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### HJB-POD feeback control of advection-diffusion equation with a model predictive control snapshot sampling

A. Alla and M. Hinze

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# HJB-POD feeback control of advection-diffusion equation with a model predictive control snapshot sampling.

**A. Alla** * **M. Hinze** *

* *Universität Hamburg, Bundesstr. 55, 20146, Hamburg, Germany (e-mail: {alessandro.alla, michael.hinze}@uni-hamburg.de).*

**Abstract:** In this paper we present the approximation of an infinite horizon optimal control problem for evolutive advection-diffusion equations. The method is based on a model reduction technique, using a Proper Orthogonal Decomposition (POD) approximation, coupled with a Hamilton-Jacobi-Bellman (HJB) equation which characterizes the value function of the corresponding control problem for the reduced system. We show that it is possible to improve the surrogate model by means of a Model Predictive Control (MPC) solver. Finally, we present numerical tests to illustrate our approach and to show the effectiveness of the method in comparison to existing approaches.

*Keywords:* Optimal Control, Proper Orthogonal Decomposition, Hamilton-Jacobi equations, Model Predictive Control, Advection-Diffusion equations

## 1. INTRODUCTION

In this report we investigate an infinite horizon optimal control problem for time-dependent linear advection-diffusion equations. The basic ingredient of the method is the coupling between a POD approximation of the equation and a Dynamic Programming scheme for the stationary Hamilton-Jacobi-Bellman equation characterizing the value function of the optimal control problem. Due to the curse of dimensionality, we need to restrict the dimension of the POD system to a rather small number (typically 3-4). This limitation naturally affects the accuracy of the POD approximation (see Gubisch et al. (2013)), and, as a consequence, the problem class which we can treat with this technique.

It is well known that the solution of the HJB equation is not an easy task from the numerical point of view since viscosity solutions of the HJB equation are usually just Lipschitz-continuous. Optimal control problems for ODEs are solved by Dynamic Programming (DP), both analytically and numerically (see Bardi et al. (1997) for a general presentation of this theory). From the numerical point of view, this approach has been developed for many classical control problems obtaining convergence results and a-priori error estimates (see the recent book by Falcone et al. (2014)).

We should mention that a first tentative approach to couple POD and HJB equations was proposed by Atwell et al. (2001) for the control of the 1D heat equation. Then, Kunisch et al. (1999, 2001) extended this approach to diffusion dominated equations and, in particular, in Kunisch et al. (2004) apply HJB-POD feedback control to the viscous Burgers equation. We also mention an adaptive POD technique for 1D advection dominated problems proposed in Alla et al. (2013). Recently with this approach HJB-POD control of Navier-Stokes equation was investigated in Alla et al. (2015).

In general the snapshot sampling plays an important role to build the reduced order model. In many cases, the control problem is initialized with a forecast. The surrogate model then is adapted through an iterative process, see e.g. Afanasiev et al. (2001), Arian et al. (2002).

In Kunisch et al. (2004), the HJB equation is solved twice with a multilevel algorithm. They set up the reduced model from an initial control input and the correspondent HJB equation on a coarse grid, in order to have a quick guess on the value function. Then, they compute HJB on a fine grid and the optimal trajectory.

The novelty of the present paper consists in the way the surrogate model is built. The snapshot sampling is done taking advantage of the MPC algorithm. MPC approximates optimal control problems by means of a repeated solution of open-loop control problems (see the monographs Grüne et al. (2011); Rawlings et al. (2009)). In this way MPC helps us to set up the surrogate model with information on the control input. It therefore is not necessary to start with a forecast. We note that we perform MPC on a short prediction horizon in order to avoid expensive offline stages. Then, we switch to the HJB-POD model in order to refine the coarse approximation.

The paper is organized as follows. We first present the optimal control problem in Section 2, then we describe the DP equation and MPC algorithm in Section 3. Proper orthogonal decomposition, applied to optimal control problems, is summarized in Section 4. Finally, the numerical tests are presented in Section 5.

## 2. THE OPTIMAL CONTROL PROBLEM

In this section we describe the optimal control problem. The gouverning equation is given by the unsteady one-dimensional advection-diffusion equation. The equation reads:

$$\left.\begin{aligned} y_t - \varepsilon y_{xx} + c y_x = u \qquad & \text{in } \Omega \times (0, \infty], \\ y(\cdot, 0) = y_0 \qquad & \text{in } \Omega, \\ y(x, \cdot) = 0 \qquad & \text{in } \partial\Omega \times (0, \infty), \end{aligned}\right\} \tag{1}$$

where $\Omega = (a, b)$ is an open interval, $y : \Omega \times [0, \infty] \to \mathbb{R}$ denotes the state, and the parameters $\varepsilon, c$ are positive real constants. The control signals are elements of $\mathcal{U} \equiv \{u : \Omega \times [0, T] \to U,\, u \in L^\infty(0, \infty, L^2(\Omega))\}$, where $U$ is defined as follows $U = \{u \in \mathbb{R} \text{ s.t. } u_a \leq u \leq u_b\}$ with given $u_a, u_b \in \mathbb{R}$. The initial value is denoted by $y_0$. Note that we deal with zero Dirichlet boundary conditions. The cost functional we want to minimize is given by

$$J(u) := \int_0^\infty \left( \|y(\cdot, t; u) - \bar{y}\|^2_{L^2(\Omega)} + \alpha |u(t)|^2 \right) e^{-\lambda t}\, dt, \tag{2}$$

where $\bar{y}$ is the desired state, $\alpha \in \mathbb{R}^+$ and $\lambda > 0$ is the discount factor. The optimal control problem can be formulated as

$$\min_{u \in \mathcal{U}} J(u) \text{ s. t. } y(u) \text{ satisfies (1).} \tag{3}$$

Existence and uniqueness results for (3) can be found in, e.g. Lions (1971).

## 3. OPTIMAL CONTROL TOOLS

In this section we present the optimal control problem in abstract form and we explain how to solve it with HJB equations and Model Predictive Control.

### *3.1 Hamilton-Jacobi-Bellman equations*

We illustrate the dynamic programming approach for abstract optimal control problems of the form

$$\min_{u \in \mathcal{U}} J_{y_0}(u) := \int_{t_0}^\infty L(y(t), u(t))\, e^{-\lambda t}\, dt \tag{4}$$

$$\text{subject to } \dot{y}(t) = f(y(t), u(t)),\ y(t_0) = y_0,$$

with system dynamics in $\mathbb{R}^n$. We assume $\lambda > 0$, and $L(\cdot, \cdot)$ and $f(\cdot, \cdot)$ to be Lipschitz-continuous, bounded functions. The control signals are, now, elements of $\mathcal{U} \equiv \{u : [0, T] \to U,\, u(\cdot) \in L^\infty(0, T)\}$, where $U$ is a compact subset of $\mathbb{R}^m$. In order to emphasize the depence on the initial condition $y_0$ we use the notation $J_{y_0}(u)$.
In this setting, a standard solution tool is the application of the dynamic programming principle, which leads to a characterization of the value function $v(y_0) := \inf_{u \in \mathcal{U}} J_{y_0}(u)$ as a unique viscosity solution of the Hamilton-Jacobi-Bellman equation (HJB)

$$\lambda v(y_0) - \inf_{u \in U} \{Dv \cdot f(y_0, u) + L(y_0, u)\} = 0\,. \tag{5}$$

To approximate equation (5), we construct a fully-discrete semi-Lagrangian scheme which is based on a discretization of the system dynamics with time step $h$, and a finite element discretization of the state space with mesh parameter $k$, leading to a fully discrete approximation $V_{h,k}(y_0)$ of the value function $v$ satisfying

$$V_{h,k}(y_0) = \min_{u \in U} \{(1 - \lambda h) I_1[V_{h,k}](y_0 + h f(y_0, u)) + L((y_0, u)\} \tag{6}$$

for every element $y_0$ of the discretized spatial domain. In general, the arrival point $y_0 + hf(y_0, u)$ is not a node of the state space grid, and therefore the value of $V_{h,k}$ at this point is approximated by means of a first-order interpolant of the data, denoted by $I_1[V_{h,k}]$ which is built, in our case, by means of the Lagrange's method (we refer the reader to (Bardi et al., 1997, Appendix A) for more details).

The goal is to find a feedback control law of the form $u(t) = \Phi(y(t), t)$ which steers the system to the desired trajectory. $\Phi$ is called *feedback map*. The computation of feedback maps is almost built in and comes straightforward from the knowledge of the value function. In fact;

$$\Phi(y_{y_0})(t)) = u^*(t) = \arg\min_{u \in U} \left\{ L(y_0, u) + \nabla v(y_0)^T f(y_0, u) \right\}.$$

The characterization of the value function is valid for all classical problems in any dimension and its approximation is based on a-priori error estimates (we refer to Falcone et al. (2014) for more details).
The request to solve an HJB in high dimensions comes up naturally whenever we want to control evolutive PDEs. However, a direct discretization is impossible, in many practically relevant situations, since the system of ODEs associated to a semi-discretization in time would have the dimension equal to the space dimension where one should solve the HJB equation. Fortunately, at the discrete level, the snapshot POD method (see Sirovich (1987)) allows us to obtain low-dimensional reduced models even for complex dynamics, and, thus, presents an oppurtunity to circunmvent the curse of dimensionality in the numerical solution of the HJB equation (see Section 5).

### *3.2 Model Predictive Control*

To introduce the MPC algorithm we consider again the abstract problem (4). This method allows to compute a state feedback law for (4) by solving a sequence of finite time horizon problems.

To begin with, we introduce the finite horizon cost functional as follows:

$$J^N_{y_0}(u; t_0) = \int_{t_0}^{t_0^N} L\big(y(t), u(t)\big) e^{-\lambda t}\, dt,$$

where $N$ is a natural number, $t_0^N = t_0 + N\Delta t$ is the final time and $N\Delta t$ denotes the length of the prediction horizon for the chosen time step $\Delta t > 0$. The state $y$ solves $\dot{y}(t) = f(y(t), u(t)),\ y(t_0) = y_0,\ t \in [t_0, t_0^N)$ and is denoted by $y(\cdot, t_0; u(\cdot))$. Note that we have added a discount factor which is not common in the MPC framework. Recent work on the stabilizing properties of MPC with discounted optimal control is presented in Gaitsgory et al. (2014).

In Algorithm 1 the method is presented and works as follows: we store the optimal control on the first subinterval $[t_0, t_0 + \Delta t]$ together with the associated optimal trajectory. Then, we initialize a new finite horizon optimal control problem whose initial condition is given by the optimal trajectory $y(t) = y(t; t_0, u^N(t))$ at $t = t_0 + \Delta t$ using the optimal control $\phi^N(y_{y_0}(t)) = u^N(t)$ for $t \in (t_0, t_0 + \Delta t]$. We iterate this process by setting $t_0 = t_0 + \Delta t$. Note that (7) is an open loop problem on a finite time horizon $[t_0, t_0 + N\Delta t]$ which can be treated by classical techniques, see e.g. Hinze et al. (2009). In general, the larger the prediction horizon, the better the feedback law one can obtain. However, one is interested in short prediction horizons (or even horizon of minimal length) while guaranteeing stabilization

**Algorithm 1** (MPC algorithm)

**Require:** time step $\Delta t > 0$, finite horizon $N \in \mathbb{N}$, weighting parameter $\lambda > 0$.

1: **for** $n = 0, 1, 2, \ldots$ **do**
2: Compute the state $y(t_n)$ of the system at $t_n = n\Delta t$.
3: Set $t_0 = t_n = n\Delta t$, $x = y(t_n)$ and compute a global solution
$$u^N := \arg\min_{u \in \mathcal{U}} J^N_{y_0}(u; t_0). \quad (7)$$
4: Define the MPC feedback value $\phi^N(y_{y_0}(t)) = u^N(t)$, $t \in (t_0, t_0 + \Delta t]$ and use this control to compute the associated state $y = y(t; t_0, u^N(t))$ by solving the dynamical system in (4) on $[t_0, t_0 + \Delta t]$.
5: **end for**

properties of the MPC scheme (see Grüne et al. (2011)). The computation of this minimal horizon is related to a relaxed dynamic programmic principle in terms of the value function for the finite horizon problem (7) defined as follows:
$$v^N(y_0, t_0) = \inf_{u \in \mathcal{U}} J^N_{y_0}(u; t_0).$$
The value function $v^N$ satisfies the DPP for the finite horizon problem for $t_0 + k\Delta t$, $0 < k < N$:

$$v^N(y_0, t_0) = \inf_{u \in \mathcal{U}} \left\{ \int_{t_0}^{t_0 + k\Delta t} L\big(y(t; t_0; u^N(t))\big)\, dt + v^{N-k}\big(y(t; t_0 + k\Delta t, u^N(t))\big) \right\}$$

The stability of the method might be expressed in terms of the relaxed Dynamic Programming Principle:
$$v^N(y_0, t_0) \geq v^N\big(y(t_0 + \Delta t; t_0, \phi^N(t), t_0 + \Delta t, )\big) + \alpha^N L\big(y_0, \phi^N(t_0))\big), \quad (8)$$
where $\alpha^N \in (0, 1]$.
In order to estimate $\alpha^N$ in the relaxed DPP we require that the system is exponentially controllable. We refer the interested readear to the monograph Grüne et al. (2011) where an explicit formula for $\alpha^N$ is given under these assumptions.

## 4. MODEL ORDER REDUCTION FOR OPTIMAL CONTROL PROBLEMS

In this section we briefly explain the Proper Orthogonal Decomposition (POD) and demonstrate how POD can be used to build reduced order models solving optimal control problems. We then show how to apply this technique to our reference problem (3).

The Reduced Order Modelling (ROM) approach to optimal control problems is based on projecting the dynamics onto a low dimensional manifold utilizing projectors that contain information from the system. A common approach here is based on the snapshot form of POD proposed in Sirovich (1987), which in the present situation works as follows. We compute the snapshots set $y_1, \ldots, y_n$ of the problem corresponding to different time instances $t_1, \ldots, t_n$ and define the POD ansatz of order $\ell$ for the state $y$ by
$$y^\ell(x, t) = \sum_{i=1}^{\ell} w_i(t)\psi_i(x), \quad (9)$$
where the basis functions $\{\psi_i\}_{i=1}^\ell$ are obtained from the singular value decomposition of the snapshot matrix $Y = [y_1, \ldots, y_n]$, i.e. $Y = \Psi\Sigma V$, and the first $\ell$ columns of $\Psi$ form the POD basis functions of rank $\ell$. Here the SVD is based on the Euclidean inner product $\langle \cdot, \cdot \rangle$. This is reasonable in our situation, since the numerical computations performed in our numerical example are based on a uniform grid, but other inner products which fit better with the physics of the PDE can be treated by a suitable modification of the snapshot matrix. Note that, for the purpose of model reduction we consider the control input as follows: $u(x, t) := \sum_{i=1}^N b_i(x)u_i(t)$ where the given functions $b_i(x) : \Omega \to \mathbb{R}$ play the role of the so called shape functions and $u(t)$ is the unkonwn input.

As already mentioned, the snapshots are computed on the basis of a stable finite difference discretization of (1) which leads to a semi-discrete ODE system of the form
$$\dot{y} = Ay + Bu, \quad y(t_0) = y_0. \quad (10)$$
where $B \in \mathbb{R}^{n \times N}$ with $(B)_{ij} = b_i(x_j)$. Then, it is clear that the optimal control problem (3) fits into the more abstract setting (4).

The reduced optimal control problem is obtained through replacing (10) by a dynamical system computed by a Galerkin approximation with ansatz (9) for the state. This leads to a $\ell-$dimensional system for the unknown coefficients $\{w_i\}_{i=1}^\ell$, namely
$$M^\ell \dot{w} = A^\ell w + B^\ell u, \quad w(t_0) = w_0. \quad (11)$$
Here the entries of the mass $M^\ell$ and the stiffness $A^\ell$ matrices are given by $\langle \psi_j, \psi_i \rangle$ and $\langle \psi_j, A\psi_i \rangle$, respectivelly. The reduced shape function is obtained by $(B^\ell)_{ij} = \langle b_i(x_j), \psi_i \rangle$. The coefficients of the initial condition $y^\ell(t_0) \in \mathbb{R}^\ell$ are determined by $w_i(t_0) = (w_0)_i = \langle y_0, \psi_i \rangle$, $1 \leq i \leq \ell$, and the solution of the reduced dynamical problem is denoted by $w(t) \in \mathbb{R}^\ell$.

Then, the POD-Galerkin approximation of (4) leads to the optimization problem
$$\min J^\ell_{w_0}(u), \quad (12)$$
where $u \in \mathcal{U}$, $w$ solves (11) and the cost functional is defined by
$$J^\ell_{w_0}(u) = \int_{t_0}^{\infty} L(w(s), u(s), s)e^{-\lambda s}\, ds.$$
The value function $v^\ell$, defined for the initial state $w_0 \in \mathbb{R}^\ell$ is given by
$$v^\ell(w_0) = \inf_{u \in \mathcal{U}} J^\ell_{w_0}(u),$$
and $w$ solves (11) with the control $u$ and initial condition $w_0$. Reduced HJB equations are defined in $\mathbb{R}^\ell$, but we need to restrict our numerically domain to a bounded subset of $\mathbb{R}^\ell$. We refer the interested reader to Alla et al. (2013) for a detailed description.
We have not discussed yet how to build the snapshot matrix $Y$. Here we propose an approach based on the MPC algorithm which provides snapshots which carry information of the controlled problem. The idea consists in first solving the full problem with Algorithm 1 and to sample state and control snapshots. In the next step the

snaposhots are used to compute the POD basis functions and to set up the reduced model which is used for the numerical solution of the HJB equation. We note that the MPC algorithm, in general, is fast for tracking problems, expecially in our case where the prediction horizion $N$ is short, i.e. $N = 2, 3$. The proposed method is summarized in Algorithm 2.

**Algorithm 2** MPC-HJB-POD Algorithm

**Require:** time step $\Delta t > 0$, finite horizon $N \in \mathbb{N}$, weighting parameter $\lambda > 0$, time step size $h > 0$ , spatial step size $k > 0$

1: Solve Algorithm 1 for a given time interval
2: Build the snapshot matrix $Y$ from the MPC output.
3: Compute POD basis of order $\ell$
4: Compute the reduced value function $v^\ell$.
5: Compute the optimal trajectory by means of the reduced value function

## 5. NUMERICAL TESTS

In this section we present our numerical tests. The first test deals with a smooth initial condition whereas in the second test we take a non-smooth initial condition. The first test also considers a time-dependent pertubation in order to investigate the robustness of the feedback control. In both examples we vary the diffusion coefficient and the advection term and we deal with 3 POD basis functions.

### *5.1 Test 1: Smooth initial condition*

In equation (3) we set: $\Omega = (0, 2), y_0 = 5(x - x^2), \lambda = 1, U = \{-3, 0, 3\}$. We only consider one shape function $b_1(x) = y_0(x)$. The desired configuration is given by $\bar{y} \equiv 0$. In (6) we take $k = 0.1$, $h = 0.01$. The optimal trajectory, in this case, is also obtained with a time stepsize of 0.01. In Figure 1, we show the solution of the uncontrolled equation (1), i.e. $u \equiv 0$.

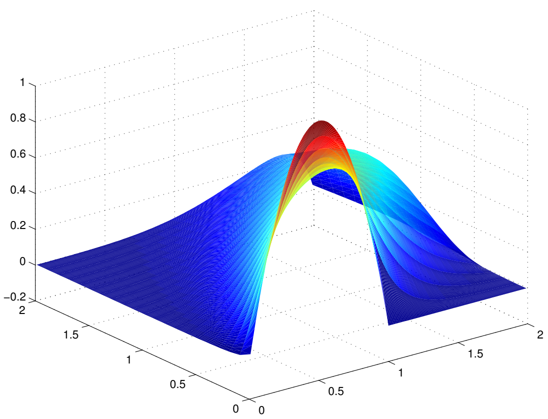

Fig. 1. Test 1: Solution of equation (1) with $\varepsilon = 0.1, c = 1, u \equiv 0$

The approximation of the optimal control problem (3) follows Algorithm 2. The snapshots are computed by a MPC Algorithm (see Algorithm 1) where, in order to avoid expensive offline computations we consider a short prediction horizon $N = 3$.

We, then, compute the POD basis functions and switch to the HJB-approach. The computation of the value function allows to have a complete overview of the problem and, with the help of the MPC algorithm, we are able to improve our approximation.

In Figure 2 we present the approximation of the optimal solution following Algorithm 2.

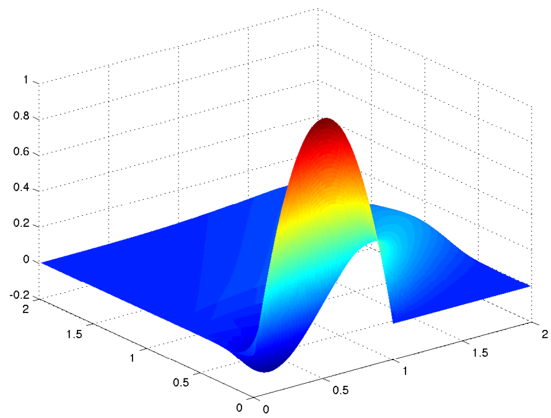

Fig. 2. Test 1: Solution of problem (3) with $\varepsilon = 0.1, c = 1$ with Algorithm 2

We make a comparison between MPC solution, Algorithm 2, and the scheme proposed by Kunisch et al. (2004), referred to as KVX in the sequel, which consists in taking snapshots from the uncontrolled equation. For this purpose we evaluate the cost functional for the obtained optimal solutions. First of all we consider the advection term $c = 0.4$ and $\varepsilon \in [0.1, 0.01]$. As we can see in Figure 3 the best result is obtained with Algorithm 2; it turns out the value of the cost functional in Algorithm 2 is always below that obtained by the other schemes. The black line represents the approach of KVX, and as we can see, as soon as the diffusion term is decreasing the related cost function becomes larger than that of the MPC algorithm due to the fact that we should enlarge the number of basis functions or enrich the snapshot set to avoid this behavior. Then, it is interesting to study the case with the advection dominated term $c = 1$. As soon as we increase the advection term our problem is *close* to hyperbolic settings where the decay of the eigenvalue is not really fast. As we can see in Figure 4, Algorithm 2 not always produces smaller cost function values then the MPC algorithm. This is due to the curse of dimensionality. We would need to increase the number of basis functions for the HJB approach. But this is not feasible at the moment. In any case we improved the solution with our scheme obtaining smaller cost function values than KVX in the presented examples.

Finally we present the solution of the optimal control problem with a perturbation of the state in Figure 5. This is important in practice since, in general, the state is only known up to a perturbation. We note that the perturbation is a random time-dependent function and

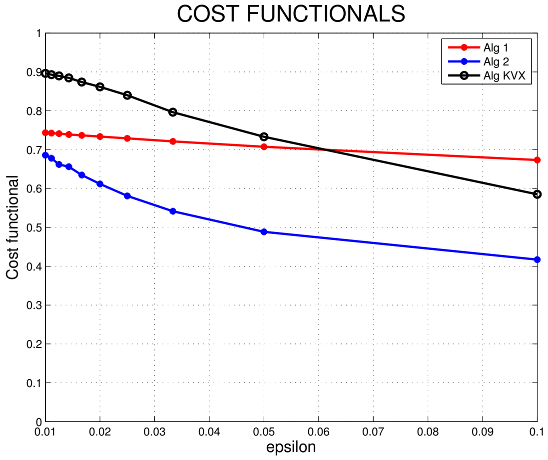


Fig. 3. Test 1: Evaluation of the Cost Functional for Algorithm 1 (red), Algorithm 2 (blue), and Algorithm KVX (black) for $c = 0.4$.

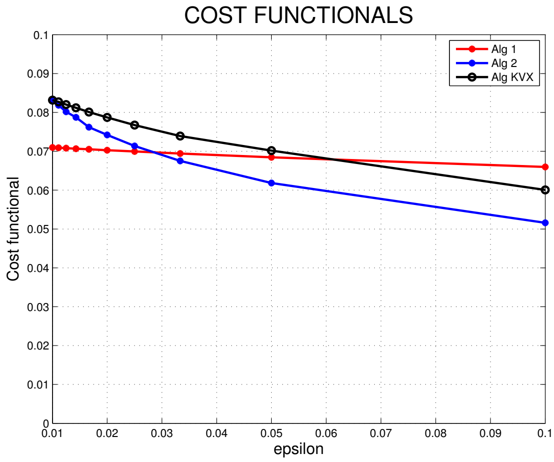


Fig. 4. Test 1: Evaluation of the Cost Functional for Algorithm 1 (red), Algorithm 2 (blue), and Algorithm KVX (black) for $c = 0.1$.

that the optimal trajectory is built upon the value function already computed without noise.

### *5.2 Test 2: Non-smooth initial condition*

In equation (3) we set: $\Omega = (0, 2), y_0 = \text{sgn}(1 - x), \lambda = 1, U = \{-3, 0, 3\}, b_1(x) = y_0(x)$. The desired configuration is given by $\bar{y} \equiv 0$. In equation (6) we take $k = 0.1, h = 0.01$ whereas the optimal trajectory is obtained with a time stepsize of 0.01.
In Figure 6 we show the solution of the uncontrolled equation (1).

The snapshots, shown in Figure 7, are computed by a MPC Algorithm (see Algorithm 1) with $N = 5$.

In Figure 8 we present the approximation of the optimal solution following Algorithm 2.

Finally, we present the evaluation of the cost functional in Figure 9. As exepcted, the behaviour is not so exiciting as in the previous tests. The non-smooth initial condition would need a greater number of POD basis functions to

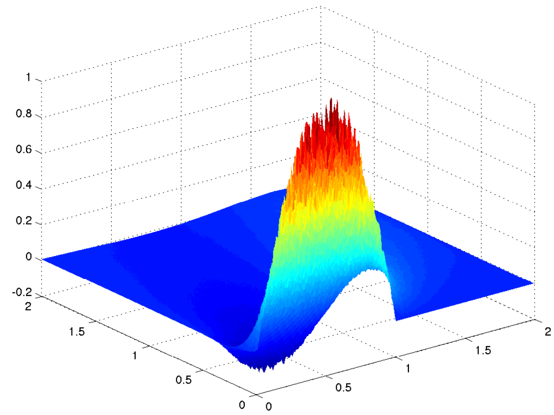

Fig. 5. Test 1: Solution of problem (3) with $\varepsilon = 0.1, c = 0.4$ and perturbation on the intial condition following Algorithm 2

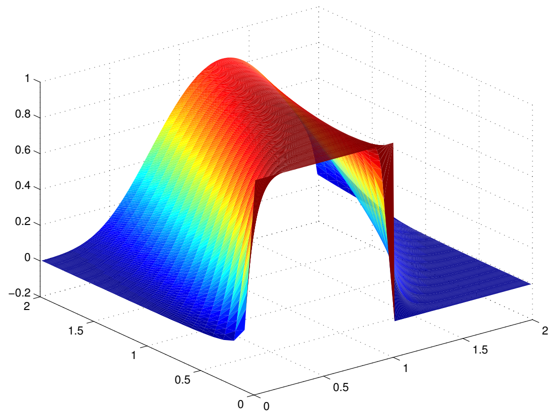

Fig. 6. Test 2: Solution of equation (1) with $\varepsilon = 0.01, c = 1, u \equiv 0$

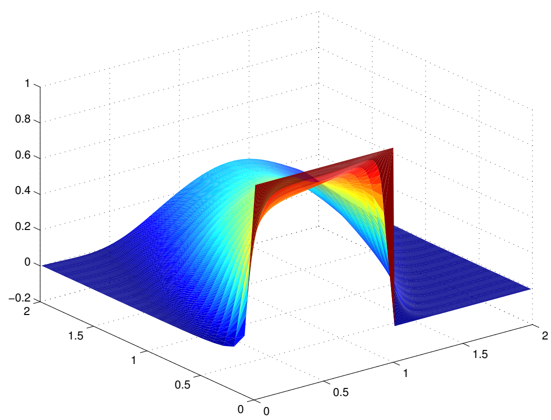

Fig. 7. Test 2: Solution of problem (3) with $\varepsilon = 0.1, c = 1$ with Algorithm 1

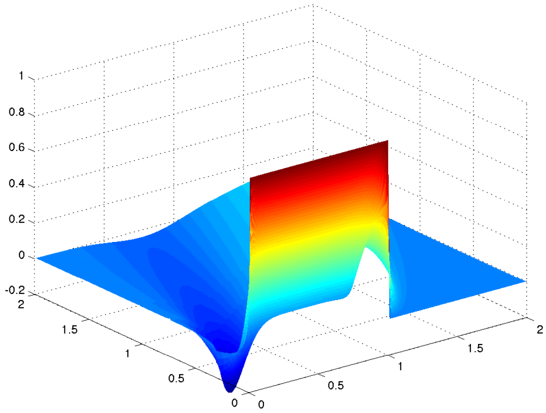

Fig. 8. Test 2: Solution of problem (3) with $\varepsilon = 0.1, c = 0.4$ with Algorithm 2

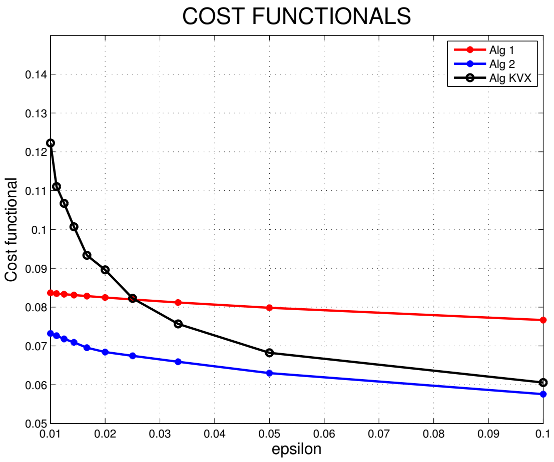


Fig. 9. Test 2: Evaluation of the Cost Functional for Algorithm 1 (red), Algorithm 2 (blue), and Algorithm KVX (black) for $c = 0.4$

describe the systems better. Nevertheless, if the diffusion term is less than $\frac{1}{40}$ our approximation, in this example, delivers a lower value than the MPC solver, and, also as, the algorithm provided by Kunisch et al. (2004).

## 6. CONCLUSION

We present an algorithm for the solution of the optimal control problem (3) by means of feedback control. To deal with HJB equation we need to work with low dimensional model and the POD method helps us in the reduction. We observe that the POD works better with good information about the system and for this purpose we run a MPC solver to compute the snapshots. In this way, for the examples considered, we provide an improvement of the low-dimensional approximation of the HJB equation compared to the approach proposed in Kunisch et al. (2004).

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