

Hamburger Beiträge

zur Angewandten Mathematik

A Posteriori Error Representations for Elliptic Optimal Control Problems with Control and State Constraints

Andreas Günther, Michael Hinze
and Moulay Hicham Tber

Nr. 2010-03
April 2010

A Posteriori Error Representations for Elliptic Optimal Control Problems with Control and State Constraints

Andreas Günther, Michael Hinze and Moulay Hicham Tber

Abstract. In this work we develop an adaptive algorithm for solving elliptic optimal control problems with simultaneously appearing state and control constraints. Building upon the concept proposed in [9] the algorithm applies a Moreau-Yosida regularization technique for handling state constraints. The state and co-state variables are discretized using continuous piecewise linear finite elements while a variational discretization concept is applied for the control. To perform the adaptive mesh refinement cycle we derive local error representations which extend the goal-oriented error approach to our setting. The performance of the overall adaptive solver is demonstrated by a numerical example.

Mathematics Subject Classification (2000). 49J20; 65N30; 65N50.

Keywords. elliptic optimal control problem, control constraints, state constraints, goal-oriented adaptivity, error estimates.

1. Introduction

Optimal control problems with state constraints have been the topic of an increasing number of theoretical and numerical studies. The challenging character of these problems has its origin in the fact that state constraints feature Lagrange multipliers of low regularity only [4, 7]. In the presence of additional control constraints, the solution may exhibit subsets where both control and state are active simultaneously. In this case, the Lagrange multipliers associated to the control and state constraints may not be unique [20]. To overcome this difficulty several techniques in the literature have been proposed. Very popular are relaxation concepts for state constraints such as Lavrentiev, interior point and Moreau-Yosida regularization. The former one is investigated in [21] and [17]. Barrier methods

This work was supported by the DFG priority program 1253 grant No. DFG 06/382.

in function space [24] applied to state constrained optimal control problems are considered in [18]. Relaxation by Moreau-Yosida regularization is considered for the fully discrete case in [3, 5], and in function space in [13]. Residual-type a posteriori error estimators for mixed control-state constrained problems are derived in [16]. The dual weighted residual method proposed in [1] is investigated in [11, 22] the presence of control constraints and for state constraints in [2, 9]. Within the framework of goal-oriented adaptive function space algorithms a Lavrentiev regularization approach is considered in [16], and an adaptive interior point method is proposed in [19, 25].

In this note we combine results from [9] and [10] to design an adaptive finite element algorithm for solving elliptic optimal control problems with pointwise control and state constraints. Following [14], our algorithm combines a Moreau-Yosida regularization approach with a semi-smooth Newton solver [12]. We apply variational discretization [8, 15] to the regularized optimal control problem. For a fixed regularization parameter, we develop a goal-oriented a posteriori error estimate by extending the error representation obtained in [9] to the control and state constrained case. Let us note that in this work we are not interested in controlling the error contribution stemming from the regularization.

The rest of this paper is organized as follows: In the next section we present the optimal control problem under consideration and recall its first order necessary optimality system. Section 3 is devoted to the purely state-constrained case and collects results from [9] which in Section 4 are extended to simultaneously appearing control and state constraints. We introduce a regularized version of the original problem and derive an error representation in terms of the objective functional. Finally, a numerical experiment is reported in Section 5.

2. Optimal Control Problem

Let Ω be a bounded polygonal and convex domain in \mathbb{R}^d ($d = 2, 3$) with boundary $\partial\Omega$. We consider the general elliptic partial differential operator $\mathcal{A} : H^1(\Omega) \rightarrow H^1(\Omega)^*$ defined by

$$\mathcal{A}y := \sum_{i,j=1}^d \partial_{x_j} (a_{ij}y_{x_i}) + \sum_{i=1}^d b_i y_{x_i} + cy$$

along with its formal adjoint operator \mathcal{A}^*

$$\mathcal{A}^*y = \sum_{i=1}^d \partial_{x_i} \left(\sum_{j=1}^d a_{ij}y_{x_j} + b_i y \right) + cy.$$

We subsequently assume the coefficients a_{ij}, b_i and c ($i, j = 1, \dots, d$) to be sufficiently smooth functions on $\bar{\Omega}$. Moreover we suppose that there exists $c_0 > 0$ such that $\sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \geq c_0$ for almost all x in Ω and all ξ in \mathbb{R}^d . Corresponding to

the operator \mathcal{A} we associate the bilinear form $a(\cdot, \cdot) : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}$ with

$$a(y, v) := \int_{\Omega} \left(\sum_{i,j=1}^d a_{ij} y_{x_i} v_{x_j} + \sum_{i=1}^d b_i y_{x_i} v + cyv \right).$$

Suppose that the form a is coercive on $H^1(\Omega)$, i.e. there exists $c_1 > 0$ such that $a(v, v) \geq c_1 \|v\|_{H^1(\Omega)}^2$ for all v in $H^1(\Omega)$. This follows for instance if

$$\inf \operatorname{ess}_x \in \Omega \left(c - \frac{1}{2} \sum_{i=1}^d \partial_{x_i} b_i \right) > 0 \text{ and } \inf \operatorname{ess}_x \in \partial\Omega \left(\sum_{i=1}^d b_i \nu_i \right) \geq 0$$

holds. Here ν denotes the unit outward normal at $\partial\Omega$.

For given $u \in L^2(\Omega)$ and fixed $f \in L^2(\Omega)$ the homogeneous Neumann boundary value problem

$$\begin{aligned} \mathcal{A}y &= u + f && \text{in } \Omega \\ \partial_{\nu_{\mathcal{A}}} y &:= \sum_{i,j=1}^d a_{ij} y_{x_i} \nu_j = 0 && \text{on } \partial\Omega \end{aligned} \quad (2.1)$$

has a unique solution $y =: \mathcal{G}(u) \in H^2(\Omega)$. Moreover, there exists a constant C depending on the domain Ω such that

$$\|\mathcal{G}(u)\|_{H^2(\Omega)} \leq C(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}).$$

The weak form of (2.1) is given by

$$a(y, v) = (u + f, v) \quad \forall v \in H^1(\Omega), \quad (2.2)$$

where (\cdot, \cdot) denotes the standard inner product in $L^2(\Omega)$.

For given $u_d, y_d \in L^2(\Omega)$, $\alpha > 0$, $u_a, u_b \in \mathbb{R}$ with $u_a < u_b$, y_a and $y_b \in C(\Omega)$ with $y_a < y_b$ we focus on the optimal control problem

$$\begin{aligned} J(y, u) &:= \frac{1}{2} \|y - y_d\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \|u - u_d\|_{L^2(\Omega)}^2 \rightarrow \min \\ \text{s.t. } &y = \mathcal{G}(u), \quad u \in U_{ad}, \quad \text{and } y_a \leq y \leq y_b \quad \text{a.e. in } \bar{\Omega}, \end{aligned} \quad (2.3)$$

where U_{ad} is the set of admissible controls given by

$$U_{ad} = \{u \in L^2(\Omega) : u_a \leq u \leq u_b \text{ in } \Omega\}.$$

We require the Slater condition

$$\exists u_s \in U_{ad} : \quad y_a < \mathcal{G}(u_s) < y_b \quad \text{in } \Omega.$$

The proof of the following theorem follows from [6, 7].

Theorem 2.1. *The optimal control problem (2.3) has a unique solution $(y, u) \in H^2(\Omega) \times U_{ad}$. Moreover there exist $p \in W^{1,s}(\Omega)$ for all $1 \leq s < d/(d-1)$, λ_a ,*

$\lambda_b \in L^2(\Omega)$ and $\mu_a, \mu_b \in \mathcal{M}(\bar{\Omega})$ satisfying the optimality system

$$\begin{aligned}
& y = \mathcal{G}(u), \\
(p, \mathcal{A}v) &= (y - y_d, v) + \langle \mu_a + \mu_b, v \rangle \quad \forall v \in W^{1,1-\frac{1}{s}}(\Omega) \text{ with } \partial_{\nu_{\mathcal{A}}} v|_{\partial\Omega} = 0, \\
& \alpha(u - u_d) + p + \lambda_a + \lambda_b = 0, \\
\lambda_a &\leq 0, \quad u \geq u_a, \quad (\lambda_a, u - u_a) = 0, \\
\lambda_b &\geq 0, \quad u \leq u_b, \quad (\lambda_b, u - u_b) = 0, \\
\mu_a &\leq 0, \quad y \geq y_a, \quad \langle \mu_a, y - y_a \rangle = 0, \\
\mu_b &\geq 0, \quad y \leq y_b, \quad \langle \mu_b, y - y_b \rangle = 0.
\end{aligned} \tag{2.4}$$

3. The Purely State-constrained Case

First let us consider problem (2.3) without control constraints, i.e. $U_{ad} = U = L^2(\Omega)$. Then clearly $\lambda_a = \lambda_b = 0$, $p = -\alpha(u - u_d)$ and μ_a, μ_b in (2.4) are uniquely determined.

3.1. Finite Element Discretization

In the sequel we consider a shape-regular simplicial triangulation \mathcal{T}_h of Ω . Since Ω is assumed to be a polyhedral, the boundary $\partial\Omega$ is exactly represented by the boundaries of simplices $T \in \mathcal{T}_h$. We refer to $\mathcal{N}_h = \cup_{i=1}^{np} \{x_i\}$ as the set of nodes of \mathcal{T}_h . The overall mesh size is defined by $h := \max_{T \in \mathcal{T}_h} \text{diam } T$. Further, we associate with \mathcal{T}_h the continuous piecewise linear finite element space

$$V_h = \{v \in C_0(\bar{\Omega}) : v|_T \in P_1(T), \forall T \in \mathcal{T}_h\},$$

where $P_1(T)$ is the space of first-order polynomials on T . The standard nodal basis of V_h denoted by $\{\phi_i\}_{i=1}^{np}$ satisfies $\phi_i(x_j) = \delta_{ij}$ for all x_j in \mathcal{N}_h and $i, j \in \{1, \dots, np\}$. Furthermore for all $v \in C^0(\bar{\Omega})$ we denote by $I_h v := \sum_{i=1}^{np} v(x_i) \phi_i$ the Lagrange interpolation of v . In analogy to (2.2) we define for given $u \in L^2(\Omega)$ the discrete solution operator \mathcal{G}_h by

$$y_h =: \mathcal{G}_h(u) \iff y_h \in V_h \text{ and } a(y_h, v_h) = (u + f, v_h) \quad \forall v_h \in V_h.$$

Problem (2.3) is now approximated by the following sequence of variational discrete control problems depending on the mesh parameter h :

$$\begin{aligned}
\min_{u \in U} J_h(y_h, u) &:= \frac{1}{2} \|y_h - y_d\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \|u - u_{d,h}\|_U^2 \\
\text{subject to } y_h &= \mathcal{G}_h(u) \text{ and } y_a(x_j) \leq y_h(x_j) \leq y_b(x_j) \text{ for } j = 1, \dots, np.
\end{aligned} \tag{3.1}$$

Here, $u_{d,h}$ denotes an approximation to u_d which is assumed to satisfy

$$\|u_d - u_{d,h}\| \leq Ch.$$

Problem (3.1) represents a convex infinite-dimensional optimization problem of similar structure as problem (2.3), but with only finitely many equality and inequality constraints for the state, which define a convex set of admissible functions. Since for $h > 0$ small enough we have $y_a < \mathcal{G}_h(u_s) < y_b$ such that [6, 7] can again be applied to obtain

Lemma 3.1. *Problem (3.1) has a unique solution $u_h \in U$. There exist unique $\mu_1^a, \dots, \mu_{np}^a, \mu_1^b, \dots, \mu_{np}^b \in \mathbb{R}$ and a unique function $p_h \in V_h$ such that with $y_h = \mathcal{G}_h(u_h)$, $\mu_{a,h} = \sum_{j=1}^{np} \mu_j^a \delta_{x_j}$ and $\mu_{b,h} = \sum_{j=1}^{np} \mu_j^b \delta_{x_j}$ we have*

$$a(v_h, p_h) = \int_{\Omega} (y_h - y_d)v_h + \int_{\Omega} v_h d(\mu_{a,h} + \mu_{b,h}) \quad \forall v_h \in V_h, \quad (3.2)$$

$$p_h + \alpha(u_h - u_{d,h}) = 0, \quad (3.3)$$

$$\mu_j^a \leq 0, \quad y_h(x_j) \geq y_a(x_j), \quad j = 1, \dots, np \text{ and } \int_{\Omega} (y_h - I_h y_a) d\mu_{a,h} = 0, \quad (3.4)$$

$$\mu_j^b \geq 0, \quad y_h(x_j) \leq y_b(x_j), \quad j = 1, \dots, np \text{ and } \int_{\Omega} (y_h - I_h y_b) d\mu_{b,h} = 0. \quad (3.5)$$

Here, δ_x denotes the Dirac measure concentrated at x .

Remark 3.2. Problem (3.1) is still an infinite-dimensional optimization problem, but with finitely many state constraints. By (3.3) it follows that $u_h \in V_h$, i.e. the optimal discrete solution is discretized implicitly through the optimality condition of the discrete problem. Hence in (3.1) U may be replaced by V_h to obtain the same discrete solution u_h , which results in a finite-dimensional discrete optimization problem instead.

3.2. Local error indicators

From here onwards we assume $u_d = u_{d,h}$, so that $J = J_h$ holds. This assumption is fulfilled by affine linear functions u_d . Including more general desired controls u_d would lead to additional weighted data oscillation contributions $(u_d - u_{d,h}, \cdot)$ in the error representation (4.5). For their treatment in the context of residual type a posteriori estimators we refer to [16].

Let us abbreviate

$$\mu := \mu_a + \mu_b, \quad \mu_h := \mu_{a,h} + \mu_{b,h}.$$

Following [1] we introduce the dual, control and primal residual functionals determined by the discrete solution y_h, u_h, p_h, μ_h^a and μ_h^b of (3.2)-(3.5) by

$$\rho^p(\cdot) := J_y(y_h, u_h)(\cdot) - a(\cdot, p_h) + \langle \mu_h, \cdot \rangle,$$

$$\rho^u(\cdot) := J_u(y_h, u_h)(\cdot) + (\cdot, p_h) \text{ and}$$

$$\rho^y(\cdot) := -a(y_h, \cdot) + (u_h + f, \cdot).$$

In addition we set

$$e^\mu(y) := \langle \mu + \mu_h, y_h - y \rangle.$$

It follows from (3.3) that $\rho^u(\cdot) \equiv 0$. This is due to variational discretization, i.e. we do not discretize the control, so that the discrete structure of the solution u_h of problem (3.1) is induced by the optimality condition (3.3).

The proof of the following theorem can be found in [9].

Theorem 3.3. *There hold the error representations*

$$2(J(y, u) - J(y_h, u_h)) = \rho^p(y - i_h y) + \rho^y(p - i_h p) + e^\mu(y), \quad (3.6)$$

and

$$\begin{aligned}
2(J(y, u) - J(y_h, u_h)) &= J_y(y_h, u_h)(y - y_h) - a(y - y_h, p_h) \\
&\quad - a(y_h, p - i_h p) + (u_h + f, p - i_h p) \\
&\quad + (y - y_d, y - y_h) - a(y - y_h, p) \tag{3.7}
\end{aligned}$$

with arbitrary quasi-interpolants $i_h y$ and $i_h p \in V_h$.

Following the lines of Remark 3.5 in [1] we split the above equation into a cellwise representation and integrate by parts. This gives rise to define

$$\begin{aligned}
R_{|T}^{y_h} &= u_h + f - \mathcal{A}y_h \\
R_{|T}^{p_h} &= y_h - y_d - \mathcal{A}^* p_h \\
R_{|T}^p &= y - y_d - \mathcal{A}^* p \\
r_{|\Gamma}^{y_h} &= \begin{cases} \frac{1}{2}\nu \cdot [\nabla y_h \cdot (a_{ij})] & \text{for } \Gamma \subset \partial T \setminus \partial\Omega \\ \nu \cdot (\nabla y_h \cdot (a_{ij})) & \text{for } \Gamma \subset \partial\Omega \end{cases} \\
r_{|\Gamma}^{p_h} &= \begin{cases} \frac{1}{2}\nu \cdot [(a_{ij})\nabla p_h] & \text{for } \Gamma \subset \partial T \setminus \partial\Omega \\ \nu \cdot ((a_{ij})\nabla p_h + p_h b) & \text{for } \Gamma \subset \partial\Omega \end{cases} \\
r_{|\Gamma}^p &= \begin{cases} \frac{1}{2}\nu \cdot [(a_{ij})\nabla p] & \text{for } \Gamma \subset \partial T \setminus \partial\Omega \\ \nu \cdot ((a_{ij})\nabla p + pb) & \text{for } \Gamma \subset \partial\Omega \end{cases},
\end{aligned}$$

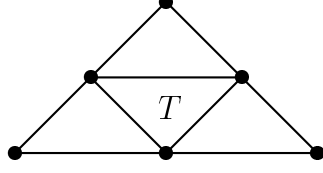
where $[\cdot]$ defines the jump across the inter-element edge Γ . Now (3.7) can be rewritten in the form

$$\begin{aligned}
2(J(y, u) - J(y_h, u_h)) &= \sum_{T \in \mathcal{T}_h} (y - y_h, R_{|T}^{p_h})_T - (y - y_h, r_{|\partial T}^{p_h})_{\partial T} \\
&\quad + (R_{|T}^{y_h}, p - i_h p)_T - (r_{|\partial T}^{y_h}, p - i_h p)_{\partial T} \\
&\quad + (y - y_h, R_{|T}^p)_T - (y - y_h, r_{|\partial T}^p)_{\partial T}.
\end{aligned}$$

Since this localized sum still contains unknown quantities, we make use of local higher order approximation ([1, Sec. 5.1]) which has shown to be a successful heuristic technique for a posteriori error estimation. More precisely we take the local higher order quadratic interpolant operator $i_{2h}^{(2)} : V_h \rightarrow P_2(T)$ for some $T \in \mathcal{T}_h$. In detail for $d = 2$ the local interpolant $i_{2h}^{(2)} v_h$ for an arbitrary function $v_h \in V_h$ on a triangle T is defined by

$$(i_{2h}^{(2)} v_h)(x_1, x_2) := a + bx_1 + cx_2 + dx_1x_2 + ex_1^2 + fx_2^2, \quad (x_1, x_2)^T \in \Omega,$$

where the coefficients $a, b, c, d, e, f \in \mathbb{R}$ are obtained by the solution of a linear system demanding the exact interpolation in the sampling nodes shown in Figure 1. The technique for computing $i_{2h}^{(2)} v_h$ for some $v_h \in V_h$ can easily be carried over to three space dimensions.


 FIGURE 1. Used sampling nodes for $i_{2h}^{(2)}$, $d = 2$.

We substitute $R_{|T}^p$ and $r_{|\Gamma}^p$ by

$$\begin{aligned} R_{|T}^{i_{2h}^{(2)} p_h} &= i_{2h}^{(2)} y_h - y_d - \mathcal{A}^* i_{2h}^{(2)} p_h \\ r_{|\Gamma}^{i_{2h}^{(2)} p_h} &= \begin{cases} 0 & \text{for } \Gamma \subset \partial T \setminus \partial \Omega \\ \nu \cdot ((a_{ij}) \nabla i_{2h}^{(2)} p_h + i_{2h}^{(2)} p_h b) & \text{for } \Gamma \subset \partial \Omega \end{cases} \end{aligned}$$

and define

$$\begin{aligned} \eta := & \frac{1}{2} \sum_{T \in \mathcal{T}_h} (i_{2h}^{(2)} y_h - y_h, R_{|T}^{p_h})_T - (i_{2h}^{(2)} y_h - y_h, r_{|\partial T}^{p_h})_{\partial T} \\ & + (R_{|T}^{y_h}, i_{2h}^{(2)} p_h - p_h)_T - (r_{|\partial T}^{y_h}, i_{2h}^{(2)} p_h - p_h)_{\partial T} \\ & + (i_{2h}^{(2)} y_h - y_h, R_{|T}^{i_{2h}^{(2)} p_h})_T - (i_{2h}^{(2)} y_h - y_h, r_{|\partial T}^{i_{2h}^{(2)} p_h})_{\partial T}. \end{aligned}$$

For numerical experiments we refer to [9] and Section 5.

4. The Control- and State-constrained Case

Let us now consider problem (2.3) with control and state constraints. The situation then becomes more involved due to the fact that Lagrange multipliers may not be unique if control and state active sets intersect [20]. To overcome difficulties arising from this fact we consider

4.1. Moreau-Yosida Regularization

In Moreau-Yosida regularization the state constraints $y_a \leq y \leq y_b$ are substituted by appropriate regularization terms which are added to the objective functional J . The corresponding regularized optimal control problem reads

$$\begin{aligned} J^\gamma(y, u) := & J(y, u) + \frac{\gamma}{2} \|\min(0, y - y_a)\|^2 + \frac{\gamma}{2} \|\max(0, y - y_b)\|^2 \rightarrow \min \\ \text{s.t. } & y = \mathcal{G}(u) \quad \text{and} \quad u \in U_{ad}, \end{aligned} \quad (4.1)$$

where $\gamma > 0$ denotes a regularization parameter tending to $+\infty$ later on. The max- and min-expressions in the regularized objective functional J^γ arise from regularizing the indicator function corresponding to the set of admissible states. Notice that (4.1) is a purely control constrained optimal control problem that has a unique solution $(y^\gamma, u^\gamma) \in H^2(\Omega) \times U_{ad}$. Furthermore, we can prove the existence

of Lagrange multipliers $(p^\gamma, \lambda_a^\gamma, \lambda_b^\gamma) \in L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ using standard theory of mathematical programming in Banach spaces such that

$$\begin{aligned} (p^\gamma, \mathcal{A}v) &= (y^\gamma - y_d, v) + (\mu_a^\gamma + \mu_b^\gamma, v) \quad \forall v \in H^2(\Omega) \text{ with } \partial_{\nu, \mathcal{A}} v|_{\partial\Omega} = 0, \\ &\quad \alpha(u^\gamma - u_d) + p^\gamma + \lambda_a^\gamma + \lambda_b^\gamma = 0, \\ &\quad \lambda_a^\gamma \leq 0, \quad u^\gamma \geq u_a, \quad (\lambda_a^\gamma, u^\gamma - u_a) = 0, \\ &\quad \lambda_b^\gamma \geq 0, \quad u^\gamma \leq u_b, \quad (\lambda_b^\gamma, u^\gamma - u_b) = 0 \end{aligned} \tag{4.2}$$

holds, where

$$\mu_a^\gamma = \gamma \min(0, y^\gamma - y_a) \quad \text{and} \quad \mu_b^\gamma = \gamma \max(0, y^\gamma - y_b).$$

For the convergence of the solutions of the regularized problems to the solution of the limit problem (2.3) we refer the reader to [10, Thm. 3.1]. To recover the solution of the optimal control problem (2.3) an overall algorithm can be designed by solving (4.1) for a sequence $\gamma \rightarrow \infty$. For (4.1) with γ fixed, a locally superlinear semi-smooth Newton method can be applied (see [14]).

4.2. Finite Element Discretization

For the convenience of the reader we assume y_a and y_b to be real numbers. Again we apply variational discretization [15], but now to problem (4.1). We therefore consider

$$\begin{aligned} J_h^\gamma(y_h, u_h) &:= J_h(y_h, u_h) + \frac{\gamma}{2} \|\min(0, y_h - y_a)\|^2 + \frac{\gamma}{2} \|\max(0, y_h - y_b)\|^2 \rightarrow \min \\ \text{s.t.} \quad y_h &= \mathcal{G}_h(u_h) \quad \text{and} \quad u_h \in U_{ad}. \end{aligned} \tag{4.3}$$

The existence of a solution of (4.3) as well as of Lagrange multipliers again follows from standard arguments. The corresponding first order optimality system associated to (4.3) leads to the variationally discretized counterpart of (4.2)

$$\begin{aligned} a(v_h, p_h^\gamma) &= (v_h, y_h^\gamma - y_d + \mu_{a,h}^\gamma + \mu_{b,h}^\gamma) \quad \forall v_h \in V_h, \\ &\quad \alpha(u_h^\gamma - u_{d,h}) + p_h^\gamma + \lambda_{a,h}^\gamma + \lambda_{b,h}^\gamma = 0, \\ &\quad \lambda_{a,h}^\gamma \leq 0, \quad u_h^\gamma \geq u_a, \quad (\lambda_{a,h}^\gamma, u_h^\gamma - u_a) = 0, \\ &\quad \lambda_{b,h}^\gamma \geq 0, \quad u_h^\gamma \leq u_b, \quad (\lambda_{b,h}^\gamma, u_h^\gamma - u_b) = 0, \end{aligned} \tag{4.4}$$

where $y_h^\gamma, p_h^\gamma \in V_h$ and $u_h^\gamma, \lambda_{a,h}^\gamma, \lambda_{b,h}^\gamma \in L^2(\Omega)$. The multipliers corresponding to the regularized state constraints $\mu_{a,h}^\gamma$ and $\mu_{b,h}^\gamma$ are given by

$$\mu_{a,h}^\gamma = \gamma \min(0, y_h^\gamma - y_a) \quad \text{and} \quad \mu_{b,h}^\gamma = \gamma \max(0, y_h^\gamma - y_b).$$

We mention here that (4.3) is a function space optimization problem and the optimal control u_h^γ is not necessarily an element of the finite element space. However, regarding (4.4), u_h^γ corresponds to the projection of a finite element function onto the admissible set U_{ad} , namely

$$u_h^\gamma = \Pi_{[u_a, u_b]} \left(-\frac{1}{\alpha} p_h^\gamma + u_{d,h} \right),$$

where $\Pi_{[u_a, u_b]}$ is the orthogonal projection onto U_{ad} .

4.3. Error Representation and Estimator

For a fixed regularization parameter γ we now derive an error representation in J for the solutions of (4.1) and (4.3) respectively.

Similar as before we define the following residuals

$$\begin{aligned}\rho^{p^\gamma}(\cdot) &:= J_y(y_h^\gamma, u_h^\gamma)(\cdot) - a(\cdot, p_h^\gamma) + (\mu_h^\gamma, \cdot), \text{ and} \\ \rho^{y^\gamma}(\cdot) &:= -a(y_h^\gamma, \cdot) + (u_h^\gamma + f, \cdot)\end{aligned}$$

with

$$\begin{aligned}\mu^\gamma &:= \gamma \min(0, y^\gamma - y_a) + \gamma \max(0, y^\gamma - y_b), \text{ and} \\ \mu_h^\gamma &:= \gamma \min(0, y_h^\gamma - y_a) + \gamma \max(0, y_h^\gamma - y_b).\end{aligned}$$

The functions μ^γ and μ_h^γ play the role of the Lagrange multipliers μ , μ_h corresponding to state constraints in the limit problem (2.3) (compare with [9, Thm. 4.1, Rem. 4.1]). Furthermore we abbreviate

$$\lambda^\gamma := \lambda_a^\gamma + \lambda_b^\gamma \quad \text{and} \quad \lambda_h^\gamma := \lambda_{a,h}^\gamma + \lambda_{b,h}^\gamma.$$

Theorem 4.1. *Let (u^γ, y^γ) and (u_h^γ, y_h^γ) be the solutions of the optimal control problems (4.1) and (4.3) with corresponding adjoint states p^γ, p_h^γ and multipliers $\lambda^\gamma, \lambda_h^\gamma, \mu^\gamma, \mu_h^\gamma$ associated to control and state constraints respectively. Then*

$$\begin{aligned}2(J(y^\gamma, u^\gamma) - J_h(y_h^\gamma, u_h^\gamma)) &= \\ \rho^{p^\gamma}(y^\gamma - i_h y^\gamma) + \rho^{y^\gamma}(p^\gamma - i_h p^\gamma) + (\mu^\gamma + \mu_h^\gamma, y_h^\gamma - y^\gamma) + (\lambda^\gamma + \lambda_h^\gamma, u_h^\gamma - u^\gamma).\end{aligned}\tag{4.5}$$

For the proof we refer the reader to [10]. Applying integration by parts the error representation (4.5) can be localized as follows,

$$\begin{aligned}2(J(y^\gamma, u^\gamma) - J_h(y_h^\gamma, u_h^\gamma)) &= \sum_{T \in \mathcal{T}_h} (y^\gamma - y_h^\gamma, R_{|T}^{p_h^\gamma})_T - (y^\gamma - y_h^\gamma, r_{|\partial T}^{p_h^\gamma})_{\partial T} \\ &\quad + (R_{|T}^{y_h^\gamma}, p^\gamma - i_h p^\gamma)_T - (r_{|\partial T}^{y_h^\gamma}, p^\gamma - i_h p^\gamma)_{\partial T} \\ &\quad + (y^\gamma - y_h^\gamma, R_{|T}^{p^\gamma})_T - (y^\gamma - y_h^\gamma, r_{|\partial T}^{p^\gamma})_{\partial T} \\ &\quad + (\lambda^\gamma + \lambda_h^\gamma, u_h^\gamma - u^\gamma)_T.\end{aligned}$$

Here the interior and edge residuals $R_{|T}^\bullet, r_{|\partial T}^\bullet$ are similarly defined as in the purely state constrained case. In order to derive a computable estimator we again replace the unknown functions y^γ and p^γ in (4.5) by $i_{2h}^{(2)} y_h^\gamma$ and $i_{2h}^{(2)} p_h^\gamma$. Since $u^\gamma = \Pi_{[u_a, u_b]}(-\frac{1}{\alpha} p^\gamma + u_d)$ holds, a reasonable locally computable approximation then is given by

$$\tilde{u}^\gamma = \Pi_{[u_a, u_b]} \left(-\frac{1}{\alpha} i_{2h}^{(2)} p_h^\gamma + u_d \right),$$

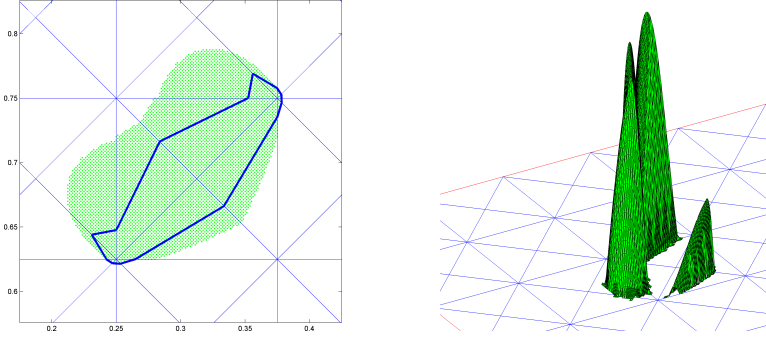


FIGURE 2. u_a active set: blue by u_h^γ , green by \tilde{u}^γ (left), integrand $(\tilde{\lambda}^\gamma + \lambda_h^\gamma)(u_h^\gamma - \tilde{u}^\gamma)$ with support on symmetric difference of active sets (right).

as is already suggested in [22]. Similarly for $\lambda^\gamma = -p^\gamma - \alpha(u^\gamma - u_d)$ we locally compute

$$\tilde{\lambda}^\gamma = -i_{2h}^{(2)} p_h^\gamma - \alpha(\tilde{u}^\gamma - u_d)$$

instead. The estimator η^γ now reads

$$\eta^\gamma = \sum_{T \in \mathcal{T}_h} \eta_T^\gamma,$$

where

$$\begin{aligned} 2\eta_T^\gamma &= (i_{2h}^{(2)} y_h^\gamma - y_h^\gamma, R_{|T}^{p_h^\gamma})_T - (i_{2h}^{(2)} y_h^\gamma - y_h^\gamma, r_{|\partial T}^{p_h^\gamma})_{\partial T} \\ &\quad + (R_{|T}^{y_h^\gamma}, i_{2h}^{(2)} p_h^\gamma - p_h^\gamma)_T - (r_{|\partial T}^{y_h^\gamma}, i_{2h}^{(2)} p_h^\gamma - p_h^\gamma)_{\partial T} \\ &\quad + (i_{2h}^{(2)} y_h^\gamma - y_h^\gamma, R_{|T}^{i_{2h}^{(2)} p_h^\gamma})_T - (i_{2h}^{(2)} y_h^\gamma - y_h^\gamma, r_{|\partial T}^{i_{2h}^{(2)} p_h^\gamma})_{\partial T} \\ &\quad + (\tilde{\lambda}^\gamma + \lambda_h^\gamma, u_h^\gamma - \tilde{u}^\gamma)_T. \end{aligned}$$

While for most quantities in η_T^γ quadrature rules of moderate order are well suited, one has to take care for the last term

$$(\tilde{\lambda}^\gamma + \lambda_h^\gamma, u_h^\gamma - \tilde{u}^\gamma)_T = \int_T (\tilde{\lambda}^\gamma + \lambda_h^\gamma)(u_h^\gamma - \tilde{u}^\gamma). \quad (4.6)$$

The integrand is continuous but has a support within the symmetric difference of the control active set of the variational discrete solution and the locally improved quantities. Such a situation is depicted in Figure 2. One recognizes that \tilde{u}^γ captures the activity structure of u_h^γ but smoothes out the control active boundary towards the exact control active boundary. The kidney-shaped green area resolves the true control active set from the example of Section 5 already very well even on a coarse mesh (compare also Figure 3 (right)). Finally for computing (4.6) we just provide

the integrand and a desired tolerance and apply the adaptive quadrature routine of [23, Algo. 31] for triangles containing the boundary of the control active set.

In order to study the effectivity of our implemented estimator, we define

$$I_{\text{eff}} := \frac{J(y^\gamma, u^\gamma) - J_h(y_h^\gamma, u_h^\gamma)}{\eta^\gamma}.$$

Remark 4.2. The adjoint variable p admits less regularity at state active sets so that higher order interpolation with regard to the adjoint variable is not completely satisfying. However this circumstance only leads to local higher weights in the estimator and therefore reasonably suggests to refine the corresponding regions. The effectivity of the estimator is not affected as we are going to see in the numerical experiment. Another possible technique to derive a computable approximation for $p^\gamma - i_h p^\gamma$ is to substitute p_h^γ for p^γ and compute $p_h^\gamma - p_h^\gamma(x_T)$, where x_T denotes the barycenter of the element T .

5. Numerical Experiment

We consider problem (2.3) with data

$$\begin{aligned} \Omega &= (0, 1)^2, \quad \mathcal{A} = -\Delta + \text{Id}, \quad y_d = \sin(2\pi x_1) \sin(2\pi x_2), \quad f = u_d = 0, \\ u_a &= -30, \quad u_b = 30, \quad y_a = -0.55, \quad y_b = 0.55, \quad \alpha = 10^{-4}. \end{aligned}$$

Its analytic solution is not known, so for obtaining the effectivity index we compute a reference solution on a uniform grid (level $l = 14$, $np = 525313$, $h = 0.00195$) which delivers an approximation of $J(y^\gamma, u^\gamma)$. The numerical solution in terms of $-\frac{1}{\alpha} p_h^\gamma$ as well as the optimal state y_h^γ is displayed in Figure 3 for $\gamma = 10^{14}$ on the mesh $l = 14$. The projection of $-\frac{1}{\alpha} p_h^\gamma$ onto $[u_a, u_b]$ corresponds to the optimal control u_h^γ which together with y_h^γ represents our best approximation to the solution of (4.1). The boundaries of the control active sets are depicted as solid lines, while the state active sets are displayed as star and cross markers. The color blue corresponds to the lower bound while the color red highlights the upper bound. We numerically approximate $J(y^\gamma, u^\gamma)$ by 0.0375586175. In Table 1 we depict the effectivity coefficient and the convergence history of the goal. Notice that the values of the effectivity coefficient are close to 1 which illustrates the good performance of our error estimator. A comparison between our adaptive finite element algorithm and a uniform mesh refinement in terms of number of degrees of freedom $N_{\text{dof}} := np$ is reported in Figure 4. The adaptive refinement process performs well even though the benefit in this example is not big since the characteristic features of the optimal solution already occupy a considerable area of the computational domain, as is illustrated by the adapted grid in Figure 4. Our motivation for including this example is to illustrate the variational discretization effect on the mesh refinement process. If variational discretization for the control would not have been used, also some refinement at the boundary of the control active set would be expected. For the details in terms of our marking strategy,

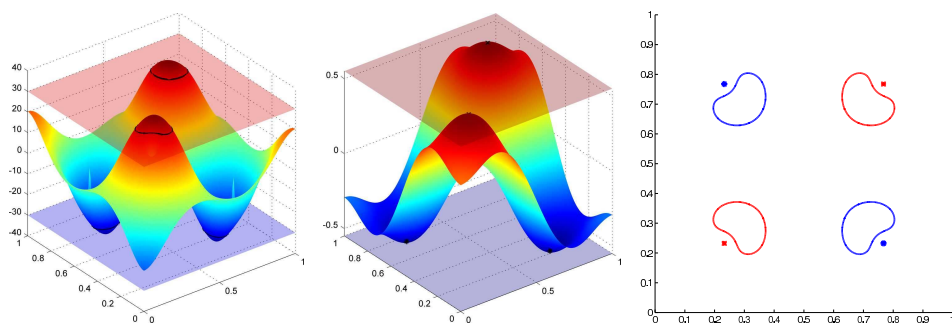


FIGURE 3. $u_a, -\frac{1}{\alpha}p_h^\gamma, u_b$ (left), $y_a \leq y_h^\gamma \leq y_b$ (middle) and active sets (right) for $l = 14$.

k	np	$J(y^\gamma, u^\gamma) - J_h(y_h^\gamma, u_h^\gamma)$	I_{eff}
1	81	$4.275 \cdot 10^{-3}$	1.622
10	3123	$7.148 \cdot 10^{-5}$	1.144
18	63389	$3.996 \cdot 10^{-6}$	1.290

TABLE 1. Adaptive refinement.

stopping criterion of the adaptive finite element algorithm and one more example we refer the reader to [10].

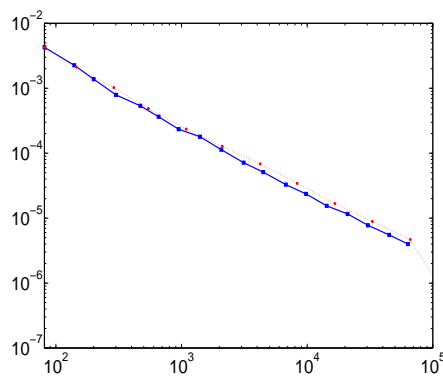
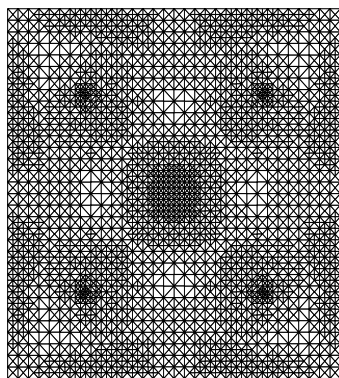


FIGURE 4. Adaptive mesh for $k = 10$ (left), comparison of error decrease in the goal (right).

References

- [1] R. Becker and R. Rannacher, *An optimal control approach to a posteriori error estimation in finite element methods*. Acta Numerica **10** (2001), 1–102.
- [2] O. Benedix and B. Vexler, *A posteriori error estimation and adaptivity for elliptic optimal control problems with state constraints*. Comput Optim Appl, DOI 10.1007/s10589-008-9200-y (2008).
- [3] M. Bergounioux, M. Haddou, M. Hintermüller, and K. Kunisch, *A comparison of a Moreau-Yosida based active strategy and interior point methods for constrained optimal control problems*. SIAM J. Optim. **11** (2000), 495–521.
- [4] M. Bergounioux and K. Kunisch, *On the structure of the Lagrange multiplier for state-constrained optimal control problems*. Syst. Control Lett. **48** (2002), 169–176.
- [5] M. Bergounioux and K. Kunisch, *Primal-dual strategy for state-constrained optimal control problems*. Comput Optim Appl **22**(2002), 193–224.
- [6] E. Casas, *Boundary control of semilinear elliptic equations with pointwise state constraints*. SIAM J. Control Optim. **31**(1993), 993–1006.
- [7] E. Casas, *Control of an elliptic problem with pointwise state constraints*. SIAM J. Control Optim. **24**(1986), 1309–1318.
- [8] K. Deckelnick and M. Hinze, *A finite element approximation to elliptic control problems in the presence of control and state constraints*. Hamburger Beiträge zur Angewandten Mathematik, Universität Hamburg, Preprint No. HBAM2007-01 (2007).
- [9] A. Günther and M. Hinze, *A posteriori error control of a state constrained elliptic control problem*. J. Numer. Math. **16**(2008), 307–322.
- [10] A. Günther and M.H. Tber, *A goal-oriented adaptive Moreau-Yosida algorithm for control- and state-constrained elliptic control problems*. DFG Schwerpunktprogramm 1253, Preprint No. SPP1253-089 (2009).
- [11] M. Hintermüller and R.H.W. Hoppe, *Goal-oriented adaptivity in control constrained optimal control of partial differential equations*. SIAM J. Control Optim. **47**(2008), 1721–1743.
- [12] M. Hintermüller, K. Ito and K. Kunisch, *The primal-dual active set strategy as a semismooth Newton method*. SIAM J. Optim. **13**(2003), 865–888.
- [13] M. Hintermüller and K. Kunisch, *Feasible and noninterior path-following in constrained minimization with low multiplier regularity*. SIAM J. Control Optim. **45**(2006), 1198–1221.
- [14] M. Hintermüller and K. Kunisch, *Pde-constrained optimization subject to pointwise constraints on the control, the state and its derivative*. SIAM J. Optim. **20**(2009), 1133–1156.
- [15] M. Hinze, *A variational discretization concept in control constrained optimization: the linear-quadratic case*. Comput Optim Appl **30**(2005), 45–63.
- [16] R.H.W. Hoppe and M. Kieweg, *Adaptive finite element methods for mixed control-state constrained optimal control problems for elliptic boundary value problems*. Comput Optim Appl, DOI 10.1007/s10589-008-9195-4 (2008).
- [17] C. Meyer, A. Rösch and F. Tröltzsch, *Optimal control of PDEs with regularized pointwise state constraints*. Comput Optim Appl **33**(2006), 209–228.

- [18] A. Schiela, *State constrained optimal control problems with states of low regularity*. SIAM J. Control Optim. **48**(2009), 2407–2432.
- [19] A. Schiela and A. Günther, *Interior point methods in function space for state constraints - inexact Newton and adaptivity*. DFG Schwerpunktprogramm 1253, Preprint No. SPP1253-08-06 (2009).
- [20] A. Shapiro, *On uniqueness of Lagrange multipliers in optimization problems subject to cone constraints*. SIAM J. Optim. **7**(1997), 508–518.
- [21] F. Tröltzsch, *Regular Lagrange multipliers for control problems with mixed pointwise control-state constraints*. SIAM J. Optim. **15**(2005) 616–634.
- [22] B. Vexler and W. Wollner, *Adaptive finite elements for elliptic optimization problems with control constraints*. SIAM J. Control Optim. **47**(2008), 509–534.
- [23] W. Vogt, *Adaptive Verfahren zur numerischen Quadratur und Kubatur*. IfMath TU Ilmenau, Preprint No. M 1/06 (2006).
- [24] M. Weiser, *Interior point methods in function space*. SIAM J. Control Optim. **44**(2005), 1766–1786.
- [25] W. Wollner, *A posteriori error estimates for a finite element discretization of interior point methods for an elliptic optimization problem with state constraints*. Comput Optim Appl, DOI 10.1007/s10589-008-9209-2 (2008).

Acknowledgment

The authors gratefully acknowledge support by the DFG Priority Program 1253 for meetings in Graz and Hamburg through grants DFG06-382, and by the Austrian Ministry of Science and Research and the Austrian Science Fund FWF under START-grant Y305 “Interfaces and Free Boundaries”.

Andreas Günther
Konrad-Zuse-Zentrum für Informationstechnik Berlin (ZIB)
Takustr. 7
D-14195 Berlin-Dahlem
Germany
e-mail: guenther@zib.de

Michael Hinze
University of Hamburg
Bundesstr. 55
D-20146 Hamburg
Germany
e-mail: michael.hinze@uni-hamburg.de

Moulay Hicham Tber
Department of mathematics
Faculty of Science and Technology
B.P. 523 Beni-Mellal
Morocco
e-mail: hicham.tber@gmail.com