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Abstract: We analyze a numerical method for solving the inverse problem of identifying the diffusion matrix in an elliptic PDE from distributed noisy measurements. We use a regularized least squares approach in which the state equations are given by a finite element discretization of the elliptic PDE. The unknown matrix parameters act as control variables and are handled with the help of variational discretization as introduced in [8]. For a suitable coupling of Tikhonov regularization parameter, finite element grid size and noise level we are able to prove L^2–convergence of the discrete solutions to the unique norm–minimal solution of the identification problem; corresponding convergence rates can be obtained provided that a suitable projected source condition is fulfilled. Finally, we present a numerical experiment which supports our theoretical findings.

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1 Introduction

In this paper we are concerned with a convergence analysis for a numerical method that identifies a diffusion matrix in the elliptic boundary value problem

$$-\text{div }(A\nabla y) = f \text{ in } \Omega,\ y = 0 \text{ on } \partial\Omega \tag{1.1}$$

from distributed noisy measurements. Here, $\Omega \subset \mathbb{R}^n$ is a bounded, polyhedral domain and $f \in H^{-1}(\Omega)$. In addition we assume that the diffusion matrix $A(x) = (a_{ij}(x))_{i,j=1}^n$ satisfies $a_{ij} \in L^\infty(\Omega)$ and is uniformly elliptic. The boundary value problem (1.1) then has a unique weak solution $y \in H_0^1(\Omega)$ which we denote by $y = T(A, f)$. Our aim is to identify the unknown diffusion matrix from distributed noisy measurements $(z_\delta^{(i)}, f_\delta^{(i)}) \in L^2(\Omega)\times H^{-1}(\Omega), 1 \le i \le N$ satisfying

$$\|z_\delta^{(i)} - z^{(i)}\| \le \delta, \quad \|f_\delta^{(i)} - f^{(i)}\|_{H^{-1}} \le \delta. \tag{1.2}$$

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Here,

$$z^{(i)} = T(A^*, f^{(i)}), \quad 1 \le i \le N \tag{1.3}$$

for some symmetric diffusion matrix A^* with

$$a|\xi|^2 \le A^*(x)\xi \cdot \xi \le b|\xi|^2 \quad \text{ for all } \xi \in \mathbb{R}^n, \text{ a.e. in } \Omega, \tag{1.4}$$

where $0 < a < b < \infty$. We shall employ a least squares approach in order to reconstruct the diffusion matrix, more precisely we consider the following optimization problem:

$$(P) \quad \min_{A \in \mathcal{M}} J(A) := \frac{1}{2} \sum_{i=1}^{N} \|y^{(i)} - z_\delta^{(i)}\|^2 + \frac{\gamma}{2}\|A\|^2 \quad \text{ subject to } y^{(i)} = T(A, f_\delta^{(i)}), 1 \le i \le N,$$

where $\gamma > 0$ and we use the symbol $\|\cdot\|$ for the L^2-norm of scalar, vector- or matrix-valued functions. The set $\mathcal{M}$ of admissible diffusion matrices will be defined in Section 2 below. One of the main difficulties in analyzing (P) is that the mapping $A \mapsto y = T(A, f)$ is not weakly (sequentially) closed in $L^2(\Omega, \mathbb{R}^{n,n})$. This can be seen with the help of the following example taken from [16, Section 3]: let $\Omega = (0,1) \subset \mathbb{R}$ and $a_k \in L^\infty(\Omega)$ be defined by

$$a_k(x) := \begin{cases} a, & \frac{m}{k} \le x < \frac{m+\frac{1}{2}}{k} \\ b, & \frac{m+\frac{1}{2}}{k} \le x < \frac{m+1}{k} \end{cases} \qquad m = 0, \ldots, k-1.$$

If we let $y_k = T(a_k, f)$ it can be shown that $y_k \to y = T(\hat{a}, f)$ in $L^2(\Omega)$, where $\hat{a} \equiv \left(\frac{1}{2}(\frac{1}{a} + \frac{1}{b})\right)^{-1}$. On the other hand, $a_k \rightharpoonup a$ in $L^2(\Omega)$ with $a \equiv \frac{1}{2}(a+b)$, so that $y \ne T(a, f)$, since $\hat{a} \ne a$.

One possibility to overcome this problem is to use a stronger norm in the Tikhonov regularization (such as $\|\cdot\|_{H^1}$, see e.g. [15], [20]), but this approach can be numerically cumbersome especially when one takes ellipticity constraints into account. In [4] we were able to prove the existence of a solution to (P) by applying the concept of H–convergence ([16]). Furthermore, we considered an approximation of (P) by discretizing (1.1) with the help of finite elements and established the convergence of corresponding minimizers to a minimum of (P) in the case that $\gamma > 0$ is fixed and $z_\delta^{(i)} = z^{(i)}, 1 \le i \le N$. The goal of the present work is to extend the convergence analysis to the case that the regularization parameter γ tends to zero, while we also take into account noisy measurements satisfying (1.2). Denoting by (P_h^δ) our approximation of (P) with corresponding minimum A_h^δ we shall prove in Section 3 that $A_h^\delta \to \bar{A}$ in L^2 as the mesh size h and the noise level δ tend to zero provided that γ is coupled to these parameters in a suitable way. Here, $\bar{A} \in \mathcal{M}$ denotes the norm-minimal (in the L^2–sense) diffusion matrix satisfying (1.3). Under a suitable projected source condition and appropriate smoothness conditions on the data we then show in Section 4 that an error bound of the form

$$\|A_h^\delta - \bar{A}\| \le Ch$$

holds. Section 5 presents a numerical test calculation which supports the rates obtained.

Let us briefly refer to related publications that have been concerned with the identification of matrix–valued parameters. In [1] and [9] a reconstructed matrix is obtained as the large–time limit of a suitable dynamical system. A stability result, which can also be used for the convergence analysis of numerical methods, is derived in [11] by Hsaio and Sprekels for the reconstruction of matrices of the form $A = \nabla p \otimes \nabla p$. In [13], Kohn and Lowe introduce a variational method involving a functional which is convex in the matrix A and the conductivities $A\nabla y^{(i)}$. Rannacher and Vexler prove in [17] error estimates for a matrix–identification problem in which a finite number of unknown parameters is estimated from finitely many pointwise observations.

The problem of identifying a scalar diffusion coefficient has been investigated much more intensively compared to the matrix case. Identifiability results have been obtained e.g. in [3], [18] and [19]. A survey of numerical methods for parameter estimation problems can be found in [14]. Error estimates for a least squares approach have been obtained by Falk in [6] and more recently by Wang and Zou [20] taking into account Tikhonov regularization. The latter paper also contains a long list of further references.

2 Notation and preliminary results

Let us denote by $\mathcal{S}_n$ the set of all symmetric $n \times n$ matrices equipped with the inner product $A \cdot B = \mathrm{trace}(AB)$. We introduce the subset

$$K := \{A \in \mathcal{S}_n \,|\, a|\xi|^2 \le A\xi \cdot \xi \le b|\xi|^2 \text{ for all } \xi \in \mathbb{R}^n\}.$$

Since K is a convex and closed subset of $\mathcal{S}_n$ we may define the orthogonal projection $P_K : \mathcal{S}_n \to K$, which is characterised by the relation

$$(A - P_K(A)) \cdot (B - P_K(A)) \le 0 \quad \text{for all } B \in K. \tag{2.1}$$

We define the admissible set for the optimization problem (P) by

$$\mathcal{M} := \{A \in L^\infty(\Omega, \mathbb{R}^{n,n}) \,|\, A(x) \in K \text{ a.e. in } \Omega\} \subset L^\infty(\Omega, \mathcal{S}_n).$$

According to [4, Theorem 2.2], (P) has a solution $A \in \mathcal{M}$. For the convenience of the reader we include the necessary first order conditions satisfied by A. To begin, consider the mappings

$$F_i : \mathcal{M} \to L^2(\Omega), \quad F_i(A) := T(A, f_\delta^{(i)}), \quad 1 \le i \le N.$$

It is not difficult to verify that

$$F_i'(A)H = w^{(i)}, \quad H \in L^\infty(\Omega, \mathcal{S}_n),$$

where $w^{(i)} \in H_0^1(\Omega)$ is the unique weak solution of the elliptic equation

$$\int_\Omega A\nabla w^{(i)} \cdot \nabla v dx = -\int_\Omega H\nabla y^{(i)} \cdot \nabla v dx \quad \text{for all } v \in H_0^1(\Omega) \tag{2.2}$$

and $y^{(i)} = T(A, f_\delta^{(i)})$. Denoting by $p^{(i)} \in H_0^1(\Omega), i = 1, \ldots, N$ the solutions of the adjoint problems

$$\int_\Omega A\nabla p^{(i)} \cdot \nabla v dx = \int_\Omega (y^{(i)} - z_\delta^{(i)}) v dx \quad \text{for all } v \in H_0^1(\Omega),$$

we have for $H \in L^\infty(\Omega, \mathcal{S}_n)$

$$
\begin{aligned}
& J'(A)H && (2.3)\\
= \;& \sum_{i=1}^N \int_\Omega (y^{(i)} - z_\delta^{(i)}) w^{(i)} dx + \gamma \int_\Omega A \cdot H dx = \sum_{i=1}^N \int_\Omega A \nabla p^{(i)} \cdot \nabla w^{(i)} dx + \gamma \int_\Omega A \cdot H dx \\
= \;& -\sum_{i=1}^N \int_\Omega H \nabla y^{(i)} \cdot \nabla p^{(i)} dx + \gamma \int_\Omega A \cdot H dx = \int_\Omega \big(-\sum_{i=1}^N \nabla y^{(i)} \otimes \nabla p^{(i)} + \gamma A \big) \cdot H dx,
\end{aligned}
$$

where $(p \otimes q)_{ij} = \frac{1}{2}(p_i q_j + p_j q_i), i, j = 1, \ldots, n$ for $p, q \in \mathbb{R}^n$. We now have

Lemma 2.1. *Let $\gamma > 0$ and $A \in \mathcal{M}$ be a solution of (P). Then*

$$
A(x) = P_K \left(\frac{1}{\gamma} \sum_{i=1}^N \nabla y^{(i)}(x) \otimes \nabla p^{(i)}(x) \right) \qquad \textit{a.e. in } \Omega.
$$

Proof. The optimality of A yields $J'(A)(B - A) \geq 0$ for all $B \in \mathcal{M}$, which can be rewritten with the help of (2.3) as

$$
\int_\Omega \big(\frac{1}{\gamma} \sum_{i=1}^N \nabla y^{(i)} \otimes \nabla p^{(i)} - A \big) \cdot \big(B - A \big) dx \leq 0, \qquad \text{for all } B \in \mathcal{M}.
$$

A localization argument then shows that

$$
\left(\frac{1}{\gamma} \sum_{i=1}^N \nabla y^{(i)}(x) \otimes \nabla p^{(i)}(x) - A(x) \right) \cdot \big(C - A(x) \big) \leq 0 \qquad \text{for all } C \in K \text{ and a.a. } x \in \Omega
$$

which implies the result. ∎

Next, let $\mathcal{T}_h$ be a regular triangulation of Ω with mesh size $h := \max_{T \in \mathcal{T}_h} \operatorname{diam}(T)$. Let us denote by

$$
X_h := \{ v_h \in C^0(\bar{\Omega}) \,|\, v_h \text{ is a linear polynomial on each } T \in \mathcal{T}_h, v_{h|\partial\Omega} = 0 \} \subset H_0^1(\Omega)
$$

the space of continuous, piecewise linear finite elements. For a given $A \in \mathcal{M}$ and $f \in H^{-1}(\Omega)$, the problem

$$
\int_\Omega A \nabla y_h \cdot \nabla v_h dx = \langle f, v_h \rangle \quad \text{ for all } v_h \in X_h
$$

has a unique solution $y_h \in X_h$ which we denote by $y_h = T_h(A, f)$. Here, $\langle \cdot, \cdot \rangle$ is the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Standard arguments yield the following bounds

$$
\|T_h(A, f)\|_{H^1} \leq C \|f\|_{H^{-1}}, \tag{2.4}
$$

$$
\|T_h(A, f) - T(A, f)\|_{H^1} \leq C \inf_{v_h \in X_h} \|T(A, f) - v_h\|_{H^1} \tag{2.5}
$$

with a constant C that is independent of $f \in H^{-1}(\Omega)$, $A \in \mathcal{M}$ and the mesh size h.
We are now in position to formulate our numerical method which is based on solving the following semi-discrete minimization problem:

$$
(P_h^\delta) \quad \min_{A \in \mathcal{M}} J_h^\delta(A) := \frac{1}{2} \sum_{i=1}^N \|y_h^{(i)} - z_\delta^{(i)}\|^2 + \frac{\gamma}{2} \|A\|^2 \quad \text{subject to } y_h^{(i)} = T_h(A, f_\delta^{(i)}), 1 \leq i \leq N. \tag{2.6}
$$

Note that following the idea of [8] the matrix parameters are not discretized, compare however Remark 2.3 below.

Lemma 2.2. *Problem* (P_h^δ) *has a solution* $A_h^\delta \in \mathcal{M}$. *Every solution* $A \in \mathcal{M}$ *of* (P_h^δ) *satisfies*

$$A(x) = P_K \left(\frac{1}{\gamma} \sum_{i=1}^{N} \nabla y_h^{(i)}(x) \otimes \nabla p_h^{(i)}(x) \right) \quad a.e.\ in\ \Omega, \tag{2.7}$$

where $y_h^{(i)} = T_h(A, f_\delta^{(i)})$ *and* $p_h^{(i)} \in X_h$ *are the solutions of the adjoint problems*

$$\int_\Omega A\nabla p_h^{(i)} \cdot \nabla v_h dx = \int_\Omega (y_h^{(i)} - z_\delta^{(i)}) v_h dx \quad \text{for all } v_h \in X_h, 1 \le i \le N.$$

Proof. Let $(A_k)_{k\in\mathbb{N}} \subseteq \mathcal{M}$ be a minimizing sequence for problem (P_h^δ) so that $J_h^\delta(A_k) \searrow \inf_{A\in\mathcal{M}} J_h^\delta(A)$ as $k \to \infty$. Since $(A_k)_{k\in\mathbb{N}}$ is bounded in $L^\infty(\Omega, \mathbb{R}^{n,n})$ there exists $A_h^\delta \in L^\infty(\Omega, \mathbb{R}^{n,n})$ such that $A_{k'} \overset{*}{\rightharpoonup} A_h^\delta$ in $L^\infty(\Omega, \mathbb{R}^{n,n})$ for some subsequence. In addition one readily verifies that $A_h^\delta \in \mathcal{M}$. The sequences $y_{k'}^{(i)} = T_h(A_{k'}, f_\delta^{(i)}), 1 \le i \le N$ are uniformly bounded in the finite-dimensional space X_h so that we may assume that $y_{k'}^{(i)} \to y_h^{(i)}$ in $H^1(\Omega), 1 \le i \le N$. Clearly $y_h^{(i)} = T_h(A_h^\delta, f_\delta^{(i)})$ and therefore

$$\begin{aligned} J_h^\delta(A_h^\delta) &= \frac{1}{2}\sum_{i=1}^{N} \|y_h^{(i)} - z_\delta^{(i)}\|^2 + \frac{\gamma}{2}\|A_h^\delta\|^2 \le \lim_{k'\to\infty} \frac{1}{2}\sum_{i=1}^{N} \|y_{k'}^{(i)} - z_\delta^{(i)}\|^2 + \frac{\gamma}{2} \liminf_{k'\to\infty} \|A_{k'}\|^2 \\ &\le \liminf_{k'\to\infty} J_h^\delta(A_{k'}) = \inf_{A\in\mathcal{M}} J_h^\delta(A). \end{aligned}$$

The relation (2.7) is obtained exactly as in the continuous case. ∎

Remark 2.3. Let us note that in view of (2.7) A_h^δ is piecewise constant on $\mathcal{T}_h$ so that a discretization of the set $\mathcal{M}$ is not required. Variational discretization automatically yields solutions to (2.6) which allow a finite-dimensional representation.

In order to analyze the convergence of the above method we shall make use of the concept of Hd-convergence. This concept was introduced in [5] in the context of finite volume discretizations of elliptic boundary value problems with the aim of adapting H–convergence results to the discrete setting. The following theorem is a finite element version of the result obtained in [5].

Theorem 2.4. *Let* $(A_k)_{k\in\mathbb{N}}$ *be a sequence in* $\mathcal{M}$ *and* $(\mathcal{T}_{h_k})_{k\in\mathbb{N}}$ *a sequence of triangulations with* $\lim_{k\to\infty} h_k = 0$. *Then there exists a subsequence* $(A_{k'})_{k'\in\mathbb{N}}$ *and* $A \in \mathcal{M}$ *such that for every* $f \in H^{-1}(\Omega)$

$$T_{h_{k'}}(A_{k'}, f) \rightharpoonup T(A, f) \text{ in } H_0^1(\Omega) \text{ and } A_{k'}\nabla T_{h_{k'}}(A_{k'}, f) \rightharpoonup A\nabla T(A, f) \text{ in } L^2(\Omega)^n.$$

We then say that the sequence $(A_{k'})_{k'\in\mathbb{N}}$ *Hd–converges to* A *and write* $A_{k'} \overset{Hd}{\to} A$.

Proof. See [4, Theorem 3.1]. ∎

Please note that the Hd–limit in general will depend on the sequence $(\mathcal{T}_{h_k})_{k\in\mathbb{N}}$.

Corollary 2.5. *Let $(A_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{M}$ and $(\mathcal{T}_{h_k})_{k\in\mathbb{N}}$ a sequence of triangulations with $\lim_{k\to\infty} h_k = 0$. Suppose that $A_k \stackrel{Hd}{\to} A_0$ and $A_k \stackrel{*}{\rightharpoonup} A_1$ in $L^\infty(\Omega,\mathbb{R}^{n,n})$. Then*

$$A_0 \le A_1 \text{ a.e. in } \Omega \qquad \text{and} \qquad \|A_0\|^2 \le \|A_1\|^2 \le \liminf_{k\to\infty} \|A_k\|^2.$$

Proof. See [4, Corollary 3.1]. ∎

3 Convergence

To begin, note that (1.3) and (1.4) imply that the set

$$M := \{B \in \mathcal{M} \mid z^{(i)} = T(B, f^{(i)}), 1 \le i \le N\} \tag{3.1}$$

is not empty. Since M is a closed, convex subset of $L^2(\Omega,\mathbb{R}^{n,n})$, there exists a uniquely determined $\bar{A} \in M$ such that

$$\|\bar{A}\|_{L^2} = \min_{B\in M} \|B\|_{L^2}. \tag{3.2}$$

The following lemma gives a sufficient criterion for a matrix function $\bar{A}$ to be norm-minimal.

Lemma 3.1. *Let $\bar{A} \in \mathcal{M}$ and $z^{(i)} \in H_0^1(\Omega)$ with $z^{(i)} = T(\bar{A}, f^{(i)}), 1 \le i \le N$. Suppose that there exist $\psi^{(i)} \in H_0^1(\Omega), 1 \le i \le N$ such that*

$$\bar{A}(x) = P_K\left(\sum_{i=1}^N \nabla z^{(i)}(x) \otimes \nabla\psi^{(i)}(x)\right) \qquad \text{a.e. in } \Omega.$$

Then $\|\bar{A}\|_{L^2} = \min_{B\in M} \|B\|_{L^2}$.

Proof. Let $B \in M$ be arbitrary. Then

$$\|B\|^2 - \|\bar{A}\|^2 = \|B-\bar{A}\|^2 + 2\int_\Omega \bar{A}\cdot(B-\bar{A})dx.$$

In view of our assumption on $\bar{A}$ and (2.1) we have

$$\big(\sum_{i=1}^N \nabla z^{(i)}(x) \otimes \nabla\psi^{(i)}(x) - \bar{A}(x)\big)\cdot\big(B(x)-\bar{A}(x)\big) \le 0, \qquad \text{a.e. in } \Omega,$$

so that

$$\begin{aligned}\|B\|^2 - \|\bar{A}\|^2 &\ge 2\sum_{i=1}^N \int_\Omega \big(\nabla z^{(i)} \otimes \nabla\psi^{(i)}\big)\cdot\big(B-\bar{A}\big)dx \\ &= 2\sum_{i=1}^N \int_\Omega B\nabla z^{(i)}\cdot\nabla\psi^{(i)}dx - 2\sum_{i=1}^N \int_\Omega \bar{A}\nabla z^{(i)}\cdot\nabla\psi^{(i)}dx = 0,\end{aligned}$$

which finishes the proof. ∎

In order to formulate our first main result we introduce

$$\rho_h^{(i)} := \inf_{v_h\in X_h} \|z^{(i)} - v_h\|_{H^1}, \quad 1 \le i \le N \text{ and } \rho_h := \max_{1\le i\le N} \rho_h^{(i)}.$$

Note that $\rho_h \to 0$ as $h \to 0$.

Theorem 3.2. *Let $A_h^\delta \in \mathcal{M}$ be a solution of (P_h^δ) and suppose that*

$$\gamma \to 0, \quad \frac{\delta}{\sqrt{\gamma}} \to 0, \quad \frac{\rho_h}{\sqrt{\gamma}} \to 0, \ \text{as } \delta \to 0, h \to 0. \tag{3.3}$$

Then $A_h^\delta \to \bar{A}$ in $L^2(\Omega, \mathbb{R}^{n,n})$ as $\delta \to 0, h \to 0$, where $\bar{A} \in \mathcal{M}$ is as in (3.2).

Proof. Given sequences $(\delta_k)_{k\in\mathbb{N}}, (h_k)_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} \delta_k = 0, \lim_{k\to\infty} h_k = 0$, choose $\gamma_k > 0$ such that $\lim_{k\to\infty} \gamma_k = 0$ and

$$\frac{\delta_k}{\sqrt{\gamma_k}} \to 0, \quad \frac{\rho_{h_k}}{\sqrt{\gamma_k}} \to 0 \text{ as } k \to \infty. \tag{3.4}$$

Furthermore, let $A_k = A_{h_k}^{\delta_k} \in \mathcal{M}$ be a solution of $(P_{h_k}^{\delta_k})$. We deduce from Theorem 2.4 that there exists a subsequence $(A_{k'})_{k'\in\mathbb{N}}$ and $A_0 \in \mathcal{M}$, $A_1 \in L^\infty(\Omega, \mathbb{R}^{n,n})$ such that

$$A_{k'} \overset{Hd}{\to} A_0, \quad A_{k'} \overset{*}{\rightharpoonup} A_1 \text{ in } L^\infty(\Omega, \mathbb{R}^{n,n}).$$

Corollary 2.5 implies that

$$A_0 \le A_1 \text{ a.e. in } \Omega, \quad \|A_0\|^2 \le \liminf_{k'\to\infty} \|A_{k'}\|^2. \tag{3.5}$$

For better readability we write again $(A_k)_{k\in\mathbb{N}}$ instead of $(A_{k'})_{k'\in\mathbb{N}}$.
We first claim that $A_0 \in M$. Since A_k is a minimum of $(P_{h_k}^{\delta_k})$ we infer that $J_{h_k}^{\delta_k}(A_k) \le J_{h_k}^{\delta_k}(\bar{A})$, so that

$$\frac{1}{2}\sum_{i=1}^N \|y_{h_k}^{(i)} - z_{\delta_k}^{(i)}\|^2 + \frac{\gamma_k}{2}\|A_k\|^2 \le \frac{1}{2}\sum_{i=1}^N \|\bar{z}_{h_k}^{(i)} - z_{\delta_k}^{(i)}\|^2 + \frac{\gamma_k}{2}\|\bar{A}\|^2.$$

Here, $y_{h_k}^{(i)} = T_{h_k}(A_k, f_{\delta_k}^{(i)})$, $\bar{z}_{h_k}^{(i)} = T_{h_k}(\bar{A}, f_{\delta_k}^{(i)}), 1 \le i \le N$. As a consequence we obtain with the help of (1.2)

$$\begin{aligned}\sum_{i=1}^N \|y_{h_k}^{(i)} - z^{(i)}\|^2 \ &\le \ 2\sum_{i=1}^N \|y_{h_k}^{(i)} - z_{\delta_k}^{(i)}\|^2 + 2\sum_{i=1}^N \|z_{\delta_k}^{(i)} - z^{(i)}\|^2 \\ &\le \ 2\sum_{i=1}^N \|\bar{z}_{h_k}^{(i)} - z_{\delta_k}^{(i)}\|^2 + 2\gamma_k\|\bar{A}\|^2 + 2N\delta_k^2. \end{aligned} \tag{3.6}$$

In order to estimate the first term on the right hand side we use (1.3), (1.2), (2.4), (2.5) and the definition of ρ_h and obtain for $i = 1, \dots, N$

$$\begin{aligned}\|\bar{z}_{h_k}^{(i)} - z_{\delta_k}^{(i)}\|_{L^2} \ &\le \ \|\bar{z}_{h_k}^{(i)} - z^{(i)}\|_{H^1} + \|z^{(i)} - z_{\delta_k}^{(i)}\|_{L^2} \\ &\le \ \|T_{h_k}(\bar{A}, f_{\delta_k}^{(i)} - f^{(i)})\|_{H^1} + \|T_{h_k}(\bar{A}, f^{(i)}) - T(\bar{A}, f^{(i)})\|_{H^1} + \delta_k \\ &\le \ C\|f_{\delta_k}^{(i)} - f^{(i)}\|_{H^{-1}} + C\rho_{h_k} + \delta_k \\ &\le \ C(\delta_k + \rho_{h_k}). \end{aligned} \tag{3.7}$$

Inserting (3.7) into (3.6) we infer that $y_{h_k}^{(i)} \to z^{(i)}$ in $L^2(\Omega)$ as $k \to \infty, 1 \le i \le N$. On the other hand we have that

$$y_{h_k}^{(i)} = T_{h_k}(A_k, f^{(i)}) + T_{h_k}(f_{\delta_k}^{(i)} - f^{(i)}) \rightharpoonup T(A_0, f^{(i)}) \quad \text{in } H_0^1(\Omega)$$

recalling (2.4), (1.2) and the fact that $A_k \overset{Hd}{\to} A_0$. As a result we infer that

$$z^{(i)} = T(A_0, f^{(i)}), \quad 1 \le i \le N$$

and therefore $A_0 \in M$.
Let us show next that $A_0 = \bar{A}$. To see this, let $B \in M$ be arbitrary. Since $J_{h_k}^{\delta_k}(A_k) \le J_{h_k}^{\delta_k}(B)$ we have

$$\frac{1}{2}\sum_{i=1}^{N} \|y_{h_k}^{(i)} - z_{\delta_k}^{(i)}\|^2 + \frac{\gamma_k}{2}\|A_k\|^2 \le \frac{1}{2}\sum_{i=1}^{N} \|z_{h_k}^{(i)} - z_{\delta_k}^{(i)}\|^2 + \frac{\gamma_k}{2}\|B\|^2, \tag{3.8}$$

where again $y_{h_k}^{(i)} = T_{h_k}(A_k, f_{\delta_k}^{(i)})$, while $z_{h_k}^{(i)} = T_{h_k}(B, f_{\delta_k}^{(i)}), 1 \le i \le N$. Similarly as in (3.7) we obtain

$$\|z_{h_k}^{(i)} - z_{\delta_k}^{(i)}\| \le C(\delta_k + \rho_{h_k})$$

so that (3.8) yields

$$\|A_k\|^2 \le C\big(\frac{\delta_k^2}{\gamma_k} + \frac{\rho_{h_k}^2}{\gamma_k}\big) + \|B\|^2. \tag{3.9}$$

Sending $k \to \infty$ we infer from (3.5) and (3.4) that $\|A_0\| \le \|B\|$ for every $B \in M$, so that we deduce that $A_0 = \bar{A}$. Finally,

$$\|A_k - \bar{A}\|^2 = \|A_k - A_0\|^2 = \|A_k\|^2 + \|A_0\|^2 - 2(A_k, A_0)_{L^2}$$

from which we infer with the help of (3.9) (with $B = A_0$) and the fact that $A_k \overset{*}{\rightharpoonup} A_1$ in $L^\infty(\Omega, \mathbb{R}^{n,n})$

$$\limsup_{k\to\infty} \|A_k - \bar{A}\|^2 \le 2\|A_0\|^2 - 2(A_1, A_0)_{L^2} \le 0,$$

since $A_0 \le A_1$ a.e. in Ω. In conclusion, $A_k \to \bar{A}, k \to \infty$ in $L^2(\Omega, \mathbb{R}^{n,n})$. Since the limit is unique, the whole sequence converges to $\bar{A}$ and the theorem is proved. ∎

4 Error bound

In order to obtain an error estimate we require stronger conditions on the data of our problem. In what follows we shall assume that Ω is a bounded, polygonal subset of $\mathbb{R}^2$ and that

$$z^{(i)} \in H_0^1(\Omega) \cap W^{2,p}(\Omega), f^{(i)} \in L^p(\Omega), 1 \le i \le N \text{ for some } p > 2. \tag{4.1}$$

Furthermore, we suppose that there exist $\psi^{(i)} \in H_0^1(\Omega) \cap W^{2,p}(\Omega), 1 \le i \le N$ such that $z^{(i)} = T(\bar{A}, f^{(i)}), 1 \le i \le N$, where the matrix function $\bar{A}$ satisfies

$$\bar{A}(x) = P_K\left(\sum_{i=1}^{N} \nabla z^{(i)}(x) \otimes \nabla \psi^{(i)}(x)\right) \qquad \text{a.e. in } \Omega. \tag{4.2}$$

In addition, we assume that there exists $\mu > 2$ such that

$$\|T(\bar{A}, f)\|_{W^{2,q}} \le C\|f\|_{L^q}, \quad f \in L^q(\Omega), 1 < q < \mu. \tag{4.3}$$

Note that $\bar{A} \in \mathcal{M}$ and satisfies $\|\bar{A}\| = \min_{B\in M} \|B\|$ in view of Lemma 3.1. Let us write $\bar{A}(x) = P_K\big(E(x)\big)$, where $E(x) = \sum_{i=1}^{N} \nabla z^{(i)}(x) \otimes \nabla\psi^{(i)}(x)$. Since $\nabla z^{(i)}, \nabla\psi^{(i)} \in W^{1,p}(\Omega, \mathbb{R}^n)$ and the embedding $W^{1,p}(\Omega) \hookrightarrow C^0(\bar{\Omega})$ is continuous we have that $E \in W^{1,p}(\Omega, \mathcal{S}_n)$, which we may extend to a function $E \in W^{1,p}(\mathbb{R}^n, \mathcal{S}_n)$. Furthermore, the projection P_K is Lipschitz continuous with Lipschitz constant 1, so that

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|\bar{a}_{kl}(x+h) - \bar{a}_{kl}(x)|^p}{|h|^p} dx &= \int_{\mathbb{R}^n} \frac{|\big(P_K\big(E(x+h)\big) - P_K\big(E(x)\big)\big)e_k \cdot e_l|^p}{|h|^p} dx \\ &\leq \int_{\mathbb{R}^n} \frac{|E(x+h) - E(x)|^p}{|h|^p} dx \leq C,\end{aligned}$$

which implies that

$$\bar{a}_{kl} \in W^{1,p}(\Omega), \quad 1 \leq k, l \leq 2. \tag{4.4}$$

Let us next interpret (4.2) as a projected source condition (see e.g. [7], which also contains further references). To do so, we assume for a moment that $f_\delta^{(i)} = f^{(i)}$ and write the objective functional in (P) in the form

$$J(A) = \frac{1}{2}\|F(A) - Z\|^2 + \frac{\gamma}{2}\|A\|^2,$$

where $F : \mathcal{M} \to L^2(\Omega)^N$ is given by $F_i(A) = T(A, f^{(i)})$ and $Z_i = z_\delta^{(i)}, 1 \leq i \leq N$. We claim:

Lemma 4.1. *There exists $\Theta \in L^2(\Omega)^N$ such that*

$$\bar{A} = P_{\mathcal{M}}\big(F'(\bar{A})^*\Theta\big),$$

where $P_{\mathcal{M}}$ denotes the orthogonal projection onto $\mathcal{M}$ in $L^2(\Omega, \mathcal{S}_n)$.

Proof. Recalling (2.2) we see that

$$F'(\bar{A})H = (\bar{w}^{(i)})_{1 \leq i \leq N}, \qquad H \in L^\infty(\Omega, \mathcal{S}_n),$$

where $\bar{w}^{(i)} \in H_0^1(\Omega)$ are the solutions of

$$\int_\Omega \bar{A}\nabla\bar{w}^{(i)} \cdot \nabla v dx = -\int_\Omega H\nabla z^{(i)} \cdot \nabla v dx \quad \text{for all } v \in H_0^1(\Omega). \tag{4.5}$$

In view of (4.4) the functions $\theta^{(i)} := \nabla \cdot \big(\bar{A}\nabla\psi^{(i)}\big)$ belong to $L^2(\Omega)$. Hence $\Theta = (\theta^{(i)})_{1 \leq i \leq N} \in L^2(\Omega)^N$ and we have for $H \in L^\infty(\Omega, \mathcal{S}_n)$

$$\begin{aligned}\langle F'(\bar{A})^*\Theta, H\rangle &= \big(\Theta, F'(\bar{A})H\big)_{L^2} = \sum_{i=1}^N \int_\Omega \nabla \cdot \big(\bar{A}\nabla\psi^{(i)}\big)\,\bar{w}^{(i)} dx \\ &= -\sum_{i=1}^N \int_\Omega \bar{A}\nabla\bar{w}^{(i)} \cdot \nabla\psi^{(i)} dx = \sum_{i=1}^N \int_\Omega H\nabla z^{(i)} \cdot \nabla\psi^{(i)} dx \\ &= \Big(\sum_{i=1}^N \nabla z^{(i)} \otimes \nabla\psi^{(i)}, H\Big)_{L^2}.\end{aligned}$$

Here we have used (4.5). As a consequence we may identify

$$F'(\bar{A})^*\Theta = \sum_{i=1}^N \nabla z^{(i)} \otimes \nabla\psi^{(i)} \in L^2(\Omega, \mathcal{S}_n)$$

and therefore we obtain for every $B \in \mathcal{M}$

$$
\begin{aligned}
&\big(F'(\bar{A})^*\Theta - \bar{A}, B - \bar{A}\big)_{L^2} \\
= \; & \int_\Omega \Big(\sum_{i=1}^N \nabla z^{(i)} \otimes \nabla \psi^{(i)} - P_K\big(\sum_{i=1}^N \nabla z^{(i)} \otimes \nabla \psi^{(i)}\big)\Big) \cdot \Big(B - P_K\big(\sum_{i=1}^N \nabla z^{(i)} \otimes \nabla \psi^{(i)}\big)\Big) dx \\
\leq \; & 0,
\end{aligned}
$$

by (2.1), which implies the result. ∎

Next, let us suppose that the sequence of triangulations $(\mathcal{T}_h)_{h>0}$ is quasiuniform. We introduce the Ritz projection $R_h : H_0^1(\Omega) \to X_h$ associated with $\bar{A}$ by

$$\int_\Omega \bar{A}\nabla R_h z \cdot \nabla v_h dx = \int_\Omega \bar{A}\nabla z \cdot \nabla v_h dx \quad \text{for all } v_h \in X_h.$$

In view of (4.3) and (4.4) we may apply [2, Theorem 8.1.11] and [2, Theorem 8.5.3] together with inequality (8.5.5) and deduce that there exists $h_0 > 0$ so that for all $0 < h \leq h_0$

$$
\begin{aligned}
\|R_h z\|_{W^{1,\infty}} &\leq C\|z\|_{W^{1,\infty}}, \quad z \in W^{1,\infty}(\Omega) &(4.6)\\
\|z - R_h z\| + h\|\nabla(z - R_h z)\| &\leq Ch^2\|z\|_{H^2}, \quad z \in H^2(\Omega) \cap H_0^1(\Omega). &(4.7)
\end{aligned}
$$

In particular we obtain for all $f \in L^2(\Omega)$

$$\|T_h(\bar{A}, f) - T(\bar{A}, f)\| \leq Ch^2\|f\|. \tag{4.8}$$

Theorem 4.2. *Let the conditions (4.1)–(4.3) be satisfied and $A_h^\delta \in \mathcal{M}$ be a solution of (P_h^δ). If $\sqrt{\delta} < h \leq h_0$ and $\gamma = \rho h^2$ for some suitable $\rho > 0$, then*

$$
\begin{aligned}
\|A_h^\delta - \bar{A}\| &\leq Ch, \\
\|y_h^{(i)} - z^{(i)}\| + h\|\nabla(y_h^{(i)} - z^{(i)})\| &\leq Ch^2, \quad 1 \leq i \leq N,
\end{aligned}
$$

where $y_h^{(i)} = T_h(A_h^\delta, f_\delta^{(i)}), 1 \leq i \leq N$.

Proof. Clearly,

$$
\begin{aligned}
\frac{\gamma}{2}\|A_h^\delta - \bar{A}\|^2 &= \frac{\gamma}{2}\|A_h^\delta\|^2 - \gamma(A_h^\delta, \bar{A})_{L^2} + \frac{\gamma}{2}\|\bar{A}\|^2 &(4.9)\\
&= J_h^\delta(A_h^\delta) - \frac{1}{2}\sum_{i=1}^N \|y_h^{(i)} - z_\delta^{(i)}\|^2 + \gamma(\bar{A} - A_h^\delta, \bar{A})_{L^2} - \frac{\gamma}{2}\|\bar{A}\|^2.
\end{aligned}
$$

Since A_h^δ is a solution of (P_h^δ) we infer

$$J_h^\delta(A_h^\delta) \leq J_h^\delta(\bar{A}) = \frac{\gamma}{2}\|\bar{A}\|^2 + \frac{1}{2}\sum_{i=1}^N \|z_h^{(i)} - z_\delta^{(i)}\|^2,$$

where $z_h^{(i)} = T_h(\bar{A}, f_\delta^{(i)})$. Furthermore, recalling (4.2) and (2.1) we have

$$(\bar{A} - A_h^\delta, \bar{A})_{L^2} \leq \sum_{i=1}^N \int_\Omega (\bar{A} - A_h^\delta) \cdot \nabla z^{(i)} \otimes \nabla \psi^{(i)} dx = \sum_{i=1}^N \int_\Omega (\bar{A} - A_h^\delta)\nabla z^{(i)} \cdot \nabla \psi^{(i)} dx.$$

Inserting the above estimates into (4.9) we obtain

$$
\begin{aligned}
&\frac{\gamma}{2}\|A_h^\delta - \bar{A}\|^2 && (4.10)\\
\le\ & \frac{1}{2}\sum_{i=1}^N \|z_h^{(i)} - z_\delta^{(i)}\|^2 - \frac{1}{2}\sum_{i=1}^N \|y_h^{(i)} - z_\delta^{(i)}\|^2 + \gamma \sum_{i=1}^N \int_\Omega (\bar{A} - A_h^\delta)\nabla z^{(i)} \cdot \nabla \psi^{(i)} dx.
\end{aligned}
$$

In order to estimate the first term we employ (2.4), (4.8) and (1.2)

$$
\begin{aligned}
\|z_h^{(i)} - z_\delta^{(i)}\| &\le \|T_h(\bar{A}, f_\delta^{(i)} - f^{(i)})\| + \|T_h(\bar{A}, f^{(i)}) - T(\bar{A}, f^{(i)})\| + \|z^{(i)} - z_\delta^{(i)}\| \\
&\le C\|f_\delta^{(i)} - f^{(i)}\|_{H^{-1}} + Ch^2\|f^{(i)}\| + \delta \le C\delta + Ch^2.
\end{aligned}
$$

If we use this bound in (4.10) we deduce

$$
\frac{\gamma}{2}\|A_h^\delta - \bar{A}\|^2 + \frac{1}{2}\sum_{i=1}^N \|y_h^{(i)} - z_\delta^{(i)}\|^2 \le C\delta^2 + Ch^4 + \gamma\sum_{i=1}^N S^{(i)}, \tag{4.11}
$$

where

$$
\begin{aligned}
S^{(i)} &= \int_\Omega (\bar{A} - A_h^\delta)\nabla z^{(i)} \cdot \nabla\psi^{(i)} dx = \int_\Omega (\bar{A} - A_h^\delta)\nabla z^{(i)} \cdot \nabla(\psi^{(i)} - R_h\psi^{(i)}) dx \\
&\quad + \int_\Omega (\bar{A} - A_h^\delta)\nabla(z^{(i)} - y_h^{(i)}) \cdot \nabla R_h\psi^{(i)} dx + \int_\Omega (\bar{A} - A_h^\delta)\nabla y_h^{(i)} \cdot \nabla R_h\psi^{(i)} dx \\
&\equiv S_1^{(i)} + S_2^{(i)} + S_3^{(i)}.
\end{aligned}
$$

Using the embedding $W^{2,p}(\Omega) \hookrightarrow C^1(\bar{\Omega})$ together with (4.7) we have

$$
\begin{aligned}
|S_1^{(i)}| &\le \|A_h^\delta - \bar{A}\|\|\nabla z^{(i)}\|_{L^\infty}\|\nabla(\psi^{(i)} - R_h\psi^{(i)})\| \\
&\le Ch\|\psi^{(i)}\|_{H^2}\|z^{(i)}\|_{W^{2,p}}\|A_h^\delta - \bar{A}\| \le Ch\|A_h^\delta - \bar{A}\|.
\end{aligned} \tag{4.12}
$$

Next, (4.6) implies

$$
\begin{aligned}
|S_2^{(i)}| &\le \|A_h^\delta - \bar{A}\|\|\nabla(z^{(i)} - y_h^{(i)})\|\|\nabla R_h\psi^{(i)}\|_{L^\infty} \\
&\le C\|\psi^{(i)}\|_{W^{1,\infty}}\|A_h^\delta - \bar{A}\|\|\nabla(z^{(i)} - y_h^{(i)})\| \\
&\le C\|A_h^\delta - \bar{A}\|\|\nabla(z^{(i)} - y_h^{(i)})\|.
\end{aligned} \tag{4.13}
$$

Abbreviating $\tilde{z}_h^{(i)} = T_h(\bar{A}, f^{(i)})$ we deduce from (2.5), an inverse estimate, (1.2) and (4.8) that

$$
\begin{aligned}
\|\nabla(z^{(i)} - y_h^{(i)})\| &\le \|\nabla(z^{(i)} - \tilde{z}_h^{(i)})\| + \|\nabla(\tilde{z}_h^{(i)} - y_h^{(i)})\| \\
&\le Ch\|z^{(i)}\|_{H^2} + Ch^{-1}\|\tilde{z}_h^{(i)} - y_h^{(i)}\| \\
&\le Ch + Ch^{-1}\Big(\|\tilde{z}_h^{(i)} - z^{(i)}\| + \|z^{(i)} - z_\delta^{(i)}\| + \|z_\delta^{(i)} - y_h^{(i)}\|\Big) \\
&\le Ch + Ch^{-1}\delta + Ch^{-1}\|z_\delta^{(i)} - y_h^{(i)}\|.
\end{aligned}
$$

Inserting this bound into (4.13) we obtain

$$
|S_2^{(i)}| \le C\|A_h^\delta - \bar{A}\|\Big(h + h^{-1}\delta + h^{-1}\|z_\delta^{(i)} - y_h^{(i)}\|\Big). \tag{4.14}
$$

Finally, the definition of R_h together with the fact that $y_h^{(i)} = T_h(A_h^\delta, f_\delta^{(i)}), z^{(i)} = T(\bar{A}, f^{(i)})$ imply

$$\begin{aligned}
S_3^{(i)} &= \int_\Omega \bar{A}\nabla y_h^{(i)} \cdot \nabla R_h\psi^{(i)} dx - \int_\Omega A_h^\delta \nabla y_h^{(i)} \cdot \nabla R_h \psi^{(i)} dx \\
&= \int_\Omega \bar{A}\nabla y_h^{(i)} \cdot \nabla \psi^{(i)} dx - \langle f_\delta^{(i)}, R_h\psi^{(i)}\rangle \\
&= -\int_\Omega \nabla\cdot\left(\bar{A}\nabla\psi^{(i)}\right) y_h^{(i)} dx - \int_\Omega \bar{A}\nabla z^{(i)}\cdot\nabla R_h\psi^{(i)} dx + \langle f^{(i)} - f_\delta^{(i)}, R_h\psi^{(i)}\rangle \\
&= -\int_\Omega \nabla\cdot\left(\bar{A}\nabla\psi^{(i)}\right) y_h^{(i)} dx - \int_\Omega \bar{A}\nabla z^{(i)}\cdot\nabla(R_h\psi^{(i)} - \psi^{(i)}) dx - \int_\Omega \bar{A}\nabla z^{(i)}\cdot\nabla\psi^{(i)} dx \\
&\quad + \langle f^{(i)} - f_\delta^{(i)}, R_h\psi^{(i)}\rangle \\
&= \int_\Omega \nabla\cdot\left(\bar{A}\nabla\psi^{(i)}\right)(z^{(i)} - y_h^{(i)}) dx + \int_\Omega \bar{A}\nabla(z^{(i)} - R_h z^{(i)})\cdot\nabla(\psi^{(i)} - R_h\psi^{(i)}) dx \\
&\quad + \langle f^{(i)} - f_\delta^{(i)}, R_h\psi^{(i)}\rangle.
\end{aligned}$$

Hence, we may estimate with the help of (4.7) and (1.2)

$$\begin{aligned}
|S_3^{(i)}| &\le \|\nabla\cdot\left(\bar{A}\nabla\psi^{(i)}\right)\|\,\|z^{(i)} - y_h^{(i)}\| + \|f^{(i)} - f_\delta^{(i)}\|_{H^{-1}}\|R_h\psi^{(i)}\|_{H^1} \\
&\quad + \|\bar{A}\|_{L^\infty}\|\nabla(z^{(i)} - R_h z^{(i)})\|\,\|\nabla(\psi^{(i)} - R_h\psi^{(i)})\| \\
&\le C\left(\|z^{(i)} - z_\delta^{(i)}\| + \|z_\delta^{(i)} - y_h^{(i)}\|\right) + C\delta\|\psi^{(i)}\|_{H^1} + Ch^2\|z^{(i)}\|_{H^2}\|\psi^{(i)}\|_{H^2} \\
&\le C\|z_\delta^{(i)} - y_h^{(i)}\| + C\delta + Ch^2. \qquad (4.15)
\end{aligned}$$

Inserting (4.12), (4.14) and (4.15) into (4.11) we deduce that

$$\begin{aligned}
&\frac{\gamma}{2}\|A_h^\delta - \bar{A}\|^2 + \frac{1}{2}\sum_{i=1}^N \|y_h^{(i)} - z_\delta^{(i)}\|^2 \qquad (4.16) \\
&\le C\delta^2 + Ch^4 + C\gamma\|A_h^\delta - \bar{A}\|\left(h + h^{-1}\delta + h^{-1}\sum_{i=1}^N \|z_\delta^{(i)} - y_h^{(i)}\|\right) \\
&\quad + C\gamma\sum_{i=1}^N \|z_\delta^{(i)} - y_h^{(i)}\| + C\gamma\delta + C\gamma h^2 \\
&\le \frac{\gamma}{4}\|A_h^\delta - \bar{A}\|^2 + (C\gamma h^{-2} + \frac{1}{8})\sum_{i=1}^N \|y_h^{(i)} - z_\delta^{(i)}\|^2 \\
&\quad + C\left(\delta^2 + h^4 + \gamma h^2 + \gamma h^{-2}\delta^2 + \gamma\delta + \gamma^2\right).
\end{aligned}$$

If we choose $\gamma = \rho h^2$ with $\rho = \frac{1}{8C}$ we finally obtain

$$\|A_h^\delta - \bar{A}\|^2 \le C\left(\frac{\delta^2}{\gamma} + \frac{h^4}{\gamma} + h^2 + \frac{\delta^2}{h^2} + \delta + \gamma\right) \le Ch^2$$

since $\delta \le h^2$. Returning to (4.16) we infer $\sum_{i=1}^N \|y_h^{(i)} - z_\delta^{(i)}\|^2 \le Ch^4$, so that

$$\|y_h^{(i)} - z^{(i)}\| \le \|y_h^{(i)} - z_\delta^{(i)}\| + \|z_\delta^{(i)} - z^{(i)}\| \le Ch^2 + \delta \le Ch^2, \quad 1 \le i \le N \qquad (4.17)$$

in view of the above relations betweeen γ, h and δ. Finally, (4.7), an inverse estimate and (4.17) yield

$$\begin{aligned}
\|\nabla(y_h^{(i)} - z^{(i)})\| &\leq \|\nabla(y_h^{(i)} - R_h z^{(i)})\| + \|\nabla(R_h z^{(i)} - z^{(i)})\| \\
&\leq Ch^{-1}\|y_h^{(i)} - R_h z^{(i)}\| + Ch\|z^{(i)}\|_{H^2} \\
&\leq Ch^{-1}\big(\|y_h^{(i)} - z^{(i)}\| + \|z^{(i)} - R_h z^{(i)}\|\big) + Ch \\
&\leq Ch,
\end{aligned}$$

which finishes the proof. ∎

5 Numerical example

Let $\Omega := (-1,1)^2 \subset \mathbb{R}^2$ and consider for $N = 1$ the data $(z, f) \in H_0^1(\Omega) \times H^{-1}(\Omega)$ with

$$\begin{aligned}
z(x) &:= (1 - x_1^2)(1 - x_2^2), \\
f(x) &:= -\nabla \cdot \big(\bar{A}(x)\nabla z(x)\big)
\end{aligned}$$

for $x = (x_1, x_2) \in \Omega$. The diffusion matrix is given by

$$\bar{A}(x) := P_K\big(\nabla z(x) \otimes \nabla z(x)\big),$$

where $K = \{A \in \mathcal{S}_2 \,|\, a|\xi|^2 \leq A\xi \cdot \xi \leq b|\xi|^2 \text{ for all } \xi \in \mathbb{R}^2\}$. By construction we have $z = T(\bar{A}, f)$, while Lemma 3.1 implies that

$$\|\bar{A}\|_{L^2} = \min_{B \in M} \|B\|_{L^2}.$$

It is not difficult to verify that for a given matrix $A \in \mathcal{S}_2$ we have

$$P_K(A) = S^t \mathrm{diag}\big(P_{[a,b]}(\lambda_1(A)), P_{[a,b]}(\lambda_2(A))\big)S,$$

where $\lambda_1(A), \lambda_2(A)$ denote the eigenvalues of A, S is an orthogonal matrix such that $S^tAS = \mathrm{diag}\big(\lambda_1(A), \lambda_2(A)\big)$ and $P_{[a,b]}(\kappa) := \max(a, \min(\kappa, b))$. A calculation shows that

$$\bar{A}(x) = aI_2 + \frac{P_{[a,b]}(\eta(x)) - a}{\eta(x)} \nabla z(x) \otimes \nabla z(x),$$

where $\eta(x) = 4\big(x_1^2(1 - x_2^2)^2 + x_2^2(1 - x_1^2)^2\big)$.

For the numerical verification of the estimates in Theorem 4.2 we choose a sequence of quasi-uniform triangulations $\{\mathcal{T}_{h_l}\}$ generated with the POIMESH and REFINEMESH environment of MATLAB, where $h_l = 2^{-l}, l \in \mathbb{N}$ denotes the gridsize of $\mathcal{T}_{h_l}$. We consider the minimization problem (P_h^δ) for $\delta = \delta_l$, $\gamma = \gamma_l = 0.01h_l^2$ and take $z_\delta = z_{\delta_l}$ as the Lagrange interpolant of z, and $f_\delta = f_{\delta_l}$ as piecewise constant approximation to f on Ω defined through the function values in the barycenters of the triangles in $\mathcal{T}_{h_l}$. We note, that since f is discontinuous (1.2) in general can not be satisfied with $\delta \sim h^2$ for choice of f_δ. However, this fact seems not to have significant effects on the convergence history of our numerical solutions presented in Tab. 1. The constants a and b in the definition of K are chosen as $a = 0.5$ and $b = 10$. The discrete problems (P_h^δ) are solved using the projected steepest descent method with Armijo step size rule (see e.g. [12]), which we briefly describe for the convenience of the reader. In view of Remark 2.3 it is sufficient to iterate within the class of matrices in $\mathcal{M}$ that are piecewise constant on $\mathcal{T}_h$. Given such an A the new iterate is computed according to

$$A^+ = A(\tau) \text{ with } \tau = \max_{r\in\mathbb{N}}\{\beta^r; J_l(A(\beta^r)) - J_l(A) \le -\frac{\sigma}{\beta^r}\|A(\beta^r) - A\|^2\}$$

where we have abbreviated $J_l = J_{h_l}^{\delta_l}$. Furthermore, $\beta \in (0,1)$ and

$$A(\tau)_{|T} := P_K\Big(A_{|T} + \tau\big(\nabla y_{h|T} \otimes \nabla p_{h|T} - \gamma A_{|T}\big)\Big), \quad T \in \mathcal{T}_h.$$

Here, $y_h = T_h(A, f_{\delta_l})$ and $p_h \in X_h$ is the solution of the adjoint problem

$$\int_\Omega A_h \nabla v_h \cdot \nabla p_h dx = \int_\Omega (y_h - z_l) v_h dx \quad \text{for all } v_h \in X_h.$$

In our calculations we chose as initial matrix

$$A^0 := \mathrm{diag}(1.01, 1.01) P_h^{L^2}(A),$$

where $P_h^{L^2}$ denotes the L^2-projection onto the space of piecewise constant functions over the grid $\mathcal{T}_h$. The iteration on refinement level l was stopped if

$$\|A^+ - A(1)\| \le \tau_a + \tau_r\|A^0 - A^0(1)\|$$

or the maximum number of $l * 100$ iterations was reached, where $\tau_a = 10^{-3} * h_l$ and $\tau_r = 10^{-2} * h_l$. Furthermore, we choose $\beta := 0.5$ and $\sigma := 10^{-4}$. For $h = h_l$ the numerical solution is denoted by $\bar{A}_h$ with associated optimal state $\bar{y}_h$. The numerical results are summarized in Tab. 1, where we display the refinement level l, the number of iterations, the value of γ_l, the final L^2-errors in the parameter, the final L^2-errors in the states, both together with their experimental order of convergence (EOC), and the convergence history of the steepest descent algorithm. As predicted by Theorem 4.2, we observe quadratic convergence for the states. The matrix parameters in the present example seem to converge faster than predicted by the numerical analysis.
Fig. 1 from left to right shows $\bar{y}_h$, the Lagrange interpolant of $I_h z$ and $\bar{y}_h - I_h z$ for refinement level 5. As in the numerical experiments of [4] one observes that the difference between $\bar{y}_h$ and $I_h z$ is comparatively large in regions where ∇z (and thus $\nabla I_h z$) is small which is in agreement with classical results on the identifiability of scalar diffusion coefficients, see e.g. [18].

Acknowledgements

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l	it	γ_l	$\|\bar{A} - \bar{A}_h\|$	EOC	$\|\bar{y}_h - z\|$	EOC	$\|\bar{A}_h - A_h(1)\|$	$\tau_a + \tau_r r_0$
1	85	1.e-2	1.26	-	1.41e-1	-	3.88e-3	3.84e-3
2	200	2.5e-3	4.42e-1	1.51	2.87e-2	2.30	6.39e-4	7.58e-4
3	300	6.25e-4	1.10e-1	2.00	6.36e-3	2.17	2.37e-4	8.37e-4
4	400	1.56e-4	3.03e-2	1.86	1.51e-3	2.07	1.26e-4	3.00e-4
5	500	3.91e-5	1.09e-2	1.47	4.17e-4	1.87	6.44e-5	7.70e-5
				1/5: 1.71		1/5: 2.10		
				mean 1.71		mean 2.10		

Table 1: Mesh parameters, errors, experimental order of convergence, and convergence history of the solution algorithm. The table is supplemented with the EOC between finest and coarsest level (1/5) and with the mean value of the EOC over the refinement levels.

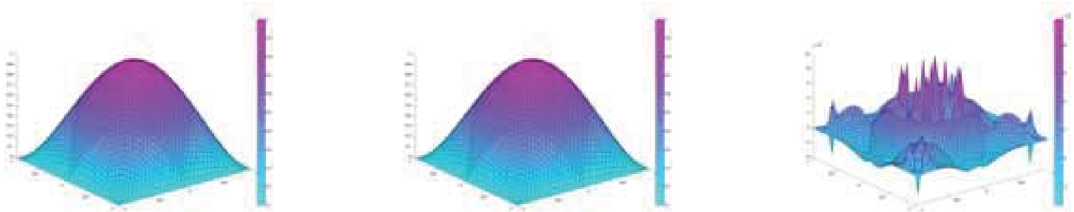

Figure 1: Numerical solution, desired state, and error $\bar{y}_h - I_h z$ on refinement level 5.

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