$$\left| \Upsilon_h^{\rho,\delta'}(M)H - \Upsilon_h^{\rho,\delta'}(N)H \right| = \left| \int_\Omega H \nabla \mathcal{U}_h(N) \cdot \nabla \mathcal{U}_h(N) - \int_\Omega H \nabla \mathcal{U}_h(M) \cdot \nabla \mathcal{U}_h(M) \right.$$
$$\left. + 2\rho \int_\Omega H \cdot M - 2\rho \int_\Omega H \cdot N \right|.$$

Thus

$$\left| \Upsilon_h^{\rho,\delta'}(M)H - \Upsilon_h^{\rho,\delta'}(N)H \right|$$
$$= \left| \int_\Omega H \nabla (\mathcal{U}_h(N) - \mathcal{U}_h(M)) \cdot \nabla (\mathcal{U}_h(N) + \mathcal{U}_h(M)) + 2\rho \int_\Omega H \cdot (M - N) \right|$$
$$\le \left(\int_\Omega |H \nabla (\mathcal{U}_h(N) - \mathcal{U}_h(M))|^2 \right)^{1/2} \left(\int_\Omega |\nabla (\mathcal{U}_h(N) + \mathcal{U}_h(M))|^2 \right)^{1/2}$$
$$+ 2\rho \left(\int_\Omega H \cdot H \right)^{1/2} \left(\int_\Omega (M - N) \cdot (M - N) \right)^{1/2}$$
$$\le d \|H\|_{L^\infty(\Omega)^{d \times d}} \|\mathcal{U}_h(N) - \mathcal{U}_h(M)\|_{H^1(\Omega)} \|\mathcal{U}_h(N) + \mathcal{U}_h(M)\|_{H^1(\Omega)}$$
$$+ 2\rho d \|H\|_{L^\infty(\Omega)^{d \times d}} |\Omega|^{1/2} \|M - N\|_{L^2(\Omega)^{d \times d}}.$$

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From the estimates (3.2), (6.1) and (6.2) we now get

$$\left| \Upsilon_h^{\rho,\delta'}(M)H - \Upsilon_h^{\rho,\delta'}(N)H \right|$$

$$\leq d\|H\|_{L^\infty(\Omega)^{d\times d}}\|\mathcal{U}_h(N) - \mathcal{U}_h(M)\|_{H^1(\Omega)} \left(\|\mathcal{U}_h(N)\|_{H^1(\Omega)} + \|\mathcal{U}_h(M)\|_{H^1(\Omega)} \right)$$

$$+ 2\rho d\|H\|_{L^\infty(\Omega)^{d\times d}}|\Omega|^{1/2}\|M - N\|_{L^2(\Omega)^{d\times d}}$$

$$\leq 2\overline{C}_h d \left(\frac{\overline{C}_h d}{\kappa^3}\|f\|_{L^2(\Omega)}^2 + \rho|\Omega|^{1/2} \right) \|H\|_{L^2(\Omega)^{d\times d}}\|M - N\|_{L^2(\Omega)^{d\times d}}.$$

The lemma is proved. □

Lemma 6.3 ([17]). *Let X be a non-empty, closed and convex subset of a Hilbert space $\mathcal{X}$ and $\mathfrak{F} : X \to R$ be a convex Fréchet differentiable functional with the gradient $\nabla\mathfrak{F}$ being L-Lipschitzian. Assume that the problem*

$$\min_{x \in X} \mathfrak{F}(x) \tag{6.3}$$

is consistent and let S denote its solution set. Let $(\alpha_m)_m, (\beta_m)_m$ and $(\gamma_m)_m$ be real sequences satisfying $(\alpha_m)_m \subset (0,1)$, $\overline{(\beta_m)_m} \subset (0,1)$, $(\gamma_m)_m \subset (0, L/2)$ and the following additional condition

$$\lim_m \alpha_m = 0, \ \sum_{m=1}^\infty \alpha_m = \infty \ \text{and} \ 0 < \liminf_m \gamma_m \leq \limsup_m \gamma_m < L/2.$$

Then, for any given $x^* \in X$ the iterative sequence $(x_m)_m$ is generated by $x_1 \in X$,

$$x_{m+1} := (1 - \beta_m)x_m + \beta_m P_X(x_m - \gamma_m \nabla\mathfrak{F}(x_m)) + \alpha_m(x^* - x_m) \tag{6.4}$$

converges strongly to the minimizer $x^\dagger = P_S x^*$ of the problem (6.3).

To identify a stopping criterion for the iteration (6.4) we adopt the following result.

Lemma 6.4. *Let X be a non-empty, closed and convex subset of a Hilbert space $\mathcal{X}$ and $\mathfrak{F} : X \to R$ be a convex Fréchet differentiable functional with the gradient $\nabla\mathfrak{F}$. Assume that the problem*

$$\min_{x \in X} \mathfrak{F}(x) \tag{6.5}$$

is consistent. Then $x^\dagger$ is a solution to (6.5) if and only if the equation

$$x^\dagger = P_X\left(x^\dagger - \gamma\nabla\mathfrak{F}(x^\dagger)\right)$$

holds, where γ is an arbitrary positive constant.

Proof. Since $\mathfrak{F}$ is convex differentiable, we have for all $\gamma > 0$ that

$$x^\dagger \text{solves (6.5)} \Leftrightarrow \left\langle \gamma\nabla\mathfrak{F}(x^\dagger), x - x^\dagger \right\rangle_\mathcal{X} \geq 0 \quad \text{for all } x \in X$$

$$\Leftrightarrow \left\langle \left(x^\dagger - \gamma\nabla\mathfrak{F}(x^\dagger)\right) - x^\dagger, x - x^\dagger \right\rangle_\mathcal{X} \leq 0 \quad \text{for all } x \in X$$

$$\Leftrightarrow x^\dagger = P_X\left(x^\dagger - \gamma\nabla\mathfrak{F}(x^\dagger)\right),$$

which finishes the proof. □

Now we state the main result of this section on the strong convergence of iterative solutions to that of our identification problem.

Theorem 6.5. *Let $(\mathcal{T}_{h_n})_n$ be a sequence of triangulations with mesh sizes $(h_n)_n$. For any positive sequence $(\delta_n)_n$, let $\rho_n := \rho(\delta_n)$ be such that*

$$\rho_n \to 0, \ \frac{\delta_n^2}{\rho_n} \to 0, \ \frac{\sigma_{h_n}^2(Q^\dagger)}{\rho_n} \to 0 \ and \ \frac{\gamma_{h_n}^2(Q^\dagger)}{\rho_n} \to 0$$

and $\left(z^{\delta_n}\right)_n$ be observations satisfying $\left\|\overline{u} - z^{\delta_n}\right\|_{H_0^1(\Omega)} \leq \delta_n$.

Moreover, for any fixed n let $(\alpha_m^n)_m, (\beta_m^n)_m$ and $(\gamma_m^n)_m$ be real sequences satisfying

$$(\alpha_m^n)_m \subset (0,1), \ \overline{(\beta_m^n)_m} \subset (0,1), \ (\gamma_m^n)_m \subset (0, L_{h_n}/2),$$
$$\lim_m \alpha_m^n = 0, \ \sum_{m=1}^\infty \alpha_m^n = \infty \ and$$
$$0 < \liminf_m \gamma_m^n \leq \limsup_m \gamma_m^n < L_{h_n}/2$$

with

$$L_{h_n} := 2\overline{C}_{h_n} d\left(\frac{\overline{C}_{h_n} d}{\kappa^3}\|f\|_{L^2(\Omega)}^2 + \rho_n|\Omega|^{1/2}\right). \tag{6.6}$$

Let Q^* be a prior estimate of the sought matrix $Q^\dagger$ and let $(Q_m^n)_m$ be the sequence of iterates generated by

$$
\begin{aligned}
&Q_0^n \in \mathcal{Q}_{ad} \cap \mathcal{V}_{h_n} \\
&Q_m^n := (1 - \beta_{m-1}^n)Q_{m-1}^n + \alpha_{m-1}^n(Q^* - Q_{m-1}^n) \\
&\quad + \beta_{m-1}^n P_{\mathcal{Q}_{ad}\cap\mathcal{V}_{h_n}}\big(Q_{m-1}^n - \gamma_{m-1}^n\big(\nabla\Pi_{h_n} z^{\delta_n} \otimes \nabla\Pi_{h_n} z^{\delta_n} - \nabla\mathcal{U}_{h_n}(Q_{m-1}^n) \otimes \nabla\mathcal{U}_{h_n}(Q_{m-1}^n) \\
&\quad + 2\rho_n Q_{m-1}^n\big)\big).
\end{aligned}
\tag{6.7}
$$

Then $(Q_m^n)_m$ converges strongly to the unique minimizer $Q_{h_n}^{\rho_n,\delta_n}$ of $\left(\mathcal{P}_{h_n}^{\rho_n,\delta_n}\right)$,

$$\lim_m \|Q_m^n - Q_{h_n}^{\rho_n,\delta_n}\|_{L^2(\Omega)^{d\times d}} = 0.$$

Furthermore, $(Q_m^n)_m^n$ converges strongly to the minimum norm solution $Q^\dagger$ of the identification problem,

$$\lim_n \left(\lim_m \|Q_m^n - Q^\dagger\|_{L^2(\Omega)^{d\times d}}\right) = 0.$$

Proof. Since
$$\nabla\Upsilon_h^{\rho,\delta}(Q) = \nabla\Pi_h z^\delta \otimes \nabla\Pi_h z^\delta - \nabla\mathcal{U}_h(Q) \otimes \nabla\mathcal{U}_h(Q) + 2\rho Q$$

for all $Q \in \mathcal{Q}_{ad} \cap \mathcal{V}_h$, the conclusion of the theorem follows directly from Theorem 4.3, Lemma 6.2 and Lemma 6.3. $\qquad\square$

7 Numerical tests

In this section we illustrate the theoretical result with two numerical examples. The first one is provided to compare with the numerical results obtained in [13], while the second one aims to illustrate the discontinuous coefficient identification problem.

For this purpose we consider the Dirichlet problem

$$-\text{div}(Q^\dagger \nabla \overline{u}) = f \text{ in } \Omega, \tag{7.1}$$

$$\overline{u} = 0 \text{ on } \partial\Omega \tag{7.2}$$

with $\Omega = \{x = (x_1, x_2) \in R^2 \mid -1 < x_1, x_2 < 1\}$ and

$$\overline{u}(x) = (1 - x_1^2)(1 - x_2^2). \tag{7.3}$$

Now we divide the interval $(-1, 1)$ into ℓ equal segments and so that the domain $\Omega = (-1, 1)^2$ is divided into $2\ell^2$ triangles, where the diameter of each triangle is $h_\ell = \frac{\sqrt{8}}{\ell}$. In the minimization problem $\left(\mathcal{P}_h^{\rho,\delta}\right)$ we take $h = h_\ell$ and $\rho = \rho_\ell = 0.001h_\ell$. For observations with noise we assume that

$$z^{\delta_\ell} = \overline{u} + \frac{x_1}{\ell} + \frac{x_2}{\ell} \text{ and } \Pi_{h_\ell} z^{\delta_\ell} = I_{h_\ell}^1 \left(\overline{u} + \frac{x_1}{\ell} + \frac{x_2}{\ell}\right)$$

so that

$$\delta_\ell = \left\| z^{\delta_\ell} - \overline{u} \right\|_{H^1(\Omega)} = \left\| \frac{x_1}{\ell} + \frac{x_2}{\ell} \right\|_{H^1(\Omega)} = \sqrt{\frac{32}{3}} \frac{1}{\ell} = \frac{2}{\sqrt{3}} h_\ell.$$

The constants $\underline{q}$ and $\overline{q}$ in the definition of the set $\mathcal{K}$ are respectively chosen as 0.05 and 10. We use the gradient-projection algorithm which is described in Theorem 6.5 for computing the solution of the problem $\left(\mathcal{P}_h^{\rho,\delta}\right)$.

Note that in (6.6) $d = 2$ and $\overline{C}_{h_\ell} = \frac{\ell}{\sqrt{2}}$, where for all $Q \in \mathcal{Q}_{ad}$ and $v \in H_0^1(\Omega)$

$$\|v\|_{H^1(\Omega)}^2 = \int_\Omega |\nabla v|^2 + \int_\Omega |v|^2$$

$$\leq \int_\Omega |\nabla v|^2 + \left(\sqrt{\frac{3}{2}}\right)^{(d+2)/2} |\Omega|^{1/d} \int_\Omega |\nabla v|^2$$

$$\leq \frac{1}{\underline{q}} \left(1 + \left(\sqrt{\frac{3}{2}}\right)^{(d+2)/2} |\Omega|^{1/d}\right) \int_\Omega Q\nabla v \cdot \nabla v.$$

So we can choose

$$\kappa = \frac{\underline{q}}{1 + \left(\sqrt{\frac{3}{2}}\right)^{(d+2)/2} |\Omega|^{1/d}}.$$

As the initial matrix Q_0 in (6.7) we choose

$$Q_0 := \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

The prior estimate is chosen with $Q^* := I_{h_\ell}^1 Q^\dagger$. Furthermore, The sequences $(\alpha_m)_m, (\beta_m)_m$ and $(\gamma_m)_m$ are chosen with

$$\alpha_m = \frac{1}{100m}, \quad \beta_m = \frac{100m\rho_\ell}{3m+1}, \text{ and } \gamma_m = \frac{100m\rho_\ell}{2m+1}.$$

Let $Q_{h_\ell}^{\rho_\ell,\delta_\ell}$ denote the computed numerical matrix with respect to ℓ and the iteration (6.7). According to the lemma 6.4, the iteration was stopped if

$$\text{Tolerance} := \left\| Q_{h_\ell}^{\rho_\ell,\delta_\ell} - P_{\mathcal{Q}_{ad} \cap \mathcal{V}_{h_\ell}} \left(Q_{h_\ell}^{\rho_\ell,\delta_\ell} - \gamma_m \nabla \Upsilon_{h_\ell}^{\rho_\ell,\delta_\ell}(Q_{h_\ell}^{\rho_\ell,\delta_\ell})\right) \right\|_{L^2(\Omega)^{d \times d}} < 10^{-6}$$

with

$$\nabla \Upsilon_{h_\ell}^{\rho_\ell,\delta_\ell}(Q_{h_\ell}^{\rho_\ell,\delta_\ell}) = \nabla \Pi_{h_\ell} z^{\delta_\ell} \otimes \nabla \Pi_{h_\ell} z^{\delta_\ell} - \nabla \mathcal{U}_{h_\ell}(Q_{h_\ell}^{\rho_\ell,\delta_\ell}) \otimes \nabla \mathcal{U}_{h_\ell}(Q_{h_\ell}^{\rho_\ell,\delta_\ell}) + 2\rho_\ell Q_{h_\ell}^{\rho_\ell,\delta_\ell}$$

or the number of iterations was reached 500.

7.1 Example 1

We now assume that

$$Q^\dagger(x) = P_\mathcal{K}\left(\nabla\bar{u}(x) \otimes \nabla\bar{u}(x)\right).$$

Let us denote $\eta(x) = 4(x_1^2(1-x_2^2)^2 + x_2^2(1-x_1^2)^2)$ and $P_{[\underline{q},\bar{q}]}(\eta(x)) = \max\left(\underline{q}, \min(\eta(x), \bar{q})\right)$. A calculation shows

$$Q^\dagger(x) = \begin{cases} \underline{q}I_2 & \text{if } \eta(x) = 0, \\ \underline{q}I_2 + \frac{P_{[\underline{q},\bar{q}]}(\eta(x))-\underline{q}}{\eta(x)}\nabla\bar{u}(x) \otimes \nabla\bar{u}(x) & \text{if } \eta(x) \neq 0. \end{cases}$$

Then along with $\bar{u}$ given in the equation (7.3) one can compute the right hand side f in the equation (7.1).

The numerical results are summarized in Table 1, where we present the refinement level ℓ, regularization parameter ρ_ℓ, mesh size h_ℓ of the triangulation, noise level δ_ℓ, number of iterations, value of tolerances, the final L^2 and L^∞-error in the coefficients, the final L^2 and H^1-error in the states. Their experimental order of convergence (EOC) is presented in Table 2, where

$$\text{EOC}_\Phi := \frac{\ln \Phi(h_1) - \ln \Phi(h_2)}{\ln h_1 - \ln h_2}$$

and $\Phi(h)$ is an error functional with respect to the mesh size h.

In Table 3 we present the numerical result for $\ell = 96$, where the value of tolerances, the final L^2 and L^∞-error in the coefficients, the final L^2 and H^1-error in the states are displayed for each one hundred iteration. The convergence history given in Table 1, Table 2 and Table 3 shows that the gradient-projection algorithm performs well for our identification problem.

All figures are presented here corresponding to $\ell = 96$. Figure 1 from left to right shows the graphs of the interpolation $I_{h_\ell}^1\bar{u}$, computed numerical state of the algorithm at the 500th iteration, and the difference to $I_{h_\ell}^1\bar{u}$. We write

$$Q^\dagger = \begin{pmatrix} q_{11}^\dagger & q_{12}^\dagger \\ q_{12}^\dagger & q_{22}^\dagger \end{pmatrix} \text{ and } Q_{h_\ell}^{\rho_\ell,\delta_\ell} = \begin{pmatrix} q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{11} & q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{12} \\ q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{12} & q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{22} \end{pmatrix}.$$

Figure 2 from left to right we display $I_{h_\ell}^1 q_{11}^\dagger, I_{h_\ell}^1 q_{12}^\dagger$ and $I_{h_\ell}^1 q_{22}^\dagger$. Figures 3 shows $q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{11}, q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{12}$ and $q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{22}$. Figure 4 from left to right we display differences $q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{11} - I_{h_\ell}^1 q_{11}^\dagger$, $q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{12} - I_{h_\ell}^1 q_{12}^\dagger$ and $q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{22} - I_{h_\ell}^1 q_{22}^\dagger$.

For the simplicity of the notation we denote by

$$\Gamma := \|Q_{h_\ell}^{\rho_\ell,\delta_\ell} - I_{h_\ell}^1 Q^\dagger\|_{L^2(\Omega)^{d\times d}}, \ \ \Delta := \|Q_{h_\ell}^{\rho_\ell,\delta_\ell} - I_{h_\ell}^1 Q^\dagger\|_{L^\infty(\Omega)^{d\times d}}, \tag{7.4}$$

$$\Sigma := \|\mathcal{U}_{h_\ell}(Q_{h_\ell}^{\rho_\ell,\delta_\ell}) - I_{h_\ell}^1\bar{u}\|_{L^2(\Omega)} \text{ and } \Xi := \|\mathcal{U}_{h_\ell}(Q_{h_\ell}^{\rho_\ell,\delta_\ell}) - I_{h_\ell}^1\bar{u}\|_{H^1(\Omega)}. \tag{7.5}$$

Convergence history									
ℓ	ρ_ℓ	h_ℓ	δ_ℓ	Ite.	Tol.	Γ	Δ	Σ	Ξ
6	4.7140e-4	0.4714	0.5443	500	0.0165	1.0481e-3	1.0381e-3	0.041001	0.20961
12	2.3570e-4	0.2357	0.2722	500	0.0057	1.3471e-4	1.0825e-4	0.012848	0.070352
24	1.1785e-4	0.1179	0.1361	500	0.0014	1.6826e-5	1.2084e-5	4.1855e-3	0.023265
48	5.8926e-5	0.0589	0.0680	500	3.5653e-4	2.0962e-6	1.4577e-6	9.8725e-4	6.3264e-3
96	2.9463e-5	0.0295	0.0340	500	8.9059e-5	2.6177e-7	1.8007e-7	2.6475e-4	1.8936e-3

Table 1: Refinement level ℓ, regularization parameter ρ_ℓ, mesh size h_ℓ of the triangulation, noise level δ_ℓ, number of iterations, value of tolerances, errors Γ, Δ, Σ and Ξ.

Experimental order of convergence				
ℓ	$\mathbf{EOC_\Gamma}$	$\mathbf{EOC_\Delta}$	$\mathbf{EOC_\Sigma}$	$\mathbf{EOC_\Xi}$
6	–	–	–	–
12	2.9598	3.2615	1.6741	1.5750
24	3.0011	3.1632	1.6181	1.5964
48	3.0048	3.0513	2.0839	1.8787
96	3.0014	3.0171	1.8988	1.7403
Mean of EOC	2.9918	3.1233	1.8187	1.6976

Table 2: Experimental order of convergence between finest and coarsest level for Γ, Δ, Σ and Ξ.

Numerical result for $\ell = 96$					
Iterations	**Tolerances**	Γ	Δ	Σ	Ξ
100	0.1875	5.0541	4.5433	4.6756	27.2404
200	9.8522e-3	0.01224	0.58589	6.6362e-3	0.021169
300	8.9476e-5	5.8712e-7	6.0808e-7	2.6475e-4	1.8936e-3
400	8.9386e-5	3.0944e-7	2.4408e-7	2.6475e-4	1.8936e-3
500	8.9059e-5	2.6177e-7	1.8007e-7	2.6475e-4	1.8936e-3

Table 3: Errors Γ, Δ, Σ and Ξ for $\ell = 96$.
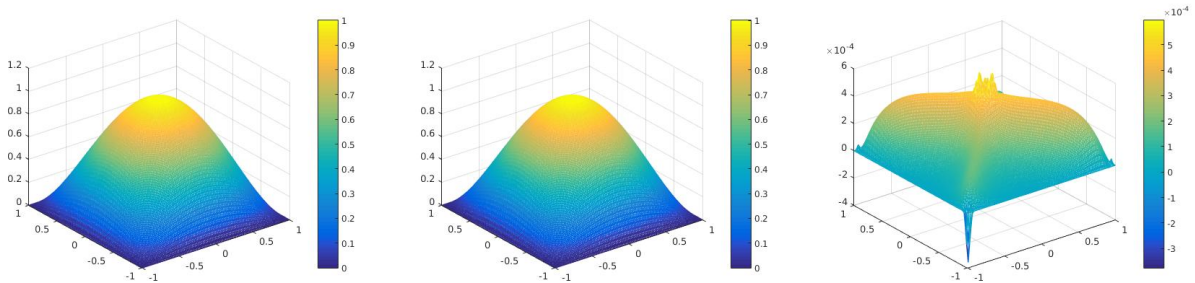

Figure 1: Interpolation $I_{h_\ell}^1 \overline{u}$, computed numerical state of the algorithm at the 500th iteration, and the difference to $I_{h_\ell}^1 \overline{u}$.
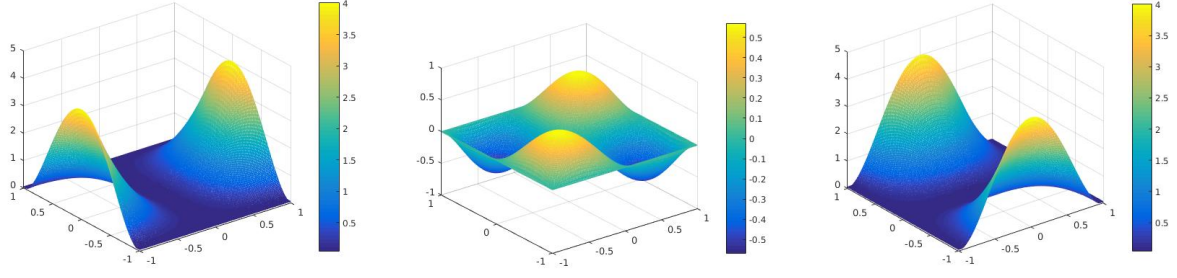
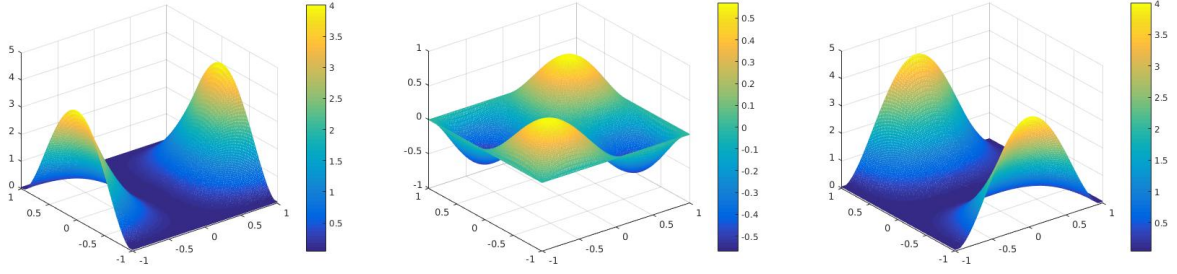Figure 2: Graphs of $I_{h_\ell}^1 q_{11}^\dagger, I_{h_\ell}^1 q_{12}^\dagger$ and $I_{h_\ell}^1 q_{22}^\dagger$.

Figure 3: Graphs of $q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{11}, q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{12}$ and $q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{22}$.
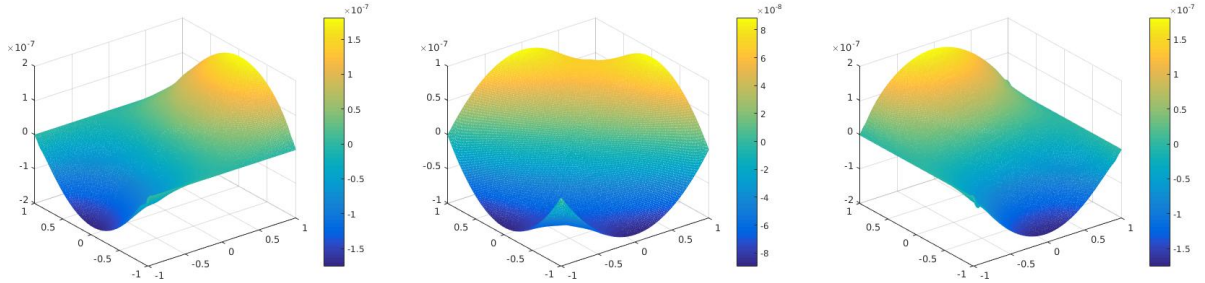

Figure 4: Differences $q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{11} - I_{h_\ell}^1 q_{11}^\dagger$, $q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{12} - I_{h_\ell}^1 q_{12}^\dagger$ and $q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{22} - I_{h_\ell}^1 q_{22}^\dagger$.

7.2 Example 2

We next assume that entries of the symmetric matrix $Q^\dagger \in \mathcal{Q}_{ad}$ are discontinuous which are defined as

$$q_{11}^\dagger(x) = \begin{cases} 3 & \text{if } x \in \Omega_{11} \\ 1 & \text{if } x \in \Omega \setminus \Omega_{11} \end{cases}, \quad q_{12}^\dagger(x) = \begin{cases} 1 & \text{if } x \in \Omega_{12} \\ 0 & \text{if } x \in \Omega \setminus \Omega_{12} \end{cases} \text{ and } q_{22}^\dagger(x) = \begin{cases} 4 & \text{if } x \in \Omega_{22} \\ 2 & \text{if } x \in \Omega \setminus \Omega_{22} \end{cases}$$

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with

$$\Omega_{11} := \left\{ (x_1, x_2) \in \Omega \,\Big|\, |x_1| \leq \frac{1}{2} \text{ and } |x_2| \leq \frac{1}{2} \right\},$$

$$\Omega_{12} := \left\{ (x_1, x_2) \in \Omega \,\Big|\, |x_1| + |x_2| \leq \frac{1}{2} \right\} \text{ and}$$

$$\Omega_{22} := \left\{ (x_1, x_2) \in \Omega \,\Big|\, x_1^2 + x_2^2 \leq \left(\frac{1}{2}\right)^2 \right\}.$$

Since the entries of the matrix $Q^\dagger$ are discontinuous, the right hand side f in the equation (7.1) is now given in the form of a load vector

$$F = KU,$$

where $K = (k_{ij})_{1 \leq i,j \leq N_\ell}$ with

$$k_{ij} := \int_\Omega Q^\dagger \nabla \phi_i \cdot \nabla \phi_j$$

and $\{\phi_1, \cdots, \phi_{N_\ell}\}$ being the basis for the approximating subspace $\mathcal{V}_{h_\ell}^1$, while the vector U is the nodal values of the functional $\overline{u}$.

With the notation on errors Γ, Δ, Σ and Ξ as in the equations (7.4)-(7.5) the numerical results of Example 7.2 are summarized in Table 4. For clarity we also present additionally the $H^1(\Omega)$-semi-norm error

$$\Lambda := \|\nabla \mathcal{U}_{h_\ell}(Q_{h_\ell}^{\rho_\ell, \delta_\ell}) - \nabla I_{h_\ell}^1 \overline{u}\|_{L^2(\Omega)}.$$

For simplicity in Table 4 we do not restate the regularization parameter ρ_ℓ, mesh size h_ℓ of the triangulation and noise level δ_ℓ again, since they have been given in Table 1 of Example 7.1.

The experimental order of convergence for Γ, Δ, Σ, Ξ and Λ is presented in Table 5.

All figures are presented corresponding to $\ell = 96$. Figure 5 from left to right contains graphs of the entries $q_{h_\ell}^{\rho_\ell, \delta_\ell}{}_{11}, q_{h_\ell}^{\rho_\ell, \delta_\ell}{}_{12}, q_{h_\ell}^{\rho_\ell, \delta_\ell}{}_{22}$ of the computed numerical matrix $Q_{h_\ell}^{\rho_\ell, \delta_\ell}$ and the computed numerical state $\mathcal{U}_{h_\ell}(Q_{h_\ell}^{\rho_\ell, \delta_\ell})$ of the algorithm at the 500$^{\text{th}}$ iteration, while Figure 6 from left to right we display differences $q_{h_\ell}^{\rho_\ell, \delta_\ell}{}_{11} - I_{h_\ell}^1 q_{11}^\dagger$, $q_{h_\ell}^{\rho_\ell, \delta_\ell}{}_{12} - I_{h_\ell}^1 q_{12}^\dagger$, $q_{h_\ell}^{\rho_\ell, \delta_\ell}{}_{22} - I_{h_\ell}^1 q_{22}^\dagger$ and $\mathcal{U}_{h_\ell}(Q_{h_\ell}^{\rho_\ell, \delta_\ell}) - I_{h_\ell}^1 \overline{u}$.

Convergence history							
ℓ	Ite.	Tol.	Γ	Δ	Σ	Ξ	Λ
6	500	0.020795	9.7270e-4	6.4331e-4	0.058024	0.058024	2.1247e-4
12	500	0.005585	1.3063e-4	8.6203e-5	0.014679	0.014679	3.7208e-5
24	500	0.001423	1.6639e-5	1.1138e-5	3.6807e-3	3.6807e-3	5.4592e-6
48	500	3.5555e-4	2.0900e-6	1.4148e-6	9.2086e-4	9.2086e-4	7.2313e-7
96	500	8.9001e-5	2.6157e-7	1.7825e-7	2.3026e-4	2.3026e-4	9.2168e-8

Table 4: Refinement level ℓ, number of iterations, value of tolerances, errors $\Gamma, \Delta, \Sigma, \Xi$ and Λ.

Experimental order of convergence				
ℓ	$\mathbf{EOC_\Gamma}$	$\mathbf{EOC_\Delta}$	$\mathbf{EOC_\Sigma = EOC_\Xi}$	$\mathbf{EOC_\Lambda}$
6	–	–	–	–
12	2.8965	2.8997	1.9829	2.5136
24	2.9728	2.9522	1.9957	2.7689
48	2.9930	2.9768	1.9989	2.9164
96	2.9982	2.9886	1.9997	2.9719
Mean of EOC	2.9651	2.9543	1.9943	2.7927

Table 5: Experimental order of convergence for Γ, Δ, Σ, Ξ and Λ.
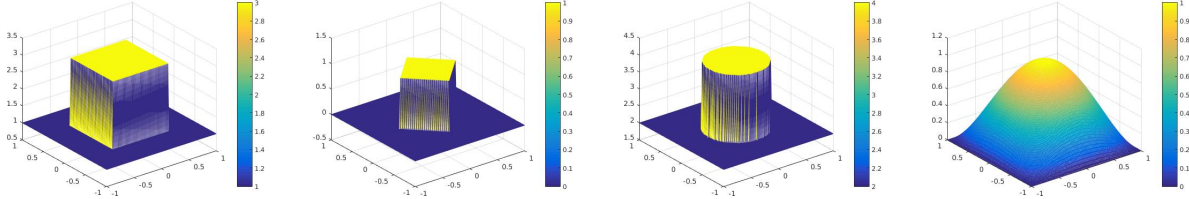
Figure 5: Graphs of $q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{11}$, $q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{12}$, $q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{22}$ and $\mathcal{U}_{h_\ell}\big(Q_{h_\ell}^{\rho_\ell,\delta_\ell}\big)$ at the 500th iteration.
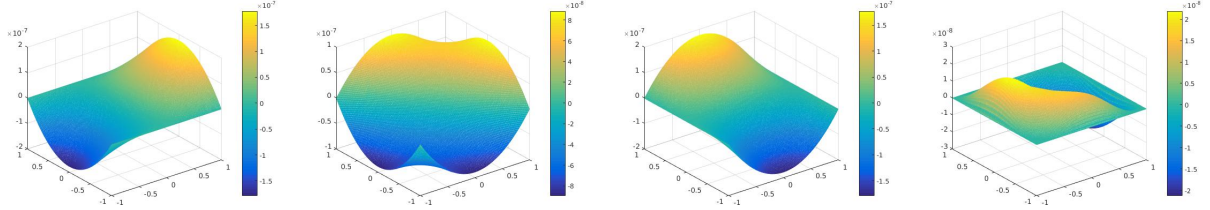

Figure 6: Graphs of $q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{11} - I_{h_\ell}^1 q_{11}^\dagger$, $q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{12} - I_{h_\ell}^1 q_{12}^\dagger$, $q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{22} - I_{h_\ell}^1 q_{22}^\dagger$ and $\mathcal{U}_{h_\ell}\big(Q_{h_\ell}^{\rho_\ell,\delta_\ell}\big) - I_{h_\ell}^1 \overline{u}$ at the 500th iteration.

Finally, Figure 7 from left to right we perform graphs of $q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{11}$, $q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{12}$, $q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{22}$ and $\mathcal{U}_{h_\ell}\big(Q_{h_\ell}^{\rho_\ell,\delta_\ell}\big)$ at the 50th iteration. At this iteration the value of tolerance is 4.1052, while errors $\Gamma, \Delta, \Sigma, \Xi$ and Λ are 6.9093, 3.9500, 9.9183, 46.2302 and 45.1537, respectively.
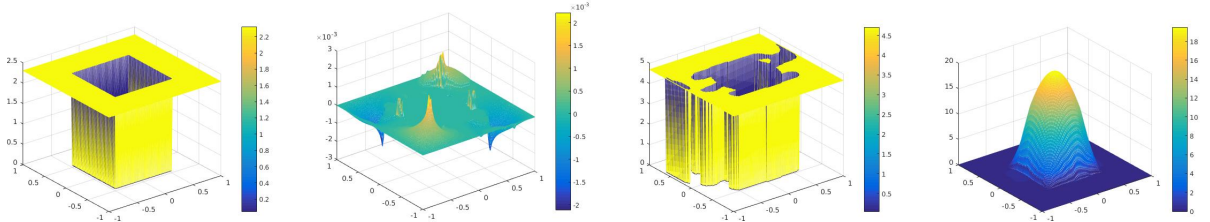

Figure 7: Graphs of $q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{11}$, $q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{12}$, $q_{h_\ell}^{\rho_\ell,\delta_\ell}{}_{22}$ and $\mathcal{U}_{h_\ell}\big(Q_{h_\ell}^{\rho_\ell,\delta_\ell}\big)$ at the 50th iteration.

We close this section by noting that the proposed method may be extended to the case where the

observation z^δ is only available in a compact subset Ω_{obs} of the domain Ω, i.e. $\Omega_{\text{obs}} \Subset \Omega$. We then use a suitable $H_0^1(\Omega)$-extension $\widehat{z}^\delta$ of z^δ as measurement in our cost functional. We then consider the following strictly convex minimization problem:

$$\min_{Q \in \mathcal{Q}_{ad}} \int_\Omega Q \nabla \big(\mathcal{U}(Q) - \widehat{z}^\delta \big) \cdot \nabla \big(\mathcal{U}(Q) - \widehat{z}^\delta \big) + \rho \| Q \|^2_{L^2(\Omega)^{d \times d}} \qquad \left(\widehat{\mathcal{P}}^{\rho,\delta} \right)$$

instead of $\left(\mathcal{P}^{\rho,\delta} \right)$. This problem then attains a unique solution $\widehat{Q}^{\rho,\delta}$, as in the case with full observations.

Acknowledgments

References

[1] H. Attouch, G. Buttazzo and G. Michaille, *Variational Analysis in Sobolev and BV Space*, SIAM Mathematical Programming Society Philadelphia, 2006.

[2] C. Bernardi, Optimal finite element interpolation on curved domain, *SIAM J. Numer. Anal.* 26(1989), 1212–1240.

[3] C. Bernardi and V. Girault, A local regularization operator for triangular and quadrilateral finite elements, *SIAM J. Numer. Anal.* 35(1998), 1893–1916.

[4] S. Brenner and R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, New York, 1994.

[5] T. F. Chan and X. C. Tai, Identification of discontinuous coefficients in elliptic problems using total variation regularization, *SIAM J. Sci. Comput.* 25(2003), 881–904.

[6] G. Chavent and K. Kunisch, The output least squares identifiability of the diffusion coefficient from an H^1-observation in a 2-D elliptic equation, *ESAIM Control Optim. Calc. Var.* 8(2002), 423–440.

[7] Z. Chen and J. Zou, An augmented Lagrangian method for identifying discontinuous parameters in elliptic systems, *SIAM J. Control Optim.* 37(1999), 892–910.

[8] I. Cherlenyak, Numerische Lösungen inverser Probleme bei elliptischen Differentialgleichungen. *Dr. rer. nat. Dissertation*, Universität Siegen, 2009, Verlag Dr. Hut, München 2010.

[9] C. Chicone and J. Gerlach, A note on the identifiability of distributed parameters in elliptic equations, *SIAM J. Math. Anal.* 18(1987), 1378–1384.

[10] P. G. Ciarlet, *Basis Error Estimates for Elliptic Problems,* Handbook of Numerical Analisis, Vol. II, P. G. Ciarlet and J. -L. Lions, eds., North-Holland, Amsterdam, 1991.

[11] P. Clément, Approximation by finite element functions using local regularization, *RAIRO Anal. Numér.* 9(1975), 77–84.

[12] K. Deckelnick and M. Hinze, Identification of matrix parameters in elliptic PDEs, *Control & Cybernetics* 40(2011), 957–970.

[13] K. Deckelnick and M. Hinze, Convergence and error analysis of a numerical method for the identification of matrix parameters in elliptic PDEs, *Inverse Problems* 28(2012), 115015 (15pp).

[14] H. W. Engl, M. Hanke and A. Neubauer, *Regularization of Inverse Problems*, Mathematics and its Applications, 375. Kluwer Academic Publishers Group, Dordrecht, 1996.

[15] H. W. Engl, K. Kunisch and A. Neubauer, Convergence rates for Tikhonov regularization of nonlinear ill-posed problems, *Inverse Problems* 5(1989), 523–540.

[16] H. W. Engl and J. Zou, A new approach to convergence rate analysis of Tikhonov regularization for parameter identification in heat conduction, *Inverse Problems* 16(2000), 1907–1923.

[17] C. D. Enyi and M. E. Soh, Modified gradient-projection algorithm for solving convex minimization problem in Hilbert spaces, *International Journal of Applied Mathematics* 44(2014), 144–150.

[18] R. S. Falk, Error estimates for the numerical identification of a variable coefficient, *Math. Comp.* 40(1983), 537–546.

[19] J. Flemming and B. Hofmann, Convergence rates in constrained Tikhonov regularization: equivalence of projected source conditions and variational inequalities, *Inverse Problems* 27(2011), 085001 (11pp).

[20] M. Grasmair, Generalized Bregman distances and convergence rates for non-convex regularization methods, *Inverse problems* 26(2010), 115014 (16pp).

[21] D. N. Hào and T. N. T. Quyen, Convergence rates for Tikhonov regularization of coefficient identification problems in Laplace-type equations, *Inverse Problems* 26(2010), 125014 (23pp).

[22] D. N. Hào and T. N. T. Quyen, Convergence rates for total variation regularization of coefficient identification problems in elliptic equations I, *Inverse Problems* 27(2011), 075008 (28pp).

[23] D. N. Hào and T. N. T. Quyen, Convergence rates for total variation regularization of coefficient identification problems in elliptic equations II, *J. Math. Anal. Appl.* 388(2012), 593–616.

[24] D. N. Hào and T. N. T. Quyen, Convergence rates for Tikhonov regularization of a two-coefficient identification problem in an elliptic boundary value problem, *Numer. Math.* 120(2012), 45–77.

[25] D. N. Hào and T. N. T. Quyen, Finite element methods for coefficient identification in an elliptic equation, *Appl. Anal.* 93(2014), 1533–1566.

[26] M. Hinze, A variational discretization concept in control constrained optimization: the linear-quadratic case, *Comput. Optim. Appl.* 30(2005), 45–61.

[27] K. H. Hoffmann and J. Sprekels, On the identification of coefficients of elliptic problems by asymptotic regularization, *Numer. Funct. Anal. Optim.* 7(1985), 157–177.

[28] B. Hofmann, B. Kaltenbacher C. Pöschl and O Scherzer, A convergence rates result for Tikhonov regularization in Banach spaces with non-smooth operators, *Inverse Problems* 23(2007), 987–1010.

[29] T. Hohage and F. Weidling, Verification of a variational source condition for acoustic inverse medium scattering problems, *Inverse Problems* 31(2015), 075006 (14pp).

[30] T. Hohage and F. Werner, Convergence Rates for Inverse Problems with Impulsive Noise, *SIAM J. Numer. Anal.* 52(2014), 1203–1221.

[31] B. Kaltenbacher and J. Schöberl, A saddle point variational formulation for projection-regularized parameter identification, *Numer. Math.* 91(2002), 675–697.

[32] Y. L. Keung and J. Zou, Numerical identifications of parameters in parabolic systems, *Inverse Problems* 14(1998), 83–100.

[33] Y. L. Keung and J. Zou, An efficient linear solver for nonlinear parameter identification problems, *SIAM J. Sci. Comput.* 22(2000), 1511–1526.

[34] I. Knowles, Uniqueness for an elliptic inverse problem, *SIAM J. Appl. Math.* 59(1999), 1356–1370.

[35] R. V. Kohn and B. D. Lowe, A variational method for parameter identification, *RAIRO Modél Math. Anal. Numér.* 22(1988), 119–158.

[36] R. V. Kohn and M. Vogelius, Determining Conductivity by Boundary Measurements, *Comm. Pure Appl. Math.* 37(1984), 289–298.

[37] R. V. Kohn and M. Vogelius, Relaxation of a variational method for impedance computed tomography, *Comm. Pure Appl. Math.* 40(1987), 745–777.

[38] K. Kunisch, *Numerical methods for parameter estimation problems*, Inverse problems in diffusion processes (Lake St. Wolfgang, 1994), 199–216, SIAM, Philadelphia, PA, 1995.

[39] F. Murat and L. Tartar, *H-Convergence.* Topics in the Mathematical Modelling of composite materials, Andrej Charkaev and Robert Kohn, eds., pp. 21–43, Birkhäuser, 1997.

[40] R. Rannacher and B. Vexler, A priori error estimates for the finite element discretization of elliptic parameter identification problems with pointwise measurements, *SIAM J. Control Optim.* 44(2005), 1844-1863.

[41] G. R. Richter, An inverse problem for the steady state diffusion equation *SIAM J. Appl. Math.*, 41(1981), 210–221.

[42] R. Scott and S. Y. Zhang, Finite element interpolation of nonsmooth function satisfying boundary conditions, *Math. Comp.* 54(1990), 483–493.

[43] S. Spagnolo, Sulla convergenza di soluzioni di equazioni paraboliche ed ellittiche. *Ann. Sc. Norm. Sup. Pisa* 22(1968), 571–597.

[44] L. Tartar, *Estimation of homegnized coefficients.* Topics in the Mathematical Modelling of composite materials, Andrej Charkaev and Robert Kohn, eds., pp. 9–20, Birkhäuser, 1997.

[45] G. M. Troianiello, *Elliptic Differential Equations and Obstacle Problems*, Plenum, New York, 1987.

[46] G. Vainikko and K. Kunisch, Identifiability of the transmissivity coefficient in an elliptic boundary value problem, *Z. Anal. Anwendungen* 12(1993), 327–341.

[47] L. Wang and J. Zou, Error estimates of finite element methods for parameter identification problems in elliptic and parabolic systems, *Discrete Contin. Dyn. Syst. Ser. B.* 14(2010), 1641–1670.

[48] F. Werner and T. Hohage, Convergence rates in expectation for Tikhonov-type regularization of inverse problems with Poisson data, *Inverse problems* 28(2012), 104004 (15pp).