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# SPACE-TIME FINITE ELEMENT APPROXIMATION OF PARABOLIC OPTIMAL CONTROL PROBLEMS 

WEI GONG * , MICHAEL HINZE ${ }^{\dagger}$, AND ZHAOJIE ZHOU ${ }^{\diamond}$


#### Abstract

In this paper we investigate a space-time finite element approximation of parabolic optimal control problems. The first order optimality conditions are transformed into an elliptic equation of fourth order in space and second order in time involving only the state or the adjoint state in the space-time domain. We derive a priori and a posteriori error estimates for the time discretization of the state and the adjoint state. Furthermore, we also propose a space-time mixed finite element discretization scheme to approximate the space-time elliptic equations, and derive a priori error estimates for the state and the adjoint state. Numerical examples are presented to illustrate our theoretical findings and the performance of our approach.


Key words. Parabolic optimal control problem, space-time finite elements, mixed finite elements, a priori error estimates, a posteriori error estimates.

Subject Classification: 65N15, 65N30.

## 1. Introduction

Optimal control problems governed by time-dependent partial differential equations play an important role in many practical applications. Their numerical approximation forms a hot research topic and there exist lots of contributions to error analysis and numerical algorithms for timedependent optimal control problems. For recent works on this topic we refer to, e.g., [16, 20, 21, $22,23]$ and the references cited therein. In the present work we build the numerical analysis on reformulations of the optimality conditions as fourth order in space and second order in time elliptic boundary value problems for the state and the adjoint state which are valid under natural regularity assumptions on the data. This approach has recently been used in [25] to tackle parabolic optimal control problems numerically, and was motivated in e.g. [4], where a multi-grid method in the spirit of Hackbusch $[7,8]$ is proposed to solve parabolic optimal control problems. For a detailed discussion of multigrid methods in the context of optimization problems with PDE constraints we refer the reader to [2]. Multigrid methods are also applied to the numerical solution of optimal control problems with nonlinear PDE systems, see e.g. [15] for their application in flow control.

In this paper we present a discrete formulation of parabolic optimal control problems based on the reformulation of their optimality conditions as second order in time and fourth order in space elliptic boundary value problems. Since time and space have different physical meanings we separate temporal and spatial discretization and put our focus on a priori and a posteriori error analysis for the temporal discretization. In the a posteriori error analysis part we construct residual based error estimators for the time discretization and keep the space variable continuous. The key idea here consists in applying residual-based a posteriori error estimation techniques for two-point boundary value problems to the a posteriori error estimation of space-time elliptic boundary value problems for the state and the adjoint state. Furthermore, we prove a priori error estimates for

[^0]temporal semi-discretization with piecewise linear, continuous finite elements, and propose a spacetime mixed finite element method to approximate the state and the adjoint state, for which we also prove a priori error estimates. Finally, numerical examples are presented to illustrate the theoretical findings. We note that in [13] an a posteriori space-time finite element approach is presented which also is based on a reformulation of the optimality system as second order in time and fourth order in space elliptic equation.

The outline of this paper is as follows. In section 2 we present the first order optimality conditions for parabolic optimal control problems and derive the space-time domain elliptic systems for the state and the adjoint state. We prove existence and uniqueness of solutions to the space-time elliptic boundary value problems. In section 3 we derive a priori and a posteriori error estimates for the time discretization scheme. Section 4 is devoted to the numerical analysis of the space-time mixed finite element approximations of the state and the adjoint state. Numerical examples are presented in section 5 to illustrate our analytical findings.

Let $\Omega \subset \mathbb{R}^{n}(1 \leqslant n \leqslant 3)$ be an open bounded domain with sufficiently smooth boundary $\Gamma:=\partial \Omega, \Omega_{T}=\Omega \times(0, T], \Sigma_{T}=\Gamma \times(0, T]$. Throughout this paper we denote by $H^{m}(\Omega)$ and $H^{m}\left(\Omega_{T}\right)$ the usual Sobolev space on $\Omega$ and $\Omega_{T}$ of integer order $m \geqslant 0$ with norm $\|\cdot\|_{m, \Omega}$ and $\|\cdot\|_{m, \Omega_{T}}$, respectively. $H^{m}(\Gamma)$ and $H^{m}\left(\Sigma_{T}\right)$ are defined accordingly. For $m=0$ we have $H^{0}(\Omega)=L^{2}(\Omega), H^{0}\left(\Omega_{T}\right)=L^{2}\left(\Omega_{T}\right), H^{0}(\Gamma)=L^{2}(\Gamma)$, and $H^{0}\left(\Sigma_{T}\right)=L^{2}\left(\Sigma_{T}\right)$. For the analysis we need $H^{2,1}\left(\Omega_{T}\right)=L^{2}\left(0, T ; H^{2}(\Omega) \cap H_{0}^{1}(\Omega)\right) \cap H^{1}\left(0, T ; L^{2}(\Omega)\right)$ equipped with the norm

$$
\|w\|_{2,1, \Omega_{T}}=\left(\|w\|_{L^{2}\left(0, T ; H^{2}(\Omega)\right)}^{2}+\|w\|_{H^{1}\left(0, T ; L^{2}(\Omega)\right)}^{2}\right)^{\frac{1}{2}}
$$

For the state space we take $X \equiv W(0, T)=\left\{v \in L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right) ; \frac{\partial v}{\partial t} \in L^{2}\left(0, T ; H^{-1}(\Omega)\right)\right\}$. For the control space we take $U:=L^{2}\left(\Omega_{T}\right)$. Throughout the paper $c$ and $C$ denote generic positive constants.

## 2. Optimal control problem

In this paper we consider the optimal control problem

$$
\begin{equation*}
\min _{(y, u) \in X \times U} J(y, u)=\frac{1}{2}\left\|y-y_{d}\right\|_{0, \Omega_{T}}^{2}+\frac{\alpha}{2}\|u\|_{0, \Omega_{T}}^{2} \tag{2.1}
\end{equation*}
$$

subject to

$$
\left\{\begin{align*}
\frac{\partial y}{\partial t}-\Delta y=u & \text { in } \Omega_{T}  \tag{2.2}\\
y=0 & \text { on } \Sigma_{T} \\
y(0)=y_{0} & \text { in } \Omega
\end{align*}\right.
$$

where $\alpha>0, y_{0} \in H_{0}^{1}(\Omega), y_{d} \in L^{2}\left(\Omega_{T}\right)$ and $T>0$ are fixed. The analysis of this optimal control problem is well understood. In e.g. [18] among other things the following theorem is proven.

Theorem 2.1. The control problem (2.1)-(2.2) admits a unique solution ( $y, u) \in X \times U$. The pair $(y, u)$ is the solution of (2.1)-(2.2) if and only if there exists a unique adjoint state $p \in X$ such that the triplet $(y, p, u)$ satisfies the optimality system

$$
\begin{array}{rlrlrl}
\frac{\partial y}{\partial t}-\Delta y & =u & & \text { in } \Omega_{T}, y=0 & \text { on } \Sigma_{T}, y(0)=y_{0} & \\
\text { in } \Omega \\
-\frac{\partial p}{\partial t}-\Delta p & =y-y_{d} & & \text { in } \Omega_{T}, p=0 & \text { on } \Sigma_{T}, p(T)=0 &  \tag{2.5}\\
\text { in } \Omega \\
\alpha u+p & =0, & & \text { in } \Omega_{T} & &
\end{array}
$$

For the state $y$ and adjoint state $p$ we have the following regularity results.
Lemma 2.2. If $y_{0} \in H_{0}^{1}(\Omega)$ and $y_{d} \in L^{2}\left(\Omega_{T}\right)$, we according to [10] have

$$
y, p \in H_{2}^{2,1}\left(\Omega_{T}\right)
$$

Furthermore, if in addition $y_{0} \in H^{3}(\Omega)$ and the following compatibility conditions

$$
g_{0}:=y_{0} \in H_{0}^{1}(\Omega), g_{1}:=u(0)+\Delta g_{0} \in H_{0}^{1}(\Omega)
$$

hold, we by [10] also have

$$
\text { (a) } \quad y \in L^{2}\left(0, T ; H^{4}(\Omega)\right) \cap H^{2}\left(0, T ; L^{2}(\Omega)\right) \cap H^{1}\left(0, T ; H^{2}(\Omega)\right) \text {. }
$$

Similarly, if we suppose that $y_{d} \in H^{2,1}\left(\Omega_{T}\right)$ and that the following compatibility conditions

$$
\widetilde{g}_{0}:=p(T) \in H_{0}^{1}(\Omega), \widetilde{g}_{1}:=y(T)-y_{d}(T)+\Delta \widetilde{g}_{0} \in H_{0}^{1}(\Omega)
$$

hold, [10] delivers

$$
\text { (b) } \quad p \in L^{2}\left(0, T ; H^{4}(\Omega)\right) \cap H^{2}\left(0, T ; L^{2}(\Omega)\right) \cap H^{1}\left(0, T ; H^{2}(\Omega)\right) .
$$

Remark 2.3. Since we assume that our domain $\Omega$ is sufficiently smooth, the regularities of the optimal state, the optimal control, and the associated adjoint are limited through the regularities of the initial state $y_{0}$ and of the desired state $y_{d}$. By a bootstrap argument we from (2.3)-(2.5) infer for the state $y$ the regularity
(c) $\quad y \in L^{2}\left(0, T ; H^{6}(\Omega)\right) \cap H^{1}\left(0, T ; H^{4}(\Omega)\right) \cap H^{2}\left(0, T ; H^{2}(\Omega)\right) \cap H^{3}\left(0, T ; L^{2}(\Omega)\right)$.

For the adjoint $p$ we obtain with the assumption $y_{d} \in L^{2}\left(0, T ; H^{4}(\Omega)\right) \cap H^{1}\left(0, T ; H^{2}(\Omega)\right) \cap H^{2}\left(0, T ; L^{2}(\Omega)\right)$ that
(d) $\quad p \in L^{2}\left(0, T ; H^{6}(\Omega)\right) \cap H^{1}\left(0, T ; H^{4}(\Omega)\right) \cap H^{2}\left(0, T ; H^{2}(\Omega)\right) \cap H^{3}\left(0, T ; L^{2}(\Omega)\right)$
holds.
We now show that the optimal state $y$ and the associated adjoint state $p$ under natural regularity assumptions on the data also form solutions to certain $2 n d$-order in time and $4 t h$-order in space elliptic partical differential equations. More precisely, we shall show that $y$ solves

$$
\left\{\begin{array}{rc}
-\frac{\partial^{2} y}{\partial t^{2}}+\Delta^{2} y+\frac{1}{\alpha} y=\frac{1}{\alpha} y_{d} & \text { in } \Omega_{T}  \tag{2.6}\\
y=0 & \text { on } \Sigma_{T} \\
\Delta y=0 & \text { on } \Sigma_{T} \\
\left(\frac{\partial y}{\partial t}-\Delta y\right)(T)=0 & \text { in } \Omega \\
y(0)=y_{0} & \text { in } \Omega
\end{array}\right.
$$

while $p$ forms a solution to

$$
\left\{\begin{align*}
-\frac{\partial^{2} p}{\partial t^{2}}+\Delta^{2} p+\frac{1}{\alpha} p=-\frac{\partial y_{d}}{\partial t}+\Delta y_{d} & \text { in } \Omega_{T}  \tag{2.7}\\
p=0 & \text { on } \Sigma_{T}, \\
\Delta p=y_{d} & \text { on } \Sigma_{T}, \\
\left(\frac{\partial p}{\partial t}+\Delta p\right)(0)=y_{d}(0)-y_{0} & \text { in } \Omega \\
p(T)=0 & \text { in } \Omega
\end{align*}\right.
$$

We now provide the notions of weak solutions for (2.6) and (2.7), respectively. For this purpose let us define the spaces

$$
\begin{aligned}
& H_{0}^{2,1}\left(\Omega_{T}\right):=\left\{v \in H^{2,1}\left(\Omega_{T}\right): v(0)=0 \text { in } \Omega\right\}, \\
& \widetilde{H}_{0}^{2,1}\left(\Omega_{T}\right):=\left\{v \in H^{2,1}\left(\Omega_{T}\right): v(T)=0 \text { in } \Omega\right\},
\end{aligned}
$$

the bilinear forms

$$
\begin{aligned}
& A_{T}: \\
& A_{0}: H_{0}^{2,1}\left(\Omega_{T}\right) \times H_{0}^{2,1}\left(\Omega_{T}\right) \longrightarrow \mathbb{R} \\
&: c \\
& 3
\end{aligned}
$$

as well as linear forms

$$
\begin{aligned}
& L_{T}: \\
& L_{0}: H_{0}^{2,1}\left(\Omega_{T}\right) \longrightarrow \mathbb{R}, \\
& 0
\end{aligned}
$$

where

$$
\begin{aligned}
A_{T}\left(v_{1}, v_{2}\right) & :=\int_{\Omega_{T}}\left(\frac{\partial v_{1}}{\partial t} \frac{\partial v_{2}}{\partial t}+\frac{1}{\alpha} v_{1} v_{2}\right)+\int_{\Omega_{T}} \Delta v_{1} \Delta v_{2}+\int_{\Omega} \nabla v_{1}(T) \nabla v_{2}(T), \\
A_{0}\left(v_{1}, v_{2}\right) & :=\int_{\Omega_{T}}\left(\frac{\partial v_{1}}{\partial t} \frac{\partial v_{2}}{\partial t}+\frac{1}{\alpha} v_{1} v_{2}\right)+\int_{\Omega_{T}} \Delta v_{1} \Delta v_{2}+\int_{\Omega} \nabla v_{1}(0) \nabla v_{2}(0), \\
L_{T}(v) & :=\int_{\Omega_{T}} \frac{1}{\alpha} y_{d} v, \text { and } \\
L_{0}(v) & :=\int_{0}^{T}\left\langle-\frac{\partial y_{d}}{\partial t}+\Delta y_{d}, v\right\rangle_{H^{-1}(\Omega) \times H_{0}^{1}(\Omega)}-\int_{\Omega}\left(y_{d}(0)-y_{0}\right) v(0)+\int_{\Sigma_{T}} y_{d} \nabla v \cdot n .
\end{aligned}
$$

Definition 2.4. We call $y \in H^{2,1}\left(\Omega_{T}\right)$ with $y(0)=y_{0} \in H_{0}^{1}(\Omega)$ a weak solution to (2.6), if $y$ satisfies

$$
\begin{equation*}
A_{T}(y, v)=L_{T}(v) \quad \forall v \in H_{0}^{2,1}\left(\Omega_{T}\right) \tag{2.8}
\end{equation*}
$$

Let $y_{d} \in H^{2,1}\left(\Omega_{T}\right)$. We call $p \in \widetilde{H}_{0}^{2,1}\left(\Omega_{T}\right)$ a weak solution to (2.7), if $p$ satisfies

$$
\begin{equation*}
A_{0}(p, \phi)=L_{0}(\phi) \quad \forall \phi \in \widetilde{H}_{0}^{2,1}\left(\Omega_{T}\right) \tag{2.9}
\end{equation*}
$$

In the following we prove existence and uniqueness of solutions to (2.8) and (2.9). For this purpose we equip $H^{2,1}\left(\Omega_{T}\right)$ with the inner product

$$
\lfloor v, w\rceil:=\int_{\Omega_{T}} \frac{\partial v}{\partial t} \frac{\partial w}{\partial t}+\int_{\Omega_{T}} v w+\int_{\Omega_{T}} \Delta v \Delta w
$$

which induces the norm

$$
\left\|\|v\|:=\left(\left\|\frac{\partial v}{\partial t}\right\|_{0, \Omega_{T}}^{2}+\|\Delta v\|_{0, \Omega_{T}}^{2}+\|v\|_{0, \Omega_{T}}^{2}\right)^{\frac{1}{2}}\right.
$$

For the two norms $\|\cdot\|_{2,1, \Omega_{T}}$ and $\|\|\cdot\|\|$ we have the following equivalence result (compare [24]).
Lemma 2.5. The norms $\|\cdot\|_{2,1, \Omega_{T}}$ and $|\|\cdot\||$ are equivalent on $H_{0}^{2,1}\left(\Omega_{T}\right)$, i.e., there exist positive constants $c_{1}$ and $c_{2}$ such that

$$
c_{1}\| \| v\| \| \leqslant\|v\|_{2,1, \Omega_{T}} \leqslant c_{2}\| \| v\| \| .
$$

Proof. For $v \in H_{0}^{2,1}\left(\Omega_{T}\right)$ we set $u_{v}:=\frac{\partial v}{\partial t}-\Delta v \in L^{2}\left(\Omega_{T}\right)$. Then $v$ forms a weak solution to

$$
\left\{\begin{aligned}
\frac{\partial v}{\partial t}-\Delta v=u_{v} & \text { in } \Omega_{T}, \\
v=0 & \text { on } \Sigma_{T}, \\
v(0)=0 & \text { in } \Omega
\end{aligned}\right.
$$

which depends continuously on $u_{v}$, i.e. we have

$$
\begin{aligned}
\|v\|_{2,1, \Omega_{T}}^{2} & \leqslant C\left\|u_{v}\right\|_{0, \Omega_{T}}^{2} \\
& \leqslant C\left\|\frac{\partial v}{\partial t}-\Delta v\right\|_{0, \Omega_{T}}^{2} \\
& \leqslant C\left(\left\|\frac{\partial v}{\partial t}\right\|_{0, \Omega_{T}}^{2}+\|\Delta v\|_{0, \Omega_{T}}^{2}\right) \\
& \leqslant c_{2}\|v\|^{2} .
\end{aligned}
$$

The estimate

$$
\|\|v\|\|^{2}=\left\|\frac{\partial v}{\partial t}\right\|_{0, \Omega_{T}}^{2}+\|\Delta v\|_{0, \Omega_{T}}^{2}+\|v\|_{0, \Omega_{T}}^{2} \leqslant\|v\|_{2,1, \Omega_{T}}^{2}
$$

follows directly from the definition of $\|v\|_{2,1, \Omega_{T}}$. This gives the claim.
Similarly one can prove that the norm $\|\cdot\|_{2,1, \Omega_{T}}$ is also equivalent to $\|\|\cdot\|\|$ on $\widetilde{H}_{0}^{2,1}\left(\Omega_{T}\right)$. We are now in a position to prove

Theorem 2.6. There exists a unique weak solution $y \in H^{2,1}\left(\Omega_{T}\right)$ to problem (2.8). If in addition $y_{d} \in W(0, T)$, (2.9) admits a unique weak solution $p \in \widetilde{H}_{0}^{2,1}\left(\Omega_{T}\right)$.
Proof. Since $H^{2,1}\left(\Omega_{T}\right) \hookrightarrow C\left([0, T] ; H^{1}(\Omega)\right)$,

$$
\begin{aligned}
\int_{\Omega} \nabla v_{1}(T) \nabla v_{2}(T) & \leqslant\left\|\nabla v_{1}(T)\right\|_{0, \Omega}\left\|\nabla v_{2}(T)\right\|_{0, \Omega} \\
& \leqslant\left\|v_{1}\right\|_{C\left([0, T] ; H^{1}(\Omega)\right)}\left\|v_{2}\right\|_{C\left([0, T] ; H^{1}(\Omega)\right)} \\
& \leqslant C\left\|v_{1}\right\|_{2,1, \Omega_{T}}\left\|v_{2}\right\|_{2,1, \Omega_{T}}
\end{aligned}
$$

holds for $v_{1}, v_{2} \in H^{2,1}\left(\Omega_{T}\right)$. This implies

$$
\begin{aligned}
& A_{T}\left(v_{1}, v_{2}\right) \\
\leqslant & \left\|\frac{\partial v_{1}}{\partial t}\right\|_{0, \Omega_{T}}\left\|\frac{\partial v_{2}}{\partial t}\right\|_{0, \Omega_{T}}+\frac{1}{\alpha}\left\|v_{1}\right\|_{0, \Omega_{T}}\left\|v_{2}\right\|_{0, \Omega_{T}}+\left\|\Delta v_{1}\right\|_{0, \Omega_{T}}\left\|\Delta v_{2}\right\|_{0, \Omega_{T}}+C\left\|v_{1}\right\|_{2,1, \Omega_{T}}\left\|v_{2}\right\|_{2,1, \Omega_{T}} \\
\leqslant & C\left\|v_{1}\right\|_{2,1, \Omega_{T}}\left\|v_{2}\right\|_{2,1, \Omega_{T}}
\end{aligned}
$$

with a positive constant $C$. Moreover, we have

$$
A_{T}(v, v) \geqslant \int_{\Omega_{T}}\left(\left(\frac{\partial v}{\partial t}\right)^{2}+\frac{1}{\alpha} v^{2}\right)+\int_{\Omega_{T}}(\Delta v)^{2} .
$$

Lemma 2.5 now yields

$$
A_{T}(v, v) \geqslant C\|v\|_{2,1, \Omega_{T}}^{2}
$$

which implies that $A_{T}$ is coercive. Note that $L_{T}$ is linear and bounded. Therefore, the Lax-Milgram theorem implies that the weak formulation (2.8) admits a unique solution $y \in H^{2,1}\left(\Omega_{T}\right)$.

Similarly we can prove that the bilinear form $A_{0}(\cdot, \cdot)$ is bounded and coercive, i.e.,

$$
A_{0}\left(v_{1}, v_{2}\right) \leqslant C\left\|v_{1}\right\|_{2,1, \Omega_{T}}\left\|v_{2}\right\|_{2,1, \Omega_{T}}
$$

and

$$
A_{0}(v, v) \geqslant C\|v\|_{2,1, \Omega_{T}}^{2}
$$

Since $L_{0}$ is linear and bounded, i.e.,

$$
\left|L_{0}(v)\right| \leqslant C\|v\|_{2,1, \Omega_{T}}
$$

the Lax-Milgram theorem again gives the existence and uniqueness of a solution $p \in \widetilde{H}_{0}^{2,1}\left(\Omega_{T}\right)$ to the weak formulation (2.9). This completes the proof.

For our optimal control problem (2.1) we now can prove
Theorem 2.7. Let $(y, u) \in X \times U$ denote the solution to problem (2.1)-(2.2) with associated adjoint state $p \in X$. Assume that $y$ satisfies (a) of Lemma 2.2. Then y satisfies (2.6) a.e. in space time, and is a weak solution to (2.6). If p satisfies (b) of Lemma 2.2, then p solves (2.7) a.e. in space time, and is a weak solution to (2.7).
Proof. Since the solution $y$ of (2.1)-(2.2) together with adjoint state $p$ satisfy the regularity of Lemma 2.2, we may insert (2.5) into (2.3) and take the derivative with respect to time. This yields

$$
\frac{\partial^{2} y}{\partial t^{2}}-\Delta y_{t}=-\frac{1}{\alpha} p_{t}
$$

Inserting this equation into the adjoint equation we obtain

$$
\frac{\partial^{2} y}{\partial t^{2}}-\Delta y_{t}=\frac{1}{\alpha}\left(\Delta p+y-y_{d}\right)
$$

Now, we use the state equation to replace $p$ in the previous equation. This gives

$$
\begin{aligned}
\frac{\partial^{2} y}{\partial t^{2}}-\Delta y_{t} & =\frac{1}{\alpha}\left(\Delta p+y-y_{d}\right) \\
& =\frac{1}{\alpha}\left(y-y_{d}\right)+\Delta\left(-\frac{\partial y}{\partial t}+\Delta y\right) \\
& =-\Delta y_{t}+\Delta^{2} y+\frac{1}{\alpha}\left(y-y_{d}\right) .
\end{aligned}
$$

Thus

$$
\begin{equation*}
-\frac{\partial^{2} y}{\partial t^{2}}+\Delta^{2} y+\frac{1}{\alpha} y=\frac{1}{\alpha} y_{d} \tag{2.10}
\end{equation*}
$$

The boundary conditions together with initial value for state variable then read

$$
\begin{equation*}
y=0 \quad \text { on } \Sigma_{T}, y(0)=y_{0} \quad \text { in } \Omega . \tag{2.11}
\end{equation*}
$$

From the boundary condition of adjoint state $p$ we obtain

$$
0=p=\alpha\left(\Delta y-\frac{\partial y}{\partial t}\right) \quad \text { on } \Sigma_{T}
$$

which implies

$$
\begin{equation*}
\Delta y=0 \quad \text { on } \Sigma_{T} . \tag{2.12}
\end{equation*}
$$

Note that $p(T)=0$ in $\Omega$. Thus

$$
\begin{equation*}
0=p(T)=\alpha\left(\Delta y-\frac{\partial y}{\partial t}\right)(T) \quad \text { in } \Omega \tag{2.13}
\end{equation*}
$$

Collecting (2.10)-(2.13) gives (2.6). Therefore, we can conclude that the state $y$ satisfies the spacetime elliptic boundary value problem (2.6) a.e. in space-time. Now let $y \in L^{2}\left(0, T ; H^{4}(\Omega)\right) \cap$ $H^{2}\left(0, T ; L^{2}(\Omega)\right) \cap H^{1}\left(0, T ; H^{2}(\Omega)\right)$. Then $y \in H^{2,1}\left(\Omega_{T}\right)$ and by Green's formula one can easily prove that $y$ satisfies (2.8), which implies that $y$ is also a weak solution to (2.6).

By similar arguments we can prove that the adjoint state $p$ satisfies the space-time elliptic equations (2.7) a.e. in space-time under the assumption $y_{d} \in H^{2,1}\left(\Omega_{T}\right)$, and also forms a weak solution to (2.7), where we only have to require $y_{d} \in W(0, T)$.

## 3. Error estimates for the time discretization scheme

In this section we present a priori and a posteriori error analysis for the temporal discretization, while the space variable is kept continuous.

Let $0=t_{0}<t_{1}<\cdots<t_{M-1}<t_{M}=T$ be a time grid with $k_{n}=t_{n}-t_{n-1}, n=1,2, \cdots, M$. Set $I_{n}=\left[t_{n-1}, t_{n}\right]$ and $k=\max _{1 \leqslant n \leqslant M} k_{n}$. For the time discretization of the state $y$ we define

$$
V_{t}^{k}=\left\{v \in H^{2,1}\left(\Omega_{T}\right) ;\left.v(\cdot)\right|_{I_{n}} \in P_{1}\left(I_{n}\right)\right\}, \quad \bar{V}_{t}^{k}=V_{t}^{k} \cap H_{0}^{2,1}\left(\Omega_{T}\right)
$$

and consider the semi-discrete problem: Find $y_{k} \in V_{t}^{k}$ with $y_{k}(0)=y_{0}$ and

$$
\begin{equation*}
A_{T}\left(y_{k}, v_{k}\right)=L_{T}\left(v_{k}\right) \quad \forall v_{k} \in \bar{V}_{t}^{k} \tag{3.1}
\end{equation*}
$$

Since $V_{t}^{k}$ is a closed subspace of $H^{2,1}\left(\Omega_{T}\right)$, the Lax-Milgram theorem implies that (3.1) admits a unique solution $y_{k} \in V_{t}^{k}$. For this scheme we have

Theorem 3.1. Let $y \in H^{2,1}\left(\Omega_{T}\right)$ and $y_{k} \in V_{t}^{k}$ be the solutions to (2.8) and (3.1), respectively. Assume that $y \in H^{2}\left(0, T ; H^{2}(\Omega)\right) \cap W^{2, \infty}\left(0, T ; H^{1}(\Omega)\right)$. Then we have

$$
\left\|y-y_{k}\right\|_{0, \Omega_{T}}+k\left\|y-y_{k}\right\|_{H^{1}\left(0, T ; L^{2}(\Omega)\right)} \leqslant C k^{2}
$$

Proof. From (2.8) and (3.1) we obtain the following error equation:

$$
\begin{align*}
\int_{\Omega_{T}}\left(\frac{\partial\left(y-y_{k}\right)}{\partial t} \frac{\partial v_{k}}{\partial t}+\right. & \left.\frac{1}{\alpha}\left(y-y_{k}\right) v_{k}\right)+\int_{\Omega_{T}} \Delta\left(y-y_{k}\right) \Delta v_{k} \\
& +\int_{\Omega} \nabla\left(y-y_{k}\right)(T) \nabla v_{k}(T)=0 \quad \forall v_{k} \in \bar{V}_{t}^{k} \tag{3.2}
\end{align*}
$$

Let $R_{k} y \in V_{t}^{k}$ denote the temporal Ritz projection ([3]) of $y$, which is defined by

$$
\int_{\Omega_{T}} \frac{\partial\left(y-R_{k} y\right)}{\partial t} \frac{\partial v_{k}}{\partial t}=0, \quad \forall v_{k} \in \bar{V}_{t}^{k}
$$

Decompose $y-y_{k}=y-R_{k} y+R_{k} y-y_{k}=\xi_{y}+\eta_{y}$. Then we can rewrite (3.2) as

$$
\begin{align*}
& \int_{\Omega_{T}}\left(\frac{\partial \eta_{y}}{\partial t} \frac{\partial v_{k}}{\partial t}+\frac{1}{\alpha} \eta_{y} v_{k}\right)+\int_{\Omega_{T}} \Delta \eta_{y} \Delta v_{k}+\int_{\Omega} \nabla \eta_{y}(T) \nabla v_{k}(T) \\
= & -\int_{\Omega_{T}} \frac{1}{\alpha} \xi_{y} v_{k}-\int_{\Omega} \nabla \xi_{y}(T) \nabla v_{k}(T)-\int_{\Omega_{T}} \Delta \xi_{y} \Delta v_{k} . \tag{3.3}
\end{align*}
$$

Testing (3.3) with $v_{k}=\eta_{y}$ leads to

$$
\begin{aligned}
& \int_{\Omega_{T}}\left(\left(\frac{\partial \eta_{y}}{\partial t}\right)^{2}+\frac{1}{\alpha} \eta_{y}^{2}\right)+\int_{\Omega_{T}} \Delta \eta_{y}^{2}+\int_{\Omega} \nabla \eta_{y}(T)^{2} \\
= & -\int_{\Omega_{T}} \frac{1}{\alpha} \xi_{y} \eta_{y}-\int_{\Omega} \nabla \xi_{y}(T) \nabla \eta_{y}(T)-\int_{\Omega_{T}} \Delta \xi_{y} \Delta \eta_{y}
\end{aligned}
$$

Using the Young's inequality and error estimates for the Ritz projection ([27]) we obtain

$$
\begin{aligned}
& \frac{1}{\alpha}\left\|\eta_{y}\right\|_{0, \Omega_{T}}^{2}+\left\|\frac{\partial \eta_{y}}{\partial t}\right\|_{0, \Omega_{T}}^{2}+\left\|\Delta \eta_{y}\right\|_{0, \Omega_{T}}^{2}+\left\|\nabla \eta_{y}(T)\right\|_{0, \Omega}^{2} \\
\leqslant & C\left(\left\|\xi_{y}\right\|_{0, \Omega_{T}}^{2}+\left\|\nabla \xi_{y}(T)\right\|_{0, \Omega}^{2}+\left\|\Delta \xi_{y}\right\|_{0, \Omega_{T}}^{2}\right) \\
\leqslant & C k^{4}\left(\|y\|_{H^{2}\left(0, T ; L^{2}(\Omega)\right)}^{2}+\|y\|_{W^{2, \infty}\left(0, T ; H^{1}(\Omega)\right)}^{2}+\|\Delta y\|_{H^{2}\left(0, T ; L^{2}(\Omega)\right)}^{2}\right) .
\end{aligned}
$$

Then triangle inequality and error estimates for the Ritz projection finally give the estimates

$$
\left\|y-y_{k}\right\|_{H^{1}\left(0, T ; L^{2}(\Omega)\right)} \leqslant C k, \quad\left\|y-y_{k}\right\|_{0, \Omega_{T}} \leqslant C k^{2}
$$

For the temporal discretization of the adjoint state $p$ we proceed similarly as above. We seek $p_{k} \in \widetilde{V}_{t}^{k}$ such that

$$
\begin{equation*}
A_{0}\left(p_{k}, \phi_{k}\right)=L_{0}\left(\phi_{k}\right) \quad \forall \phi_{k} \in \widetilde{V}_{t}^{k} \tag{3.4}
\end{equation*}
$$

holds. Here $\widetilde{V}_{t}^{k}:=V_{t}^{k} \cap \widetilde{H}_{0}^{2,1}\left(\Omega_{T}\right)$. Then a proof similar to that of the previous theorem gives
Theorem 3.2. Let $p \in \widetilde{H}_{0}^{2,1}\left(\Omega_{T}\right)$ and $p_{k} \in \widetilde{V}_{t}^{k}$ be the solutions to (2.9) and (3.4), respectively. Suppose that $p \in H^{2}\left(0, T ; H^{2}(\Omega)\right) \cap W^{2, \infty}\left(0, T ; H^{1}(\Omega)\right)$. Then we have

$$
\left\|p-p_{k}\right\|_{0, \Omega_{T}}+k\left\|p-p_{k}\right\|_{H^{1}\left(0, T ; L^{2}(\Omega)\right)} \leqslant C k^{2}
$$

Now we are in the position to derive temporal residual type a posteriori error estimates for $y$ and $p$. We adopt standard Lagrange interpolation in the a posteriori error estimates since $H^{1}(0, T) \hookrightarrow C([0, T])$.
Theorem 3.3. Let $y \in H^{2,1}\left(\Omega_{T}\right)$ and $y_{k} \in V_{t}^{k}$ denote the solutions to (2.8) and (3.1), respectively. Then we obtain

$$
\left\|y-y_{k}\right\|_{2,1, \Omega_{T}}^{2} \leqslant C \eta_{y}^{2}
$$

where

$$
\eta_{y}^{2}=\sum_{n} k_{n}^{2} \int_{I_{n}}\left\|\frac{1}{\alpha} y_{d}+\frac{\partial^{2} y_{k}}{\partial t^{2}}-\frac{1}{\alpha} y_{k}-\Delta^{2} y_{k}\right\|_{0, \Omega}^{2}+\sum_{n} \int_{I_{n}}\left\|\Delta y_{k}\right\|_{0, \Gamma}^{2}
$$

Proof. Let $e^{y}=y-y_{k}$, and let $\pi_{k} e^{y}$ denote the standard Lagrange type temporal interpolation of $e^{y}$. By (2.8), (3.1), Lemma 2.5 and the definition of Lagrange interpolation we have

$$
\begin{aligned}
& c\left\|y-y_{k}\right\|_{2,1, \Omega_{T}}^{2} \\
\leqslant & \frac{1}{\alpha}\left\|y-y_{k}\right\|_{0, \Omega_{T}}^{2}+\left\|\frac{\partial\left(y-y_{k}\right)}{\partial t}\right\|_{0, \Omega_{T}}^{2}+\left\|\Delta\left(y-y_{k}\right)\right\|_{0, \Omega_{T}}^{2} \\
\leqslant & \int_{\Omega_{T}} \frac{\partial\left(y-y_{k}\right)}{\partial t} \frac{\partial e^{y}}{\partial t}+\int_{\Omega_{T}} \frac{1}{\alpha}\left(y-y_{k}\right) e^{y}+\int_{\Omega} \nabla\left(y-y_{k}\right)(T) \nabla e^{y}(T)+\int_{\Omega_{T}} \Delta\left(y-y_{k}\right) \Delta e^{y} \\
= & \int_{\Omega_{T}} \frac{\partial\left(y-y_{k}\right)}{\partial t} \frac{\partial\left(e^{y}-\pi_{k} e^{y}\right)}{\partial t}+\int_{\Omega_{T}} \frac{1}{\alpha}\left(y-y_{k}\right)\left(e^{y}-\pi_{k} e^{y}\right) \\
& +\int_{\Omega_{T}} \Delta\left(y-y_{k}\right) \Delta\left(e^{y}-\pi_{k} e^{y}\right)+\int_{\Omega} \nabla\left(y-y_{k}\right)(T) \nabla\left(e^{y}-\pi_{k} e^{y}\right)(T) \\
= & \frac{1}{\alpha} \int_{\Omega_{T}} y_{d}\left(e^{y}-\pi_{k} e^{y}\right)-\int_{\Omega_{T}} \frac{\partial y_{k}}{\partial t} \frac{\partial\left(e^{y}-\pi_{k} e^{y}\right)}{\partial t}-\int_{\Omega_{T}} \frac{1}{\alpha} y_{k}\left(e^{y}-\pi_{k} e^{y}\right) \\
& -\int_{\Omega_{T}} \Delta y_{k} \Delta\left(e^{y}-\pi_{k} e^{y}\right)-\int_{\Omega} \nabla y_{k}(T) \nabla\left(e^{y}-\pi_{k} e^{y}\right)(T) \\
= & \frac{1}{\alpha} \int_{\Omega_{T}} y_{d}\left(e^{y}-\pi_{k} e^{y}\right)-\int_{\Omega_{T}} \frac{\partial y_{k}}{\partial t} \frac{\partial\left(e^{y}-\pi_{k} e^{y}\right)}{\partial t}-\int_{\Omega_{T}} \frac{1}{\alpha} y_{k}\left(e^{y}-\pi_{k} e^{y}\right)-\int_{\Omega_{T}} \Delta y_{k} \Delta\left(e^{y}-\pi_{k} e^{y}\right)
\end{aligned}
$$

Integrating by parts on each time interval yields

$$
\begin{aligned}
& c\left\|y-y_{k}\right\|_{2,1, \Omega_{T}}^{2} \\
\leqslant & \sum_{n} \int_{I_{n}} \int_{\Omega}\left(\frac{1}{\alpha} y_{d}+\frac{\partial^{2} y_{k}}{\partial t^{2}}-\frac{1}{\alpha} y_{k}-\Delta^{2} y_{k}\right)\left(e^{y}-\pi_{k} e^{y}\right) \\
& +\int_{\Sigma_{T}} \Delta y_{k} \nabla\left(\pi_{k} e^{y}-e^{y}\right) \cdot n
\end{aligned}
$$

Using error estimates for Lagrange interpolation, the trace inequality, and Young's inequality we obtain

$$
\begin{aligned}
\left\|y-y_{k}\right\|_{2,1, \Omega_{T}}^{2} \leqslant & C \sum_{n} k_{n}^{2} \int_{I_{n}}\left\|\frac{1}{\alpha} y_{d}+\frac{\partial^{2} y_{k}}{\partial t^{2}}-\frac{1}{\alpha} y_{k}-\Delta^{2} y_{k}\right\|_{0, \Omega}^{2} \\
& +C \sum_{n} \int_{I_{n}}\left\|\Delta y_{k}\right\|_{0, \Gamma}^{2} .
\end{aligned}
$$

We note that the previous theorem in particular implies

$$
\begin{aligned}
\left\|y-y_{k}\right\|_{H^{1}\left(0, T ; L^{2}(\Omega)\right)}^{2} \leqslant & C \sum_{n} k_{n}^{2} \int_{I_{n}}\left\|\frac{1}{\alpha} y_{d}+\frac{\partial^{2} y_{k}}{\partial t^{2}}-\frac{1}{\alpha} y_{k}-\Delta^{2} y_{k}\right\|_{0, \Omega}^{2} \\
& +C \sum_{n} \int_{I_{n}}\left\|\Delta y_{k}\right\|_{0, \Gamma}^{2}
\end{aligned}
$$

Remark 3.4. Since $\Delta y=0$ on $\Sigma_{T}$, the term $\sum_{n} \int_{I_{n}}\left\|\Delta y_{k}\right\|_{0, \Gamma}^{2}$ accounts for the violation of this boundary condition on the time-discrete level.

For the adjoint state $p$ we proceed similarly.
Theorem 3.5. Let $p \in \widetilde{H}_{0}^{2,1}\left(\Omega_{T}\right)$ and $p_{k} \in \widetilde{V}_{t}^{k}$ be the solutions to (2.9) and (3.4), respectively. Then we have

$$
\left\|p-p_{k}\right\|_{2,1, \Omega_{T}}^{2} \leqslant C \eta_{p}^{2},
$$

where

$$
\eta_{p}^{2}=\sum_{n} k_{n}^{2} \int_{I_{n}}\left\|-\frac{\partial y_{d}}{\partial t}+\Delta y_{d}+\frac{\partial^{2} p_{k}}{\partial t^{2}}-\frac{1}{\alpha} p_{k}-\Delta^{2} p_{k}\right\|_{0, \Omega}^{2}+\sum_{n} \int_{I_{n}}\left\|y_{d}-\Delta p_{k}\right\|_{0, \Gamma}^{2} .
$$

Proof. Let $e^{p}=p-p_{k}$, and $\pi_{k} e^{p}$ be the standard Lagrange type temporal interpolation of $e^{p}$. Using (2.9), (3.4), Lemma 2.5 and the definition of Lagrange interpolation we deduce

$$
\begin{aligned}
& c\left\|p-p_{k}\right\|_{2,1, \Omega_{T}}^{2} \\
\leqslant & \frac{1}{\alpha}\left\|p-p_{k}\right\|_{0, \Omega_{T}}^{2}+\left\|\frac{\partial\left(p-p_{k}\right)}{\partial t}\right\|_{0, \Omega_{T}}^{2}+\left\|\Delta\left(p-p_{k}\right)\right\|_{0, \Omega_{T}}^{2} \\
\leqslant & \int_{\Omega_{T}} \frac{\partial\left(p-p_{k}\right)}{\partial t} \frac{\partial e^{p}}{\partial t}+\int_{\Omega_{T}} \frac{1}{\alpha}\left(p-p_{k}\right) e^{p}+\int_{\Omega} \nabla\left(p-p_{k}\right)(0) \nabla e^{p}(0)+\int_{\Omega_{T}} \Delta\left(p-p_{k}\right) \Delta e^{p} \\
= & \int_{\Omega_{T}} \frac{\partial\left(p-p_{k}\right)}{\partial t} \frac{\partial\left(e^{p}-\pi_{k} e^{p}\right)}{\partial t}+\int_{\Omega_{T}} \frac{1}{\alpha}\left(p-p_{k}\right)\left(e^{p}-\pi_{k} e^{p}\right) \\
& +\int_{\Omega_{T}} \Delta\left(p-p_{k}\right) \Delta\left(e^{p}-\pi_{k} e^{p}\right) \\
= & \int_{\Omega_{T}}\left(-\frac{\partial y_{d}}{\partial t}+\Delta y_{d}\right)\left(e^{p}-\pi_{k} e^{p}\right)+\int_{\Sigma_{T}} y_{d} \nabla\left(e^{p}-\pi_{k} e^{p}\right) \cdot n \\
& -\int_{\Omega_{T}} \frac{\partial p_{k}}{\partial t} \frac{\partial\left(e^{p}-\pi_{k} e^{p}\right)}{\partial t}-\int_{\Omega_{T}} \frac{1}{\alpha} p_{k}\left(e^{p}-\pi_{k} e^{p}\right)-\int_{\Omega_{T}} \Delta p_{k} \Delta\left(e^{p}-\pi_{k} e^{p}\right) .
\end{aligned}
$$

Integrating by parts on each element yields

$$
\begin{aligned}
& c\left\|p-p_{k}\right\|_{2,1, \Omega_{T}}^{2} \\
\leqslant & \sum_{n} \int_{I_{n}} \int_{\Omega}\left(-\frac{\partial y_{d}}{\partial t}+\Delta y_{d}+\frac{\partial^{2} p_{k}}{\partial t^{2}}-\frac{1}{\alpha} p_{k}-\Delta^{2} p_{k}\right)\left(e^{p}-\pi_{k} e^{p}\right) \\
& +\int_{\Sigma_{T}}\left(y_{d}-\Delta p_{k}\right) \nabla\left(e^{p}-\pi_{k} e^{p}\right) \cdot n
\end{aligned}
$$

Error estimates of Lagrange interpolation, the trace inequality and Young's inequality then give

$$
\begin{aligned}
& \left\|p-p_{k}\right\|_{2,1, \Omega_{T}}^{2} \\
\leqslant & C \sum_{n} k_{n}^{2} \int_{I_{n}}\left\|-\frac{\partial y_{d}}{\partial t}+\Delta y_{d}+\frac{\partial^{2} p_{k}}{\partial t^{2}}-\frac{1}{\alpha} p_{k}-\Delta^{2} p_{k}\right\|_{0, \Omega}^{2} \\
& +C \sum_{n} \int_{I_{n}}\left\|y_{d}-\Delta p_{k}\right\|_{0, \Gamma}^{2} .
\end{aligned}
$$

Theorem 3.5 also implies

$$
\begin{aligned}
& \left\|p-p_{k}\right\|_{H^{1}\left(0, T ; L^{2}(\Omega)\right)}^{2} \\
\leqslant & C \sum_{n} k_{n}^{2} \int_{I_{n}}\left\|-\frac{\partial y_{d}}{\partial t}+\Delta y_{d}+\frac{\partial^{2} p_{k}}{\partial t^{2}}-\frac{1}{\alpha} p_{k}-\Delta^{2} p_{k}\right\|_{0, \Omega}^{2} \\
& +C \sum_{n} \int_{I_{n}}\left\|y_{d}-\Delta p_{k}\right\|_{0, \Gamma}^{2} .
\end{aligned}
$$

Remark 3.6. Similarly as above, $\sum_{n} \int_{I_{n}}\left\|y_{d}-\Delta p_{k}\right\|_{0, \Gamma}^{2}$ accounts for the violation of the boundary condition $\Delta p=y_{d}$ on $\Sigma_{T}$ on the time-discrete level.

## 4. Space-time mixed finite element discretization

Several options can be used to tackle the space-time elliptic problems (2.6) and (2.7) numerically. In order to use piecewise linear, continuous finite elements for the spatial discretization we in this section propose a space-time mixed finite element method to treat (2.6) and (2.7) numerically and prove corresponding a priori error estimates.

In order to derive a mixed formulation of (2.6) let $w:=-\Delta y$, where $y$ denotes the unique solution to (2.6). Then (2.6) motivates the following mixed formulation: Find $(y, w)$ which satisfies

$$
\left\{\begin{align*}
-\frac{\partial^{2} y}{\partial t^{2}}-\Delta w+\frac{1}{\alpha} y=\frac{1}{\alpha} y_{d} & \text { in } \Omega_{T}  \tag{4.1}\\
\Delta y+w=0 & \text { in } \Omega_{T} \\
y=0 & \text { on } \Sigma_{T} \\
w=0 & \text { on } \Sigma_{T} \\
\left(\frac{\partial y}{\partial t}-\Delta y\right)(T)=0 & \text { in } \Omega \\
y(0)=y_{0} & \text { in } \Omega
\end{align*}\right.
$$

Let

$$
Y:=\left\{v \in H^{1}\left(0, T ; H_{0}^{1}(\Omega)\right): v(0)=0 \text { in } \Omega\right\}, W:=L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right)
$$

Then the mixed variational form for (4.1) is to find $y \in H^{1}\left(0, T ; H_{0}^{1}(\Omega)\right)$ satisfying $y(0)=y_{0}$ and $w \in W$ such that

$$
\left\{\begin{align*}
\int_{\Omega_{T}}\left(\frac{\partial y}{\partial t} \frac{\partial v}{\partial t}+\frac{1}{\alpha} y v\right)+\int_{\Omega_{T}} \nabla w \nabla v+ & \int_{\Omega} \nabla y(T) \nabla v(T)=\int_{\Omega_{T}} \frac{1}{\alpha} y_{d} v \forall v \in Y  \tag{4.2}\\
& -\int_{\Omega_{T}} \nabla y \nabla \phi+\int_{\Omega_{T}} w \phi=0 \quad \forall \phi \in W
\end{align*}\right.
$$

In the following we prove that the pair $(y, w)$ with $w:=-\Delta y$ and $y$ the unique smooth solution to (2.6) is a solution to the mixed variational form (4.2), and also that this mixed form has at most one solution $(y, w) \in Y \times W$, so that the unique smooth solution $y$ to (2.6) defines the mixed variational solution.

We start with proving that (4.2) admits at most one solution. Suppose that $\left(y_{1}, w_{1}\right)$ and $\left(y_{2}, w_{2}\right)$ are two different solutions to problem (4.2). Then $(\widetilde{y}, \widetilde{w})=\left(y_{1}-y_{2}, w_{1}-w_{2}\right)$ satisfies the following homogeneous system

$$
\left\{\begin{align*}
& \int_{\Omega_{T}}\left(\frac{\partial \widetilde{y}}{\partial t} \frac{\partial v}{\partial t}+\frac{1}{\alpha} \widetilde{y} v\right)+\int_{\Omega_{T}} \nabla \widetilde{w} \nabla v+\int_{\Omega} \nabla \widetilde{y}(T) \nabla v(T)=0 \quad \forall v \in Y  \tag{4.3}\\
&-\int_{\Omega_{T}} \nabla \widetilde{y} \nabla \phi+\int_{\Omega_{T}} \widetilde{w} \phi=0 \quad \forall \phi \in W
\end{align*}\right.
$$

Testing (4.3) with $v=\widetilde{y}$ and $\phi=\widetilde{w}$, and adding the resulting equations yields

$$
\int_{\Omega_{T}}\left(\left(\frac{\partial \widetilde{y}}{\partial t}\right)^{2}+\frac{1}{\alpha} \widetilde{y}^{2}\right)+\int_{\Omega}(\nabla \widetilde{y}(T))^{2}+\int_{\Omega_{T}}(\widetilde{w})^{2}=0
$$

Hence,

$$
\widetilde{y} \equiv 0, \quad \widetilde{w} \equiv 0 .
$$

Thus, (4.2) admits at most one solution.
Now let $y \in H^{2,1}\left(\Omega_{T}\right) \cap L^{2}\left(0, T ; H^{3}(\Omega)\right)$ be the smooth solution to (2.6). Then $w:=-\Delta y \in$ $L^{2}\left(0, T ; H^{1}(\Omega)\right)$, and $y, w$ satisfy (4.2), as is shown in the following. For $v \in H^{1}\left(0, T ; C_{0}^{\infty}(\Omega)\right)$ we have from (2.8)

$$
\int_{\Omega_{T}}\left(\frac{\partial y}{\partial t} \frac{\partial v}{\partial t}+\frac{1}{\alpha} y v\right)+\int_{\Omega_{T}} \nabla w \nabla v+\int_{\Omega} \nabla y(T) \nabla v(T)=\int_{\Omega_{T}} \frac{1}{\alpha} y_{d} v
$$

According to the density argument we derive the first equation of (4.2). Moreover, we have

$$
-\int_{\Omega_{T}} \nabla y \nabla \phi+\int_{\Omega_{T}} w \phi=0 \quad \forall \phi \in W
$$

holds. Thus $(y, w)$ solves (4.2), and consequently, it is the unique solution to (4.2).
Now we are in a position to consider the discretization of (4.2). We will consider space-time discretization and pure time discretization, respectively.

In the case of space-time discretization, let $\mathscr{T}^{h}$ be a quasi-uniform partitioning of $\Omega_{T}$ into disjoint regular $(n+1)$-simplices $\tau$ or rectangles, so that $\overline{\Omega_{T}}=\bigcup_{\tau \in \mathscr{T} h} \bar{\tau}$, where element edges lying on the boundary may be curved. Let $h_{\tau}$ denote the diameter of $\tau$. Set $h=\max _{\tau} h_{\tau}$. Associated with $\mathscr{T}^{h}$ is a finite dimensional subspace $V^{h}$ of $C\left(\overline{\Omega_{T}}\right)$, such that $\left.\chi\right|_{\tau}$ are linear or bilinear polynomials for all $\chi \in Y^{h}$ and $\tau \in \mathscr{T}^{h}$. Let us set $Y^{h}=V^{h} \cap H^{1}\left(0, T ; H_{0}^{1}(\Omega)\right), Y_{0}^{h}=V^{h} \cap Y$ and $W_{0}^{h}=V^{h} \cap W$ in this case.

In the case of pure time discretization, let $0=t_{0}<t_{1}<\cdots<t_{N-1}<t_{N}=T$ denote a time grid on $[0, T]$ with grid size $h=\max _{n} h_{n}$, where $h_{n}=t_{n}-t_{n-1}$. Let us set $\mathscr{T}^{h}=\left\{I_{n}\right\}_{n=1}^{N}$ with $I_{n}=\left[t_{n-1}, t_{n}\right]$ and

$$
V^{h}=\left\{v \in H^{1}\left(0, T ; H^{1}(\Omega) ;\left.v(\cdot)\right|_{I_{n}} \in P_{1}\left(I_{n}\right)\right\} .\right.
$$

We in this case set $Y^{h}=V^{h} \cap H^{1}\left(0, T ; H_{0}^{1}(\Omega), Y_{0}^{h}=V^{h} \cap Y\right.$ and $W_{0}^{h}=V^{h} \cap W$.
The discrete approximation to (4.2) is to find $y_{h} \in Y^{h}, w_{h} \in W_{0}^{h}$ such that $y_{h}(0)=y_{0 h}$ and

$$
\text { (4.4) }\left\{\begin{aligned}
\int_{\Omega_{T}}\left(\frac{\partial y_{h}}{\partial t} \frac{\partial v_{h}}{\partial t}+\frac{1}{\alpha} y_{h} v_{h}\right)+\int_{\Omega_{T}} \nabla w_{h} \nabla v_{h} & +\int_{\Omega} \nabla y_{h}(T) \nabla v_{h}(T)=\int_{\Omega_{T}} \frac{1}{\alpha} y_{d} v_{h} \forall v_{h} \in Y_{0}^{h}, \\
& -\int_{\Omega_{T}} \nabla y_{h} \nabla \phi_{h}+\int_{\Omega_{T}} w_{h} \phi_{h}=0 \forall \phi_{h} \in W_{0}^{h} .
\end{aligned}\right.
$$

holds. Here $y_{0 h}=: y_{0}$ for pure time discretization and $y_{0 h}=: \widetilde{y}_{0}(0)$ for space-time discretization, where $\widetilde{y}_{0}$ is an approximation of $y_{0}$ in $Y^{h}$, whose approximation properties can be adjusted to the error bounds which we shall prove below.

Now we are in a position to estimate the errors between the solutions $(y, w)$ of (4.2) and ( $y_{h}, w_{h}$ ) of (4.4).
Theorem 4.1. Let $(y, w) \in Y \times W$ and $\left(y_{h}, w_{h}\right) \in Y^{h} \times W_{0}^{h}$ be the solutions to (4.2) and (4.4), respectively. In the case of pure time discretization we for $y \in H^{2}\left(0, T ; H^{4}(\Omega)\right) \cap W^{2, \infty}\left(0, T ; H^{1}(\Omega)\right)$ have

$$
\left\|y-y_{h}\right\|_{H^{1}\left(0, T ; L^{2}(\Omega)\right)} \leqslant C h, \quad\left\|y-y_{h}\right\|_{0, \Omega_{T}} \leqslant C h^{2} \quad \text { and } \quad\left\|w-w_{h}\right\|_{0, \Omega_{T}} \leqslant C h^{2}
$$

In the fully discrete case we for $y \in H^{2}\left(0, T ; H^{4}(\Omega)\right)$ have

$$
\left\|y-y_{h}\right\|_{H^{1}\left(0, T ; L^{2}(\Omega)\right)} \leqslant C h \text { and }\left\|w-w_{h}\right\|_{0, \Omega_{T}} \leqslant C h .
$$

Proof. From (4.2) and (4.4) we deduce the following error equations

$$
\begin{align*}
\int_{\Omega_{T}}\left(\frac{\partial\left(y-y_{h}\right)}{\partial t} \frac{\partial v_{h}}{\partial t}+\right. & \left.\frac{1}{\alpha}\left(y-y_{h}\right) v_{h}\right)+\int_{\Omega_{T}} \nabla\left(w-w_{h}\right) \nabla v_{h} \\
& +\int_{\Omega} \nabla\left(y-y_{h}\right)(T) \nabla v_{h}(T)=0 \quad \forall v_{h} \in Y_{0}^{h}  \tag{4.5}\\
-\int_{\Omega_{T}} \nabla\left(y-y_{h}\right) \nabla \phi_{h}+ & \int_{\Omega_{T}}\left(w-w_{h}\right) \phi_{h}=0 \quad \forall \phi_{h} \in W_{0}^{h} . \tag{4.6}
\end{align*}
$$

Let $y^{I} \in Y^{h}, w^{I} \in W_{0}^{h}$ and set $\psi^{h}=y^{I}-y_{h}$. Then (4.5) yields

$$
c\left\|y^{I}-y_{h}\right\|_{H^{1}\left(0, T ; L^{2}(\Omega)\right)}^{2}+\left|\left(y^{I}-y_{h}\right)(T)\right|_{1, \Omega}^{2}
$$

$$
\begin{align*}
\leqslant & \int_{\Omega_{T}} \frac{\partial\left(y^{I}-y_{h}\right)}{\partial t} \frac{\partial \psi^{h}}{\partial t}+\frac{1}{\alpha}\left(y^{I}-y_{h}\right) \psi^{h}+\int_{\Omega} \nabla\left(y^{I}-y_{h}\right)(T) \nabla \psi^{h}(T) \\
= & \int_{\Omega_{T}} \frac{\partial\left(y^{I}-y\right)}{\partial t} \frac{\partial \psi^{h}}{\partial t}+\frac{1}{\alpha}\left(y^{I}-y\right) \psi^{h}+\int_{\Omega} \nabla\left(y^{I}-y\right)(T) \nabla \psi^{h}(T) \\
& +\int_{\Omega_{T}} \frac{\partial\left(y-y_{h}\right)}{\partial t} \frac{\partial \psi^{h}}{\partial t}+\frac{1}{\alpha}\left(y-y_{h}\right) \psi^{h}+\int_{\Omega} \nabla\left(y-y_{h}\right)(T) \nabla \psi^{h}(T) \\
= & \int_{\Omega_{T}} \frac{\partial\left(y^{I}-y\right)}{\partial t} \frac{\partial \psi^{h}}{\partial t}+\frac{1}{\alpha}\left(y^{I}-y\right) \psi^{h}+\int_{\Omega} \nabla\left(y^{I}-y\right)(T) \nabla \psi^{h}(T)-\int_{\Omega_{T}} \nabla\left(w-w_{h}\right) \nabla \psi^{h} \\
= & \int_{\Omega_{T}} \frac{\partial\left(y^{I}-y\right)}{\partial t} \frac{\partial \psi^{h}}{\partial t}+\frac{1}{\alpha}\left(y^{I}-y\right) \psi^{h}+\int_{\Omega} \nabla\left(y^{I}-y\right)(T) \nabla \psi^{h}(T) \\
& -\int_{\Omega_{T}} \nabla\left(w-w^{I}\right) \nabla \psi^{h}-\int_{\Omega_{T}} \nabla\left(w^{I}-w_{h}\right) \nabla \psi^{h} . \tag{4.7}
\end{align*}
$$

Similarly, from equation (4.6) we have

$$
\begin{aligned}
& -\int_{\Omega_{T}} \nabla\left(w^{I}-w_{h}\right) \nabla \psi^{h} \\
= & -\int_{\Omega_{T}} \nabla\left(w^{I}-w_{h}\right) \nabla\left(y^{I}-y\right)-\int_{\Omega_{T}} \nabla\left(w^{I}-w_{h}\right) \nabla\left(y-y_{h}\right) \\
= & -\int_{\Omega_{T}} \nabla\left(w^{I}-w_{h}\right) \nabla\left(y^{I}-y\right)-\int_{\Omega_{T}}\left(w^{I}-w_{h}\right)\left(w-w_{h}\right) \\
= & -\int_{\Omega_{T}} \nabla\left(w^{I}-w_{h}\right) \nabla\left(y^{I}-y\right)-\int_{\Omega_{T}}\left(w^{I}-w_{h}\right)\left(w-w^{I}\right)-\int_{\Omega_{T}}\left(w^{I}-w_{h}\right)\left(w^{I}-w_{h}\right) \\
(4.8)= & -\int_{\Omega_{T}} \nabla\left(w^{I}-w_{h}\right) \nabla\left(y^{I}-y\right)-\int_{\Omega_{T}}\left(w^{I}-w_{h}\right)\left(w-w^{I}\right)-\left\|w^{I}-w_{h}\right\|_{0, \Omega_{T}}^{2} .
\end{aligned}
$$

Combining (4.7) and (4.8) we get

$$
\begin{aligned}
& c\left\|y^{I}-y_{h}\right\|_{H^{1}\left(0, T ; L^{2}(\Omega)\right)}^{2}+\left\|w^{I}-w_{h}\right\|_{0, \Omega_{T}}^{2}+\left|\left(y^{I}-y_{h}\right)(T)\right|_{1, \Omega}^{2} \\
= & \int_{\Omega_{T}} \frac{\partial\left(y^{I}-y\right)}{\partial t} \frac{\partial \psi^{h}}{\partial t}+\frac{1}{\alpha}\left(y^{I}-y\right) \psi^{h}-\int_{\Omega_{T}} \nabla\left(w-w^{I}\right) \nabla \psi^{h}+\int_{\Omega} \nabla\left(y^{I}-y\right)(T) \nabla \psi^{h}(T) \\
& -\int_{\Omega_{T}} \nabla\left(w^{I}-w_{h}\right) \nabla\left(y^{I}-y\right)-\int_{\Omega_{T}}\left(w^{I}-w_{h}\right)\left(w-w^{I}\right) \\
(4.9)= & \sum_{i=1}^{6} E_{i} .
\end{aligned}
$$

Now it remains to estimate $E_{i}, i=1, \cdots, 6$. We note that these error representations are valid for both, the pure time-discrete as well as for the fully discrete case. We now distinguish these cases and proceed with considering the pure time-discrete case. Let $y^{I}$ and $w^{I}$ denote the temporal Ritz projection ([3]) of $y$ and the temporal Lagrange interpolation of $w$, respectively. Then we have the following error estimates:

$$
\begin{align*}
\left|E_{1}\right| & =\left|\int_{\Omega_{T}}\left(y^{I}-y\right)_{t}\left(y^{I}-y_{h}\right)_{t}\right|=0  \tag{4.10}\\
\left|E_{2}\right| & =\left|\frac{1}{\alpha} \int_{\Omega_{T}}\left(y^{I}-y\right)\left(y^{I}-y_{h}\right)\right| \leqslant \alpha^{-1} C h^{2}\|y\|_{H^{2}\left(0, T ; L^{2}(\Omega)\right)}\left\|y^{I}-y_{h}\right\|_{0, \Omega_{T}}  \tag{4.11}\\
\left|E_{6}\right| & =\left|\int_{\Omega_{T}}\left(w^{I}-w_{h}\right)\left(w-w^{I}\right)\right| \leqslant C h^{2}\|w\|_{H^{2}\left(0, T ; L^{2}(\Omega)\right)}\left\|w^{I}-w_{h}\right\|_{0, \Omega_{T}} \tag{4.12}
\end{align*}
$$

Note that $w^{I}$ and $y^{I}$ are continuous w.r.t the space variable. By Green's formula we have

$$
\begin{aligned}
E_{3} & =\int_{\Omega_{T}} \nabla\left(w-w^{I}\right) \nabla\left(y^{I}-y_{h}\right) \\
& =-\int_{\Omega_{T}} \Delta\left(w-w^{I}\right)\left(y^{I}-y_{h}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
E_{5} & =\int_{\Omega_{T}} \nabla\left(y-y^{I}\right) \nabla\left(w^{I}-w_{h}\right) \\
& =-\int_{\Omega_{T}} \Delta\left(y-y^{I}\right)\left(w^{I}-w_{h}\right) .
\end{aligned}
$$

Therefore we deduce

$$
\begin{equation*}
\left|E_{3}\right| \leqslant C h^{2}\|\Delta w\|_{H^{2}\left(0, T ; L^{2}(\Omega)\right)}\left\|y^{I}-y_{h}\right\|_{0, \Omega_{T}} \tag{4.13}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|E_{5}\right| \leqslant C h^{2}\|\Delta y\|_{H^{2}\left(0, T ; L^{2}(\Omega)\right)}\left\|w^{I}-w_{h}\right\|_{0, \Omega_{T}} . \tag{4.14}
\end{equation*}
$$

It remains to estimate $E_{4}$. Following [27] we obtain

$$
\begin{align*}
\left|E_{4}\right| & =\left|\int_{\Omega} \nabla\left(y^{I}-y\right)(T) \nabla \psi^{h}(T)\right| \\
& \leqslant C h^{2}\|y\|_{W^{2, \infty}\left(0, T ; H^{1}(\Omega)\right)}\left|\psi^{h}(T)\right|_{1, \Omega} . \tag{4.15}
\end{align*}
$$

Combining (4.10)-(4.15) we deduce

$$
\begin{aligned}
& \left\|y^{I}-y_{h}\right\|_{H^{1}\left(0, T ; L^{2}(\Omega)\right)}^{2}+\left\|w^{I}-w_{h}\right\|_{0, \Omega_{T}}^{2}+\left|\left(y^{I}-y_{h}\right)(T)\right|_{1, \Omega}^{2} \\
\leqslant & C \alpha^{-1} h^{2}\|y\|_{H^{2}\left(0, T ; L^{2}(\Omega)\right)}\left\|y^{I}-y_{h}\right\|_{0, \Omega_{T}}+C h^{2}\|\Delta w\|_{H^{2}\left(0, T ; L^{2}(\Omega)\right)}\left\|y^{I}-y_{h}\right\|_{0, \Omega_{T}} \\
& +C h^{2}\|y\|_{W^{2, \infty}\left(0, T ; H^{1}(\Omega)\right)}\left|\left(y^{I}-y_{h}\right)(T)\right|_{1, \Omega}+C h^{2}\|\Delta y\|_{H^{2}\left(0, T ; L^{2}(\Omega)\right)}\left\|w^{I}-w_{h}\right\|_{0, \Omega_{T}} \\
& +C h^{2}\|w\|_{H^{2}\left(0, T ; L^{2}(\Omega)\right)}\left\|w^{I}-w_{h}\right\|_{0, \Omega_{T}} .
\end{aligned}
$$

Applications of the triangle inequality and of Young's inequality, combined with error estimates for the Ritz projection and Lagrange interpolation, lead to the following estimate

$$
\left\|y-y_{h}\right\|_{H^{1}\left(0, T ; L^{2}(\Omega)\right)} \leqslant C h, \quad\left\|y-y_{h}\right\|_{0, \Omega_{T}} \leqslant C h^{2}
$$

and

$$
\left\|w-w_{h}\right\|_{0, \Omega_{T}} \leqslant C h^{2}
$$

Now we are in a position to derive the error estimates for the space-time discretization case. Let $y^{I}=R_{h} y \in Y^{h}$ and $w^{I}=R_{h} w \in W_{0}^{h}$ denote the Ritz projections ([3]), which are defined as follows:

$$
\int_{\Omega_{T}} \nabla\left(\varphi-R_{h} \varphi\right) \nabla v_{h}=0, \quad \forall v_{h} \in W_{0}^{h}
$$

for either $\varphi=y$ or $\varphi=w$. Then we have
(4.16) $\left|E_{1}\right|=\left|\int_{\Omega_{T}} \frac{\partial\left(y^{I}-y\right)}{\partial t} \frac{\partial\left(y^{I}-y_{h}\right)}{\partial t}\right| \leqslant C h\|y\|_{H^{2}\left(0, T ; H^{1}(\Omega)\right)}\left\|\frac{\partial\left(y^{I}-y_{h}\right)}{\partial t}\right\|_{0, \Omega_{T}}$,
(4.17) $\left|E_{2}\right|=\left|\frac{1}{\alpha} \int_{\Omega_{T}}\left(y^{I}-y\right)\left(y^{I}-y_{h}\right)\right| \leqslant \alpha^{-1} C h^{2}\|y\|_{H^{2}\left(0, T ; H^{2}(\Omega)\right)}\left\|y^{I}-y_{h}\right\|_{0, \Omega_{T}}$,
(4.18) $\left|E_{3}\right|=\left|\int_{\Omega_{T}} \nabla\left(w-w^{I}\right) \nabla\left(y^{I}-y_{h}\right)\right|=0$,
(4.19) $\left|E_{4}\right|=\left|\int_{\Omega} \nabla\left(y^{I}-y\right)(T) \nabla\left(y^{I}-y_{h}\right)(T)\right| \leqslant C h\|y\|_{H^{2}\left(0, T ; H^{2}(\Omega)\right)}\left|\left(y^{I}-y_{h}\right)(T)\right|_{1, \Omega}$,
(4.20) $\left|E_{5}\right|=\left|\int_{\Omega_{T}} \nabla\left(w^{I}-w_{h}\right) \nabla\left(y^{I}-y\right)\right|=0$,
(4.21) $\left|E_{6}\right|=\left|\int_{\Omega_{T}}\left(w^{I}-w_{h}\right)\left(w-w^{I}\right)\right| \leqslant C h^{2}\|w\|_{H^{2}\left(0, T ; H^{2}(\Omega)\right)}\left\|w^{I}-w_{h}\right\|_{0, \Omega_{T}}$.

From (4.16)-(4.21) we deduce

$$
\begin{aligned}
& \left\|y^{I}-y_{h}\right\|_{H^{1}\left(0, T ; L^{2}(\Omega)\right)}^{2}+\left\|w^{I}-w_{h}\right\|_{0, \Omega_{T}}^{2}+\left|\left(y^{I}-y_{h}\right)(T)\right|_{1, \Omega}^{2} \\
\leqslant & C h\|y\|_{H^{2}\left(0, T ; H^{2}(\Omega)\right)}\left\|\frac{\partial\left(y^{I}-y_{h}\right)}{\partial t}\right\|_{0, \Omega_{T}}+C \alpha^{-1} h^{2}\|y\|_{H^{2}\left(0, T ; H^{2}(\Omega)\right)}\left\|y^{I}-y_{h}\right\|_{0, \Omega_{T}} \\
& +C h\|y\|_{H^{2}\left(0, T ; H^{2}(\Omega)\right)}\left|\left(y^{I}-y_{h}\right)(T)\right|_{1, \Omega}+C h^{2}\|w\|_{H^{2}\left(0, T ; H^{2}(\Omega)\right)}\left\|w^{I}-w_{h}\right\|_{0, \Omega_{T}}
\end{aligned}
$$

Several applications of the triangle inequality and of Young's inequality combined with error estimates for the Ritz projection lead to

$$
\left\|y-y_{h}\right\|_{H^{1}\left(0, T ; L^{2}(\Omega)\right)} \leqslant C h \text { and }\left\|w-w_{h}\right\|_{0, \Omega_{T}} \leqslant C h
$$

Remark 4.2. (Superconvergence, see [28] for results and details) If we could expect superconvergence properties on uniform structured meshes (for example, on rectangular meshes, on three directional triangular meshes in two spatial dimensions, and on uniform brick meshes in three spatial dimensions), in the fully discrete case we expect the improved estimates

$$
\left\|y-y_{h}\right\|_{0, \Omega_{T}} \leqslant C h^{2}|\log h| \quad \text { and } \quad\left\|w-w_{h}\right\|_{0, \Omega_{T}} \leqslant C h^{2}|\log h|
$$

In the following we consider the space-time mixed finite element approximation of the adjoint state $p$. Let $\vartheta:=-\Delta p$, where $p$ denotes the unique solution to (2.7). Then (2.7) motivates the following mixed formulation for the adjoint state $p$ :

$$
\left\{\begin{align*}
-\frac{\partial^{2} p}{\partial t^{2}}-\Delta \vartheta+\frac{1}{\alpha} p=-\frac{\partial y_{d}}{\partial t}+\Delta y_{d} & \text { in } \Omega_{T}  \tag{4.22}\\
\vartheta+\Delta p=0 & \text { in } \Omega_{T} \\
p=0 & \text { on } \Sigma_{T} \\
-\vartheta=y_{d} & \text { on } \Sigma_{T} \\
\left(\frac{\partial p}{\partial t}+\Delta p\right)(0)=y_{d}(0)-y_{0} & \text { in } \Omega \\
p(T)=0 & \text { in } \Omega
\end{align*}\right.
$$

Let

$$
\begin{aligned}
\widetilde{Y}: & =\left\{p \in H^{1}\left(0, T ; H_{0}^{1}(\Omega)\right): p(T)=0 \text { in } \Omega\right\} \\
\widetilde{W}: & =L^{2}\left(0, T ; H^{1}(\Omega)\right)
\end{aligned}
$$

The mixed variational formulation related to (4.22) is to find $p \in \tilde{Y}, \vartheta+y_{d} \in W$ such that:

$$
\left\{\begin{array}{c}
\int_{\Omega_{T}}\left(\frac{\partial p}{\partial t} \frac{\partial v}{\partial t}+\frac{1}{\alpha} p v\right)+\int_{\Omega_{T}} \nabla \vartheta \nabla v+\int_{\Omega} \nabla p(0) \nabla v(0)  \tag{4.23}\\
=\int_{\Omega_{T}}\left(-\frac{\partial y_{d}}{\partial t}+\Delta y_{d}\right) v-\int_{\Omega}\left(y_{d}(0)-y_{0}\right) v(0) \quad \forall v \in \tilde{Y} \\
-\int_{\Omega_{T}} \nabla p \nabla \phi+\int_{\Omega_{T}} \vartheta \phi=0 \quad \forall \phi \in W
\end{array}\right.
$$

According to (4.23) the mixed finite element approximation of the adjoint state $p$ is to find $p_{h} \in$ $\widetilde{Y}_{0}^{h}, \vartheta_{h} \in V^{h}$ and $\left.\vartheta_{h}\right|_{\Gamma}=-\widetilde{y}_{d}$ such that

$$
\left\{\begin{array}{r}
\int_{\Omega_{T}}\left(\frac{\partial p_{h}}{\partial t} \frac{\partial v_{h}}{\partial t}+\frac{1}{\alpha} p_{h} v_{h}\right)+\int_{\Omega_{T}} \nabla \vartheta_{h} \nabla v_{h}+\int_{\Omega} \nabla p_{h}(0) \nabla v_{h}(0)  \tag{4.24}\\
=\int_{\Omega_{T}}\left(-\frac{\partial y_{d}}{\partial t}+\Delta y_{d}\right) v_{h}-\int_{\Omega}\left(y_{d}(0)-y_{0}\right) v_{h}(0), \quad \forall v_{h} \in \widetilde{Y}_{0}^{h} \\
-\int_{\Omega_{T}} \nabla p_{h} \nabla \phi_{h}+\int_{\Omega_{T}} \vartheta_{h} \phi_{h}=0, \quad \forall \phi_{h} \in W_{0}^{h}
\end{array}\right.
$$

holds, where $\widetilde{Y}_{0}^{h}=V^{h} \cap \tilde{Y}$ and $\widetilde{y}_{d} \in V^{h}$ is an approximation of $y_{d}$.
By arguments similar to those applied in the analysis of the state $y$ we get
Theorem 4.3. Let $(p, \vartheta) \in \widetilde{Y} \times \widetilde{W}$ and $\left(p_{h}, \vartheta_{h}\right) \in \widetilde{Y}_{0}^{h} \times V^{h}$ denote the solutions to (4.23) and (4.24), respectively. In the pure time-discrete case, we for $p \in H^{2}\left(0, T ; H^{4}(\Omega)\right) \cap W^{2, \infty}\left(0, T ; H^{1}(\Omega)\right)$ have

$$
\left\|p-p_{h}\right\|_{H^{1}\left(0, T ; L^{2}(\Omega)\right)} \leqslant C h, \quad\left\|p-p_{h}\right\|_{0, \Omega_{T}} \leqslant C h^{2} \quad \text { and } \quad\left\|\vartheta-\vartheta_{h}\right\|_{0, \Omega_{T}} \leqslant C h^{2} .
$$

In the fully discrete case we for $p \in H^{2}\left(0, T ; H^{4}(\Omega)\right)$ have

$$
\left\|p-p_{h}\right\|_{H^{1}\left(0, T ; L^{2}(\Omega)\right)} \leqslant C h \quad \text { and } \quad\left\|\vartheta-\vartheta_{h}\right\|_{0, \Omega_{T}} \leqslant C h .
$$

Remark 4.4. In allignement to Remark 4.2 we for the fully discrete space in the case of superconvergencemeshes expect the estimates

$$
\left\|p-p_{h}\right\|_{0, \Omega_{T}} \leqslant C h^{2}|\log h| \quad \text { and } \quad\left\|\vartheta-\vartheta_{h}\right\|_{0, \Omega_{T}} \leqslant C h^{2}|\log h| .
$$

Remark 4.5. By exploiting the respective second identites in (4.2) and (4.4), the estimates of Theorem 4.1 also yield

$$
\left\|y-y_{h}\right\|_{H^{1}\left(\Omega_{T}\right)} \leqslant C h
$$

in the pure time-discrete case, and

$$
\left\|y-y_{h}\right\|_{H^{1}\left(\Omega_{T}\right)} \leqslant C h^{\frac{1}{2}}
$$

for space-time discretization. Furthermore, we can expect

$$
\left\|y-y_{h}\right\|_{H^{1}\left(\Omega_{T}\right)} \leqslant C h \sqrt{|\log h|}
$$

for space-time discretization on uniform meshes by using the results of Remark 4.4. We note that similar results also apply to the error estimates for the adjoint state $p$.

## 5. Numerical examples

In this section, we will carry out some numerical examples to confirm our theoretical results. Firstly, we test the convergence order of the space-time mixed finite element scheme presented in Section 4. Secondly, we investigate the effectiveness of the temporal a posteriori error estimates $\eta_{y}$ and $\eta_{p}$ developed in Section 3. In both cases a space-time mixed finite element method is used to solve the space-time elliptic equations. The adaptive procedure is built upon an initial time mesh, together with a refinement strategy based on the local a posteriori error estimator $\eta_{\phi, I_{n}}$ ( $\phi=y$ or $p$ ) related to the interval $I_{n}$ together with bulk marking, i.e. all intervalls $I_{n}$ satisfying

$$
\eta_{\phi, I_{n}} \geqslant \theta \max \left\{\eta_{\phi, I_{n}}\right\},
$$

are refined, where $\theta \in(0,1)$. The effectivity index $E I$ is defined as

$$
E I=\frac{\eta_{\phi}}{\left\|\phi-\phi_{k}\right\|_{H^{1}\left(0, T ; L^{2}(\Omega)\right)}}
$$

where $\phi=y$ or $p$.

Table 1. Errors in $y$ and $w$ for Example 5.1 on a uniform rectangular mesh with fixed space mesh size.

| Dof | $\left\\|w-w_{h}\right\\|_{0, \Omega_{T}}$ | order | $\left\\|y-y_{h}\right\\|_{0, \Omega_{T}}$ | order | $\left\\|y-y_{h}\right\\|_{H^{1}\left(0, T ; L^{2}(\Omega)\right)}$ | order |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 2565 | 0.130512029211 | $\backslash$ | 0.013224854142 | $\backslash$ | 0.362590027287 | $\backslash$ |
| 4617 | 0.027382620124 | 2.2529 | 0.002775737620 | 2.2523 | 0.179149675402 | 1.0172 |
| 8721 | 0.006451392985 | 2.0856 | 0.000654996906 | 2.0833 | 0.089179789894 | 1.0064 |
| 16929 | 0.001594433608 | 2.0166 | 0.000162900612 | 2.0075 | 0.044535562504 | 1.0018 |

TABLE 2. Errors of state $y$ and $w$ for Example 5.1 on a uniform rectangle mesh.

| Dof | $\left\\|w-w_{h}\right\\|_{0, \Omega_{T}}$ | order | $\left\\|y-y_{h}\right\\|_{0, \Omega_{T}}$ | order | $\left\\|y-y_{h}\right\\|_{H^{1}\left(0, T ; L^{2}(\Omega)\right)}$ | order |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 25 | 0.238351935729 | $\backslash$ | 0.047527619881 | $\backslash$ | 0.365097724548 | $\backslash$ |
| 81 | 0.058703812693 | 2.0216 | 0.012226170429 | 1.9588 | 0.179271813135 | 1.0261 |
| 289 | 0.014624795001 | 2.0050 | 0.003078755763 | 1.9896 | 0.089187811733 | 1.0072 |
| 1089 | 0.003657974813 | 1.9993 | 0.000771587704 | 1.9964 | 0.044536635124 | 1.0019 |

Table 3. Errors of state $y$ and $w$ for Example 5.1 on general triangle meshes.

| Dof | $\left\\|w-w_{h}\right\\|_{0, \Omega_{T}}$ | order | $\left\\|y-y_{h}\right\\|_{0, \Omega_{T}}$ | order | $\left\\|y-y_{h}\right\\|_{H^{1}\left(0, T ; L^{2}(\Omega)\right)}$ | order |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 141 | 0.038492500552 | $\backslash$ | 0.008262001121 | $\backslash$ | 0.200428019476 | $\backslash$ |
| 521 | 0.015827469735 | 1.3599 | 0.002168764945 | 2.0467 | 0.099641508687 | 1.0697 |
| 2001 | 0.007415718014 | 1.1268 | 0.000556931706 | 2.0205 | 0.049566630052 | 1.0378 |
| 7841 | 0.003527879682 | 1.0879 | 0.000141382932 | 2.0077 | 0.024676235560 | 1.0214 |

TABLE 4. Errors of state $y$ and $w$ for Example 5.1 on uniform triangle meshes.

| Dof | $\left\\|w-w_{h}\right\\|_{0, \Omega_{T}}$ | order | $\left\\|y-y_{h}\right\\|_{0, \Omega_{T}}$ | order | $\left\\|y-y_{h}\right\\|_{H^{1}\left(0, T ; L^{2}(\Omega)\right)}$ | order |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 25 | 0.586731483264 | $\backslash$ | 0.097380683188 | $\backslash$ | 0.623236266193 | $\backslash$ |
| 81 | 0.187433063539 | 1.6463 | 0.029508172382 | 1.7225 | 0.315028169663 | 0.9843 |
| 289 | 0.060557215241 | 1.6300 | 0.007845736087 | 1.9111 | 0.155997022609 | 1.0140 |
| 1089 | 0.021473108657 | 1.4958 | 0.001991686518 | 1.9779 | 0.077531227695 | 1.0087 |

Example 5.1. Let $\Omega=[0,1], T=1$ so that $\Omega_{T}=(0,1) \times(0,1)$. We choose an exact solution ( $y, u$ ) of our optimal control problem (2.1) together with an exact adjoint state $p$ as follows

$$
y(x, t)=\sin (\pi x) \cos (\pi t), \quad p(x, t)=\sin (\pi x) \sin (\pi t), \quad u(x, t)=-\sin (\pi x) \sin (\pi t)
$$

Here, the corresponding desired state $y_{d}$ and the right hand side $f$ have to be chosen accordingly.
We first consider the behavior of the error with respect to time. The domain is triangulated into rectangles and bilinear finite elements are applied to approximate the state y. The uniform space mesh has fixed size $1 / 512$, while the time mesh is refined uniformly. In Table 1 the results for the experimental order of convergence are presented for the errors $\left\|y-y_{h}\right\|_{H^{1}\left(0, T ; L^{2}(\Omega)\right)},\left\|w-w_{h}\right\|_{0, \Omega_{T}}$ and $\left\|y-y_{h}\right\|_{0, \Omega_{T}}$. We find errors of order $O(h), O\left(h^{2}\right)$ and $O\left(h^{2}\right)$, respectively, which is in agreement with the theoretical analysis of Theorem 4.1.

Next, in Table 2 we present the error of the states under uniform refinement with respect to a uniform rectangular space-time mesh. In this case, superconvergence properties may be expected, compare Remark 4.2. In fact, we observe $O\left(h^{2}\right)$ convergence for $\left\|w-w_{h}\right\|_{0, \Omega_{T}}$ and also for $\left\|y-y_{h}\right\|_{0, \Omega_{T}}$. In the subsequent tables 3 and 4 we present numerical results for the space-time discretization on general triangle meshes, and on uniform triangle meshes, respectively. One can see that we for general triangle meshes obtain the convergence predicted by Theorem 4.1, whereas on uniform triangle meshes we obtain an error of order $O\left(h^{3 / 2}\right)$.

Table 5. Errors of state $y$ for Example 5.2 on uniform and adaptive meshes with the same fixed space mesh size and $\varepsilon=1.0 e-003$.

| Uniform | $\left\\|y-y_{h}\right\\|_{0, \Omega_{T}}$ | $\left\\|w-w_{h}\right\\|_{0, \Omega_{T}}$ | Adaptive | $\left\\|y-y_{h}\right\\|_{0, \Omega_{T}}$ | $\left\\|w-w_{h}\right\\|_{0, \Omega_{T}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 2121 | 0.1847 | 1.8231 | 2121 | 0.0158 | 0.1557 |
| 4141 | 0.1283 | 1.2671 | 4343 | 0.0011 | 0.0108 |
| 8181 | 0.0755 | 0.7451 | 8181 | $7.2029 \mathrm{e}-004$ | 0.0068 |
| 10201 | 0.0555 | 0.5481 | 10605 | $2.7602 \mathrm{e}-004$ | 0.0022 |



Figure 1. The surfaces and contour lines for the states $y_{h}$ and $w_{h}$ on uniform meshes for Example 5.2 with $\varepsilon=1.0 e-3$.

Example 5.2. In this example we investigate the effectiveness of the a posteriori error estimator for the state $y$ presented in Theorem 3.3. Let $\Omega_{T}=(0,1) \times(0,1)$. The exact solutions are chosen as

$$
y(x, t)=\sin (\pi x) \operatorname{atan}((t-1 / 2) / \varepsilon), \quad p(x, t)=\sin (\pi x) \sin (\pi t), \quad u(x, t)=-\sin (\pi x) \sin (\pi t)
$$

where the corresponding desired state $y_{d}$ and right hand side $f$ have to be chosen accordingly.
The state $y$ developes an interior layer at $t=1 / 2$ for small $\varepsilon$. We use the a posteriori error estimator $\eta_{y}$ of Theorem 3.3 to construct the adaptive mesh for the state $y$.

The surfaces and contour lines for the state $y$ on the uniform and on the adaptive mesh are presented in Figure 1 and 2, respectively. The errors of the state $y$ on uniform and adaptive meshes are displayed in Table 5. The error reduction for the state $y$ on the uniform and on the adaptive mesh is shown in Figure 4. We observe that the adaptive mesh largely improves the quality of the numerical solutions. Figure 3 displays the adaptive mesh and documents the behaviors of the error estimators. We deduce, that our a posteriori error indicator $\eta_{y}$ captures the singular layer exactly, with an effectivity index $E I$ close to 2 .

All the numerical results clearly indicate that the a posteriori error estimator for the state $y$ is reliable and efficient, and that the use of adaptive meshes can heavily improve the behavior of numerical solutions and also save substantially calculation time.
Example 5.3. In this example we investigate the behavior of the a posteriori error estimator for the adjoint state $p$ developed in Theorem 3.5. Again $\Omega_{T}=(0,1) \times(0,1)$. The exact solutions are


Figure 2. The surfaces and contour lines of the states $y_{h}$ and $w_{h}$ on adaptive meshes for Example 5.2 with $\varepsilon=1.0 e-3$.


Figure 3. The adaptive mesh (left) and the performance of the a posteriori error estimator (right) for the state $y$.

Table 6. Values of $\eta_{p}^{i}$ and $\eta_{p}^{b}$ for Example 5.3 on time-adaptive meshes with a fixed space mesh (left with size $\frac{1}{100}$, right with size $\frac{1}{150}$ ) and $\varepsilon=1.0 e-004$.

| Adaptive | $\eta_{p}^{i}$ | $\eta_{p}^{b}$ | Adaptive | $\eta_{p}^{i}$ | $\eta_{p}^{b}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 3200 | 2.4717 | $3.4169 \mathrm{e}-004$ | 6600 | 1.5436 | $1.2586 \mathrm{e}-004$ |
| 5000 | 1.0833 | $4.8034 \mathrm{e}-005$ | 8607 | 0.9511 | $4.3404 \mathrm{e}-005$ |
| 7100 | 0.6553 | $2.7031 \mathrm{e}-005$ | 10268 | 0.7285 | $2.9835 \mathrm{e}-005$ |
| 9600 | 0.4549 | $2.2352 \mathrm{e}-005$ | 14647 | 0.4605 | $2.2354 \mathrm{e}-005$ |

chosen as

$$
y(x, t)=\sin (\pi x) \sin (\pi t), \quad p(x, t)=x(x-1)\left(t-\frac{e^{(t-1) / \varepsilon}-e^{-1 / \varepsilon}}{1-e^{-1 / \varepsilon}}\right), \quad u(x, t)=-p(x, t)
$$

where the desired state $y_{d}$ and right hand side $f$ have to be chosen accordingly.


Figure 4. The error reduction for the state $y$ on uniform and adaptive meshes.
Table 7. Errors of adjoint state $p$ for Example 5.3 on uniform and adaptive time-meshes with a fixed space mesh and $\varepsilon=1.0 e-003$.

| Uniform | $\left\\|p-p_{h}\right\\|_{0, \Omega_{T}}$ | $\left\\|\vartheta-\vartheta_{h}\right\\|_{0, \Omega_{T}}$ | Adaptive | $\left\\|p-p_{h}\right\\|_{0, \Omega_{T}}$ | $\left\\|\vartheta-\vartheta_{h}\right\\|_{0, \Omega_{T}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 2626 | 0.0200 | 0.2196 | 2727 | $4.6762 \mathrm{e}-005$ | $4.8235 \mathrm{e}-004$ |
| 4141 | 0.0175 | 0.1873 | 4343 | $1.7571 \mathrm{e}-005$ | $1.4911 \mathrm{e}-004$ |
| 6161 | 0.0124 | 0.1337 | 6161 | $1.4466 \mathrm{e}-005$ | $1.1573 \mathrm{e}-004$ |
| 9191 | 0.0087 | 0.0955 | 9090 | $1.0441 \mathrm{e}-005$ | $6.7456 \mathrm{e}-005$ |

The adjoint state $p$ in this example developes a boundary layer at $t=1$ if $\varepsilon$ is small. We use the a posteriori error estimator $\eta_{p}$ of Theorem 3.5 to construct adaptive time meshes for the adjoint state $p$. We denote by $\eta_{p}^{i}$ and $\eta_{p}^{b}$, respectively the first and second contribution of the a posteriori error estimator $\eta_{p}$. The values of $\eta_{p}^{i}$ and $\eta_{p}^{b}$ on adaptive times meshes on a fixed space mesh are presented in Table 6. We observe that $\eta_{p}^{i}$ dominates the a posteriori error estimator $\eta_{p}$ and that the contribution of $\eta_{p}^{b}$ is, as expected, negligible.

The surfaces and contour lines for the adjoint state $p$ on uniform and adaptive meshes are shown in Figure 5 and 6. The errors of the adjoint $p$ on uniform and adaptive meshes are presented in Table 7. The error reduction for the adjoint state $p$ on uniform and adaptive meshes is documented in Figure 8. We can conclude that the use of adaptive meshes improves the numerical solutions and saves substantially computation time.

The adaptive mesh and the behaviors of errors and error estimators are displayed in Figure 7. We see that the a posteriori error estimator $\eta_{p}$ exactly captures the singular boundary layer with effectivity index $E I$ of approximately 2 .

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Figure 5. The surfaces and contour lines of the adjoint state $p_{h}$ (top) and of $\vartheta_{h}$ (bottom) on uniform a uniform mesh for Example 5.3 with $\varepsilon=1.0 e-4$.


Figure 6. The surfaces and contour lines of $p_{h}($ top $)$ and $\vartheta_{h}$ (bottom) on adaptive meshes for Example 5.3 with $\varepsilon=1.0 e-4$.
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Figure 7. The adaptive mesh (left) and the performance of the a posteriori error estimator (right) for the adjoint state $p$.


Figure 8. The error reduction for adjoint state $p$ on uniform and adaptive meshes.

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