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# Optimal control of fluid - structure interaction problem: linear viscous fluid is coupled with elastic body 

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#### Abstract

In the present paper we analyze the optimal control problem governed by a dynamical fluid-solid interaction model. We consider linear fluid and solid models, described by the Stokes equation and linear elasticity equation, respectively. Along the boundary, on which the fluid and the solid interact, we assume the velocity and stress vectors to be continuous. Moreover, we assume the interface boundary to be fixed. For a justification and an analysis of such type of fluid-solid interaction model see [3]. The cost function for the optimal control problem is considered to be of the tracking type.

In the first, theoretical, part of the paper we prove that the optimal control problem, introduced above, admits at least one solution $u \in U_{a d}$. Further, using both, sensitivity and adjoint approach, we derive the first order necessary optimality conditions for the optimal control problem.

The second, practical, part of the paper is devoted to the description of the numerical realization for finding solution of the optimal control problem and numerical examples. Theoretical results from the first part of the paper, in particular, the presentation of the adjoint system, are here exploited.

For the spatial discretization of both, primal and adjoint problems, we apply the Finite Volume Particle Method (FVPM) (see e.g. [4, 9]). During the implementation we also use an extension of this method for incompressible flows, i.e. an adaptation of the FVPM for the solution of the Poisson equation and the pressure-correction algorithm (see [6]). For the time discretization of the Stokes and linear elasticity problem we apply the implicit and the Newmark time stepping schemes, respectively. The solution of the optimal control problem is determined using the Gradient Algorithm.

In the final part of the paper we present some numerical examples for the solution of the optimal control problem using theoretical and numerical techniques introduced above.


Keywords: Optimal control, Fluid-structure interaction, Finite Volume Particle Method

## 1 Introduction

In the present paper we analyze the optimal control problem governed by a dynamical fluid-solid interaction model. To begin with, we consider linear fluid and solid models, described by the Stokes equation and linear elasticity equation, respectively, and derive the first order necessary optimality conditions for a specific form of the objective function.

[^0]Let $\Omega=\Omega_{1} \bigcup \Omega_{2}$ be a bounded set in $\mathbb{R}^{d}, d=2$, with the boundary $\Gamma=$ $\Gamma_{1} \bigcup \Gamma_{2} \bigcup \Gamma_{0} . \Gamma$ is a $C^{\infty}$ - manifold of dimension $(d-1)$. We assume, that $\Omega$ lies locally on one side of $\Gamma$. Fluid occupies the $\Omega_{1}$ domain, with the boundary $\Gamma_{1} \bigcup \Gamma_{0}$. The solid occupies the domain $\Omega_{2}$ with the boundary $\Gamma_{2} \bigcup \Gamma_{0} . \Gamma_{0}$ is the common boundary for $\Omega_{1}$ and $\Omega_{2}$, so that alongside the boundary $\Gamma_{0}$ the fluid and the solid interact. For the better understanding the situation is schematically represented in Figure 1.


Figure 1: Schematic illustration of physical domains for the fluid-structure interaction

The mathematical model for the Stokes problem is given by:

$$
\left.\begin{array}{rl}
\rho_{1} y_{t}+\nabla p-\mu_{1} \nabla \cdot\left(\nabla y+\nabla y^{T}\right) & =\rho_{1} f_{1}+B_{y} u_{y} \text { in }(0, T) \times \Omega_{1}, \\
-\operatorname{div} y & =0 \text { in }(0, T) \times \Omega_{1}, \\
y & =0 \text { on }(0, T) \times \Gamma_{1}, \\
\left.y\right|_{t=0} & =y_{0} \text { in } \Omega_{1} .
\end{array}\right\}
$$

Here $y=\left(y_{1}, y_{2}\right)$ denotes the fluid velocity, $p$ the fluid pressure, $f_{1}$ the given body force per unit mass, $B_{y}$ a given control operator, $u_{y} \in U_{y}$ denotes a control variable of the fluid, $\rho_{1}$ and $\mu_{1}$ the constant fluid density and viscosity, $y_{0}$ the given initial velocity, $T>0$ the terminal time.

The following equations describe the linear elasticity problem:

$$
\begin{align*}
\rho_{2} x_{t t}-\mu_{2} \nabla \cdot\left(\nabla x+\nabla x^{T}\right)-\lambda_{2} \nabla(\nabla \cdot x) & =\rho_{2} f_{2}+B_{x} u_{x} \text { in }(0, T) \times \Omega_{2}, \\
x & =0 \text { on }(0, T) \times \Gamma_{2}, \\
\left.x\right|_{t=0} & =x_{0} \text { in } \Omega_{2}  \tag{LE}\\
\left.x_{t}\right|_{t=0} & =x_{1} \text { in } \Omega_{2} .
\end{align*}
$$

In the model (LE) $x=\left(x_{1}, x_{2}\right)$ denotes the displacement of the solid, $f_{2}$ the given loading force per unit mass, $B_{x}$ a given control operator, $u_{x} \in U_{x}$ denotes a control variable for the solid, $\mu_{2}$ and $\lambda_{2}$ are the Lamé constants, $\rho_{2}$ the constant solid density, $x_{0}$ and $x_{1}$ the given initial data. $U_{y}$ and $U_{x}$ are the Hilbert spaces of controls.

Furthermore, along the boundary $\Gamma_{0}$ on which the fluid and the solid interact we assume the velocity and stress vectors to be continuous. This can be described mathematically by the following interface conditions:

$$
\left.\begin{array}{rlrl}
x_{t} & =y \quad \text { on } \Gamma_{0}, &  \tag{IC}\\
\mu_{2}\left(\nabla x+\nabla x^{T}\right) \cdot n_{2}+\lambda_{2}(\nabla \cdot x) n_{2} & =p n_{1}-\mu_{1}\left(\nabla y+\nabla y^{T}\right) \cdot n_{1} & \text { on } \Gamma_{0} .
\end{array}\right\}
$$

Here, $n_{i}, i=1,2$, are the unit normal vectors to the boundary $\partial \Omega_{i}$ pointing toward he exterior of $\Omega_{i}, i=1,2$. Throughout we assume the interface boundary $\Gamma_{0}$ to be fixed.

In addition to the state equations (SP) and (LE) we consider the cost function $J\left(y, x, u_{y}, u_{x}\right)$. Then, the optimal control problem will be read as:

$$
\left.\begin{array}{l}
\min J\left(y, x, u_{y}, u_{x}\right) \\
\begin{array}{l}
\text { over }
\end{array}\left(y, x, u_{y}, u_{x}\right) \\
\text { subject to }
\end{array}\right\} \quad(\mathrm{SP}),(\mathrm{LE}),(\mathrm{IC}) . \quad \text { (OCP) }
$$

We assume that the objective function $J\left(y, x, u_{y}, u_{x}\right)$ is at least once Frechetdifferentiable, weakly lower semi-continuous and radially unbounded in $u_{y}$ and $u_{x}$. The last means that

$$
J\left(y, x, u_{y}, u_{x}\right) \rightarrow \infty \quad \text { as } \quad\left\|u_{y}\right\|_{U_{y}} \rightarrow \infty \quad \text { or } \quad\left\|u_{x}\right\|_{U_{x}} \rightarrow \infty
$$

In the following we specify the objective as

$$
\begin{equation*}
J=\frac{\alpha_{1}}{2} \int_{0}^{T}\left\{\int_{\Omega_{1}}\left|y-y_{d}\right|^{2} d x+\int_{\Omega_{2}}\left|x-x_{d}\right|^{2} d x\right\} d t+\frac{\alpha_{4 y}}{2}\left\|u_{y}\right\|_{U_{y}}^{2}+\frac{\alpha_{4 x}}{2}\left\|u_{x}\right\|_{U_{x}}^{2} \tag{1}
\end{equation*}
$$

This function is of tracking type and obviously satisfy the required properties. The task is to establish the necessary optimality conditions for the optimal control problem (OCP) with the objective function defined by (1).

To begin with, we introduce the required functional spaces:

$$
\begin{align*}
& V_{y}=\left\{f \in H_{1}^{0}\left(\Omega_{1}\right),\left.\operatorname{div} f\right|_{\Omega_{1}}=0\right\}  \tag{2}\\
& H_{y}=\left\{f \in L^{2}\left(\Omega_{1}\right),\left.\operatorname{div} f\right|_{\Omega_{1}}=0\right\}  \tag{3}\\
& V_{x}=\left\{f \in H_{1}^{0}\left(\Omega_{2}\right)\right\}  \tag{4}\\
& H_{x}=\left\{f \in L^{2}\left(\Omega_{2}\right)\right\},  \tag{5}\\
& W_{y}(0, T)=\left\{f \in L^{2}\left(0, T ; V_{y}\right) ; f^{\prime} \in L^{2}\left(0, T ; V_{y}^{\star}\right)\right\}  \tag{6}\\
& W_{x}(0, T)=\left\{f \in L^{2}\left(0, T ; V_{x}\right) ; f^{\prime} \in L^{2}\left(0, T ; H_{x}\right) ; f^{\prime \prime} \in L^{2}\left(0, T ; V_{x}^{\star}\right)\right\},  \tag{7}\\
& V=\left\{f \in H_{1}^{0}(\Omega),\left.\operatorname{div} f\right|_{\Omega_{1}}=0\right\}  \tag{8}\\
& H=\left\{f \in L^{2}(\Omega),\left.\operatorname{div} f\right|_{\Omega_{1}}=0\right\}  \tag{9}\\
& W= W_{y} \times W_{x},  \tag{10}\\
& U_{y}, U_{x}= \operatorname{Hilbert} \text { spaces of controls, }  \tag{11}\\
& U= U_{y} \times U_{x}  \tag{12}\\
& V \subset H \subset V^{\star}, \quad H^{\star}=H .
\end{align*}
$$

Similar to [3], we introduce continuous bilinear forms
$a_{y}(t, \phi, \psi)=\frac{1}{2} \int_{\Omega_{1}} \mu_{1}\left(\nabla \phi+\nabla \phi^{T}\right):\left(\nabla \psi+\nabla \psi^{T}\right) d \Omega, \forall \phi, \psi \in V_{y}$,
$a_{x}(t, \phi, \psi)=\int_{\Omega_{2}} \frac{\mu_{2}}{2}\left(\nabla \phi+\nabla \phi^{T}\right):\left(\nabla \psi+\nabla \psi^{T}\right)+\lambda_{2}(\nabla \cdot \phi)(\nabla \cdot \psi) d \Omega, \forall \phi, \psi \in\left(V_{4}\right)$
where the relation: denotes $A: B=\operatorname{tr}\left(A^{\top} B\right) . a_{y}(t, \phi, \psi)$ and $a_{x}(t, \phi, \psi)$ are continuous, coercive, bilinear forms on $V_{y}$ and $V_{x}$, respectively. Moreover, they satisfy the following relations:

$$
\begin{align*}
a_{y}(t, \phi, \phi) & \geq K_{y}\|\phi\|_{V_{y}}^{2}, \forall \phi \in V_{y}, \text { if } \operatorname{meas}\left(\Gamma_{1}\right) \neq 0,  \tag{15}\\
a_{x}(t, \phi, \phi) & \geq K_{x}\|\phi\|_{V_{x}}^{2}, \forall \phi \in V_{x}, \text { if } \operatorname{meas}\left(\Gamma_{2}\right) \neq 0  \tag{16}\\
a_{y}(t, \phi, \phi)+(\phi, \phi)_{\Omega_{1}} & \geq K_{y}\|\phi\|_{V_{y}}^{2}, \forall \phi \in V_{y}, \text { if } \operatorname{meas}\left(\Gamma_{1}\right)=0  \tag{17}\\
a_{x}(t, \phi, \phi)+(\phi, \phi)_{\Omega_{2}} & \geq K_{x}\|\phi\|_{V_{x}}^{2}, \forall \phi \in V_{x}, \text { if } \operatorname{meas}\left(\Gamma_{2}\right)=0 . \tag{18}
\end{align*}
$$

(These conditions will be needed to prove the existence of $y$ and $x$, compare [7], e.g. p.105).

Since the forms $a_{y}(t, \phi, \psi)$ and $a_{x}(t, \phi, \psi)$ are linear and continuous, they can be interpreted using the operators $A_{y}(t)$ and $A_{x}(t)$ in the following way:

$$
\begin{array}{ll}
a_{y}(t, \phi, \psi)=\left(A_{y}(t) \phi, \psi\right), & A_{y}(t) \phi \in V_{y}^{\star} \\
a_{x}(t, \phi, \psi)=\left(A_{x}(t) \phi, \psi\right), & A_{x}(t) \phi \in V_{x}^{\star}
\end{array}
$$

This defines linear operators:

$$
A_{y} \in \mathcal{L}\left(V_{y}, V_{y}^{\star}\right), \quad A_{x} \in \mathcal{L}\left(V_{x}, V_{x}^{\star}\right)
$$

The control operators $B_{y}$ and $B_{x}$ are defined as:

$$
\begin{equation*}
B_{y} \in \mathcal{L}\left(U_{y}, L^{2}\left(0, T ; V_{y}^{\star}\right)\right), \quad B_{x} \in \mathcal{L}\left(U_{x}, L^{2}\left(0, T ; H_{x}\right)\right) \tag{h1}
\end{equation*}
$$

Moreover, we assume that the body forces $f_{1}$ and $f_{2}$ and the initial data $y_{0}, x_{0}$ and $x_{1}$ satisfy:

$$
\begin{align*}
& f_{1} \in L^{2}\left(0, T ; V_{y}^{\star}\right), \quad f_{2} \in L^{2}\left(0, T ; H_{x}\right),  \tag{h2}\\
& y_{0} \in H_{y}, \quad x_{0} \in V_{x}, \quad x_{1} \in H_{x}
\end{align*}
$$

Using the introduced linear operators $A_{y}$ and $A_{x}$ the evolution problems (SP) and (LE) can be rewritten in the form:

$$
\begin{align*}
\rho_{1} y_{t}+A_{y}(t) y & =\rho_{1} f_{1}+B_{y} u_{y} \text { in } L^{2}\left(0, T ; V_{y}^{\star}\right)  \tag{SPo}\\
\left.y\right|_{t=0} & =y_{0}, y_{0} \text { given in } H_{y} \\
\rho_{2} x_{t t}+A_{x}(t) x & =\rho_{2} f_{2}+B_{x} u_{x} \text { in } L^{2}\left(0, T ; H_{x}\right)  \tag{LEo}\\
\left.x\right|_{t=0} & =x_{0}, x_{0} \text { given in } V_{x} \\
\left.x_{t}\right|_{t=0} & =x_{1}, x_{1} \text { given in } H_{x}
\end{align*}
$$

Using the results in [3], we can prove:
Theorem 1.1. Assuming the hypotheses (h1) and (h2) be satisfied, the fluidstructure interaction problem (SPo) $+($ LEo $)+(I C)$ admits a unique solution $(y, x) \in W_{y} \times W_{x}$. The mapping

$$
\left\{f_{1}, u_{y}, y_{0}, f_{2}, u_{x}, x_{0}, x_{1}\right\} \rightarrow\left\{y, x, \frac{d x}{d t}\right\}
$$

is a linear continuous map of
$L^{2}\left(0, T ; V_{y}^{\star}\right) \times U_{y} \times H_{y} \times L^{2}\left(0, T ; H_{x}\right) \times U_{x} \times V_{x} \times H_{x} \rightarrow L^{2}\left(0, T ; V_{y}\right) \times L^{2}\left(0, T ; V_{x}\right) \times L^{2}\left(0, T ; H_{x}\right)$.

## 2 Set of inequalities defining the optimal control

Consider the sets of admissible controls $U_{y_{a d}}$ for the fluid and $U_{x_{a d}}$ for the solid to be closed convex subsets of $U_{y}$ and $U_{x}$, respectively. Introduce the abbreviations

$$
U=U_{y} \times U_{x}, \quad U_{a d}=U_{y_{a d}} \times U_{x_{a d}}, \quad u=\left(u_{y}, u_{x}\right) \in U_{y} \times U_{x}
$$

and
$\Lambda_{u}=$ canonical isomorphism of $U$ into $U^{\star}$,
$\Lambda_{y}=$ canonical isomorphism of $V_{y}$ into $V_{y}^{\star}$,
$\Lambda_{x}=$ canonical isomorphism of $V_{x}$ into $V_{x}^{\star}$.
Using the introduced notations and the existence results of Theorem 1.1 we rewrite the (OCP) into the reduced setting as:

$$
\left.\begin{array}{l}
\min \hat{J}(u)=J\left(y(u), x(u), u_{y}, u_{x}\right)  \tag{OCPr}\\
\text { subject to } \quad u=\left(u_{y}, u_{x}\right) \in U_{a d} .
\end{array} \quad \text { over } \quad\left(u_{y}, u_{x}\right)\right\}
$$

Here, $y\left(u_{y}\right)$ and $x\left(u_{x}\right)$ is the unique solution of the problem (SPo) $+(\mathrm{LEo})+(\mathrm{IC})$ corresponding to the controls $u_{y}$ and $u_{x}$. This solution exists due to Theorem 1.1. Now, we are in a position to prove the following theorem:

Theorem 2.1. There exists an element $u \in U_{a d}$ being a solution of the reduced optimal control problem (OCPr).
Proof. Since $\hat{J}(u) \geq 0$ there exists an infimum $j=\inf _{u \in U_{a d}} \hat{J}(u)$. Therefore, there is a minimizing sequence $\left\{u_{n}\right\}_{n=1}^{\infty} \subset U_{a d}$, such that

$$
\hat{J}\left(u_{n}\right) \rightarrow j \quad \text { as } n \rightarrow \infty
$$

Because of the radial unboundness of $\hat{J}(u)$ relative to $u$ and the fact that $\hat{J}\left(u_{n}\right) \rightarrow j$ as $n \rightarrow \infty$, the sequence $\left\{u_{n}\right\}_{n=1}^{\infty}$ is bounded, i.e.

$$
\left\|u_{n}\right\|_{U} \leq \text { const }
$$

Since $U=U_{y} \times U_{x}$ is a Hilbert space, it is reflexive. Therefore, from the sequence $\left\{u_{n}\right\}_{n=1}^{\infty}$ we may extract a weakly convergent subsequence $\left\{u_{n_{k}}\right\}_{k=1}^{\infty}$, such that

$$
u_{n_{k}} \rightharpoonup \bar{u} \quad \text { in } U
$$

Because $U_{a d}$ is closed and convex, it is weakly closed. Therefore, $\bar{u} \in U_{a d}$. Due to the weakly lower semi-continuity of $\hat{J}(u)$ we obtain

$$
\hat{J}(\bar{u}) \leq \lim _{k \rightarrow \infty} \inf \hat{J}\left(u_{n_{k}}\right)=j
$$

Since $j$ is the infimum of all possible controls on $U_{a d}$, we have $\hat{J}(\bar{u})=j$. Thus, $\bar{u} \in U_{a d}$ is the optimal control for (OCPr).

The control $u$ is an optimal control for the (OCPr) and thus also for (OCP) if and only if the following optimality condition is satisfied:

$$
\left\langle\hat{J}^{\prime}(u),(v-u)\right\rangle_{U^{\star}, U} \geq 0, \quad \forall v \in U_{a d}
$$

This means

$$
\left.\begin{array}{l}
\left\langle\hat{J}_{u_{y}}(u),\left(v_{y}-u_{y}\right)\right\rangle_{U_{y}^{\star}, U_{y}} \geq 0, \quad \forall v_{y} \in U_{y_{a d}},  \tag{OC}\\
\left\langle\hat{J}_{u_{x}}(u),\left(v_{x}-u_{x}\right)\right\rangle_{U_{x}^{\star}, U_{x}} \geq 0, \quad \forall v_{x} \in U_{x_{a d}} .
\end{array}\right\}
$$

Calculating the derivatives $\hat{J}_{u_{y}}(u)$ and $\hat{J}_{u_{x}}(u)$ we obtain

$$
\left.\begin{array}{l}
\int_{0}^{T}\left\langle\alpha_{1} \Lambda_{y}\left(y(u)-y_{d}\right), y(v)-y(u)\right\rangle_{V_{y}^{\star}, V_{y}} d t+\alpha_{4 y}\left(u_{y}, v_{y}-u_{y}\right)_{U_{y}} \geq 0, \quad \forall v_{y} \in U_{y_{a d}}, \\
\int_{0}^{T}\left\langle\alpha_{1} \Lambda_{x}\left(x(u)-x_{d}\right), x(v)-x(u)\right\rangle_{V_{x}^{\star}, V_{x}} d t+\alpha_{4 x}\left(u_{x}, v_{x}-u_{x}\right)_{U_{x}} \geq 0, \quad \forall v_{x} \in U_{x_{a d}} .
\end{array}\right\}
$$

We formally introduce the adjoint states $\bar{y}(u)$ and $\bar{x}(u)$ for the fluid and solid by:

$$
\left.\begin{array}{rlrl}
-\bar{y}_{t}+A_{y}^{\star}(t) \bar{y} & =\alpha_{1} \Lambda_{y}\left(y(u)-y_{d}\right) & & \text { in } \Omega_{1} \times(0, T), \\
\bar{y}(T, u) & =0, & & \text { in } \Omega_{1} . \tag{ALE}
\end{array}\right\}
$$

If we impose a stronger hypotheses on $\Lambda_{x}\left(x(u)-x_{d}\right)$, namely:

$$
\Lambda_{x}\left(x(u)-x_{d}\right) \in L^{2}\left(0, T ; H_{x}\right), \quad\left(\text { instead of } L^{2}\left(0, T ; V_{x}^{\star}\right)\right)
$$

and reverse the flow of time (change $t$ to $T-t$ ), then by applying Theorem1.1 the adjoint problem (ASP) + (ALE) admits a unique solution $\left\{\bar{y}, \bar{x}, \frac{d \bar{x}}{d t}\right\} \in L^{2}\left(0, T ; V_{y}\right) \times$ $L^{2}\left(0, T ; V_{x}\right) \times L^{2}\left(0, T ; H_{x}\right)$.

To proceed further we multiply the both sides of (ASP) and (ALE) by $(y(v)$ $y(u))$ and $(x(v)-x(u))$, respectively. We integrate the obtained products over the time interval $[0, T]$. Noting that

$$
\begin{gathered}
\int_{0}^{T}\left(-\frac{d}{d t} \bar{y}(u), y(v)-y(u)\right) d t=\int_{0}^{T}\left(\bar{y}(u), \frac{d}{d t} y(v)-\frac{d}{d t} y(u)\right) d t \\
\int_{0}^{T}\left(A_{y}^{\star}(t) \bar{y}(u), y(v)-y(u)\right) d t=\int_{0}^{T}\left(\bar{y}(u), A_{y}(t) y(v)-A_{y}(t) y(u)\right) d t
\end{gathered}
$$

and
$\int_{0}^{T}\left(\phi^{\prime \prime}(t), \psi(t)\right) d t=\left(\phi^{\prime}(T), \psi(T)\right)-\left(\phi^{\prime}(0), \psi(0)\right)-\left(\phi(T), \psi^{\prime}(T)\right)+\left(\phi(0), \psi^{\prime}(0)\right)+\int_{0}^{T}\left(\phi(t), \psi^{\prime \prime}(t)\right) d t$.
The last equality is valid for e.g. $\phi, \psi \in L^{2}\left(0, T ; V_{x}\right) ; \phi^{\prime}, \psi^{\prime} \in L^{2}\left(0, T ; H_{x}\right) ; \phi^{\prime \prime}, \psi^{\prime \prime} \in$ $L^{2}\left(0, T ; V_{x}^{\star}\right)$. Using these three relations we obtain for the (ASP)
$\int_{0}^{T} \alpha_{1}\left(\Lambda_{y}\left(y(u)-y_{d}\right), y(v)-y(u)\right)_{V_{y}^{\star}, V_{y}} d t=\int_{0}^{T}\left(\bar{y}(u),\left(\frac{d}{d t}+A_{y}(t)\right)(y(v)-y(u))\right) d t=$ $=\int_{0}^{T}\left(\bar{y}(u), B_{y} v_{y}-B_{y} u_{y}\right) d t=\left(B_{y}^{\star} \bar{y}(u), v_{y}-u_{y}\right)_{U_{y}^{\star}, U_{y}}=\left(\Lambda_{U_{y}}^{-1} B_{y}^{\star} \bar{y}(u), v_{y}-u_{y}\right)_{U_{y}}$.

For the adjoint problem (ALE) the reformulations yield:

$$
\begin{aligned}
& \int_{0}^{T} \alpha_{1}\left(\Lambda_{x}\left(x(u)-x_{d}\right), x(v)-x(u)\right)_{V_{x}^{\star}, V_{x}} d t=\int_{0}^{T}\left(\bar{x}(u),\left(\frac{d^{2}}{d t^{2}}+A_{x}(t)\right)(x(v)-x(u))\right) d t= \\
& =\int_{0}^{T}\left(\bar{x}(u), B_{x} v_{x}-B_{x} u_{x}\right) d t=\left(B_{x}^{\star} \bar{x}(u), v_{x}-u_{x}\right)_{U_{x}^{\star}, U_{x}}=\left(\Lambda_{U_{x}}^{-1} B_{x}^{\star} \bar{x}(u), v_{x}-u_{x}\right)_{U_{x}} .
\end{aligned}
$$

Therefore, the optimality conditions ( $\mathrm{OC}^{\prime}$ ) can be rewritten as

$$
\left.\begin{array}{l}
\left(\Lambda_{U_{y}}^{-1} B_{y}^{\star} \bar{y}(u)+\alpha_{4 y} u_{y}, v_{y}-u_{y}\right)_{U_{y}} \geq 0, \quad \forall v_{y} \in U_{y_{a d}}  \tag{OC"}\\
\left(\Lambda_{U_{x}}^{-1} B_{x}^{\star} \bar{x}(u)+\alpha_{4 x} u_{x}, v_{x}-u_{x}\right)_{U_{x}} \geq 0, \quad \forall v_{x} \in U_{x_{a d}}
\end{array}\right\}
$$

At this point we are in a position to formulate the following theorem:
Theorem 2.2. First order necessary optimality conditions. The optimal control $u$ for the (OCP) can be characterized by the following system of equations and inequalities:

$$
\begin{gathered}
(S P o)+(L E o)+(A S P)+(A L E)+\left(O C^{\prime \prime}\right) \text { with } \\
y(u), \bar{y}(u) \in L^{2}\left(0, T ; V_{y}\right) ; \\
x(u), \bar{x}(u) \in L^{2}\left(0, T ; V_{x}\right) ; \quad x^{\prime}(u), \bar{x}^{\prime}(u) \in L^{2}\left(0, T ; H_{x}\right) .
\end{gathered}
$$

The well-posed adjoint boundary value problem, i.e. with boundary and interface conditions, is given by:

$$
\left.\begin{array}{rl}
-\rho_{1} \bar{y}_{t}+\nabla \bar{p}-\mu_{1} \nabla \cdot\left(\nabla \bar{y}+\nabla \bar{y}^{T}\right) & =\alpha_{1} \Lambda_{y}\left(y(u)-y_{d}\right) \text { in }(0, T) \times \Omega_{1}, \\
-\operatorname{div} \bar{y} & =0 \text { in }(0, T) \times \Omega_{1}, \\
\bar{y}(u) & =0 \text { on }(0, T) \times \Gamma_{1}, \\
\bar{y}(T, u) & =0 \text { in } \Omega_{1} .
\end{array}\right\}
$$

$$
\left.\left.\begin{array}{rl}
\rho_{2} \bar{x}_{t t}-\mu_{2} \nabla \cdot\left(\nabla \bar{x}+\nabla \bar{x}^{T}\right)-\lambda_{2} \nabla(\nabla \cdot \bar{x}) & =\alpha_{1} \Lambda_{x}\left(x(u)-x_{d}\right) \text { in }(0, T) \times \Omega_{2},  \tag{ALE'}\\
\bar{x} & =0 \text { on }(0, T) \times \Gamma_{2}, \\
\bar{x}(T, u) & =0 \text { in } \Omega_{2}, \\
\bar{x}_{t}(T, u) & =0 \text { in } \Omega_{2} . \\
\bar{p} n_{1}-\mu_{1}\left(\nabla \bar{y}+\nabla \bar{y}^{T}\right) \cdot n_{1} & =-\bar{y} \quad \text { on }(0, T) \times \Gamma_{0}, \\
\mu_{2}\left(\nabla \bar{x}+\nabla \bar{x}^{T}\right) \cdot n_{2}+\lambda_{2}(\nabla \cdot \bar{x}) n_{2} & =\bar{x}_{t} \quad \text { on }(0, T) \times \Gamma_{0} .
\end{array}\right\} \quad(\text { AIC })\right\}
$$

## 3 Numerical realization

The discretization of the fluid-solid interaction problem (SP) $+(\mathrm{LE})+(\mathrm{IC})$ is described firstly in this section. Then, an optimization algorithm is presented to solve the (OCP). A description for its numerical realization follows after that. Finally, a numerical example is presented.

### 3.1 Discrete solution of the fluid-solid interaction problem

At the beginning we concentrate on the fluid modeled by the Stokes problem. For convenience, we rewrite its mathematical formulation once more:

$$
\left.\begin{array}{rlrl}
\rho_{1} y_{t}+\nabla p-\mu_{1} \nabla \cdot\left(\nabla y+\nabla y^{T}\right) & =\rho_{1} f_{1}+B_{y} u_{y} \text { in }(0, T) \times \Omega_{1}, & \\
-\operatorname{div} y & =0 \text { in }(0, T) \times \Omega_{1}, & \\
y & =0 \text { on }(0, T) \times \Gamma_{1}, &  \tag{3}\\
\left.y\right|_{t=0} & =y_{0} \text { in } \Omega_{1} . & \\
x_{t} & =y \quad \text { on } \Gamma_{0}, & \\
\mu_{2}\left(\nabla x+\nabla x^{T}\right) \cdot n_{2}+\lambda_{2}(\nabla \cdot x) n_{2} & =p n_{1}-\mu_{1}\left(\nabla y+\nabla y^{T}\right) \cdot n_{1} & \text { on } \Gamma_{0} .
\end{array}\right\}
$$

After division of the time interval $(0, T)$ into $n$ equidistant parts, a single time step become the length $\Delta t$. The time discretization of the Stokes problem (SP) is of semi-implicit type, i.e. the velocity in the mass conservation equation as well as in the diffusive part of the momentum equation is considered in the new time point. The same is true for the pressure contribution in the momentum equation. The r.h.s. of this equation is considered in contrast at the old time point. Particularly, at each time step we solve the following Quasi-Stokes problem:

$$
\left.\begin{array}{rl}
\rho_{1} \frac{y^{i+1}-y^{i}}{\Delta t}+\nabla p^{i+1}-\mu_{1} \nabla \cdot\left(\nabla y^{i+1}+\nabla\left(y^{i+1}\right)^{T}\right) & =\rho_{1} f_{1}^{i}+B_{y} u_{y}^{i}  \tag{DSP}\\
& -\operatorname{div} y^{i+1}
\end{array}\right\}
$$

For the solution of the discrete problem (DSP) a Schurcomplement-method is applied, which is realized using the preconditioned CG-Algorithm (for details see $[5,8])$. Details of the implementation: For solving the Quasi-Stokes problem we use the $\mathrm{BC}(1)+(2)$.

The spatial discretization of the Stokes problem (SP) is realized by the Finite Volume Particle Method (FVPM). Details of this method for conservation laws see e.g. in $[4,9]$. An extension of this method for incompressible flows, i.e. an adaptation of the FVPM for the solution of the Poisson equation and the pressurecorrection algorithm see [6]. Also, we give some details how this method should be applied for the PCG-Algorithm later for the concrete numerical example.

In the following we consider the solid which motion is modeled by the linear elasticity problem:

$$
\begin{align*}
\rho_{2} x_{t t}-\mu_{2} \nabla \cdot\left(\nabla x+\nabla x^{T}\right)-\lambda_{2} \nabla(\nabla \cdot x) & =\rho_{2} f_{2}+B_{x} u_{x} \text { in }(0, T) \times \Omega_{2}, \\
x & =0 \text { on }(0, T) \times \Gamma_{2},  \tag{LE}\\
\left.x\right|_{t=0} & =x_{0} \text { in } \Omega_{2}, \\
\left.x_{t}\right|_{t=0} & =x_{1} \text { in } \Omega_{2} .
\end{align*}
$$

$$
\left.\begin{array}{rlrl}
x_{t} & =y \quad \text { on } \Gamma_{0}, &  \tag{IC}\\
\mu_{2}\left(\nabla x+\nabla x^{T}\right) \cdot n_{2}+\lambda_{2}(\nabla \cdot x) n_{2} & =p n_{1}-\mu_{1}\left(\nabla y+\nabla y^{T}\right) \cdot n_{1} & \text { on } \Gamma_{0} .
\end{array}\right\}
$$

For its time discretization we apply the "Newmark" time integration scheme (follow description in the book [2]). The crucial idea is to use the Taylor expansion for the displacement $x$ and its velocity $x_{t}$ at the new time point $(t+\Delta t)$.

$$
\begin{aligned}
x\left(t+\Delta_{t}\right) & \approx x(t)+\Delta t x_{t}(t)+\frac{\Delta t^{2}}{2}\left[\left(1-\beta_{2}\right) x_{t t}(t)+\beta_{2} x_{t t}(t+\Delta t)\right] \\
x_{t}\left(t+\Delta_{t}\right) & \approx x_{t}(t)+\Delta t\left[\left(1-\beta_{1}\right) x_{t t}(t)+\beta_{1} x_{t t}(t+\Delta t)\right]
\end{aligned}
$$

where $\beta_{1}$ and $\beta_{2}$ are appropriate parameters, which monitor the implicity or the explicity of the integration scheme. In the following numerical computations we use parameters $\beta_{1}=1 / 2$ and $\beta_{2}=1 / 2$, which produce an unconditionally stable time stepping scheme. For simplicity of notations we rewrite the system (LE) in an operator setting form as:

$$
\rho_{2} x_{t t}+A_{x}(t) x=\rho_{2} f_{2}+B_{x} u_{x}
$$

with a linear continuous operator $A_{x} \in \mathcal{L}\left(V_{x}, V_{x}^{\star}\right)$ introduced in Section 1. Given the initial values for the displacement $x(0)$ and its velocity $x_{t}(0)$ at the time point $t=0$, we obtain the acceleration $x_{t t}(0)$ from the equation:

$$
x_{t t}(0)=-\frac{1}{\rho_{2}} A_{x}(0) x(0)+f_{2}+\frac{1}{\rho_{2}} B_{x} u_{x}
$$

For the successive time steps we calculate $x_{t t}(t+\Delta t)$ from the equation:

$$
\left.\begin{array}{l}
\rho_{2} x_{t t}(t+\Delta t)+A_{x}(t+\Delta t) x(t+\Delta t)= \\
\rho_{2} x_{t t}(t+\Delta t)+A_{x}(t+\Delta t)\left[x(t)+\Delta t x_{t}(t)+\frac{\Delta t^{2}}{2}\left[\left(1-\beta_{2}\right) x_{t t}(t)+\beta_{2} x_{t t}(t+\Delta t)\right]\right]= \\
{\left[\rho_{2}+\frac{\Delta t^{2}}{2} \beta_{2} A_{x}(t+\Delta t)\right] x_{t t}(t+\Delta t)+A_{x}(t+\Delta t)\left[x(t)+\Delta t x_{t}(t)+\frac{\Delta t^{2}}{2}\left(1-\beta_{2}\right) x_{t t}(t)\right]=} \\
\rho_{2} f_{2}+B_{x} u_{x}
\end{array}\right\}
$$

and substitute its value if required in:

$$
\begin{aligned}
x(t+\Delta t) & \approx x(t)+\Delta t x_{t}(t)+\frac{\Delta t^{2}}{2}\left[\left(1-\beta_{2}\right) x_{t t}(t)+\beta_{2} x_{t t}(t+\Delta t)\right] \\
x_{t}(t+\Delta t) & \approx x_{t}(t)+\Delta t\left[\left(1-\beta_{1}\right) x_{t t}(t)+\beta_{1} x_{t t}(t+\Delta t)\right]
\end{aligned}
$$

The interface boundary condition on $\Gamma_{0}$ are considered once in $(\star)$ by the application of $A_{x}(t+\Delta t)$ in the second addend as a flux contribution. The space discretization is implemented, similar to the fluid case, using the FVPM.

Now follows the discretization of the adjoint Stokes system (ASP'). Following the argumentation in [1] we apply the next scheme for the time discretization:

$$
\left.\begin{array}{rl}
\rho_{1} \frac{\bar{y}^{i-1}-\bar{y}^{i}}{\Delta t}+\nabla \bar{p}^{i-1}-\mu_{1} \nabla \cdot\left(\nabla \bar{y}^{i-1}+\nabla\left(\bar{y}^{i-1}\right)^{T}\right) & =\alpha_{1} \Lambda_{y}\left(y^{i-1}-y_{d}^{i-1}\right),  \tag{DASP}\\
& -\operatorname{div} \bar{y}^{i-1}
\end{array}\right\}
$$

Starting with the final time $t=T$ we apply the discretization scheme (DASP) to obtain values for $\bar{y}$ and $\bar{p}$ for the preceeding time point $(t-\Delta t)$, which corresponds to the notation $(i-1)$ in (DASP). For their computation we apply also the preconditioned CG-algorithm with the schurcomplement-method (see above).

For the time discretization of the adjoint linear elasticity problem we apply the Newmark scheme described above. The difference is only in the starting time point, $t=T$ and $\Delta t$ is replaced by $-\Delta t$ in the computations concerning the adjoint problem. Spatial discretization for both adjoint problems is provided using FVPM.

Next, we describe the optimization strategy to solve the optimal control problem (OCP). For this purpose we apply the following gradient algorithm:

## Algorithm 3.1. Optimization algorithm (Gradient algorithm)

$k=0, u_{0}$ is given.
S0. If $\left\|\nabla \hat{J}\left(u^{k}\right)\right\|_{2} \leq$ tol Stop, else:
S1. Compute $y_{k}, p_{k}, x_{k}, \dot{x}_{k}$ for given $u_{k}$ using state equations.
S2. Compute $\bar{y}_{k}, \bar{p}_{k}, \bar{x}_{k}, \dot{\bar{x}}_{k}$ using adjoint equations.
S3. Compute the descent direction $v_{k}$ as an antigradient

$$
v_{k}=-\hat{J}\left(u^{k}\right) .
$$

S4. Step length computation:

$$
s_{k}=\operatorname{argmin}_{s>0} \hat{J}\left(u_{k}+s v_{k}\right) .
$$

S5. Set $u_{k+1}=u_{k}+s_{k} v_{k}, k=k+1$, goto $S 0$.
Details for application of Algorithm 3.1 will be described in the next subsection, where we consider a concrete numerical example.

### 3.2 Example

As a numerical example we consider the following two domains $\Omega_{1} \bigcup \Omega_{2}=\Omega$. For better understanding the situation is illustrated in Figure 2. The fluid occupies


Figure 2: Schematic illustration of physical domains for the numerical example
the domain $\Omega_{1}$ with the boundary $\Gamma_{0} \bigcup \Gamma_{1}$, the solid occupies the domain $\Omega_{2}$ with the boundary $\Gamma_{0} \bigcup \Gamma_{2} \bigcup \Gamma_{2}^{\prime} . \Gamma_{0}$ is the common boundary for the fluid and for the solid. Each domain is a quadrat with the side length one. We consider $U_{a d}=U$, which simplifies the calculation of the descent direction $v_{k}$ in the step S 3 of the Optimization Algorithm 3.1.

For the realization of the spatial discretization with FVPM $N \times N$ particles $\psi_{I}(x, t), I=(i, j), i, j=1, . ., N$ are equidistantly positioned within each domain, i.e. into $\Omega_{1}$ and $\Omega_{2}$. Theoretically, particles can move with their own velocities, but since the interface $\Gamma_{0}$ is fixed due to the assumptions, velocities of particles are zero. The flux function $\mathcal{F}(y, t)$ for the fluid is

$$
\mathcal{F}(y, t)=-\frac{1}{\rho_{1}}\left(\begin{array}{r|r}
2 \mu_{1} \frac{\partial y_{x}}{\partial x}-p & \mu_{1}\left(\frac{\partial y_{x}}{\partial y}+\frac{\partial y_{y}}{\partial x}\right) \\
\hline \mu_{1}\left(\frac{\partial y_{x}}{\partial y}+\frac{\partial y_{y}}{\partial x}\right) & 2 \mu_{1} \frac{\partial y_{y}}{\partial y}-p
\end{array}\right) .
$$



Figure 3: Displacement field for the desired solid and the velocity field for the desired fluid for $t=0.25, t=0.5, t=0.75, t=1.0$.

The flux function $\mathcal{F}(x, t)$ for the solid is

$$
\mathcal{F}(x, t)=-\frac{1}{\rho_{2}}\left(\begin{array}{l|l}
2 \mu_{2} \frac{\partial x_{x}}{\partial x}+\lambda_{2}\left(\frac{\partial x_{x}}{\partial x}+\frac{\partial x_{y}}{\partial y}\right) & \mu_{2}\left(\frac{\partial x_{x}}{\partial y}+\frac{\partial x_{y}}{\partial x}\right) \\
\hline \mu_{2}\left(\frac{\partial x_{x}}{\partial y}+\frac{\partial x_{y}}{\partial x}\right) & 2 \mu_{2} \frac{\partial x_{y}}{\partial y}+\lambda_{2}\left(\frac{\partial x_{x}}{\partial x}+\frac{\partial x_{y}}{\partial y}\right)
\end{array}\right)
$$

In the two formulas above the fluid velocity and solid displacement vectors $y$ and $x$ are defined as $y=\binom{y_{x}}{y_{y}}$ and $x=\binom{x_{x}}{x_{y}}$. Note that the derivatives $\frac{\partial \cdot}{\partial x}$ and $\frac{\partial}{\partial y}$ within the flux functions are evaluated at the consistency points $x_{i j}$ in order to obtain a consistent approximation (for details see [6]). This approach is similar to the staggered grid approach.

Boundary conditions on $\Gamma_{2}^{\prime}$ for this numerical example slightly differ from the BC assumed so far. For the sake of stability in this example we consider:

$$
\left.\mu_{2}\left(\nabla x+\nabla x^{T}\right) \cdot n_{2}+\lambda_{2}(\nabla \cdot x) n_{2}=0 \quad \text { on } \Gamma_{2}^{\prime} \cdot\right\}
$$

Parameters $\alpha_{1}, \alpha_{4 x}$ and $\alpha_{4 y}$ are set to one, one and 0.01 , respectively. Final time $T=1$. The desired state is evaluated with controls $u_{x}$ and $u_{y}$ are set to zero and 0.1 , respectively, for all $t \in[0, T]$. The parameter $N$ for the spatial discretization is set to $N=10$. The time step interval is $\Delta t=0.25$. Initial state for the solid displacement and velocity is $x(0)=x_{t}(0)=0$ in $\Omega_{2}$. Initial fluid velocity is

$$
y(x, 0)=e\left[\begin{array}{c}
\left(\cos 2 \pi x_{1}-1\right) \sin 2 \pi x_{2} \\
-\left(\cos 2 \pi x_{2}-1\right) \sin 2 \pi x_{1}
\end{array}\right] \text { in } \Omega_{1}
$$

In Figure 3 the displacement field for the desired solid and the velocity field for the desired fluid are illustrated for several time points. For the solution of the (OCP) with the parameters described above, we apply the Gradient Algorithm 3.1 from the previous section. Starting control variables are $u_{x}(t)=1.0, u_{y}(t)=0$ for all $t \in[0, T]$. Velocity and displacement fields produced with these controls and serving as an initial state for the optimization are illustrated in Figure 4. The convergence behavior of this numerical test is listed in Table 1.


Figure 4: Displacement field for the initial solid and the velocity field for the initial fluid for $t=0.25, t=0.5, t=0.75, t=1.0$.

| Iteration number | Step length | $J\left(x, y, u_{x}, u_{y}\right)$ | $\left\\|\nabla \hat{J}\left(u_{x}, u_{y}\right)\right\\|_{0}$ |
| :---: | :---: | :---: | :---: |
| 0 |  | 0.168569 |  |
| 1 | 1.00 | 0.158357 | 0.020238 |
| 2 | 1.00 | 0.158202 | 0.000175 |
| 3 | 1.00 | 0.158088 | 0.000129 |
| 4 | 1.00 | 0.158004 | 0.000095 |

Table 1: Convergence behavior of the Algorithm 3.1 for the numerical example


Figure 5: Displacement field for the optimized solid and the velocity field for the optimized fluid for $t=0.25, t=0.5, t=0.75, t=1.0$.

Figure 5 illustrates displacement and velocity fields for the solid and for the fluid, respectively, obtained after the optimization for different time points.

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