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VARIATIONAL DISCRETIZATION OF PARABOLIC CONTROL PROBLEMS ON EVOLVING SURFACES WITH POINTWISE STATE CONSTRAINTS

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Abstract. We consider a linear-quadratic pde constrained optimal control problem on an evolving surface with pointwise state constraints. We reformulate the optimization problem on a fixed surface and approximate the reformulated problem by a discrete control problem based on a discretization of the state equation by linear finite elements in space and a discontinuous Galerkin scheme in time. We prove error bounds for control and state.

Key words. linear-quadratic optimization problem; linear parabolic pde; two-dimensional surface; finite elements

1. Introduction. In applications the situation of a moving hypersurface seperating two moving regions is a widespread setting to model various phenomena. In this general setting one may think of biological processes happening in these regions or on the interface between these regions. Examples for this scenario are cell membranes seperating the environment from the cell interior, or the interface between the two phases of a two-phase flow where soluble surfactants in the bulk regions affect a certain interfacial surfactant concentration, see [11] and the references therein for a two-phase flow example.

It is a natural to consider optimization problems where the surfactant density on the surface plays the role of the state variable and to assume certain pointwise bounds for the state. To address control of the general setting above we consider in our paper a linear-quadratic PDE-constrained optimization problem on the moving hypersurface (and not phenomena or interactions in or with the regions outside the moving hypersurface). By using the variational discretization from [12] with linear finite elements in space and a discontinuous Galerkin scheme in time we discretize the optimization problem and prove error estimates for the control and the state.

The corresponding optimization problem in an Euclidean setting is treated in [4] and we will follow the argumentation therein closely. We reformulate our constraint which is a linear advection-diffusion equation on the moving surface treated numerically in [8, 9, 15, 10] as a linear parabolic pde on the initial surface. We refer to [1] for details concerning the reformulation and to [16] for an error estimate for a finite element approximation of the reformulated equation.

There are only few papers which deal with the numerics of linear-quadratic, pde constrained optimization problems on surfaces. In [14] an optimal control problem for the Lapace-Beltrami on surfaces is considered and in [18] a linear-quadratic parabolic control problem on evolving surfaces with pointwise box constraints is considered.

Our paper is organized as follows. In Section 2 we present the linear parabolic state equation which serves as a constraint in our optimization problem. In Section 3 we formulate the optimization problem. Section 4 contains general material about finite elements on surfaces, in Section 5 the state equation is discretized, in Section 6 the optimization problem is discretized and in Section 7 we prove an estimate for the

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discretization error of the optimal control problem.

2. State equation. Let Γ_0 be a smooth two-dimensional, embedded, orientable, closed hypersurface in $\mathbb{R}^3$. We let $\Psi = \Psi(x,t) : \Omega_T \to \mathbb{R}^3$, $\Omega_T = \Gamma_0 \times (0,T)$, be a smooth 'motion', i.e. a smooth mapping so that $\Psi(\cdot,t)$ is an embedding. We assume $\Psi(\cdot,0) = id$ (this is our convention and has no serious reason). We define

$$G_T = \bigcup_{t\in[0,T]} \Gamma(t) \times \{t\} \tag{2.1}$$

where $T > 0$,

$$H^1(G_T) = \{u : G_T \to \mathbb{R} | (x,t) \mapsto u(\psi(x,t),t) \text{ is of class } H^1(\Omega_T)\} \tag{2.2}$$

and $L^2(G_T)$ etc. similarly. For given $f \in L^2(G_T)$, $y_0 \in H^1(G_T)$ we consider the initial value problem

$$\dot{y} + y\nabla_\Gamma \cdot v - \Delta_\Gamma y = f, \quad y(\cdot,0) = y_0, \tag{2.3}$$

where Δ_Γ is the Laplace-Beltrami operator on $\Gamma(t) = \Psi(\cdot,t)(\Gamma_0)$, $v(x,t) = \frac{d}{dt}\Psi(x,t)$ is the speed of the surface and the dot stands for the material derivative. The variational formulation of (2.3) is given by

$$\frac{d}{dt}\int_{\Gamma(t)} y\varphi + \int_{\Gamma(t)} \langle Dy, D\varphi\rangle = \int_{\Gamma(t)} y\dot{\varphi} \quad \forall \varphi \in C^\infty(G_T). \tag{2.4}$$

Initial value problem (2.3) has been studied numerically intensively, see e.g. [8] where the evolving surface finite element method (ESFEM) is introduced and the sequential papers [9, 10, 15]. We reformulate (2.3) on a fixed surface and will thereafter consider the state equation always in this reformulated form.

Therefore we introduce the quantity

$$\hat{y}(x,t) = y(\Psi(x,t),t) \tag{2.5}$$

and let $g_{ij} = g_{ij}(x,t)$ be the induced metric of $\Gamma(t)$ in $\Psi(x,t)$, $g^{ij} = g^{ij}(x,t)$ its inverse, $g(x,t) = \det(g_{ij}(x,t))$ and $\Gamma^k_{ij}(t)$ the Christoffel symbols of $\Gamma(t)$. We stipulate that the local coordinates of $\Gamma(t)$ are related with the local coordinates of $\Gamma(0)$ via the diffeomorphism $\Psi(\cdot,t)$.

Denoting the Levi-Civita connection of $\Gamma(t)$ by $\nabla^{\Gamma(t)}$ (and omitting the superscript in case $t = 0$) and setting

$$\hat{c} = \nabla^{\Gamma(t)} \cdot v \tag{2.6}$$

the initial value problem (2.3) transforms into the following initial value problem for $\hat{y}$

$$\begin{aligned} &\frac{d}{dt}\hat{y} - \nabla_i(g^{ij}(t)\nabla_j\hat{y}) \\ &\qquad + (g^{ij}(t)(\Gamma(t)^k_{ij} - \Gamma(0)^k_{ij}) + \nabla_j g^{kj}(t))\nabla_i\hat{y} + \hat{c}\hat{y} \quad = \hat{f}, \\ &\hat{y}(\cdot,0) = \hat{y}_0 \end{aligned} \tag{2.7}$$

where we use summation convention. In the following we will always work with this reformulated form of the state equation, omit the hat in the notation for the transformed quantities and abbreviate the coefficients in an obvious way so that we can rewrite (2.7) as

$$Ay = \frac{d}{dt}y - \nabla_i \left(a^{ij}\nabla_j y\right) + b^i \nabla_i y + cy = f, \quad y(\cdot, 0) = y_0. \tag{2.8}$$

We will use a backward equation for which we formally introduce the following differential operator

$$\tilde{A}w := -\frac{d}{dt}w - a^{ij}\nabla_i \nabla_j w - (\nabla_j a^{ij} + b^i)\nabla_i w + (c - \nabla_i b^i)w. \tag{2.9}$$

It is well-known that for given $f \in L^2(0,T;L^2(\Gamma_0))$ and $y_0 \in H^1(\Gamma_0)$ problem (2.8) has a unique solution $y \in C^0([0,T];H^1(\Gamma_0)) \cap L^2(0,T;H^2(\Gamma_0))$ which we denote by $G(f) = y$. The solution of (2.8) with f replaced by zero is denoted by y^0. The solution of (2.8) with y_0 replaced by zero is denoted by $G_0(f)$. There holds

$$G(f) = y^0 + G_0(f). \tag{2.10}$$

If $f \in L^2(0,T;H^1(\Gamma_0))$ and $y_0 \in H^2(\Gamma_0)$ then

$$y \in W := \left\{ w \in C^0([0,T];H^2(\Gamma_0)) : \frac{d}{dt}w \in L^2(0,T;H^1(\Gamma_0)) \right\} \subset C^0(\overline{\Omega_T}), \tag{2.11}$$

and

$$\max_{0\le t\le T} \|y(t)\|^2_{H^2(\Gamma_0)} + \int_0^T \|y_t(t)\|^2_{H^1(\Gamma_0)} dt \le c(\|y_0\|^2_{H^2(\Gamma_0)} + \int_0^T \|f(t)\|^2_{H^1(\Gamma_0)}). \tag{2.12}$$

Suppose that the functions $f_1, ..., f_m \in H^1(\Gamma_0) \cap L^\infty(\Gamma_0)$ are given and define $U = L^2(0,T;\mathbb{R}^m)$ as well as $B : U \to L^2(0,T;H^1(\Gamma_0))$ by

$$(Bu)(x,t) := \sum_{i=1}^m u_i(t) f_i(x), \quad (x,t) \in \Omega_T \tag{2.13}$$

then (2.12) implies that for $u \in U$, $y = G(Bu) \in W$, there holds

$$\max_{0\le t\le T} \|y(t)\|^2_{H^2(\Gamma_0)} + \int_0^T \|y_t(t)\|^2_{H^1(\Gamma_0)} dt \le c(\|y_0\|^2_{H^2(\Gamma_0)} + \int_0^T |u(t)|^2) \tag{2.14}$$

where the constant c depends in addition on the H^1-norms of $f_1, ..., f_m$. Let $M(\overline{\Omega_T})$ denote the space of Borel regular measures on $\overline{\Omega_T}$. Given $\mu \in M(\overline{\Omega_T})$ we consider the following backward parabolic problem

$$\begin{aligned} \tilde{A}\varphi &= \mu_{\Omega_T} \quad \text{in } \Omega_T \\ \varphi(\cdot, T) &= \mu_T \quad \text{in } \Omega. \end{aligned} \tag{2.15}$$

Here, $\mu_{\Omega_T} := \mu_{|\Omega_T}$, $\mu_T := \mu_{|\Gamma_0 \times \{T\}}$.

THEOREM 2.1. *There exists a unique function $\varphi \in L^s(0,T;W^{1,\sigma}(\Gamma_0))$ for all $s, \sigma \in [1,2)$ with $\frac{2}{s} + \frac{2}{\sigma} > 3$ which solves (2.15) in the sense that*

$$\int_0^T (Aw, \varphi)dt = \int_{\overline{\Omega_T}} w d\mu \quad \forall w \in W_0^\infty \tag{2.16}$$

where

$$W_0^\infty = \{w \in W : w(\cdot,0) = 0 \textit{ in } \Gamma_0, Aw \in L^\infty(\Omega_T)\} \tag{2.17}$$

and $(\cdot,\cdot)$ denotes the inner product in $L^2(\Gamma_0)$.

Proof. The proof is along the lines of the Euclidean setting for the heat equation, cf. [5, Theorem 6.3]. ◻

Note, that $\varphi \in L^1(0,T;W^{1,1}(\Gamma_0))$ so that the integral in (2.16) exists.

3. Optimization problem. We remark that we can transform Bu in (2.13) via Ψ into a function $\overline{Bu}$ which is defined on G_T and which will act as the right-hand side of our optimization problem in its (original) formulation on the moving surface. The solution operator corresponding to (2.3) is denoted by $\tilde{G}$, so that $\tilde{G}(\overline{Bu})$ is defined. We consider the following optimization problem on the moving surface

$$\begin{cases} \min_{u\in U} J(u) := \frac{1}{2}\int_0^T \|\bar{y}(\cdot,t) - y_g(\Psi(\cdot,t)^{-1},t)\|^2_{L^2(\Gamma(t))} dt \\ +\frac{\alpha}{2}\int_0^T |u(t)|^2 dt \\ \text{s.t. } \bar{y} = \tilde{G}(\overline{Bu}) \quad \text{and} \quad \bar{y} \geq 0 \end{cases} \tag{3.1}$$

where $y_g \in H^1(0,T;L^2(\Gamma_0))$ is given. Optimization problem (3.1) can be written equivalently as

$$\begin{cases} \min_{u\in U} J(u) := \frac{1}{2}\int_0^T \|(y(\cdot,t) - y_g(\cdot,t))\left(\frac{g(\cdot,t)}{g(\cdot,0)}\right)^{\frac{1}{4}}\|^2_{L^2(\Gamma_0)} dt \\ +\frac{\alpha}{2}\int_0^T |u(t)|^2 dt \\ \text{s.t. } y = G(Bu) \quad \text{and} \quad y \geq 0. \end{cases} \tag{3.2}$$

From now on we shall assume $y_0 \in H^2(\Gamma_0)$ and that $\min_{x\in\Gamma_0} y_0(x) > 0$ and hence

$$y^0 > 0 \tag{3.3}$$

in view of the maximum principle.

Since the state constraints form a convex set and the set of admissible controls is closed and convex one obtains the existence of a unique solution $u \in U$ to problem (3.2) by standard arguments. We characterize the property of being a solution in the following theorem.

THEOREM 3.1. *A function $u \in U$ is the solution of (3.2) if and only if there exist $\mu \in M(\overline{\Omega_T})$ and a function $p \in L^s(0,T;W^{1,\sigma}(\Gamma_0))$, $s,\sigma \in [1,2)$, $\frac{2}{s}+\frac{3}{\sigma} > 3$, such that with $y = G(Bu)$ there holds*

$$\int_0^T (Aw,p)dt = \int_0^T \left(\left(\frac{g(\cdot,t)}{g(\cdot,0)}\right)^{\frac{1}{4}}(y-y_g), w\right)dt + \int_{\overline{\Omega_T}} w\,d\mu \quad \forall w \in W_0^\infty, \tag{3.4}$$

$$\begin{aligned} &\alpha u(t) + (p(\cdot,t), f_i)_{i=1,\dots,m} = 0, \qquad \textit{a.e. in } (0,T) \\ &\mu \leq 0, y \geq 0 \textit{ and } \int_{\overline{\Omega_T}} y\,d\mu = 0. \end{aligned} \tag{3.5}$$

Proof. See [4, Theorem 2.2] and note, that the same argumentation as in the Euclidean case can be used and that our operators A and $\tilde{A}$, respectively replace the heat operator and the corresponding backward operator there. ◻

4. Finite Elements on Surfaces. In this section we introduce the space of continuous and piecewise linear finite element functions on a polyhedral approximation of $\Gamma_0(= S)$. Throughout the paper we assume that S is covered by a fixed finite atlas. We triangulate S by a family T_h of flat triangles with corners (i.e. nodes) lying on S. We denote the surface of class $C^{0,1}$ given by the union of the triangles $\tau \in T_h$ by $\Gamma_h = S_h$; the union of the corresponding nodes is denoted by N_h. Here, $h > 0$ denotes a discretization parameter which is related to the triangulation in the following way. For $\tau \in T$ we define the diameter $\rho(\tau)$ of the smallest disc containing τ, the diameter $\sigma(\tau)$ of the largest disc contained in τ and

$$h = \max_{\tau \in T_h} \rho(\tau), \quad \gamma_h = \min_{\tau \in T_h} \frac{\sigma(\tau)}{h}. \tag{4.1}$$

We assume that the family $(T_h)_{h>0}$ is quasi-uniform, i.e. $\gamma_h \geq \gamma_0 > 0$. We let

$$V_h = X_h = \{v \in C^0(S_h) : v_{|\tau} \text{ linear for all } \tau \in T_h\} \tag{4.2}$$

be the space of continuous piecewise linear finite elements. Let N be a tubular neighborhood of S in which the Euclidean metric of N can be written in the coordinates $(x^0, x) = (x^0, x^i)$ of the tubular neighborhood as

$$\bar{g}_{\alpha\beta} = (dx^0)^2 + \sigma_{ij}(x) dx^i dx^j. \tag{4.3}$$

Here, x^0 denotes the globally (in N) defined signed distance to S and $x = (x^i)_{i=1,2}$ local coordinates for S.

For small h we can write S_h as graph (with respect to the coordinates of the tubular neighborhood) over S, i.e.

$$S_h = \operatorname{graph} \psi = \{(x^0, x) : x^0 = \psi(x), x \in S\} \tag{4.4}$$

where $\psi = \psi_h \in C^{0,1}(S)$ suitable. Note, that

$$|D\psi|_\sigma \leq ch, \quad |\psi| \leq ch^2. \tag{4.5}$$

The induced metric of S_h is given by

$$g_{ij}(\psi(x), x) = \frac{\partial \psi}{\partial x^i}(x) \frac{\partial \psi}{\partial x^j}(x) + \sigma_{ij}(x). \tag{4.6}$$

Hence we have for the metrics, their inverses and their determinants

$$g_{ij} = \sigma_{ij} + O(h^2), \quad g^{ij} = \sigma^{ij} + O(h^2) \quad \text{and} \quad g = \sigma + O(h^2) |\sigma_{ij}\sigma^{ij}|^{\frac{1}{2}} \tag{4.7}$$

where we use summation convention.

For a function $f : S \to \mathbb{R}$ we define its lift $\hat{f} : S_h \to \mathbb{R}$ to S_h by $f(x) = \hat{f}(\psi(x), x)$, $x \in S$. For a function $f : S_h \to \mathbb{R}$ we define its lift $\tilde{f} : S \to \mathbb{R}$ to S by $f = \hat{\tilde{f}}$. This terminus can be obviously extended to subsets. Let $f \in W^{1,p}(S)$, $g \in W^{1,p^*}(S)$, $1 \leq p \leq \infty$ and p^* Hölder conjugate of p. In local coordinates $x = (x^i)$ of S hold

$$\int_S \langle Df, Dg \rangle = \int_S \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} \sigma^{ij}(x) \sqrt{\sigma(x)} dx^i dx^j, \tag{4.8}$$

$$\int_{S_h} \left\langle D\hat{f}, D\hat{g} \right\rangle = \int_S \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} g^{ij}(\psi(x), x) \sqrt{g(\psi(x), x)} dx^i dx^j, \tag{4.9}$$

$$\int_S \langle Df, Dg\rangle = \int_{S_h} \left\langle D\hat{f}, D\hat{g}\right\rangle + O(h^2)\|f\|_{W^{1,p}(S)}\|g\|_{W^{1,p^*}(S)}, \tag{4.10}$$

and similarly,

$$\int_S f = \int_{S_h} \hat{f} + O(h^2)\|f\|_{L^1(S)} \tag{4.11}$$

where now $f \in L^1(S)$ is sufficient.

The bracket $\langle u, v\rangle$ denotes here the scalar product of two tangent vectors u, v (or their covariant counterparts). $\|\cdot\|_{W^{k,p}}$ denotes the usual Sobolev norm, $|\cdot|_{W^{k,p}} = \sum_{|\alpha|=k} \|D^\alpha \cdot\|_{L^p}$ and $H^k = W^{k,2}$.

5. Discretization of the state equation. Let $0 = t_0 < t_1 < ... < t_{N-1} < t_N = T$ be a time grid with $\tau_n = t_n - t_{n-1}$, $n = 1, ..., N$, and $\tau = \max_{1\le n\le N} \tau_n$. We set

$$\begin{aligned} W_{h,\tau} = \{\Phi :&\Gamma_h \times [0,T] \to \mathbb{R} : \\ &\Phi(\cdot, t) \in X_h \text{ and } \Phi(x, \cdot) \text{ constant in } (t_{n-1}, t_n), 1 \le n \le N\} \end{aligned} \tag{5.1}$$

and define the bilinear forms

$$a : W^{1,p}(S) \times W^{1,p^*}(S) \to \mathbb{R}, \quad a(u, v) = \int_S \langle Du, Dv\rangle + uvdx, \tag{5.2}$$

$$a_h : W^{1,p}(S_h) \times W^{1,p^*}(S_h) \to \mathbb{R}, \quad a_h(u_h, v_h) = \int_{S_h} \langle Du_h, Dv_h\rangle + u_h v_h dx, \tag{5.3}$$

$$a_h^n : W^{1,p}(S_h) \times W^{1,p^*}(S_h) \to \mathbb{R}, \quad a_h^n(u_h, v_h) = \int_{S_h} \langle Du_h, Dv_h\rangle_{\tilde{g}(t_n)} + u_h v_h dx, \tag{5.4}$$

$$(Du_h, Dv_h)_{\tilde{g}(t_n)} = \int_{S_h} \langle Du_h, Dv_h\rangle_{\tilde{g}(t_n)}. \tag{5.5}$$

The last but one equation needs a further definition. Let p_1, p_2, p_3 be the midpoints of the three edges of τ, $\tau \in T_h$, and $v, w \in C^0(\tau, T^{0,1}(\tau))$ sections then we define

$$\int_\tau \langle v, w\rangle_{\tilde{g}(t_n)} = \frac{1}{3}|\tau| \sum_{k=1}^{3} a^{ij}(\tilde{p}_k) v_i(p_k) w_j(p_k) \tag{5.6}$$

where $(a^{ij}(\tilde{p}_k))$ is a contravariant representation with respect to local coordinates (x^i) (belonging to our fixed atlas) in a neighbourhood of $\tilde{p}_k$ in S and $(v_i)(p_k)$, $(w_j)(p_k)$ are covariant representations with respect to the orthogonal projections of $\frac{\partial}{\partial x^1}(\tilde{p}_k)$ and $\frac{\partial}{\partial x^2}(\tilde{p}_k)$ on τ. (Despite similar notation $\tilde{g}$ does not refer to a metric.) Furthermore, the brackets $(\cdot, \cdot)$ and $(\cdot, \cdot)_h$ denote the inner products of $L^2(S)$ and $L^2(S_h)$, respectively, and $\|\cdot\|$ and $\|\cdot\|_h$ the corresponding norms. The semi-norm associated with the bilinear on the left-hand side of (5.5) is denoted by $\|\cdot\|_{\tilde{g}(t_n)}$.

We define a discrete operator $G_h : L^2(S) \to X_h, v \mapsto G_h v = z_h$ via

$$a_h(z_h, \varphi_h) = \int_{S_h} \hat{v}\varphi_h \quad \forall \varphi_h \in X_h. \tag{5.7}$$

We denote the interpolation operator by I_h, define $P_h : L^2(\Gamma_0) \to X_h$ by

$$(\hat{z}, \phi_h)_h = (P_h z, \phi_h)_h \quad \forall \phi_h \in X_h, \quad z \in L^2(\Gamma_0), \tag{5.8}$$

let $R_h : H^1(S) \to X_h$ be defined by

$$a_h(R_h z, \phi_h) = a_h(\hat{z}, \phi_h) \quad \forall \phi_h \in X_h, \quad z \in H^1(\Gamma_0), \tag{5.9}$$

and $R_h^n : H^1(S) \to X_h$ by

$$a_h^n(R_h^n z, \phi_h) = a_h^n(\hat{z}, \phi_h) \quad \forall \phi_h \in X_h, \quad z \in H^1(S). \tag{5.10}$$

It is well-known that

$$\|\hat{z} - R_h z\|_{L^2(S_h)} + h\|D(\hat{z} - R_h z)\|_{L^2(S_h)} \leq ch^m \|z\|_{H^m(S)} \tag{5.11}$$

and

$$\|\hat{z} - R_h^n z\|_{L^2(S_h)} + h\|D(\hat{z} - R_h^n z)\|_{L^2(S_h)} \leq ch^m \|z\|_{H^m(S)} \tag{5.12}$$

hold for all $z \in H^m(S)$, $m = 1, 2$. We conclude for $z \in H^2(S)$ that

$$\begin{aligned} \|\hat{z} - R_h z\|_{L^\infty(S_h)} &\leq \|\hat{z} - I_h z\|_{L^\infty(S_h)} + \|I_h z - R_h z\|_{L^\infty(S_h)} \\ &\leq ch\|z\|_{H^2(S)} + ch^{-1}\|I_z - R_h z\|_{L^2(S_h)} \leq ch\|z\|_{H^2(S_h)}. \end{aligned} \tag{5.13}$$

There holds

$$\|\phi_h\|_{L^\infty(S_h)} \leq \rho(h)\|\phi_h\|_{H^1(S_h)} \tag{5.14}$$

for all $\phi_h \in X_h$ where $\rho(h) = \sqrt{|\log h|}$.

For $Y, \Phi \in W_{h,\tau}$ we let

$$\begin{aligned} A(Y, \Phi) :=& \sum_{n=1}^{N} \tau_n (\nabla Y^n, \nabla \Phi^n)_{\tilde{g}(t_n)} + \sum_{n=2}^{N} (Y^n - Y^{n-1}, \Phi^n)_h + (Y_+^0, \Phi_+^0)_h \\ &+ \sum_{n=1}^{N} \tau_n (b^i(t_n)\nabla_i Y^n, \Phi^n)_h + \sum_{n=1}^{N} \tau_n (c(t_n)Y^n, \Phi^n)_h \end{aligned} \tag{5.15}$$

where $\Phi^n := \Phi_-^n$, $\Phi_\pm^n = \lim_{s \to 0\pm} \Phi(t_n + s)$.

Note, that the integrals $(b^i(t_n)\nabla_i Y^n, \Phi^n)_h$ and $(c(t_n)Y^n, \Phi^n)_h$ are defined analogously to (5.6) by using a quadrature rule of order ≥ 2.

Given $u \in U$ our approximation $Y \in W_{h,\tau}$ of the solution y of the state equation in (3.2) is obtained by the following discontinuous Galerkin scheme

$$A(Y, \Phi) = \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (\widehat{Bu(t)}, \Phi^n)_h + (\hat{y}_0, \Phi_+^0)_h \quad \forall \phi \in W_{h,\tau} \tag{5.16}$$

and will be denoted by $G_{h,\tau}(Bu) = Y$.

We have the following uniform error estimate.

THEOREM 5.1. *Let $u \in U$, $y = G(Bu)$, $Y = G_{h,\tau}(Bu)$. Then*

$$\max_{1 \leq n \leq N} \|\widehat{y}(\cdot, t_n) - Y^n\|_{L^\infty(S_h)} \leq c\rho(h)(h + \sqrt{\tau})(\|y_0\|_{H^2(S)} + \|u\|_U). \tag{5.17}$$

Proof. See [16, Theorem 4.1]. ∎

6. Discretization of the optimal control problem. In the following we assume that

$$\tau = o(\rho(h)^{-2}) \tag{6.1}$$

as $h \to 0$ which implies that the right-hand side of (5.17) converges to zero as $h \to 0$.

We abbreviate

$$\beta(x,t) = \left(\frac{g(x,t)}{g(x,0)}\right)^{\frac{1}{4}}, \quad (x,t) \in \Omega_T, \tag{6.2}$$

$y_g(t_n) = y_g(\cdot, t_n)$ and with ambiguous notation $\beta(t) = \beta(\cdot, t)$, $\beta(t_n) = \widehat{\beta(\cdot, t_n)}$. We discretize our optimal control problem as follows:

$$\begin{cases} \min_{u \in U} J_{h,\tau}(u) := \frac{1}{2} \int_0^T \sum_{n=1}^N \tau_n \|\beta(t_n)(Y^n - \widehat{y_g}(t_n))\|_h^2 + \frac{\alpha}{2} \int_0^T |u(t)|^2 dt \\ \text{s.t. } Y = G_{h,\tau}(Bu) \quad \text{and} \quad Y^n(x_j) \geq 0, 1 \leq j \leq J, 1 \leq n \leq N. \end{cases} \tag{6.3}$$

Remark 6.1. *The control problem (6.3) has a unique solution $u_h \in U$ and [3, Theorem 5.3] implies the existence of $\mu_j^n \in \mathbb{R}$, $1 \leq j \leq J$, $1 \leq n \leq N$ and $P \in W_{h,\tau}$ so that*

$$\begin{aligned} &A(\Phi, P) = \sum_{n=1}^N \tau_n (Y^n - \widehat{y_g}(t_n), \Phi^n \beta(t_n)^2)_h + \sum_{n=1}^N \sum_{j=1}^J \Phi^n(x_j)\mu_j^n \quad \forall \Phi \in W_{h,\tau} \\ &\alpha u_h(t) + ((P^n, \hat{f}_i)_h)_{i=1,\dots,m} = 0 \quad \textit{a.e. in } (t_{n-1}, t_n) \\ &\mu_j^n \leq 0, Y^n(x_j) \geq 0, \quad \textit{and} \quad \sum_{n=1}^N \sum_{j=1}^J Y^n(x_j)\mu_j^n = 0. \end{aligned} \tag{6.4}$$

We define a measure $\mu_{h,\tau} \in M(\overline{\Omega_T})$ by

$$\int_{\overline{\Omega_T}} f d\mu_{h,\tau} := \sum_{n=1}^N \sum_{j=1}^J f(x_j, t_n)\mu_j^n, \quad f \in C^0(\overline{\Omega_T}) \tag{6.5}$$

and its lift $\hat{\mu}_{h,\tau}$ by

$$\langle \hat{\mu}_{h,\tau}, \cdot \rangle = \langle \mu_{h,\tau}, \tilde{\cdot} \rangle \tag{6.6}$$

on $C^0(\Omega_T^h)$, $\Omega_T^h := S_h \times [0,T]$. Note, that the lift operator $\tilde{\cdot}$ for functions $f = f(x,t)$ being defined on Ω_T^h is considered with respect to the spatial part, i.e. $\tilde{f}(\tilde{x}, t) = (\widetilde{f(\cdot, t)})(\tilde{x})$ for $(x,t) \in \Omega_T^h$, and correspondingly for $\hat{\cdot}$.

Lemma 6.2. *Let u_h, μ_j^n, P and Y be as in Remark 6.1 and $\hat{\mu}_{h,\tau}$ as in (6.6). Then there is $h_0 > 0$ so that*

$$\sum_{n=1}^N \tau_n \|Y^n\|_h^2 + \int_0^T |u_h(t)|^2 dt + \sum_{n=1}^N \sum_{j=1}^J |\mu_j^n| \leq c \quad \textit{for all } 0 < h \leq h_0. \tag{6.7}$$

Proof. From (3.3) we know that there is $\delta > 0$ so that $y^0 \geq \delta$ in $\overline{\Omega_T}$. Setting $\check{Y} := G_{h,\tau}(0) \in W_{h,\tau}$ we conclude from Theorem 5.1 that

$$\check{Y}^n(x_j) \geq \frac{\delta}{2}, \quad 1 \leq j \leq J, 1 \leq n \leq N, 0 < h \leq h_0. \tag{6.8}$$

From (6.1) we conclude

$$
\begin{aligned}
\sum_{n=1}^{N}\sum_{j=1}^{J}\check{Y}^n(x_j)|\mu_j^n| =& \sum_{n=1}^{N}\sum_{j=1}^{J}(Y^n(x_j)-\check{Y}^n(x_j))\mu_j^n \\
=& A(Y^n-\check{Y}^n,P)-\sum_{n=1}^{N}\tau_n(Y^n-\widehat{y}_g,(Y^n-\check{Y}^n)\beta(t_n)^2)_h \\
=& \sum_{n=1}^{N}\tau_n\int_{S_h}(-(Y^n)^2+\widehat{y}_gY^n+Y^n\check{Y}^n-\widehat{y}_g\check{Y}^n)\beta(t_n)^2 \\
&+\sum_{n=1}^{N}\sum_{i=1}^{m}\tau_n u_{h,i|(t_{n-1},t_n)}(\hat{f}_i,P^n)_h \\
\le& -c_0\sum_{n=1}^{N}\tau_n\|Y^n\|_h^2-\alpha\int_0^T|u_h(t)|^2dt+C
\end{aligned}
\tag{6.9}
$$

with a constant $c_0>0$. This implies the claim together with (3.4). ▯

7. Discretization error estimate of the optimization problem. The discretization error of the optimization problem is estimated in the following Theorem.

THEOREM 7.1. *Let u be the solution of (3.2) and u_h the solution of (6.3) with corresponding states $y=G(Bu)$ and $Y=G_{h,\tau}(Bu_h)$. Then there holds*

$$
\sum_{n=1}^{N}\tau_n\|\hat{y}(\cdot,t_n)-Y^n\|_h^2+\int_0^T|u(t)-u_h(t)|^2dt\le c\rho(h)(h+\sqrt{\tau}). \tag{7.1}
$$

Proof. We write

$$
\begin{aligned}
&\alpha\int_0^T|u(t)-u_h(t)|^2dt \\
=&\int_0^T u(t)(u(t)-u_h(t))dt-\alpha\int_0^T u_h(t)(u(t)-u_h(t))dt \\
=&I_1+I_2.
\end{aligned}
\tag{7.2}
$$

The first goal is to estimate I_1. Let

$$
C_0^\infty(0,T;\mathbb{R}^m)\ni v_k\to u-u_h \tag{7.3}
$$

in $L^2(0,T;\mathbb{R}^m)$, $y^h:=G(Bu_h)$ and $z_k:=G_0(Bv_k)$. Since v_k is smooth and $f_i\in L^\infty(\Gamma_0)$, $i=1,...,m$, we have $z_k\in W_0^\infty$ and in view of (2.14) there holds

$$
\begin{aligned}
\|(y-y^h)-z_k\|_{C^0(\overline{\Omega_T})}\le& c\max_{0\le t\le T}\|(y-y^h)(\cdot,t)-z_k(\cdot,t)\|_{H^2(\Gamma_0)} \\
\le& c\left(\int_0^T|(u-u_h)(t)-v_k(t)|^2dt\right)^{\frac{1}{2}} \\
\to& 0,\quad k\to\infty.
\end{aligned}
\tag{7.4}
$$

Hence using (3.4) we conclude that

$$
\begin{aligned}
I_1 =&\alpha \lim_{k\to\infty} \int_0^T u(t)\cdot v_k(t)dt \\
=&-\lim_{k\to\infty}\int_0^T \sum_{i=1}^m v_{k,i}(t)(p(\cdot,t),f_i)dt \\
=&-\lim_{k\to\infty}\int_0^T (Bv_k,p)dt \\
=&-\lim_{k\to\infty}\int_0^T (Az_k,p)dt \\
=&-\lim_{k\to\infty}\left\{\int_0^T \left(\beta(t)^2(y-y_g),z_k\right)dt + \int_{\overline{\Omega_T}} z_k d\mu\right\} \qquad (7.5)\\
=&\int_0^T \left(\beta(t)^2(y-y_g),y^h-y\right)dt + \int_{\overline{\Omega_T}} y^h - y d\mu \\
=&\sum_{n=1}^N \tau_n(\widehat{y^n}-\widehat{y_g(t_n)},(\widehat{y^{h,n}}-\widehat{y^n})\beta(t_n)^2)_h + \int_{\overline{\Omega_T}}(y^h)^- d\mu \\
&+\sum_{n=1}^N\int_{t_{n-1}}^{t_n}\{\left(\beta(t)^2(y-y_g),y^h-y\right) \\
&-\tau_n(\widehat{y^n}-\widehat{y_g(t_n)},(\widehat{y^{h,n}}-\widehat{y^n})\beta(t_n)^2)_h\} \\
=&I_{1,1}+I_{1,2}+I_{1,3}.
\end{aligned}
$$

where we used (3.5) for the last but one equation and set $w^- = \min(w,0)$.

We estimate $I_{1,2}$. For $(x,t)\in\Gamma_0\times(t_{n-1},t_n)$ we have

$$
\begin{aligned}
|y^h(x,t)| \le&|(y^h)^-(x,t)-(y^h)^-(x,t_n)| + |(y^h)^-(x,t_n)-(Y^n)^-(\hat{x})| \\
\le&|(y^h)(x,t)-(y^h)(x,t_n)| + |(y^h)(x,t_n)-(Y^n)(\hat{x})| \\
\le&2\max_{0\le s\le T}\|y^h(\cdot,s)-R_h\widetilde{y^h(\cdot,s)}\|_{L^\infty(\Gamma_0)} \\
&+\|R_h\widetilde{y^h(\cdot,t)}-R_h\widetilde{y^h(\cdot,t_n)}\|_{L^\infty(\Gamma_0)} + \|y^{h,n}-\widetilde{Y^n}\|_{L^\infty(\Gamma_0)} \\
\le&ch\max_{0\le s\le T}\|y^h(\cdot,s)\|_{H^2(\Gamma_0)} \\
&+\rho(h)\|R_h\widetilde{y^h(\cdot,t)}-R_h\widetilde{y^h(\cdot,t_n)}\|_{H^1(\Gamma_0)} \qquad (7.6)\\
&+\rho(h)(h+\sqrt{\tau})(\|y_0\|_{H^2(\Gamma_0)}+\|u_h\|_U) \\
\le&\rho(h)(h+\sqrt{\tau})(\|y_0\|_{H^2(\Gamma_0)}+\|u_h\|_U) \\
&+\rho(h)\sqrt{\tau_n}\left(\int_{t_{n-1}}^{t_n}\|\widehat{R_h y_t^h}\|^2_{H^1(\Gamma_0)}dt\right)^{\frac{1}{2}} \\
\le&\rho(h)(h+\sqrt{\tau})(\|y_0\|_{H^2(\Gamma_0)}+\|u_h\|_U) \\
\le&\rho(h)(h+\sqrt{\tau})
\end{aligned}
$$

where we used Lemma 6.2. By continuity this estimate holds also at the points $t=t_n$,

$n = 0, ..., N$. From main theorem of calculus we get $|I_{1,3}| \le c\tau$. So we have

$$I_1 \le \sum_{n=1}^{N} \tau_n(\widehat{y^n} - \widehat{y_g(t_n)}, (\widehat{y^{h,n}} - \widehat{y^n})\beta(t_n)^2)_h + \rho(h)(h + \sqrt{\tau}) + c\tau. \tag{7.7}$$

We set $\check{Y} = G_{h,\tau}(Bu)$. Then (6.4) implies that

$$\begin{aligned}
I_2 =& \sum_{n=1}^{N}\sum_{i=1}^{m}(P^n, \hat{f}_i)_h \int_{t_{n-1}}^{t_n} (u_i - u_{h,i})(t)dt \\
=& \sum_{n=1}^{N}\sum_{i=1}^{m}(\widetilde{P^n}, f_i) + ((P^n, \hat{f}_i)_h - (\widetilde{P^n}, f_i)) \int_{t_{n-1}}^{t_n} (u_i - u_{h,i})(t)dt \\
=& \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (B(u - u_h), \widetilde{P^n})dt \\
&+ \sum_{n=1}^{N}\sum_{i=1}^{m}((P^n, \hat{f}_i)_h - (\widetilde{P^n}, f_i)) \int_{t_{n-1}}^{t_n} (u_i - u_{h,i})(t)dt \\
=& A(\check{Y} - Y, P) \\
&+ \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (B(u - u_h), \widetilde{P^n})dt - \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (B\widehat{(u - u_h)}, P^n)_h dt \\
&+ \sum_{n=1}^{N}\sum_{i=1}^{m}((P^n, \hat{f}_i)_h - (\widetilde{P^n}, f_i)) \int_{t_{n-1}}^{t_n} (u_i - u_{h,i})(t)dt \\
=& A(\check{Y} - Y, P) + N_1 + N_2 \\
=& \sum_{n=1}^{N} \tau_n(Y^n - \widehat{y_g}(t_n), (\check{Y}^n - Y^n)\beta(t_n)^2)_h \\
&+ \sum_{n=1}^{N}\sum_{j=1}^{J}(\check{Y}^n(x_j) - Y^n(x_j))\mu_j^n + N_1 + N_2 \\
\le& \sum_{n=1}^{N} \tau_n(Y^n - \widehat{y_g}(t_n), (\check{Y}^n - Y^n)\beta(t_n)^2)_h \\
&+ \max_{1\le n\le N, 1\le j\le J} |(\check{Y}^n)^-(x_j)| \sum_{n=1}^{N}\sum_{j=1}^{J} |\mu_j^n| + N_1 + N_2.
\end{aligned} \tag{7.8}$$

Recalling that $y \ge 0$ in $\overline{\Omega_T}$ we have for $1 \le j \le J$, $1 \le n \le N$

$$\begin{aligned}
|(\check{Y}^n)^-(x_j)| =& |(\check{Y}^n)^-(x_j) - y^-(x_j, t_n)| \\
\le& |\check{Y}^n(x_j) - y(x_j, t_n)| \\
\le& \|\overline{Y^n} - y(\cdot, t_n)\|_{L^\infty(\Gamma_0)} \\
\le& c\rho(h)(h + \sqrt{\tau})(\|y_0\|_{H^2(\Gamma_0)} + \|u\|_U) \\
\le& c\rho(h)(h + \sqrt{\tau})
\end{aligned} \tag{7.9}$$

where we used Theorem 5.1. We conclude that

$$I_2 \le \sum_{n=1}^{N} \tau_n (Y^n - \widehat{y_g}(t_n), (\check{Y}^n - Y^n)\beta(t_n)^2)_h + c\rho(h)(h+\sqrt{\tau}) + N_1 + N_2 \tag{7.10}$$

so that together with (7.7) we deduce from (7.2) that

$$\begin{aligned}
&\alpha \int_0^T |u(t) - u_h(t)|^2 dt \le \\
&\sum_{n=1}^{N} \tau_n (\widehat{y^n} - \widehat{y_g(t_n)}, (\widehat{y^{h,n}} - \widehat{y^n})\beta(t_n)^2)_h + c\rho(h)(h+\sqrt{\tau}) + c\tau \\
&+ \sum_{n=1}^{N} \tau_n (Y^n - \widehat{y_g(t_n)}, (\check{Y}^n - Y^n)\beta(t_n)^2)_h + N_1 + N_2 \\
&= \sum_{n=1}^{N} \tau_n \int_{S_h} \beta(t_n)^2 \\
&\Big\{ -(\widehat{y^n} - Y^n)^2 + (Y^n - \widehat{y_g(t_n)})(\check{Y}^n - \widehat{y^n}) + (\widehat{y^n} - \widehat{y_g(t_n)})(\widehat{y^{h,n}} - Y^n) \Big\} \\
&+ c\rho(h)(h+\sqrt{\tau}) + c\tau + N_1 + N_2 \\
&\le - \sum_{n=1}^{N} \tau_n \|\beta(t_n)(\widehat{y^n} - Y^n)\|_h^2 + c\rho(h)(h+\sqrt{\tau}) + c\tau + N_1 + N_2.
\end{aligned} \tag{7.11}$$

It remains to estimate N_1, N_2 for which we show that $O(h^2)\|P^n\|_h$ is small. Therefore we test (6.4) with

$$\Phi^n = \begin{cases} P^n, & 1 \le n \le l, \\ 0, & n > l \end{cases} \tag{7.12}$$

where $1 \le l \le N$ is fixed, have

$$\begin{aligned}
A(\Phi, P) \ge &\sum_{n=1}^{l} \frac{\tau_n}{2} \|\nabla P^n\|_{\tilde{g}(t_n)} + \sum_{n=2}^{l} \|P^n\|_h^2 - \sum_{n=2}^{l} (P^{n-1}, P^n)_h + \|P^1\|_h^2 \\
&- c \sum_{n=1}^{l} \tau_n \|P^n\|_h \\
\ge &\sum_{n=1}^{l} \frac{\tau_n}{2} \|\nabla P^n\|_{\tilde{g}(t_n)} + \frac{1}{2} \sum_{n=2}^{l} \|P^n\|_h^2 \\
&- \frac{1}{2} \sum_{n=2}^{l} \|P^{n-1}\|_h^2 + \|P^1\|_h^2 - c \sum_{n=1}^{l} \tau_n \|P^n\|_h \\
\ge &\sum_{n=1}^{l} \frac{\tau_n}{2} \|\nabla P^n\|_{\tilde{g}(t_n)} + \frac{1}{2} \|P^l\|_h^2 \\
&+ \frac{1}{2} \|P^1\|_h^2 - c \sum_{n=1}^{l} \tau_n \|P^n\|_h
\end{aligned} \tag{7.13}$$

and obtain

$$
\begin{aligned}
A(\Phi, P) \leq & c \max_{1\leq n\leq l} \|P^n\|_{L^\infty(S_h)} + c\sum_{n=1}^{l} \tau_n \|P^n\|_h \\
\leq & \frac{c}{h} \max_{1\leq n\leq l} \|P^n\|_h + c\sum_{n=1}^{l} \tau_n \|P^n\|_h .
\end{aligned} \tag{7.14}
$$

from which we conclude recursively for $l = 1, ..., N$ that

$$
\|P^n\|_h \leq \frac{c}{h} \tag{7.15}
$$

for $n = 1, ..., N$. □

REFERENCES

[1] Borukhava, M., Kröner, H. (2016) Alternative to evolving surface finite element method. arXiv:1501.07900 [math.NA].

[2] Brenner, S. C., Scott, L. R. (2008) *The Mathematical Theory of Finite Element Methods*. Texts in Applied Mathematics, third edition, Berlin: Springer.

[3] Casas, E. (1993) Boundary control of semi linear elliptic equations with pointwise state constraints. *SIAM J. Control. Optim.*, **31**, 993–1006.

[4] Deckelnick, K., Hinze, M. (2011) Variational discretization of parabolic control problems in the presence of pointwise state constraints. *J. Comput. Math.*, **29**, no. 1, 1–15.

[5] de Los Reyes, J.C., Merino, P., Rehberg, J., Tröltzsch, F. (2008) Optimality conditions for state-constrained PDE control problems with time-dependent controls. *Control Cybern.*, **37**, no. 1, 5–38.

[6] Demlow, A. (2009) Higher-order finite element methods and pointwise error estimates for elliptic problems on surfaces. *SIAM J. Numer. Anal.*, **47**, No. 2, 805–827.

[7] Dziuk, G. (1988) *Finite elements for the Beltrami operator on arbitrary surfaces*. Partial Differential Equations and Calculus of Variations (S. Hildebrandt & R. Leis eds). Lecture Notes in Mathematics, vol. 1357. Berlin: Springer, pp. 142–155.

[8] Dziuk, G., Elliott, C.M. (2007) Finite elements on evolving surfaces. *IMA J. Numer. Anal.*, **27**, no. 2, 262–292.

[9] G. Dziuk and C. M. Elliott. L^2-estimates for the evolving surface finite element method. *Math. Comput.*, 82, no. 281: 1-24, 2013.

[10] G. Dziuk, Ch. Lubich and D. Mansour. Runge-Kutta time discretization of parabolic differential equations on evolving surfaces. *IMA J. Numer. Anal.*, 32, no. 2: 394-416, 2012.

[11] Garcke, H., Lam, K.F., Stinner, B. (2014) Diffuse interface modelling of soluble surfactants in two-phase flow. *Commun. Math. Sci.*, **12**, no. 8, 1475-1522.

[12] Hinze, M. (2005) A variational discretization concept in control constrained optimization: The linear-quadratic case. *Comput. Optim. Appl.*, **30**(1), 45-61.

[13] Hinze, M., Pinnau, R., Ulbrich, M.,Ulbrich, S. (2009) *Optimization with pde constraints*. Mathematical Modelling: Theory and Applications, Volume 23, Berlin: Springer.

[14] Hinze, M., Vierling, M. (2012) Optimal control of the Laplace-Beltrami operator on compact surfaces: concept and numerical treatment. *J. Comput. Math.*, **30**, No. 4, 392-403.

[15] B. Kovács and C.A. Power Guerra. Maximum norm stability and error estimates for the evolving surface finite element method. *http://arxiv.org/abs/1510.00605*, 2015.

[16] Kröner, H. (2016) Error estimate for a finite element approximation of the solution of a linear parabolic equation on a two-dimensional surface. arXiv:1604.04665 [math.NA].

[17] Schatz, A.H. (1998) Pointwise error estimates and asymptotic error expansion inequalities for the finite element method on irregular grids. I: Global Estimates. *Math. Comput.*, **67** (223, 877-899.

[18] Vierling, M. (2014) Parabolic optimal control problems on evolving surfaces subject to point-wise box constraints on the control-theory and numerical realization. *Interfaces Free Bound.*, **16**, No. 2, 137-173.