

# ERROR ESTIMATE FOR A FINITE ELEMENT APPROXIMATION OF THE SOLUTION OF A LINEAR PARABOLIC EQUATION ON A TWO-DIMENSIONAL SURFACE

HEIKO KRÖNER\*

**Abstract.** We show that a certain error estimate for a fully discrete finite element approximation of the solution of the heat equation which is defined in a two-dimensional Euclidean domain carries over to the case of a general linear parabolic equation which is defined on a two-dimensional surface.

**Key words.** linear parabolic equation; two-dimensional surface; finite elements

**1. Introduction.** In many applications it is important to consider PDEs which are defined on surfaces and not in Euclidean space, especially in the case of parabolic equations it is of interest to assume that these surfaces (where the equation is defined) evolve with respect to time in a certain prescribed way. In [7] the so-called evolving surface finite element method (ESFEM) is proposed in order to solve an advection-diffusion equation on an evolving surface, cf. [7, Sections 1.1 and 1.2]. There are several papers which deal with linear parabolic equations on evolving surfaces, e.g. in [10, 8] it is shown that classical  $L^2$ - and  $L^\infty$ - estimates for a semi-discrete approximation carry over to ESFEM and in [11, 9] Runge-Kutta schemes and backward difference schemes are considered; we also mention [12, 13].

In this paper we show in details that an error estimate for a fully discrete finite element approximation of the solution of the heat equation which is stated in [3, Theorem 3.1] for the two and three dimensional Euclidean setting carries over to the case of a general linear parabolic equation which is defined on a two dimensional surface. Apart from being of interest by itself we will use our transferred error estimate together with [1] in a further paper which is joint with Michael Hinze to consider a linear-quadratic PDE-restricted optimization problem on moving surfaces for which the motion is a priori given.

Our paper is organized as follows. In Section 2 we state some general facts about finite elements on surfaces. In Section 3 we formulate the equation under consideration and its discretization. In Section 4 we state and prove our main result Theorem 4.1.

**2. Finite elements on surfaces.** Let  $\Gamma_0$  be a smooth two-dimensional, embedded, orientable, closed hypersurface in  $\mathbb{R}^3$ . Throughout the paper we choose a fixed finite atlas for  $\Gamma_0$ . We triangulate  $\Gamma_0$  by a family  $T_h$  of flat triangles with corners (i.e. nodes) lying on  $S = \Gamma_0$ . We denote the surface of class  $C^{0,1}$  given by the union of the triangles  $\tau \in T_h$  by  $\Gamma_h = S_h$ ; the union of the corresponding nodes is denoted by  $N_h$ . Here,  $h > 0$  denotes a discretization parameter which is related to the triangulation in the following way. For  $\tau \in T$  we define the diameter  $\rho(\tau)$  of the smallest disc containing  $\tau$ , the diameter  $\sigma(\tau)$  of the largest disc contained in  $\tau$  and

$$(2.1) \quad h = \max_{\tau \in T_h} \rho(\tau), \quad \gamma_h = \min_{\tau \in T_h} \frac{\sigma(\tau)}{h}.$$

We assume that the family  $(T_h)_{h>0}$  is quasi-uniform, i.e.  $\gamma_h \geq \gamma_0 > 0$ . We let

$$(2.2) \quad V_h = X_h = \{v \in C^0(S_h) : v|_\tau \text{ linear for all } \tau \in T_h\}$$

---

\*Bereich Optimierung und Approximation, Fachbereich Mathematik, Universität Hamburg, Bundesstrasse 55, 20146 Hamburg, Germany, [Heiko.Kroener@uni-hamburg.de](mailto:Heiko.Kroener@uni-hamburg.de)

be the space of continuous piecewise linear finite elements. Let  $N$  be a tubular neighborhood of  $S$  in which the Euclidean metric of  $N$  can be written in the coordinates  $(x^0, x) = (x^0, x^i)$  of the tubular neighborhood as

$$(2.3) \quad \bar{g}_{\alpha\beta} = (dx^0)^2 + \sigma_{ij}(x)dx^i dx^j.$$

Here,  $x^0$  denotes the globally (in  $N$ ) defined signed distance to  $S$ ,  $x = (x^i)_{i=1,2}$  local coordinates for  $S$  and  $\sigma_{ij} = \sigma_{ij}(x)$  the metric of  $S$ .

For small  $h$  we can write  $S_h$  as graph (with respect to the coordinates of the tubular neighborhood) over  $S$ , i.e.

$$(2.4) \quad S_h = \text{graph } \psi = \{(x^0, x) : x^0 = \psi(x), x \in S\}$$

where  $\psi = \psi_h \in C^{0,1}(S)$  suitable. Note, that

$$(2.5) \quad |D\psi|_\sigma \leq ch, \quad |\psi| \leq ch^2.$$

The induced metric of  $S_h$  is given by

$$(2.6) \quad g_{ij}(\psi(x), x) = \frac{\partial\psi}{\partial x^i}(x) \frac{\partial\psi}{\partial x^j}(x) + \sigma_{ij}(x).$$

Hence we have for the metrics, their inverses and their determinants

$$(2.7) \quad g_{ij} = \sigma_{ij} + O(h^2), \quad a^{ij} = \sigma^{ij} + O(h^2) \quad \text{and} \quad g = \sigma + O(h^2)|\sigma_{ij}\sigma^{ij}|^{\frac{1}{2}}$$

where we use summation convention.

For a function  $f : S \rightarrow \mathbb{R}$  we define its lift  $\hat{f} : S_h \rightarrow \mathbb{R}$  to  $S_h$  by  $f(x) = \hat{f}(\psi(x), x)$ ,  $x \in S$ . For a function  $f : S_h \rightarrow \mathbb{R}$  we define its lift  $\tilde{f} : S \rightarrow \mathbb{R}$  to  $S$  by  $f = \tilde{f}$ . This terminus can be obviously extended to subsets. Let  $f \in W^{1,p}(S)$ ,  $g \in W^{1,p^*}(S)$ ,  $1 \leq p \leq \infty$  and  $p^*$  Hölder conjugate of  $p$ . In local coordinates  $x = (x^i)$  of  $S$  hold

$$(2.8) \quad \int_S \langle Df, Dg \rangle = \int_S \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} \sigma^{ij}(x) \sqrt{\sigma(x)} dx^i dx^j,$$

$$(2.9) \quad \int_{S_h} \langle D\hat{f}, D\hat{g} \rangle = \int_S \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} a^{ij}(\psi(x), x) \sqrt{g(\psi(x), x)} dx^i dx^j,$$

$$(2.10) \quad \int_S \langle Df, Dg \rangle = \int_{S_h} \langle D\hat{f}, D\hat{g} \rangle + O(h^2) \|f\|_{W^{1,p}(S)} \|g\|_{W^{1,p^*}(S)},$$

and similarly,

$$(2.11) \quad \int_S f = \int_{S_h} \hat{f} + O(h^2) \|f\|_{L^1(S)}$$

where now  $f \in L^1(S)$  is sufficient.

The bracket  $\langle u, v \rangle$  denotes here the scalar product of two tangent vectors  $u, v$  (or their covariant counterparts).  $\|\cdot\|_{W^{k,p}}$  denotes the usual Sobolev norm,  $|\cdot|_{W^{k,p}} = \sum_{|\alpha|=k} \|D^\alpha \cdot\|_{L^p}$  and  $H^k = W^{k,2}$ .

Since the properties and aspects needed to prove a priori error estimates for finite element approximations are formulated in terms of integrals these observations concerning the transformation behavior of integrals essentially imply that the known error estimates from the Euclidean setting carry over to the surface case as far as convergence of at most quadratic order is concerned. Still, we present details in the following.

**3. The equation and its discretization.** Let  $T > 0$ ,  $G_T = S \times [0, T]$  and let  $a^{ij} : G_T \rightarrow T^{2,0}(S)$ ,  $b^i : G_T \rightarrow T^{1,0}(S)$ ,  $c : G_T \rightarrow T^{0,0}(S)$  be of class  $C^1$  and  $a^{ij}(\cdot, t)$ ,  $b^i(\cdot, t)$ ,  $t \in [0, T]$ , sections of the corresponding tensor bundles,  $a^{ij}$  symmetric and positive definite. We consider the initial value problem

$$(3.1) \quad \frac{d}{dt}y - \nabla_i(a^{ij}\nabla_j y) + b^i\nabla_i y + cy = f \text{ in } G_T, \quad y(\cdot, 0) = y_0$$

where  $f \in L^2(0, T; H^1(S))$ ,  $y_0 \in H^2(S)$  and  $\nabla$  denotes the connection with respect to the induced metric. Problem (3.1) has a unique solution  $y \in C^0([0, T], H^1(S)) \cap L^2(0, T; H^2(S))$  and we have

$$(3.2) \quad \max_{0 \leq t \leq T} \|y(t)\|_{H^2(S)}^2 + \int_0^T \|y_t(t)\|_{H^1(S)}^2 dt \leq c \left( \|y_0\|_{H^2(S)}^2 + \int_0^T \|f(t)\|_{H^1(S)}^2 dt \right).$$

Let  $0 = t_0 < t_1 < \dots < t_{N-1} < t_N = T$  be a time grid with  $\tau_n = t_n - t_{n-1}$ ,  $n = 1, \dots, N$  and  $\tau = \max_{1 \leq n \leq N} \tau_n$ . We set

$$(3.3) \quad W_{h,\tau} = \{ \Phi : \Gamma_h(0) \times [0, T] \rightarrow \mathbb{R} : \\ \Phi(\cdot, t) \in X_h \text{ is constant in } t \in (t_{n-1}, t_n), 1 \leq n \leq N \}$$

and define the bilinear forms

$$(3.4) \quad a : W^{1,p}(S) \times W^{1,p^*}(S) \rightarrow \mathbb{R}, \quad a(u, v) = \int_S \langle Du, Dv \rangle + uv dx,$$

$$(3.5) \quad a_h : W^{1,p}(S_h) \times W^{1,p^*}(S_h) \rightarrow \mathbb{R}, \quad a_h(u_h, v_h) = \int_{S_h} \langle Du_h, Dv_h \rangle + u_h v_h dx,$$

$$(3.6) \quad a_h^n : W^{1,p}(S_h) \times W^{1,p^*}(S_h) \rightarrow \mathbb{R}, \quad a_h^n(u_h, v_h) = \int_{S_h} \langle Du_h, Dv_h \rangle_{\tilde{g}(t_n)} + u_h v_h dx,$$

$$(3.7) \quad (Du_h, Dv_h)_{\tilde{g}(t_n)} = \int_{S_h} \langle Du_h, Dv_h \rangle_{\tilde{g}(t_n)}.$$

The last but one equation needs a further definition. Let  $p_1, \dots, p_3$  be the middle points of the three sides of  $\tau$ ,  $\tau \in T_h$ , and  $v, w \in T^{0,1}(\tau)$  then we define

$$(3.8) \quad \int_{\tau} \langle v, w \rangle_{\tilde{g}(t_n)} = \frac{1}{6} |\tau| \sum_{k=1}^3 a^{ij}(\tilde{p}_k) v_i(p_k) w_j(p_k)$$

where  $(a^{ij}(\tilde{p}_k))$  is a contravariant representation with respect to local coordinates  $(x^i)$  (belonging to our fixed atlas) in a neighbourhood of  $\tilde{p}_k$  in  $S$  and  $(v_i)(p_k)$ ,  $(w_j)(p_k)$  are covariant representations with respect to the orthogonal projections of  $\frac{\partial}{\partial x^i}(\tilde{p}_k)$  and  $\frac{\partial}{\partial x^j}(\tilde{p}_k)$  on  $\tau$ . (Despite similar notation  $\tilde{g}$  does not refer to a metric.)

We define a discrete operator  $G_h : L^2(S) \rightarrow X_h, v \mapsto G_h v = z_h$  via

$$(3.9) \quad a_h(z_h, \varphi_h) = \int_{S_h} \hat{v} \varphi_h \quad \forall \varphi_h \in X_h.$$

Furthermore, the brackets  $(\cdot, \cdot)$  and  $(\cdot, \cdot)_h$  denote the inner products of  $L^2(S)$  and  $L^2(S_h)$ , respectively, and  $\|\cdot\|$ ,  $\|\cdot\|_h$  the corresponding norms. The semi-norm associated with the bilinear on the left-hand side of (3.7) is denoted by  $\|\cdot\|_{\hat{g}(t_n)}$ . We denote the interpolation operator by  $I_h$ , define  $P_h : L^2(\Gamma_0) \rightarrow X_h$  by

$$(3.10) \quad (\hat{z}, \phi_h)_h = (P_h z, \phi_h)_h \quad \forall \phi_h \in X_h,$$

let  $R_h : H^1(S) \rightarrow X_h$  be defined by

$$(3.11) \quad a_h(R_h z, \phi_h) = a_h(\hat{z}, \phi_h) \quad \forall \phi_h \in X_h$$

and  $R_h^n : H^1(S) \rightarrow X_h$  by

$$(3.12) \quad a_h^n(R_h^n z, \phi_h) = a_h^n(\hat{z}, \phi_h) \quad \forall \phi_h \in X_h.$$

It is well-known that

$$(3.13) \quad \|\hat{z} - R_h z\|_{L^2(S_h)} + h\|D(\hat{z} - R_h z)\|_{L^2(S_h)} \leq ch^m \|z\|_{H^m(S)}$$

and

$$(3.14) \quad \|\hat{z} - R_h^n z\|_{L^2(S_h)} + h\|D(\hat{z} - R_h^n z)\|_{L^2(S_h)} \leq ch^m \|z\|_{H^m(S)}$$

hold for all  $z \in H^m(S)$ ,  $m = 1, 2$ . We conclude for  $z \in H^2(S)$  that

$$(3.15) \quad \begin{aligned} \|\hat{z} - R_h z\|_{L^\infty(S_h)} &\leq \|\hat{z} - I_h z\|_{L^\infty(S_h)} + \|I_h z - R_h z\|_{L^\infty(S_h)} \\ &\leq ch\|z\|_{H^2(S)} + ch^{-1}\|I_z - R_h z\|_{L^2(S_h)} \leq ch\|z\|_{H^2(S_h)}. \end{aligned}$$

There holds

$$(3.16) \quad \|\phi_h\|_{L^\infty(S_h)} \leq \rho(h)\|\phi_h\|_{H^1(S_h)}$$

for all  $\phi_h \in X_h$  where  $\rho(h) = \sqrt{|\log h|}$ .

For  $Y, \Phi \in W_{h,\tau}$  we let

$$(3.17) \quad \begin{aligned} A(Y, \Phi) &:= \sum_{n=1}^N \tau_n (\nabla Y^n, \nabla \Phi^n)_{\hat{g}(t_n)} + \sum_{n=2}^N (Y^n - Y^{n-1}, \Phi^n)_h + (Y_+^0, \Phi_+^0)_h \\ &+ \sum_{n=1}^N \tau_n (b^i(t_n) \nabla_i Y^n, \Phi^n)_h + \sum_{n=1}^N \tau_n (c(t_n) Y^n, \Phi^n)_h \end{aligned}$$

where  $\Phi^n := \Phi_-^n$ ,  $\Phi_+^n = \lim_{s \rightarrow 0+} \Phi(t_n + s)$ .

Note, that the integrals  $(b^i(t_n) \nabla_i Y^n, \Phi^n)_h$  and  $(c(t_n) Y^n, \Phi^n)_h$  are defined similarly to (3.8) by using a quadrature rule.

Our approximation  $Y \in W_{h,\tau}$  of the solution  $y$  of (3.1) is obtained by the following discontinuous Galerkin scheme:

$$(3.18) \quad A(Y, \Phi) = \sum_{n=1}^N \int_{t_{n-1}}^{t_n} (\hat{y}, \Phi^n)_h + (\hat{y}_0, \Phi_+^0)_h \quad \forall \Phi \in W_{h,\tau}.$$

The above solution will be denoted by  $Y = G_{h,\tau}(y)$ .

**4. The error estimate.** We have the following uniform error estimate.

THEOREM 4.1. *We have*

$$(4.1) \quad \max_{1 \leq n \leq N} \|\widehat{y}(\cdot, t_n) - Y^n\|_{L^\infty(S_h)} \leq c\rho(h)(h + \sqrt{\tau})(\|y_0\|_{H^2(S)} + \|f\|_{L^2(0,T;H^1(S))}).$$

*Proof.* We adapt the proof of [3, Theorem 3.1]. Take  $\Phi \in W_{h,\tau}$ , multiply (3.1) by  $\widetilde{\Phi}^n$  and integrate over  $\Gamma_0 \times (t_{n-1}, t_n)$ . Abbreviating  $y^n := y(\cdot, t_n)$  we have

$$(4.2) \quad \begin{aligned} & (y^n - y^{n-1}, \widetilde{\Phi}^n) + \int_{t_{n-1}}^{t_n} \int_{\Gamma_0} a^{ij} D_i y D_j \widetilde{\Phi}^n + b^i D_i y \widetilde{\Phi}^n + c y \widetilde{\Phi}^n dx dt \\ &= \int_{t_{n-1}}^{t_n} (f, \widetilde{\Phi}^n) dt. \end{aligned}$$

Next, let us introduce  $\widetilde{Y} \in W_{h,\tau}$  by

$$(4.3) \quad \widetilde{Y}(\cdot, t) := R_h^n y^n, \quad t \in (t_{n-1}, t_n), 1 \leq n \leq N.$$

Using (3.12) and (4.2) we derive

$$(4.4) \quad \begin{aligned} A(\widetilde{Y}, \Phi) &= \sum_{n=1}^N \tau_n (\nabla R_h^n y^n, \nabla \Phi^n)_{\widehat{g}(t_n)} + \sum_{n=2}^N (R_h^n y^n - R_h^{n-1} y^{n-1}, \Phi^n)_h \\ &\quad + (R_h^n y^1, \Phi_+^0)_h \\ &\quad + \sum_{n=1}^N \tau_n (b^i(t_n) \nabla_i \widehat{y}^n, \Phi^n)_h + \sum_{n=1}^N \tau_n (c(t_n) \widehat{y}^n, \Phi^n)_h \\ &= \sum_{n=1}^N \tau_n (\widehat{y}^n, \Phi^n)_h + \sum_{n=1}^N \tau_n (\nabla \widehat{y}^n, \nabla \Phi^n)_{\widehat{g}(t_n)} - \sum_{n=1}^N \tau_n (R_h^n y^n, \Phi^n)_h \\ &\quad + \sum_{n=2}^N (R_h^n y^n - R_h^{n-1} y^{n-1}, \Phi^n)_h + (R_h^1 y^1, \Phi_+^0)_h \\ &\quad + \sum_{n=1}^N \tau_n (b^i(t_n) \nabla_i \widehat{y}^n, \Phi^n)_h + \sum_{n=1}^N \tau_n (c(t_n) \widehat{y}^n, \Phi^n)_h \\ &= \sum_{n=1}^N \tau_n (\widehat{y}^n, \Phi^n)_h + \sum_{n=1}^N \tau_n (\nabla \widehat{y}^n, \nabla \Phi^n)_{\widehat{g}(t_n)} - \sum_{n=1}^N \tau_n (R_h^n y^n, \Phi^n)_h \\ &\quad + \sum_{n=2}^N (\widehat{y}^n, \Phi^n)_h + \sum_{n=2}^N (\nabla \widehat{y}^n, \nabla \Phi^n)_{\widehat{g}(t_n)} - \sum_{n=2}^N (\nabla R_h^n y^n, \nabla \Phi^n)_{\widehat{g}(t_n)} \\ &\quad - \sum_{n=2}^N (\widehat{y}^{n-1}, \Phi^n)_h - \sum_{n=2}^N (\nabla \widehat{y}^{n-1}, \nabla \Phi^n)_{\widehat{g}(t_n)} \\ &\quad + \sum_{n=2}^N (\nabla R_h^{n-1} y^{n-1}, \nabla \Phi^n)_{\widehat{g}(t_n)} + (R_h^1 y^1, \Phi_+^0)_h \\ &\quad + \sum_{n=1}^N \tau_n (b^i(t_n) \nabla_i \widehat{y}^n, \Phi^n)_h + \sum_{n=1}^N \tau_n (c(t_n) \widehat{y}^n, \Phi^n)_h \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=1}^N \tau_n \left( \nabla \widehat{y}^n - \frac{1}{\tau_n} \int_{t_{n-1}}^{t_n} \nabla \widehat{y} dt, \nabla \Phi^n \right)_{\tilde{g}(t_n)} \\
&\quad + \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \int_{\Gamma_0} a^{ij} \nabla_i y \nabla_j \widetilde{\Phi}^n dx dt \\
&\quad + \sum_{n=1}^N \left( \int_{t_{n-1}}^{t_n} \nabla \widehat{y} dt, \nabla \Phi^n \right)_{\tilde{g}(t_n)} - \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \int_{\Gamma_0} a^{ij} \nabla_i y \nabla_j \widetilde{\Phi}^n dx dt \\
&\quad + \sum_{n=1}^N \tau_n \left( b^i(t_n) \left( \nabla_i \widehat{y}^n - \frac{1}{\tau_n} \int_{t_{n-1}}^{t_n} \nabla_i \widehat{y} dt \right), \Phi^n \right)_h \\
&\quad + \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \int_{\Gamma_0} b^i \nabla_i y \widetilde{\Phi}^n dx dt \\
&\quad + \sum_{n=1}^N \left( b^i(t_n) \int_{t_{n-1}}^{t_n} \nabla_i \widehat{y} dt, \Phi^n \right)_h - \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \int_{\Gamma_0} b^i \nabla_i y \widetilde{\Phi}^n dx dt \\
&\quad + \sum_{n=1}^N \tau_n \left( c(t_n) \left( \widehat{y}^n - \frac{1}{\tau_n} \int_{t_{n-1}}^{t_n} \widehat{y} dt \right), \Phi^n \right)_h + \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \int_{\Gamma_0} cy \widetilde{\Phi}^n dx dt \\
&\quad + \sum_{n=1}^N \left( c(t_n) \int_{t_{n-1}}^{t_n} \widehat{y} dt, \Phi^n \right)_h - \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \int_{\Gamma_0} cy \widetilde{\Phi}^n dx dt \\
&\quad + \sum_{n=1}^N \tau_n (\widehat{y}^n - R_h^n y^n, \Phi^n)_h \\
&\quad + \sum_{n=2}^N \int_{t_{n-1}}^{t_n} (f(t), \widetilde{\Phi}^n) dt \\
&\quad - \sum_{n=2}^N \int_{t_{n-1}}^{t_n} \int_{\Gamma_0} a^{ij} \nabla_i y \nabla_j \widetilde{\Phi}^n - b^i \nabla_i y \widetilde{\Phi}^n - cy \widetilde{\Phi}^n dx dt \\
&\quad + \sum_{n=2}^N (\widehat{y}^n - \widehat{y}^{n-1}, \Phi^n)_h - \sum_{n=2}^N (y^n - y^{n-1}, \widetilde{\Phi}^n) \\
&\quad + \sum_{n=2}^N (R_h^n y^n - \widehat{y}^n, \Phi^n)_h - \sum_{n=2}^N (R_h^{n-1} y^{n-1} - \widehat{y}^{n-1}, \Phi^n)_h \\
&\quad + (R_h^1 y^1, \Phi_+^0)_h \\
&= J_1 + \dots + J_{19} \\
&= J_1 + J_5 + J_9 + J_{13} + J_{17} + J_{18} + J_{19} \\
&\quad + \int_{t_0}^{t_1} \int_{\Gamma_0} a^{ij} \nabla_i y \nabla_j \widetilde{\Phi}^n + b^i \nabla_i y \widetilde{\Phi}^n + cy \widetilde{\Phi}^n dx dt \\
&\quad + \sum_{n=1}^N \left( \int_{t_{n-1}}^{t_n} \nabla \widehat{y} dt, \nabla \Phi^n \right)_{\tilde{g}(t_n)} + \sum_{n=1}^N \left( b^i(t_n) \int_{t_{n-1}}^{t_n} \nabla_i \widehat{y} dt, \Phi^n \right)_h
\end{aligned}
\tag{4.5}$$

$$\begin{aligned}
& + \sum_{n=1}^N \left( c(t_n) \int_{t_{n-1}}^{t_n} \hat{y} dt, \Phi^n \right)_h \\
& - \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \int_{\Gamma_0} a^{ij} \nabla_i y \nabla_j \widetilde{\Phi}^n + b^i \nabla_i y \widetilde{\Phi}^n + cy \widetilde{\Phi}^n dx dt \\
& + \sum_{n=2}^N \int_{t_{n-1}}^{t_n} (f(t), \widetilde{\Phi}^n) dt \\
& + \sum_{n=2}^N (\widehat{y}^n - \widehat{y}^{n-1}, \Phi^n)_h - \sum_{n=2}^N (y^n - y^{n-1}, \widetilde{\Phi}^n) \\
(4.6) \quad & = \sum_{n=1}^N \int_{t_{n-1}}^{t_n} (f(t), \widetilde{\Phi}^n) dt + (y_0, \widetilde{\Phi}^1) \\
& + J_1 + J_5 + J_9 + J_{13} + J_{17} + J_{18} + (R_h^1 y^1 - \widehat{y}^1, \Phi_+^0)_h \\
& + (\widehat{y}^1, \Phi_+^0)_h - (y^1, \widetilde{\Phi}_+^0) \\
& + \sum_{n=1}^N \left( \int_{t_{n-1}}^{t_n} \nabla \hat{y} dt, \nabla \Phi^n \right)_{\bar{g}(t_n)} + \sum_{n=1}^N \left( b^i(t_n) \int_{t_{n-1}}^{t_n} \nabla_i \hat{y} dt, \Phi^n \right)_h \\
& + \sum_{n=1}^N \left( c(t_n) \int_{t_{n-1}}^{t_n} \hat{y} dt, \Phi^n \right)_h \\
& - \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \int_{\Gamma_0} a^{ij} \nabla_i y \nabla_j \widetilde{\Phi}^n + b^i \nabla_i y \widetilde{\Phi}^n + cy \widetilde{\Phi}^n dx dt \\
& + \sum_{n=2}^N (\widehat{y}^n - \widehat{y}^{n-1}, \Phi^n)_h - \sum_{n=2}^N (y^n - y^{n-1}, \widetilde{\Phi}^n)
\end{aligned}$$

To be in accordance with [3] we denote  $r_1(\Phi) := J_1 + J_5 + J_9$ ,  $r_2(\Phi) := J_{13}$ ,  $r_3(\Phi) := J_{17} + J_{18}$ ,  $r_4(\Phi) = (R_h^1 y^1 - \widehat{y}^1, \Phi_+^0)_h$  and  $r(\Phi) := \sum_{i=1}^4 r_i(\Phi)$ .

As a consequence, the error  $E := \tilde{Y} - Y$  satisfies

$$\begin{aligned}
(4.7) \quad & A(E, \Phi) = r(\Phi) \\
& + \sum_{n=1}^N \int_{t_{n-1}}^{t_n} (f(t), \widetilde{\Phi}^n) dt + (y_0, \widetilde{\Phi}^1) \\
& - \sum_{n=1}^N \int_{t_{n-1}}^{t_n} (\widehat{f}(t), \Phi^n)_h + (\hat{y}_0, \Phi_+^0)_h \\
& + (\widehat{y}^1, \Phi_+^0)_h - (y^1, \widetilde{\Phi}_+^0) \\
& + \sum_{n=1}^N \left( \int_{t_{n-1}}^{t_n} \nabla \hat{y} dt, \nabla \Phi^n \right)_{\bar{g}(t_n)} + \sum_{n=1}^N \left( b^i(t_n) \int_{t_{n-1}}^{t_n} \nabla_i \hat{y} dt, \Phi^n \right)_h
\end{aligned}$$

$$\begin{aligned}
(4.8) \quad & + \sum_{n=1}^N \left( c(t_n) \int_{t_{n-1}}^{t_n} \hat{y} dt, \Phi^n \right)_h \\
& - \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \int_{\Gamma_0} a^{ij} \nabla_i y \nabla_j \widetilde{\Phi}^n + b^i \nabla_i y \widetilde{\Phi}^n + cy \widetilde{\Phi}^n dx dt \\
& + \sum_{n=2}^N (\widehat{y}^n - \widehat{y}^{n-1}, \Phi^n)_h - \sum_{n=2}^N (y^n - y^{n-1}, \widetilde{\Phi}^n) \\
& = r(\Phi) + L_1 + \dots + L_{12}
\end{aligned}$$

Let  $R_n, S_n, n = 1, \dots, N$ , be real numbers then we use the (only formally reasonable) notation

$$(4.9) \quad \left( \sum_{n=1}^N R_n \right) * S_n := \sum_{n=1}^N R_n S_n.$$

In this sense we can consider  $A(E, \phi) * S_n$  where we assume  $A(E, \phi)$  being written as a sum from 1 to  $N$  with summands given by the terms on the right-hand side of (4.7) numbered by the time index of the test function, here  $\Phi^1 = \Phi_+^0$  is considered to have time index 1.

The idea is as follows. In order to show the error estimate we want to use that  $|c|$  is large and that  $c$  has the correct sign. Both conditions are not fulfilled in general. Therefore we define

$$(4.10) \quad \bar{Y}^n := e^{\mu t_n} Y^n$$

where  $\mu = 1$  (we write  $\mu$  instead of 1 to make the influence of this factor clear).

We derive a scheme which is fulfilled by  $\bar{Y}$ . This new scheme has compared to the scheme (3.18) an additional summand on the left-hand side and a modified right-hand side as will become clear in the following.

There holds

$$(4.11) \quad e^{\mu t_{n-1}} = e^{\mu t_n} - \mu e^{\mu t_n} \tau_n + \frac{1}{2} \mu^2 e^{\mu \xi_n} \tau_n^2$$

with some  $\xi_n \in [t_{n-1}, t_n]$  and hence

$$\begin{aligned}
(4.12) \quad A(\bar{Y}, \Phi) &= A(Y, \Phi) * e^{\mu t_n} + \sum_{n=2}^N (e^{\mu t_n} - e^{\mu t_{n-1}}) (Y^{n-1}, \Phi^n)_h \\
&= A(Y, \Phi) * e^{\mu t_n} + \sum_{n=2}^N \mu (\tau_n - \frac{1}{2} \mu \tau_n^2 e^{\mu(\xi_n - t_n)}) (\bar{Y}^n, \Phi^n)_h \\
&\quad - \sum_{n=2}^N \mu (\tau_n - \frac{1}{2} \mu \tau_n^2 e^{\mu(\xi_n - t_n)}) (\bar{R}^n, \Phi^n)_h \\
&\quad - \sum_{n=2}^N \mu (\tau_n (1 - e^{\mu \tau_n}) - \frac{1}{2} \mu \tau_n^2 e^{\mu(\xi_n - t_n)} (1 - e^{\mu \tau_n})) (\bar{Y}^{n-1}, \Phi^n)_h \\
&= A(Y, \Phi) * e^{\mu t_n} + M_1 + \dots + M_3
\end{aligned}$$



where  $\bar{R}^n = \bar{Y}^n - \bar{Y}^{n-1}$  and we remark that  $M_1, \dots, M_3$  depend on  $Y, \Phi$ . We set  $\bar{Y}^n = e^{\mu t_n} \tilde{Y}^n$ ,  $\bar{E}^n = \bar{Y}^n - \bar{Y}^n$  and have

$$(4.13) \quad A(\bar{E}, \Phi) - M_1 \dots - M_3 = A(E, \Phi) * e^{\mu t_n}$$

where

$$(4.14) \quad \begin{aligned} M_1 + M_2 &= \sum_{n=2}^N \mu(\tau_n - \frac{1}{2}\mu\tau_n^2 e^{\mu(\xi_n - t_n)})(\bar{E}^n, \Phi^n)_h \\ &\quad - \sum_{n=2}^N \mu(\tau_n - \frac{1}{2}\mu\tau_n^2 e^{\mu(\xi_n - t_n)})(\bar{E}^n - \bar{E}^{n-1}, \Phi^n)_h \end{aligned}$$

and

$$(4.15) \quad |M_3| \leq c \sum_{n=2}^N \tau_n^2 |(\bar{E}^{n-1}, \Phi^n)_h|.$$

Let us fix  $l \in \{2, \dots, N\}$  and define  $\Phi \in W_{h,\tau}$  by

$$(4.16) \quad \Phi^n := \begin{cases} 0, & n = 1 \text{ or } n > l \\ \frac{\bar{E}^n - \bar{E}^{n-1}}{\tau_n}, & 2 \leq n \leq l. \end{cases}$$

We insert  $\Phi$  in (4.13) and estimate the resulting left-hand side from below and the right-hand side from above. We begin with the first estimate. We have

$$(4.17) \quad \begin{aligned} A(\bar{E}, \Phi) &= \sum_{n=2}^l \tau_n \left( \nabla \bar{E}^n, \frac{\nabla \bar{E}^n - \nabla \bar{E}^{n-1}}{\tau_n} \right)_{\bar{g}(t_n)} \\ &\quad + \sum_{n=2}^l \left( \bar{E}^n - \bar{E}^{n-1}, \frac{\bar{E}^n - \bar{E}^{n-1}}{\tau_n} \right)_h \\ &\quad + \sum_{n=2}^l \tau_n \left( b^i(t_n) \nabla_i \bar{E}^n, \frac{\bar{E}^n - \bar{E}^{n-1}}{\tau_n} \right)_h \\ &\quad + \sum_{n=2}^l \tau_n \left( c(t_n) \bar{E}^n, \frac{\bar{E}^n - \bar{E}^{n-1}}{\tau_n} \right)_h \\ &= K_1 + \dots + K_4. \end{aligned}$$

We have

$$\begin{aligned}
& \sum_{n=2}^l (\nabla \bar{E}^n - \nabla \bar{E}^{n-1}, \nabla \bar{E}^n - \nabla \bar{E}^{n-1})_{\bar{g}(\tau_n)} \\
&= \sum_{n=2}^l (\nabla \bar{E}^n, \nabla \bar{E}^n)_{\bar{g}(t_n)} - 2 \sum_{n=2}^l (\nabla \bar{E}^n, \nabla \bar{E}^{n-1})_{\bar{g}(t_n)} \\
& \quad + \sum_{n=2}^l (\nabla \bar{E}^{n-1}, \nabla \bar{E}^{n-1})_{\bar{g}(t_n)} + (\nabla \bar{E}^1, \nabla \bar{E}^1)_{\bar{g}(t_2)} \\
(4.18) \quad & \leq \sum_{n=2}^l (\nabla \bar{E}^n, \nabla \bar{E}^n)_{\bar{g}(t_n)} (2 + c\tau_n) - 2 \sum_{n=2}^l (\nabla \bar{E}^n, \nabla \bar{E}^{n-1})_{\bar{g}(t_n)} \\
& \quad + (\nabla \bar{E}^1, \nabla \bar{E}^1)_{\bar{g}(t_2)} - (\nabla \bar{E}^l, \nabla \bar{E}^l)_{\bar{g}(t_1)} \\
& \leq 2K_1 + c \sum_{n=1}^l (\nabla \bar{E}^n, \nabla \bar{E}^n)_{\bar{g}(t_n)} \tau_n + (\nabla \bar{E}^1, \nabla \bar{E}^1)_{\bar{g}(t_1)} - (\nabla \bar{E}^l, \nabla \bar{E}^l)_{\bar{g}(t_n)}
\end{aligned}$$

and hence

$$\begin{aligned}
(4.19) \quad K_1 & \geq \frac{1}{2} \sum_{n=2}^l \|\nabla \bar{E}^n - \nabla \bar{E}^{n-1}\|_{\bar{g}(\tau_n)}^2 + \|\nabla \bar{E}^l\|_{\bar{g}(t_n)}^2 - \|\nabla \bar{E}^1\|_{\bar{g}(t_1)}^2 \\
& \quad - c \sum_{n=1}^l \|\nabla \bar{E}^n\|_{\bar{g}(t_n)}^2 \tau_n.
\end{aligned}$$

We write

$$(4.20) \quad K_2 = \sum_{n=2}^l \tau_n \left\| \frac{\bar{E}^n - \bar{E}^{n-1}}{\tau_n} \right\|_h^2.$$

We have

$$\begin{aligned}
(4.21) \quad |K_3| & \leq \sum_{n=2}^l \frac{1}{4} \tau_n \left\| \frac{\bar{E}^n - \bar{E}^{n-1}}{\tau_n} \right\|_h^2 + \sum_{n=2}^l c \|\nabla \bar{E}\|_{\bar{g}(t_n)}^2 \tau_n \\
|K_4| & \leq \sum_{n=2}^l \frac{1}{4} \tau_n \left\| \frac{\bar{E}^n - \bar{E}^{n-1}}{\tau_n} \right\|_h^2 + c \sum_{n=2}^l \tau_n \|\bar{E}^n\|_h^2.
\end{aligned}$$

Furthermore, there holds

$$\begin{aligned}
-M_1 &= \sum_{n=2}^l \mu(\tau_n + \frac{1}{2}\mu\tau_n^2 e^{\mu(\xi_n - \tau_n)})(\bar{E}^n, \Phi^n)_h \\
&= \sum_{n=2}^l \mu(1 + \frac{1}{2}\mu\tau_n e^{\mu(\xi_n - \tau_n)})(\bar{E}^n, \bar{E}^n - \bar{E}^{n-1})_h \\
&= \sum_{n=2}^l \mu(1 + \frac{1}{2}\mu\tau_n e^{\mu(\xi_n - \tau_n)}) \\
(4.22) \quad &\quad \{(\bar{E}^n, \bar{E}^n)_h - (\bar{E}^n - \bar{E}^{n-1}, \bar{E}^{n-1})_h - (\bar{E}^{n-1}, \bar{E}^{n-1})_h\} \\
&\geq \frac{\mu}{2}(\bar{E}^l, \bar{E}^l)_h - \mu(\bar{E}^1, \bar{E}^1)_h - \sum_{n=1}^{l-1} \mu c \tau e^{\mu\tau}(\bar{E}^n, \bar{E}^n)_h \\
&\quad + \frac{1}{8} \sum_{n=2}^l \tau_n \left\| \frac{\bar{E}^n - \bar{E}^{n-1}}{\tau_n} \right\|_h^2 \\
-M_2 &\geq -\frac{K_2}{4}.
\end{aligned}$$

Hence the left-hand side of (4.13) can be estimated from below by

$$\begin{aligned}
(4.23) \quad &\frac{1}{2} \sum_{n=2}^l \|\nabla \bar{E}^n - \nabla \bar{E}^{n-1}\|_{\hat{g}(\tau_n)}^2 + \frac{1}{2} \|\nabla \bar{E}^l\|_{\hat{g}(t_l)}^2 - \|\nabla \bar{E}^1\|_{\hat{g}(t_1)}^2 \\
&- \mu \|\bar{E}^1\|_h^2 - c \sum_{n=1}^{l-1} \|\nabla \bar{E}^n\|_{\hat{g}(t_n)}^2 \tau_n + \sum_{n=2}^l \frac{\tau_n}{8} \left\| \frac{\bar{E}^n - \bar{E}^{n-1}}{\tau_n} \right\|_h^2 \\
&+ \frac{\mu}{4} \|\bar{E}^l\|_h^2 - \sum_{n=1}^{l-1} \mu c \tau e^{\mu\tau} \|\bar{E}^n\|_h^2.
\end{aligned}$$

In the following we estimate the right-hand side of (4.13) from above and refer to [3, Theorem 3.1] for more details concerning the estimate of  $r_i(\Phi)$ ,  $i = 1, 2, 3$ .

We have

$$\begin{aligned}
|r_1(\Phi)| &\leq \frac{1}{8} \sum_{n=2}^l \|\nabla \bar{E}^n - \nabla \bar{E}^{n-1}\|_h^2 + c\tau \int_0^T \|\nabla y_t\|_{\bar{g}(t)}^2 dt, \\
&\quad + c\tau \|y_t\|_{L^2(0,T;L^2(\Gamma_0))}^2 + \sum_{n=2}^l \|\bar{E}^n - \bar{E}^{n-1}\|_h^2, \\
|r_2(\Phi)| &\leq \frac{1}{8} \sum_{n=2}^l \tau_n \left\| \frac{\bar{E}^n - \bar{E}^{n-1}}{\tau_n} \right\|_h^2 + ch^4 \max_{1 \leq n \leq N} \|y^n\|_{H^2(\Gamma_0)}, \\
|r_3(\Phi)| &\leq \frac{1}{8} \sum_{n=2}^l \tau_n \left\| \frac{\bar{E}^n - \bar{E}^{n-1}}{\tau_n} \right\|_h^2 + ch^2 \int_0^T \|y_t\|_{H^1(\Gamma_0)}^2 dt, \\
(4.24) \quad L_2 = L_4 = L_5 = L_6 &= 0, \\
|L_1 + L_3| &\leq O(h^2) \left( \|f\|_{L^2(0,T;L^2(\Gamma_0))}^2 + \sum_{n=1}^N \tau_n \left\| \frac{\bar{E}^n - \bar{E}^{n-1}}{\tau_n} \right\|_h^2 \right), \\
|L_7 + \dots + L_{10}| &\leq (O(\tau) + O(h^2)) \left( \|y\|_{L^2(0,T;H^1(\Gamma_0))}^2 + \sum_{n=1}^N \tau_n \left\| \frac{\bar{E}^n - \bar{E}^{n-1}}{\tau_n} \right\|_h^2 \right), \\
|L_{11} + L_{12}| &\leq O(h^2) \left( \|y_t\|_{L^2(0,T;L^2(\Gamma_0))}^2 + \sum_{n=2}^N \tau_n \left\| \frac{\bar{E}^n - \bar{E}^{n-1}}{\tau_n} \right\|_h^2 \right).
\end{aligned}$$

We estimate  $\|E^1\|_h^2$  and  $\|\nabla E^1\|_{\bar{g}(t_1)}^2$ . We choose  $\Phi \in W_{h,\tau}$  with  $\Phi^n = 0$  for  $n \geq 2$  and  $\Phi^1 = \bar{E}^1$  then (4.13) implies

$$(4.25) \quad A(\bar{E}, \Phi) = A(E, \Phi) * e^{\mu t^n}.$$

We estimate the left-hand side of (4.25) from below

$$\begin{aligned}
(4.26) \quad A(\bar{E}, \Phi) &= \tau_1 (\nabla \bar{E}^1, \nabla \Phi^1)_{\bar{g}(t_1)} + (\bar{E}^1, \Phi^1)_h + \tau_1 (b^i \nabla_i \bar{E}^1, \Phi^1)_h \\
&\quad + \tau_1 (c(t_1) \bar{E}^1, \Phi^1)_h \\
&\geq \frac{\tau_1}{2} \|\nabla \bar{E}^1\|_{\bar{g}(t_1)}^2 + \frac{1}{2} \|E^1\|_h^2.
\end{aligned}$$

Now we estimate the right-hand side of (4.25) from above and show that it is bounded from above by terms of type  $c\tau^2 + ch^4$  where  $c$  depends on  $y$  and terms which can be absorbed by the terms on the right-hand side of (4.26). W.l.o.g. we may estimate  $A(E, \Phi)$  instead of  $A(E, \Phi) * e^{\mu t}$  and for this estimate we use (4.7).

We begin with the term  $r_1(\Phi)$  and present the details only for  $J_1$ , we have

$$\begin{aligned}
(4.27) \quad J_1 &= \tau_1 \left( \nabla \hat{y}^1 - \frac{1}{\tau_1} \int_{t_0}^{t_1} \nabla \hat{y} dt, \nabla \Phi^1 \right)_{\bar{g}(t_1)} \\
&\leq \tau_1 \left\| \frac{1}{\tau_1} \int_{t_0}^{t_1} \int_t^{\xi_t} |\nabla y_t| \right\|_h \|\nabla \Phi^1\|_h \\
&\leq c\tau^{\frac{3}{2}} \|\nabla y_t\|_{L^2(0,T,L^2(\Gamma_0))} \|\nabla \Phi^1\|_h \\
&\leq c\tau^{\frac{3}{2}} \left( \frac{\epsilon}{\tau^{\frac{1}{2}}} \|\nabla \Phi^1\|_h^2 + \frac{\tau^{\frac{1}{2}}}{\epsilon} \|\nabla y_t\|_{L^2(0,T,L^2(\Gamma_0))}^2 \right).
\end{aligned}$$

Furthermore, we have

$$(4.28) \quad r_2(\Phi) = \tau_1(\widehat{y}^1 - R_h^1 y^1, \Phi^1)_h = J_{13} \leq c\tau h^2 \max_{t_0 \leq t \leq t_1} \|y(t)\|_{H^2(\Gamma_0)}^2$$

$$(4.29) \quad r_3(\Phi) = J_{17} + J_{18} = 0$$

and

$$(4.30) \quad \begin{aligned} r_4(\Phi) &= (R_h^1 y^1 - \widehat{y}^1, \Phi^1)_h \\ &\leq ch^2 \max_{t_0 \leq t \leq t_1} \|y(t)\|_{H^2(\Gamma_0)}^2 \|\Phi^1\|_h \\ &\leq ch^2 \left( \frac{\epsilon}{h^2} \|\Phi^1\|_h^2 + \frac{h^2}{\epsilon} \max_{t_0 \leq t \leq t_1} \|y(t)\|_{H^2(\Gamma_0)}^2 \right). \end{aligned}$$

There holds  $L_{11} = L_{12} = 0$ . The term  $L_7$  together with the first summand in  $L_{10}$  can be estimated from above by

$$(4.31) \quad (O(h^2) + O(\tau)) \|\nabla y\|_{L^2(0,T;L^2(\Gamma_0))} \|\bar{E}^1\|_h$$

which is sufficient and the remaining terms can be treated similarly.  $\square$

#### REFERENCES

- [1] M. BORUKHAVA AND H. KRÖNER. Alternative to evolving surface finite element method. <http://arxiv.org/abs/>, 2016.
- [2] BRENNER, S. C., SCOTT, L. R. (2008) *The Mathematical Theory of Finite Element Methods*. Texts in Applied Mathematics, third edition, Berlin: Springer.
- [3] DECKELNICK AND K., HINZE, M. (2011) Variational discretization of parabolic control problems in the presence of pointwise state constraints. *J. Comput. Math.*, **29**, no. 1, 1–15.
- [4] DZIUK, G. (1988) *Finite elements for the Beltrami operator on arbitrary surfaces*. Partial Differential Equations and Calculus of Variations (S. Hildebrandt & R. Leis eds). Lecture Notes in Mathematics, vol. 1357. Berlin: Springer, pp. 142–155.
- [5] DZIUK, G. AND ELLIOTT, C.M. (2007) Finite elements on evolving surfaces. *IMA J. Numer. Anal.*, **27**, no. 2, 262–292.
- [6] G. DZIUK. Finite elements for the Beltrami operator on arbitrary surfaces. Partial Differential Equations and Calculus of Variations (S. Hildebrandt and R. Leis eds.). Lecture Notes in Mathematics, vol. 1357, Berlin, Springer, 142-155, 1988.
- [7] G. DZIUK AND C. M. ELLIOTT. Finite elements on evolving surfaces. *IMA J. Numer. Anal.*, **27** (2): 262-292, 2007.
- [8] G. DZIUK AND C. M. ELLIOTT.  $L^2$ -estimates for the evolving surface finite element method. *Math. Comput.*, **82**, no. 281: 1-24, 2013.
- [9] G. DZIUK, CH. LUBICH AND D. MANSOUR. Runge-Kutta time discretization of parabolic differential equations on evolving surfaces. *IMA J. Numer. Anal.*, **32**, no. 2: 394-416, 2012.
- [10] B. KOVÁCS AND C.A. POWER GUERRA. Maximum norm stability and error estimates for the evolving surface finite element method. <http://arxiv.org/abs/1510.00605>, 2015.
- [11] CH. LUBICH, D. MANSOUR AND C. VENKATARAMAN. Backward difference time discretization of parabolic differential equations on evolving surfaces. *IMA J. Numer. Anal.*, **33**, no. 4: 1365-1385, 2013.
- [12] M.A OLSHANSKII AND A. REUSKEN. Error analysis of a space-time finite element method for solving PDEs on evolving surfaces. *SIAM J. Numer. Anal.*, **52**, no. 4: 2092-2120, 2014.
- [13] M.A OLSHANSKII, A. REUSKEN AND X. XU. An Eulerian space-time finite element method for diffusion problems on evolving surfaces. *SIAM J. Numer. Anal.*, **52**, no. 3: 1354-1377, 2014.