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Abstract We consider nonlinear differential-algebraic equations arising in modelling of electrical circuits using modified nodal analysis and modified loop analysis. A condensed form for such equations under the action of a constant block diagonal transformation will be derived. This form gives rise to an extraction of over- and underdetermined parts and an index analysis by means of the circuit topology. Furthermore, for linear circuits, we construct index-reduced models which preserve the structure of the circuit equations.

1 Introduction

One of the most important structural quantities in the theory of differential-algebraic equations (DAEs) is the *index*. Roughly speaking, the index measures the order of derivatives of the inhomogeneity entering to the solution. Since (numerical) differentiation is an ill-posed problem, the index can - inter alia - be regarded as a quantity that expresses the difficulty in numerical solution of DAEs. In the last three decades various index concepts have been developed in order to characterize different properties of DAEs. These are the *differentiation index* [7], the *geometric index* [26], the *perturbation index* [13], the *strangeness index* [22], and the *tractability index* [24], to mention only a few. We refer to [25] for a recent survey on all these index concepts and their role in the analysis and numerical treatment of DAEs.

In this paper, we present a structure-preserving condensed form for DAEs modelling electrical circuits with possibly nonlinear components. This form is inspired

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by the canonical forms for linear DAEs developed by KUNKEL and MEHRMANN [17, 22]. The latter forms give rise to the so-called strangeness index concept which has been successfully applied to the analysis and simulation of structural DAEs from different application areas, see the doctoral theses [2, 6, 14, 27, 31–33, 36, 38] supervised by VOLKER MEHRMANN. The great advantage of the strangeness index is that it can be defined for over- and underdetermined DAEs. Our focus is on circuit DAEs arising from *modified nodal analysis* [9, 15, 35] and *modified loop analysis* [9, 29]. We show that such DAEs have a very special structure which is preserved in the developed condensed form. In the linear case, we can, furthermore, construct index-reduced models which also preserve the special structure of circuit equations.

Nomenclature

Throughout this paper, the identity matrix of size $n \times n$ is denoted by I_n, or simply by I if it is clear from context. We write $M > N$ ($M \geq N$) if the square real matrix $M - N$ is symmetric and positive (semi-)definite. The symbol $\|x\|$ stands for the Euclidean norm of $x \in \mathbb{R}^n$. For a subspace $\mathcal{V} \subset \mathbb{R}^n$, $\mathcal{V}^\perp$ denotes the orthogonal complement of $\mathcal{V}$ with respect to the Euclidean inner product. The image and the kernel of a matrix A are denoted by $\operatorname{im} A$ and $\ker A$, respectively, and $\operatorname{rank} A$ stands for the rank of A.

2 Differential-Algebraic Equations

Consider a nonlinear DAE in general form

$$\mathcal{F}(\dot{x}(t), x(t), t) = 0, \tag{1}$$

where $\mathcal{F} : \mathbb{D}_{\dot{x}} \times \mathbb{D}_x \times \mathbb{I} \to \mathbb{R}^k$ is a continuous function, $\mathbb{D}_{\dot{x}}, \mathbb{D}_x \subseteq \mathbb{R}^n$ are open, $\mathbb{I} = [t_0, t_f] \subset \mathbb{R}$, $x : \mathbb{I} \to \mathbb{D}_x$ is a continuously differentiable unknown function, and $\dot{x}$ denotes the derivative of x with respect to t.

Definition 2.1. *A function $x : \mathbb{I} \to \mathbb{D}_x$ is said to be a* solution *of the DAE* (1) *if it is continuously differentiable for all $t \in \mathbb{I}$ and* (1) *is fulfilled pointwise for all $t \in \mathbb{I}$. This function is called a* solution of the initial value problem (1) *and $x(t_0) = x_0$ with $x_0 \in \mathbb{D}_x$ if x is the solution of* (1) *and satisfies additionally $x(t_0) = x_0$. An initial value $x_0 \in \mathbb{D}_x$ is called* consistent, *if the initial value problem* (1) *and $x(t_0) = x_0$ has a solution.*

If the function $\mathcal{F}$ has the form $\mathcal{F}(\dot{x}, x, t) = \dot{x} - f(x, t)$ with $f : \mathbb{D}_x \times \mathbb{I} \to \mathbb{R}^n$, then (1) is an ordinary differential equation (ODE). In this case, the assumption of continuity of f gives rise to the consistency of any initial value. If, moreover, f is locally Lipschitz continuous with respect to x then any initial condition determines the local solution uniquely [1, Section 7.3].

Let $\mathcal{F}(\hat{\dot{x}}, \hat{x}, \hat{t}) = 0$ for some $(\hat{\dot{x}}, \hat{x}, \hat{t}) \in \mathbb{D}_{\dot{x}} \times \mathbb{D}_x \times \mathbb{I}$. If $\mathcal{F}$ is partially differentiable with respect to $\dot{x}$ and the derivative $\frac{\partial}{\partial \dot{x}} \mathcal{F}(\hat{\dot{x}}, \hat{x}, \hat{t})$ is an invertible matrix, then by the implicit function theorem [34, Section 17.8] equation (1) can locally be solved for $\dot{x}$ resulting in an ODE $\dot{x}(t) = f(x(t),t)$. For general DAEs, however, the solvability theory is much more complex and still not as well understood as for ODEs.

A powerful framework for analysis of DAEs is provided by the derivative array approach introduced in [8]. For the DAE (1) with a sufficiently smooth function $\mathcal{F}$, the *derivative array of order* $l \in \mathbb{N}_0$ is defined by stacking equation (1) and all its formal derivatives up to order l, that is,

$$\mathcal{F}_l(x^{(l+1)}(t), x^{(l)}(t), \ldots, \dot{x}(t), x(t), t) = \begin{bmatrix} \mathcal{F}(\dot{x}(t), x(t), t) \\ \frac{d}{dt}\mathcal{F}(\dot{x}(t), x(t), t) \\ \vdots \\ \frac{d^l}{dt^l}\mathcal{F}(\dot{x}(t), x(t), t) \end{bmatrix} = 0. \tag{2}$$

Loosely speaking, the DAE (1) is said to have the *differentiation index* $\mu_d \in \mathbb{N}_0$ if $l = \mu_d$ is the smallest number of differentiations required to determine $\dot{x}$ from (2) as a function of x and t. If the differentiation index is well-defined, one can extract from the derivative array (2) a so-called *underlying ODE* $\dot{x}(t) = \phi(x(t),t)$ with the property that every solution of the DAE (1) also solves the underlying ODE.

Another index concept, called *strangeness index*, was first introduced by KUNKEL and MEHRMANN for linear DAEs [17, 19, 23] and then extended to the nonlinear case [20, 22]. The strangeness index is closely related to the differentiation index and, unlike the latter, can also be defined for over- and underdetermined DAEs [21]. For our later proposes, we restrict ourselves to a linear time-varying DAE

$$\mathcal{E}(t)\dot{x}(t) = \mathcal{A}(t)x(t) + f(t), \tag{3}$$

where $\mathcal{E}, \mathcal{A} : \mathbb{I} \to \mathbb{R}^{k,n}$ and $f : \mathbb{I} \to \mathbb{R}^k$ are sufficiently smooth functions. Such a system can be viewed as a linearization of the nonlinear DAE (1) along a trajectory. Two pairs $(\mathcal{E}_1(t), \mathcal{A}_1(t))$ and $(\mathcal{E}_2(t), \mathcal{A}_2(t))$ of matrix-valued functions are called *globally equivalent* if there exist a pointwise nonsingular continuous matrix-valued function $U : \mathbb{I} \to \mathbb{R}^{k,k}$ and a pointwise nonsingular continuously differentiable matrix-valued function $V : \mathbb{I} \to \mathbb{R}^{n,n}$ such that

$$\mathcal{E}_2(t) = U(t)\mathcal{E}_1(t)V(t), \qquad \mathcal{A}_2(t) = U(t)\mathcal{A}_1(t)V(t) - U(t)\mathcal{E}_1(t)\dot{V}(t).$$

For $(\mathcal{E}(t), \mathcal{A}(t))$ at a fixed point $t \in \mathbb{I}$, the local characteristic values r, a and s are defined as

$$r = \operatorname{rank}(\mathcal{E}), \qquad a = \operatorname{rank}(Z^\top \mathcal{A} T), \qquad s = \operatorname{rank}(S^\top Z^\top \mathcal{A} T'),$$

where the columns of Z, T, T', and S span $\ker \mathcal{E}^\top$, $\ker \mathcal{E}$, $\operatorname{im} \mathcal{E}^\top$, and $\ker T^\top \mathcal{A}^\top Z$, respectively. Considering these values pointwise, we obtain functions $r, a, s : \mathbb{I} \to \mathbb{N}_0$. It was shown in [17] that under the constant rank conditions $r(t) \equiv r$, $a(t) \equiv a$ and $s(t) \equiv s$, the DAE (3) can be transformed to the globally equivalent system

$$\begin{bmatrix} I_s & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \\ \dot{x}_5(t) \end{bmatrix} = \begin{bmatrix} 0 & A_{12}(t) & 0 & A_{14}(t) & A_{15}(t) \\ 0 & 0 & 0 & A_{24}(t) & A_{25}(t) \\ 0 & 0 & I_a & 0 & 0 \\ I_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ f_4(t) \\ f_5(t) \end{bmatrix}. \quad (4)$$

Note that the component x_1 satisfies the pure algebraic equation (the fourth equation in (4)) and its derivative is also involved in (4). Adding the differentiated fourth equation to the first one, we eliminate the derivative $\dot{x}_1$ from the first equation. The resulting system can again be transformed into the form (4) with new global characteristic values r, a and s. This procedure is repeated until s becomes zero. The minimal number μ_s of steps required to extract a DAE with $s = 0$ is called the *strangeness index* of the DAE (3). By construction, the strangeness index reduces by one for each elimination step described above. A DAE with vanishing strangeness index is called *strangeness-free*. Since the characteristic values are invariant under global equivalence transformations, μ_s is also invariant under global equivalence transformations. One can also show that the strangeness index μ_s is one below the differentiation index μ_d provided that both indices exist (except for the case, where the differentiation index is zero, then the strangeness index vanishes as well), see [17,22].

This index reduction procedure has a rather theoretical character since the global equivalence transformations are difficult to determine numerically. It was shown in [19] that the solvability properties of the DAE (3) can also be established from the associated derivative array given by

$$\mathcal{M}_l(t)\dot{z}_l(t) = \mathcal{N}_l(t)z_l(t) + g_l(t),$$

where

$$\begin{aligned} \big[\mathcal{M}_l\big]_{ij} &= \binom{i}{j}\mathcal{E}^{(i-j)} - \binom{i}{j+1}\mathcal{A}^{(i-j-1)}, \;\; i,j = 0,\ldots,l, \\ \big[\mathcal{N}_l\big]_{ij} &= \begin{cases} \mathcal{A}^{(i)} & \text{for } \; i = 0,\ldots,l, \;\; j = 0, \\ 0 & \text{else}, \end{cases} \\ \big[z_l\big]_i &= x^{(i)}, \quad \big[g_l\big]_i = f^{(i)}, \;\; i = 0,\ldots,l, \end{aligned}$$

with the convention that $\binom{i}{j} = 0$ for $i < j$. If the strangeness index μ_s is well-defined, then the DAE (3) satisfies the following hypothesis.

Hypothesis 2.2. *There exist integers μ, a, d and w such that the pair $(\mathcal{M}_\mu, \mathcal{N}_\mu)$ associated with $(\mathcal{E},\mathcal{A})$ has the following properties:*

1. *For all $t \in \mathbb{I}$, we have $\operatorname{rank}\mathcal{M}_\mu(t) = (\mu+1)k - a - w$. This implies the existence of a smooth full rank matrix-valued function Z of size $((\mu+1)k, a+w)$ satisfying $Z^\top \mathcal{M}_\mu = 0$.*
2. *For all $t \in \mathbb{I}$, we have $\operatorname{rank}\big(Z(t)^\top \mathcal{N}_\mu(t)\,\big[I_n\; 0\; \ldots\; 0\big]^\top\big) = a$ and without loss of generality Z can be partitioned as $\big[Z_2\;\; Z_3\big]$ with Z_2 of size $((\mu+1)k, a)$ and Z_3 of size $((\mu+1)k, w)$ such that $\mathcal{A}_2 = Z_2^\top \mathcal{N}_\mu\,\big[I_n\; 0\; \ldots\; 0\big]^\top$ has full row rank and*

$Z_3^\top \mathcal{N}_\mu \left[I_n\ 0\ \dots\ 0\right]^\top = 0$. Furthermore, there exists a smooth full rank matrix-valued function T_2 of size $(n, n-a)$ satisfying $\mathcal{A}_2 T_2 = 0$.

3. *For all $t \in \mathbb{I}$, we have $\operatorname{rank}\big(\mathcal{E}(t)T_2(t)\big) = d$, where $d = k - a - w_\mu$ and*

$$w_\mu = k - \operatorname{rank}\left[\mathcal{M}_\mu\ \ \mathcal{N}_\mu\right] + \operatorname{rank}\left[\mathcal{M}_{\mu-1}\ \ \mathcal{N}_{\mu-1}\right]$$

with the convention that $\operatorname{rank}\left[\mathcal{M}_{-1}\ \ \mathcal{N}_{-1}\right] = 0$. This implies the existence of a smooth full rank matrix function Z_1 of size (k,d) such that $\mathcal{E}_1 = Z_1^\top \mathcal{E}$ has full row rank.

The smallest possible μ in Hypothesis 2.2 is the strangeness index of the DAE (3) and $u = n - d - a$ defines the number of undertermined components. Introducing $\mathcal{A}_1 = Z_1^\top \mathcal{A}$, $f_1(t) = Z_1^\top f(t)$, $f_2(t) = Z_2^\top g_\mu(t)$ and $f_3(t) = Z_3^\top g_\mu(t)$, we obtain a strangeness-free DAE system

$$\begin{bmatrix} \mathcal{E}_1(t) \\ 0 \\ 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} \mathcal{A}_1(t) \\ \mathcal{A}_2(t) \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix} \tag{5}$$

which has the same solutions as (3). The DAE (3) is solvable if $f_3(t) \equiv 0$ in (5). Moreover, an initial condition $x(t_0) = x_0$ is consistent if $\mathcal{A}_2(t_0)x_0 + f_2(t_0) = 0$. The initial value problem with consistent initial condition has a unique solution if $u = 0$.

3 Modified Nodal and Modified Loop Analysis

In this section, we consider the modelling of electrical circuits by DAEs based on the Kirchhoff laws and the constitutive relations for the electrical components. Derivations of these relations from Maxwell's equations can be found in [28].

A general electrical circuit with voltage and current sources, resistors, capacitors and inductors can be modelled as a directed graph whose nodes correspond to the nodes of the circuit and whose branches correspond to the circuit elements [9–11, 28]. We refer to the aforementioned works for the graph theoretic preliminaries related to circuit theory. Let n_n, n_b and n_l be, respectively, the number of nodes, branches and loops in this graph. Moreover, let $i(t) \in \mathbb{R}^{n_\mathrm{b}}$ be the vector of currents and let $v(t) \in \mathbb{R}^{n_\mathrm{b}}$ be the vector of corresponding voltages. Then Kirchhoff's current law [11, 28] states that at any node, the sum of flowing-in currents is equal to the sum of flowing-out currents, see Fig. 1. Equivalently, this law can be written as $A_0 i(t) = 0$, where $A_0 = [a_{kl}] \in \mathbb{R}^{n_\mathrm{n} \times n_\mathrm{b}}$ is an *all-node incidence matrix* with

$$a_{kl} = \begin{cases} 1, & \text{if branch } l \text{ leaves node } k, \\ -1, & \text{if branch } l \text{ enters node } k, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, Kirchhoff's voltage law [11, 28] states that the sum of voltages along the branches of any loop vanishes, see Fig. 2. This law can equivalently be written as $B_0 v(t) = 0$, where $B_0 = [b_{kl}] \in \mathbb{R}^{n_\mathrm{l} \times n_\mathrm{b}}$ is an *all-loop matrix* with

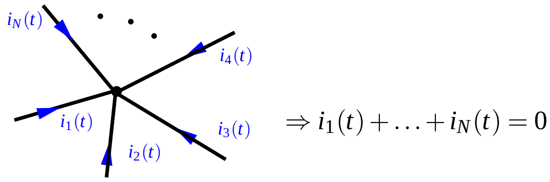

Fig. 1: Kirchhoff's current law

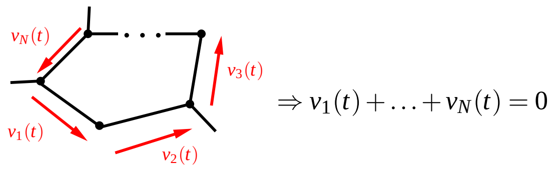

Fig. 2: Kirchhoff's voltage law

$$b_{kl} = \begin{cases} 1, & \text{if branch } l \text{ belongs to loop } k \text{ and has the same orientation,} \\ -1, & \text{if branch } l \text{ belongs to loop } k \text{ and has the contrary orientation,} \\ 0, & \text{otherwise.} \end{cases}$$

The following proposition establishes a relation between the incidence and loop matrices A_0 and B_0.

Proposition 3.1. [10, p. 213] *Let $A_0 \in \mathbb{R}^{n_\mathrm{n} \times n_\mathrm{b}}$ be an all-node incidence matrix and let $B_0 \in \mathbb{R}^{n_\mathrm{l} \times n_\mathrm{b}}$ be an all-loop matrix of a connected graph. Then*

$$\ker B_0 = \operatorname{im} A_0^\top, \qquad \operatorname{rank} A_0 = n_\mathrm{n} - 1, \qquad \operatorname{rank} B_0 = n_\mathrm{b} - n_\mathrm{n} + 1.$$

We now consider the full rank matrices $A \in \mathbb{R}^{n_\mathrm{n}-1 \times n_\mathrm{b}}$ and $B \in \mathbb{R}^{n_\mathrm{b}-n_\mathrm{n}+1 \times n_\mathrm{b}}$ obtained from A_0 and B_0, respectively, by removing linear dependent rows. The matrices A and B are called the *reduced incidence* and *reduced loop matrices*, respectively. Then the Kirchhoff laws are equivalent to

$$A\,i(t) = 0, \qquad B\,v(t) = 0. \tag{6}$$

Due to the relation $\ker B = \operatorname{im} A^\top$, we can reformulate Kirchhoff's laws as follows: there exist vectors $\eta(t) \in \mathbb{R}^{n_\mathrm{n}-1}$ and $\iota(t) \in \mathbb{R}^{n_\mathrm{b}-n_\mathrm{n}+1}$ such that

$$i(t) = B^\top \iota(t), \qquad v(t) = A^\top \eta(t). \tag{7}$$

The vectors $\eta(t)$ and $\iota(t)$ are called the vectors of *node potentials* and *loop currents*, respectively. We partition the voltage and current vectors

$$v(t) = \left[v_{\mathcal{C}}^\top(t) \;\; v_{\mathcal{L}}^\top(t) \;\; v_{\mathcal{R}}^\top(t) \;\; v_{\mathcal{V}}^\top(t) \;\; v_{\mathcal{I}}^\top(t) \right]^\top,$$
$$i(t) = \left[i_{\mathcal{C}}^\top(t) \;\; i_{\mathcal{L}}^\top(t) \;\; i_{\mathcal{R}}^\top(t) \;\; i_{\mathcal{V}}^\top(t) \;\; i_{\mathcal{I}}^\top(t) \right]^\top$$

into voltage and current vectors of capacitors, inductors, resistors, voltage and current sources of dimensions $n_{\mathcal{C}}$, $n_{\mathcal{L}}$, $n_{\mathcal{R}}$, $n_{\mathcal{V}}$ and $n_{\mathcal{I}}$, respectively. Furthermore, partitioning the incidence and loop matrices

$$A = \left[A_{\mathcal{C}} \; A_{\mathcal{L}} \; A_{\mathcal{R}} \; A_{\mathcal{V}} \; A_{\mathcal{I}} \right], \qquad B = \left[B_{\mathcal{C}} \; B_{\mathcal{L}} \; B_{\mathcal{R}} \; B_{\mathcal{V}} \; B_{\mathcal{I}} \right], \tag{8}$$

the Kirchhoff laws (6) and (7) can now be represented in two alternative ways, namely, in the incidence-based formulation

$$A_{\mathcal{C}} i_{\mathcal{C}}(t) + A_{\mathcal{L}} i_{\mathcal{L}}(t) + A_{\mathcal{R}} i_{\mathcal{R}}(t) + A_{\mathcal{V}} i_{\mathcal{V}}(t) + A_{\mathcal{I}} i_{\mathcal{I}}(t) = 0, \tag{9}$$
$$v_{\mathcal{C}}(t) = A_{\mathcal{C}}^\top \eta(t), \qquad v_{\mathcal{L}}(t) = A_{\mathcal{L}}^\top \eta(t), \qquad v_{\mathcal{R}}(t) = A_{\mathcal{R}}^\top \eta(t), \tag{10}$$
$$v_{\mathcal{V}}(t) = A_{\mathcal{V}}^\top \eta(t), \qquad v_{\mathcal{I}}(t) = A_{\mathcal{I}}^\top \eta(t), \tag{11}$$

or in the loop-based formulation

$$B_{\mathcal{R}} v_{\mathcal{R}}(t) + B_{\mathcal{C}} v_{\mathcal{C}}(t) + B_{\mathcal{L}} v_{\mathcal{L}}(t) + B_{\mathcal{V}} v_{\mathcal{V}}(t) + B_{\mathcal{I}} v_{\mathcal{I}}(t) = 0, \tag{12}$$
$$i_{\mathcal{C}}(t) = B_{\mathcal{C}}^\top \iota(t), \qquad i_{\mathcal{L}}(t) = B_{\mathcal{L}}^\top \iota(t), \qquad i_{\mathcal{R}}(t) = B_{\mathcal{R}}^\top \iota(t), \tag{13}$$
$$i_{\mathcal{V}}(t) = B_{\mathcal{V}}^\top \iota(t), \qquad i_{\mathcal{I}}(t) = B_{\mathcal{I}}^\top \iota(t). \tag{14}$$

The dynamics of electrical circuits are not only relying on the Kirchhoff laws, but their behaviour is also determined by the components being located at the branches. The branch constitutive relations for capacitors, inductors and resistors are given by

$$i_{\mathcal{C}}(t) = \frac{d}{dt} q(v_{\mathcal{C}}(t)), \tag{15}$$
$$v_{\mathcal{L}}(t) = \frac{d}{dt} \psi(i_{\mathcal{L}}(t)), \tag{16}$$
$$i_{\mathcal{R}}(t) = g(v_{\mathcal{R}}(t)), \tag{17}$$

respectively, where $q : \mathbb{R}^{n_{\mathcal{C}}} \to \mathbb{R}^{n_{\mathcal{C}}}$ is the *charge function*, $\psi : \mathbb{R}^{n_{\mathcal{L}}} \to \mathbb{R}^{n_{\mathcal{L}}}$ is the *flux function*, and $g : \mathbb{R}^{n_{\mathcal{R}}} \to \mathbb{R}^{n_{\mathcal{R}}}$ is the *conductance function*. We now give our general assumptions on the considered circuit elements. For an interpretation of these assumptions in terms of total energy of the circuit, we refer to [28].

(A1) The charge, flux and conductance functions are continuously differentiable.
(A2) The Jacobian of the charge function

$$\mathcal{C}(v_{\mathcal{C}}) := \frac{d}{dv_{\mathcal{C}}} q(v_{\mathcal{C}})$$

is symmetric and pointwise positive definite.

(A3) The Jacobian of the flux function

$$\mathcal{L}(i_{\mathcal{L}}) := \frac{d}{di_{\mathcal{L}}}\psi(i_{\mathcal{L}})$$

is symmetric and pointwise positive definite.

(A4) The conductance function satisfies $g(0) = 0$ and there exists a constant $c > 0$ such that

$$(v_{\mathcal{R},1} - v_{\mathcal{R},2})^\top \big(g(v_{\mathcal{R},1}) - g(v_{\mathcal{R},2})\big) \geq c\|v_{\mathcal{R},1} - v_{\mathcal{R},2}\|^2 \tag{18}$$

for all $v_{\mathcal{R},1}, v_{\mathcal{R},2} \in \mathbb{R}^{n_\mathcal{R}}$.

Using the chain rule, the relations (15) and (16) can equivalently be written as

$$i_{\mathcal{C}}(t) = \mathcal{C}(v_{\mathcal{C}}(t))\frac{d}{dt}v_{\mathcal{C}}(t), \tag{19}$$

$$v_{\mathcal{L}}(t) = \mathcal{L}(i_{\mathcal{L}}(t))\frac{d}{dt}i_{\mathcal{L}}(t). \tag{20}$$

Furthermore, the property (18) implies that the Jacobian of the conductance function

$$\mathcal{G}(v_{\mathcal{R}}) := \frac{d}{dv_{\mathcal{R}}}g(v_{\mathcal{R}})$$

fulfils

$$\mathcal{G}(v_{\mathcal{R}}) + \mathcal{G}^\top(v_{\mathcal{R}}) \geq 2cI > 0 \qquad \text{for all } \ v_{\mathcal{R}} \in \mathbb{R}^{n_\mathcal{R}}. \tag{21}$$

Thus, the matrix $\mathcal{G}(v_{\mathcal{R}})$ is invertible for all $v_{\mathcal{R}} \in \mathbb{R}^{n_\mathcal{R}}$. Applying the Cauchy-Schwarz inequality to (18) and taking into account that $g(0) = 0$, we have

$$\|g(v_{\mathcal{R}})\|\|v_{\mathcal{R}}\| \geq v_{\mathcal{R}}^\top g(v_{\mathcal{R}}) \geq c\|v_{\mathcal{R}}\|^2 \quad \text{for all } \ v_{\mathcal{R}} \in \mathbb{R}^{n_\mathcal{R}}$$

and, hence, $\|g(v_{\mathcal{R}})\| \geq c_r\|v_{\mathcal{R}}\|$. Then it follows from [37, Corollary, p. 201] that g has a global inverse function. This inverse is denoted by $r = g^{-1}$ and referred to as the *resistance function*. Consequently, the relation (17) is equivalent to

$$v_{\mathcal{R}}(t) = r(i_{\mathcal{R}}(t)). \tag{22}$$

Moreover, we obtain from (18) that

$$\begin{aligned}\big(i_{\mathcal{R},1} - i_{\mathcal{R},2}\big)^\top\big(r(i_{\mathcal{R},1}) - r(i_{\mathcal{R},2})\big) &= \big(g(r(i_{\mathcal{R},1})) - g(r(i_{\mathcal{R},2}))\big)^\top\big(r(i_{\mathcal{R},1}) - r(i_{\mathcal{R},2})\big)\\ &= \big(r(i_{\mathcal{R},1}) - r(i_{\mathcal{R},2})\big)^\top\big(g(r(i_{\mathcal{R},1})) - g(r(i_{\mathcal{R},2}))\big) \geq c\,\|r(i_{\mathcal{R},1}) - r(i_{\mathcal{R},2})\|^2\end{aligned}$$

holds for all $i_{\mathcal{R},1}, i_{\mathcal{R},2} \in \mathbb{R}^{n_\mathcal{R}}$. Then the inverse function theorem implies that the Jacobian

$$\mathcal{R}(i_{\mathcal{R}}) := \frac{d}{di_{\mathcal{R}}}r(i_{\mathcal{R}})$$

fulfils $\mathcal{R}(i_{\mathcal{R}}) = (\mathcal{G}(r(i_{\mathcal{R}})))^{-1}$. In particular, $\mathcal{R}(i_{\mathcal{R}})$ is invertible for all $i_{\mathcal{R}} \in \mathbb{R}^{n_{\mathcal{R}}}$, and the relation (21) yields

$$\mathcal{R}(i_{\mathcal{R}}) + \mathcal{R}^{\top}(i_{\mathcal{R}}) > 0 \qquad \text{for all } \ i_{\mathcal{R}} \in \mathbb{R}^{n_{\mathcal{R}}}.$$

Having collected all physical laws for an electrical circuit, we are now able to set up a circuit model. This can be done in two different ways. The first approach is based on the formulation of Kirchhoff's laws via the incidence matrices given in (9)-(11), whereas the second approach relies on the equivalent representation of Kirchhoff's laws with the loop matrices given in (12)-(14).

a) **Modified nodal analysis (MNA)**
Starting with Kirchhoff's current law (9), we eliminate the resistive and capacitive currents and voltages by using (17) and (19) as well as Kirchhoff's voltage law in (10) for resistors and capacitors. This results in

$$A_{\mathcal{C}}\mathcal{C}(A_{\mathcal{C}}^{\top}\eta(t))A_{\mathcal{C}}^{\top}\tfrac{d}{dt}\eta(t) + A_{\mathcal{R}}g(A_{\mathcal{R}}^{\top}\eta(t)) + A_{\mathcal{L}}i_{\mathcal{L}}(t) + A_{\mathcal{V}}i_{\mathcal{V}}(t) + A_{\mathcal{I}}i_{\mathcal{I}}(t) = 0.$$

Kirchhoff's voltage law in (10) for the inductive voltages and the component relation (20) for the inductors give

$$-A_{\mathcal{L}}^{\top}\eta(t) + \mathcal{L}(i_{\mathcal{L}}(t))\tfrac{d}{dt}i_{\mathcal{L}}(t) = 0.$$

Using Kirchhoff's voltage law in (11) for voltage sources, we obtain finally the MNA system

$$\begin{aligned} A_{\mathcal{C}}\mathcal{C}(A_{\mathcal{C}}^{\top}\eta(t))A_{\mathcal{C}}^{\top}\tfrac{d}{dt}\eta(t) + A_{\mathcal{R}}g(A_{\mathcal{R}}^{\top}\eta(t)) + A_{\mathcal{L}}i_{\mathcal{L}}(t) + A_{\mathcal{V}}i_{\mathcal{V}}(t) + A_{\mathcal{I}}i_{\mathcal{I}}(t) &= 0,\\ -A_{\mathcal{L}}^{\top}\eta(t) + \mathcal{L}(i_{\mathcal{L}}(t))\tfrac{d}{dt}i_{\mathcal{L}}(t) &= 0,\\ -A_{\mathcal{V}}^{\top}\eta(t) + v_{\mathcal{V}}(t) &= 0. \end{aligned} \tag{23}$$

In this formulation, voltages of voltage sources $v_{\mathcal{V}}$ and currents of current sources $i_{\mathcal{I}}$ are assumed to be given, whereas node potentials η, inductive currents $i_{\mathcal{L}}$ and currents of voltage sources $i_{\mathcal{V}}$ are unknown. The remaining physical variables such as voltages of the resistive, capacitive and inductive elements as well as resistive and capacitive currents can be algebraically reconstructed from the solution of system (23).

b) **Modified loop analysis (MLA)**
Using the loop matrix based formulation of Kirchhoff's voltage law (12), the constitutive relations (20) and (22) for inductors and resistors, and the loop matrix based formulation of Kirchhoff's current law in (13) for the inductive and resistive currents, we obtain

$$B_{\mathcal{L}}\mathcal{L}(B_{\mathcal{L}}^{\top}\iota(t))B_{\mathcal{L}}^{\top}\tfrac{d}{dt}\iota(t) + B_{\mathcal{R}}r(B_{\mathcal{R}}^{\top}\iota(t)) + B_{\mathcal{C}}v_{\mathcal{C}}(t) + B_{\mathcal{I}}v_{\mathcal{I}}(t) + B_{\mathcal{V}}v_{\mathcal{V}}(t) = 0.$$

Moreover, Kirchhoff's voltage law in (13) for capacitors together with the component relation (19) for capacitors gives

$$-B_{\mathcal{C}}^{\top}\iota(t)+\mathcal{C}(v_{\mathcal{C}}(t))\tfrac{d}{dt}v_{\mathcal{C}}(t)=0.$$

Combining these two relations together with Kirchhoff's voltage law in (14) for voltage sources, we obtain the MLA system

$$\begin{aligned}
B_{\mathcal{L}}\mathcal{L}(B_{\mathcal{L}}^{\top}\iota(t))B_{\mathcal{L}}^{\top}\tfrac{d}{dt}\iota(t)+B_{\mathcal{R}}r(B_{\mathcal{R}}^{\top}\iota(t))+B_{\mathcal{C}}v_{\mathcal{C}}(t)+B_{\mathcal{I}}v_{\mathcal{I}}(t)+B_{\mathcal{V}}v_{\mathcal{V}}(t)&=0,\\
-B_{\mathcal{C}}^{\top}\iota(t)+\mathcal{C}(v_{\mathcal{C}}(t))\tfrac{d}{dt}v_{\mathcal{C}}(t)&=0,\\
-B_{\mathcal{I}}^{\top}\iota(t)+i_{\mathcal{I}}(t)&=0.
\end{aligned}$$

Here, the unknown variables are loop currents ι, capacitive voltages $v_{\mathcal{C}}$ and voltages of current sources $v_{\mathcal{I}}$, and, as before, $v_{\mathcal{V}}$ and $i_{\mathcal{I}}$ are assumed to be known.

Thus, the overall circuit is described by the resistance law $i_{\mathcal{R}}(t)=g(v_{\mathcal{R}}(t))$ or $v_{\mathcal{R}}(t)=r(i_{\mathcal{R}}(t))$, the differential equations (19) and (20) for capacitors and inductors, and the Kirchhoff laws either in the form (9)-(11) or (12)-(14). By setting

$$x(t)=\begin{bmatrix}\eta(t)\\ i_{\mathcal{L}}(t)\\ i_{\mathcal{V}}(t)\end{bmatrix}\qquad\left(\text{resp. } x(t)=\begin{bmatrix}\iota(t)\\ v_{\mathcal{C}}(t)\\ v_{\mathcal{I}}(t)\end{bmatrix}\right)$$

in the MNA (resp. MLA) case, we obtain a nonlinear DAE of the form (1).

In the linear case, the capacitance matrix $\mathcal{C}(v_{\mathcal{C}}(t))\equiv\mathcal{C}$ and the inductance matrix $\mathcal{L}(i_{\mathcal{L}}(t))\equiv\mathcal{L}$ are both constant, and the component relations (17) and (22) for resistors read

$$i_{\mathcal{R}}(t)=\mathcal{G}v_{\mathcal{R}}(t),\qquad v_{\mathcal{R}}(t)=\mathcal{R}\,i_{\mathcal{R}}(t),$$

respectively, with $\mathcal{R}=\mathcal{G}^{-1}\in\mathbb{R}^{n_{\mathcal{R}}\times n_{\mathcal{R}}}$, $\mathcal{G}+\mathcal{G}^{\top}>0$ and $\mathcal{R}+\mathcal{R}^{\top}>0$. Then the circuit equations can be written as a linear DAE system

$$\mathcal{E}\dot{x}(t)=\mathcal{A}x(t)+\mathcal{B}u(t),\tag{24}$$

where $u(t)=\left[i_{\mathcal{I}}^{\top}(t),\,v_{\mathcal{V}}^{\top}(t)\right]^{\top}$, and the system matrices have the form

$$\mathcal{E}=\begin{bmatrix}A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^{\top}&0&0\\0&\mathcal{L}&0\\0&0&0\end{bmatrix},\ \mathcal{A}=\begin{bmatrix}-A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^{\top}&-A_{\mathcal{L}}&-A_{\mathcal{V}}\\A_{\mathcal{L}}^{\top}&0&0\\A_{\mathcal{V}}^{\top}&0&0\end{bmatrix},\ \mathcal{B}=\begin{bmatrix}-A_{\mathcal{I}}&0\\0&0\\0&-I_{n_{\mathcal{V}}}\end{bmatrix}\tag{25}$$

in the MNA case and

$$\mathcal{E}=\begin{bmatrix}B_{\mathcal{L}}\mathcal{L}B_{\mathcal{L}}^{\top}&0&0\\0&\mathcal{C}&0\\0&0&0\end{bmatrix},\ \mathcal{A}=\begin{bmatrix}-B_{\mathcal{R}}\mathcal{R}B_{\mathcal{R}}^{\top}&-B_{\mathcal{C}}&-B_{\mathcal{I}}\\B_{\mathcal{C}}^{\top}&0&0\\B_{\mathcal{I}}^{\top}&0&0\end{bmatrix},\ \mathcal{B}=\begin{bmatrix}0&-B_{\mathcal{V}}\\0&0\\-I_{n_{\mathcal{I}}}&0\end{bmatrix}\tag{26}$$

in the MLA case.

4 Differential-Algebraic Equations of Circuit Type

In this section, we study a special class of DAEs. First of all note that both the MNA and MLA systems can be written in a general form as

$$\begin{aligned} 0 &= E\Phi\big(E^\top x_1(t)\big)E^\top \dot{x}_1(t) \;+\; F\rho\big(F^\top x_1(t)\big) + G_2x_2(t) + G_3x_3(t) \;+\; f_1(t),\\ 0 &= \Psi\big(x_2(t)\big)\,\dot{x}_2(t) \;-\; G_2^\top x_1(t) \;+\; f_2(t),\\ 0 &= -\; G_3^\top x_1(t) \;+\; f_3(t), \end{aligned} \tag{27}$$

with the matrices $E \in \mathbb{R}^{n_1\times m_1}$, $F \in \mathbb{R}^{n_1\times m_2}$, $G_2 \in \mathbb{R}^{n_1\times n_2}$, $G_3 \in \mathbb{R}^{n_1\times n_3}$ and the continuously differentiable functions $\Phi : \mathbb{R}^{m_1} \to \mathbb{R}^{m_1\times m_1}$, $\Psi : \mathbb{R}^{n_2} \to \mathbb{R}^{n_2\times n_2}$ and $\rho : \mathbb{R}^{m_2} \to \mathbb{R}^{m_2}$ satisfying

$$\Phi(z_1) > 0 \qquad \text{for all } z_1 \in \mathbb{R}^{m_1}, \tag{28}$$

$$\Psi(z_2) > 0 \qquad \text{for all } z_2 \in \mathbb{R}^{n_2}, \tag{29}$$

$$\frac{d}{dz}\rho(z) + \left(\frac{d}{dz}\rho(z)\right)^\top > 0 \qquad \text{for all } z \in \mathbb{R}^{m_2}. \tag{30}$$

We now investigate the differentiation index of the DAE (27). The following result has been proven in [28] with the additional assumption $f_2(t) = 0$. However, this assumption has not been required in the proof.

Theorem 4.1. *[28, Theorem 6.6] Let a DAE* (27) *be given and assume that the functions* $\Phi : \mathbb{R}^{m_1} \to \mathbb{R}^{m_1\times m_1}$, $\Psi : \mathbb{R}^{n_2} \to \mathbb{R}^{n_2\times n_2}$ *and* $\rho : \mathbb{R}^{m_2} \to \mathbb{R}^{m_2\times m_2}$ *satisfy* (28)–(30)*. Further, assume that the matrices* $E \in \mathbb{R}^{n_1\times m_1}$*,* $F \in \mathbb{R}^{n_1\times m_2}$*,* $G_2 \in \mathbb{R}^{n_1\times n_2}$ *and* $G_3 \in \mathbb{R}^{n_1\times n_3}$ *fulfil*

$$\operatorname{rank}\begin{bmatrix}E & F & G_2 & G_3\end{bmatrix} = n_1, \qquad \operatorname{rank} G_3 = n_3. \tag{31}$$

Then the differentiation index μ_d *of* (27) *is well-defined and it holds*

a) $\mu_d = 0$*, if and only if* $n_3 = 0$ *and* $\operatorname{rank} E = n_1$.
b) $\mu_d = 1$*, if and only if it is not zero and*

$$\operatorname{rank}\begin{bmatrix}E & F & G_3\end{bmatrix} = n_1, \qquad \ker\begin{bmatrix}E & G_3\end{bmatrix} = \ker E \times \{0\}. \tag{32}$$

c) $\mu_d = 2$*, if and only if* $\mu_d \notin \{0,1\}$.

The additional assumptions (31) ensure that the DAE (27) is neither over- nor underdetermined, i.e, a solution of (27) exists for sufficiently smooth f_1, f_2 and f_3, and it is unique for any consistent initial value. Note that the assumptions (31) will not be made in the following. We will show that from any DAE of the form (27) one can extract a DAE of differentiation index one which has the same structure as (27). This extraction will be done by a special linear coordinate transformation.

To this end, we first introduce the matrices W_1, W_1', W_{11}, W_{11}', W_{12}, W_{12}', W_2, W_2', W_3, W_3', W_{31}, W_{31}', W_{32} and W_{32}' which have full column rank and satisfy the following conditions:

$$
\begin{array}{lll}
\text{(C1)} & \operatorname{im} W_1 = \ker E^\top, & \operatorname{im} W_1' = \operatorname{im} E, \\
\text{(C2)} & \operatorname{im} W_{11} = \ker \begin{bmatrix} F & G_3 \end{bmatrix}^\top W_1, & \operatorname{im} W_{11}' = \operatorname{im} W_1^\top \begin{bmatrix} F & G_3 \end{bmatrix}, \\
\text{(C3)} & \operatorname{im} W_{12} = \ker G_2^\top W_1 W_{11}, & \operatorname{im} W_{12}' = \operatorname{im} W_{11}^\top W_1^\top G_2, \\
\text{(C4)} & \operatorname{im} W_2 = \ker W_{11}^\top W_1^\top G_2, & \operatorname{im} W_2' = \operatorname{im} G_2^\top W_1 W_{11}, \\
\text{(C5)} & \operatorname{im} W_3 = \ker W_1^\top G_3, & \operatorname{im} W_3' = \operatorname{im} G_3^\top W_1, \\
\text{(C6)} & \operatorname{im} W_{31} = \ker G_3 W_3, & \operatorname{im} W_{31}' = \operatorname{im} W_3^\top G_3^\top, \\
\text{(C7)} & \operatorname{im} W_{32} = \ker W_3^\top G_3^\top W_1', & \operatorname{im} W_{32}' = \operatorname{im} W_1'^\top G_3 W_3.
\end{array}
$$

The following lemma provides some useful properties for these matrices.

Lemma 4.2. *Let $E \in \mathbb{R}^{n_1 \times m_1}$, $F \in \mathbb{R}^{n_1 \times m_2}$, $G_2 \in \mathbb{R}^{n_1 \times n_2}$ and $G_3 \in \mathbb{R}^{n_1 \times n_3}$ be given, and let W_j and W_j' for $j \in J := \{1, 11, 12, 2, 3, 31, 32\}$ be matrices of full column rank satisfying the conditions* (C1)–(C7)*. Then the following holds true:*

a) The relations $(\operatorname{im} W_j)^\perp = \operatorname{im} W_j'$ are fulfilled for $j \in J$.
b) The matrix $W_1 W_{11}$ has full column rank with

$$\operatorname{im} W_1 W_{11} = \ker \begin{bmatrix} E & F & G_3 \end{bmatrix}^\top. \tag{33}$$

c) The matrix $W_1 W_{11} W_{12}$ has full column rank with

$$\operatorname{im} W_1 W_{11} W_{12} = \ker \begin{bmatrix} E & F & G_2 & G_3 \end{bmatrix}^\top. \tag{34}$$

d) The matrix $W_3 W_{31}$ has full column rank with

$$\operatorname{im} W_3 W_{31} = \ker G_3. \tag{35}$$

e) The matrix $W_{31}'^\top W_3^\top G_3^\top W_1' W_{32}'$ is square and invertible.
f) The matrix $W_{12}'^\top W_{11}^\top W_1^\top G_2 W_2'$ is square and invertible.

Proof. The proof mainly relies on the simple fact that $\ker M^\top = (\operatorname{im} M)^\perp$ holds for any matrix $M \in \mathbb{R}^{m \times n}$.

a) The case $j = 1$ simply follows from

$$(\operatorname{im} W_1)^\perp = (\ker E^\top)^\perp = \operatorname{im} E = \operatorname{im} W_1'.$$

The remaining relations can be proved analogously.

b) The matrix $W_1 W_{11}$ has full column rank as a product of matrices with full column rank. Furthermore, the subset relation "$\subseteq$" in (33) is a consequence of $\begin{bmatrix} E & F & G_3 \end{bmatrix}^\top W_1 W_{11} = 0$ which follows from (C1) and (C2). To prove the reverse inclusion, assume that $x \in \ker \begin{bmatrix} E & F & G_3 \end{bmatrix}^\top$. Then

$$x \in \ker E^\top = \operatorname{im} W_1 \qquad \text{and} \qquad x \in \ker \begin{bmatrix} F & G_3 \end{bmatrix}^\top.$$

Hence, there exists a vector y such that $x = W_1 y$. We have

$$\begin{bmatrix} F & G_3 \end{bmatrix}^\top W_1 y = \begin{bmatrix} F & G_3 \end{bmatrix}^\top x = 0.$$

The definition of W_{11} gives rise to the existence of a vector z satisfying $y = W_{11} z$. Thus, $x = W_1 y = W_1 W_{11} z \in \operatorname{im} W_1 W_{11}$.

c) The matrix $W_1 W_{11} W_{12}$ has full column rank as a product of matrices with full column rank. The inclusion "$\subseteq$" in (34) follows from

$$\begin{bmatrix} E & F & G_2 & G_3 \end{bmatrix}^\top W_1 W_{11} W_{12} = 0$$

which can be proved using (C1)–(C3). For the proof of the reverse inclusion, assume that $x \in \ker \begin{bmatrix} E & F & G_2 & G_3 \end{bmatrix}^\top$. Then $x \in \ker \begin{bmatrix} E & F & G_3 \end{bmatrix}^\top$. Hence, due b) there exists a vector y such that $x = W_1 W_{11} y$. Consequently, $G_2^\top W_1 W_{11} y = G_2^\top x = 0$. The definition of W_{12} gives rise to the existence of a vector z such hat $y = W_{12} z$, and, thus, $x = W_1 W_{11} y = W_1 W_{11} W_{12} z \in \operatorname{im} W_1 W_{11} W_{12}$.

d) The matrix $W_3 W_{31}$ has full column rank as a product of matrices with full column rank. The inclusion "$\subseteq$" in (35) follows from $G_3 W_3 W_{31} = 0$. For the proof of the reverse inclusion, assume that $x \in \ker G_3$. Then $x \in \ker W_1^\top G_3$, whence, by definition of W_3, there exists a vector y with $x = W_3 y$. Then $0 = G_3 x = G_3 W_3 y$ and, by definition of W_{31}, there exists a vector z such that $y = W_{31} z$. This gives $x = W_3 y = W_3 W_{31} z \in \operatorname{im} W_3 W_{31}$.

e) First, we show that

$$\ker W_{31}'^\top W_3^\top G_3^\top W_1' W_{32}' = \{0\}. \tag{36}$$

Assume that $x \in \ker W_{31}'^\top W_3^\top G_3^\top W_1' W_{32}'$. Then

$$W_3^\top G_3^\top W_1' W_{32}' x \in \ker W_{31}'^\top = (\operatorname{im} W_{31}')^\perp = (\operatorname{im} W_3^\top G_3^\top)^\perp,$$

and, hence, $W_3^\top G_3^\top W_1' W_{32}' x \in \operatorname{im} W_3^\top G_3^\top \cap (\operatorname{im} W_3^\top G_3^\top)^\perp = \{0\}$. Thus, we have

$$W_{32}' x \in \ker W_3^\top G_3^\top W_1' = (\operatorname{im} W_1'^\top G_3 W_3)^\perp = (\operatorname{im} W_{32}')^\perp,$$

and, therefore, $W_{32}' x = 0$. Since W_{32}' has full column rank, we obtain that $x = 0$. Next, we show that

$$\ker W_{32}'^\top W_1'^\top G_3 W_3 W_{31}' = \{0\}. \tag{37}$$

Assume that $x \in \ker W_{32}'^\top W_1'^\top G_3 W_3 W_{31}'$. Then

$$W_1'^\top G_3 W_3 W_{31}' x \in \ker W_{32}'^\top = (\operatorname{im} W_{32}')^\perp = (\operatorname{im} W_1'^\top G_3 W_3)^\perp$$

and, therefore, $W_1'^\top G_3 W_3 W_{31}' x = 0$. This gives

$$G_3 W_3 W_{31}' x \in \ker W_1'^\top = (\operatorname{im} W_1')^\perp = \operatorname{im} W_1 = \ker W_3^\top G_3^\top = (\operatorname{im} G_3 W_3)^\perp,$$

whence $G_3 W_3 W_{31}' x = 0$. From this we obtain

$$W_{31}' x \in \ker G_3 W_3 = (\operatorname{im} W_3^\top G_3^\top)^\perp = (\operatorname{im} W_{31}')^\perp.$$

Thus, $W_{31}'x = 0$. The property that W_{31}' has full column rank leads to $x = 0$. Finally, (36) and (37) together imply that $W_{31}'^{\top} W_3^{\top} G_3^{\top} W_1' W_{32}'$ is nonsingular.

f) First, we show that

$$\ker W_{12}'^{\top} W_{11}^{\top} W_1^{\top} G_2 W_2' = \{0\}. \tag{38}$$

Assuming that $x \in \ker W_{12}'^{\top} W_{11}^{\top} W_1^{\top} G_2 W_2'$, we have

$$W_{11}^{\top} W_1^{\top} G_2 W_2' x \in \ker W_{12}'^{\top} = (\operatorname{im} W_{12}')^{\perp} = (\operatorname{im} W_{11}^{\top} W_1^{\top} G_2)^{\perp},$$

whence $W_{11}^{\top} W_1^{\top} G_2 W_2' x = 0$. This gives rise to

$$W_2' x \in \ker W_{11}^{\top} W_1^{\top} G_2 = (\operatorname{im} G_2^{\top} W_1 W_{11})^{\perp} = (\operatorname{im} W_2')^{\perp},$$

and, therefore, $W_2' x = 0$. The fact that W_2' has full column rank leads to $x = 0$. We now show that

$$\ker W_2'^{\top} G_2^{\top} W_1 W_{11} W_{12}' = \{0\}. \tag{39}$$

Let $x \in \ker W_2'^{\top} G_2^{\top} W_1 W_{11} W_{12}'$. Then

$$G_2^{\top} W_1 W_{11} W_{12}' x \in \ker W_2'^{\top} = (\operatorname{im} W_2')^{\perp} = (\operatorname{im} G_2^{\top} W_1 W_{11})^{\perp},$$

and, thus, $G_2^{\top} W_1 W_{11} W_{12}' x = 0$. Then we have

$$W_{12}' x \in \ker G_2^{\top} W_1 W_{11} = (\operatorname{im} W_{11}^{\top} W_1^{\top} G_2)^{\perp} = (\operatorname{im} W_{12}')^{\perp},$$

whence $W_{12}' x = 0$. Since W_{12}' has full column rank, we obtain that $x = 0$. Finally, it follows from (38) and (39) that $W_{12}'^{\top} W_{11}^{\top} W_1^{\top} G_2 W_2'$ is nonsingular.

□

We use the previously introduced matrices and their properties to decompose the vectors $x_1(t)$, $x_2(t)$ and $x_3(t)$ in the DAE (27) as

$$\begin{aligned} x_1(t) &= W_1' W_{32}' x_{11}(t) + W_1' W_{32} x_{21}(t) + W_1 W_{11}' x_{31}(t) \\ &\qquad + W_1 W_{11} W_{12}' (W_2'^{\top} G_2^{\top} W_1 W_{11} W_{12}')^{-1} x_{41}(t) + W_1 W_{11} W_{12} x_{51}(t), \\ x_2(t) &= W_2' x_{12}(t) + W_2 x_{22}(t), \\ x_3(t) &= W_3' x_{13}(t) + W_3 W_{31}' (W_{32}'^{\top} W_1'^{\top} G_3 W_3 W_{31}')^{-1} x_{23}(t) + W_3 W_{31} x_{33}(t). \end{aligned} \tag{40}$$

Introducing the vector-valued functions and matrices

$$\widetilde{x}_1(t) = \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \\ x_{31}(t) \\ x_{41}(t) \\ x_{51}(t) \end{bmatrix}, \qquad T_1 = \begin{bmatrix} W_{32}'^{\top} W_1'^{\top} \\ W_{32}^{\top} W_1'^{\top} \\ W_{11}'^{\top} W_1^{\top} \\ (W_{12}'^{\top} W_{11}^{\top} W_1^{\top} G_2 W_2')^{-1} W_{12}'^{\top} W_{11}^{\top} W_1^{\top} \\ W_{12}^{\top} W_{11}^{\top} W_1^{\top} \end{bmatrix}, \tag{41}$$

$$\widetilde{x}_2(t) = \begin{bmatrix} x_{12}(t) \\ x_{22}(t) \end{bmatrix}, \qquad T_2 = \begin{bmatrix} W_2'^\top \\ W_2^\top \end{bmatrix},$$
$$\widetilde{x}_3(t) = \begin{bmatrix} x_{13}(t) \\ x_{23}(t) \\ x_{33}(t) \end{bmatrix}, \qquad T_3 = \begin{bmatrix} W_3'^\top \\ (W_{31}'^\top W_3^\top G_3^\top W_1' W_{32}')^{-1} W_{31}'^\top W_3^\top \\ W_{31}^\top W_3^\top \end{bmatrix}, \tag{42}$$

equations (40) can be written as

$$x_1(t) = T_1^\top \widetilde{x}_1(t), \qquad x_2(t) = T_2^\top \widetilde{x}_2(t), \qquad x_3(t) = T_3^\top \widetilde{x}_3(t). \tag{43}$$

Note that, by construction of the matrices W_j and W_j', $j \in J$, the matrices T_1, T_2 and T_3 are nonsingular, and, hence, the vectors $\widetilde{x}_1(t)$, $\widetilde{x}_2(t)$ and $\widetilde{x}_3(t)$ are uniquely determined by $x_1(t)$, $x_2(t)$ and $x_3(t)$, respectively. Further, we define

$$\widetilde{f}_1(t) = \begin{bmatrix} f_{11}(t) \\ f_{21}(t) \\ f_{31}(t) \\ f_{41}(t) \\ f_{51}(t) \end{bmatrix} = T_1 f_1(t), \quad \widetilde{f}_2(t) = \begin{bmatrix} f_{12}(t) \\ f_{22}(t) \end{bmatrix} = T_2 f_2(t), \quad \widetilde{f}_3(t) = \begin{bmatrix} f_{13}(t) \\ f_{23}(t) \\ f_{33}(t) \end{bmatrix} = T_3 f_3(t).$$

Multiplying the DAE (27) from the left by $\operatorname{diag}(T_1, T_2, T_3)$ and substituting the vectors $x_1(t)$, $x_2(t)$ and $x_3(t)$ as in (43), we obtain an equivalent DAE

$$\begin{aligned} 0 &= \widetilde{E}\Phi\big(\widetilde{E}^\top \widetilde{x}_1(t)\big)\widetilde{E}^\top \dot{\widetilde{x}}_1(t) + \widetilde{F}\rho\big(\widetilde{F}^\top \widetilde{x}_1(t)\big) + \widetilde{G}_2 \widetilde{x}_2(t) + \widetilde{G}_3 \widetilde{x}_3(t) + \widetilde{f}_1(t), \\ 0 &= \Psi\big(\widetilde{x}_2(t)\big)\dot{\widetilde{x}}_2(t) - \widetilde{G}_2^\top \widetilde{x}_1(t) + \widetilde{f}_2(t), \\ 0 &= - \widetilde{G}_3^\top \widetilde{x}_1(t) + \widetilde{f}_3(t), \end{aligned} \tag{44}$$

with the matrices

$$\widetilde{E} = \begin{bmatrix} E_1 \\ E_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \widetilde{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ 0 \\ 0 \end{bmatrix}, \quad \widetilde{G}_2 = \begin{bmatrix} G_{2,11} & G_{2,12} \\ G_{2,21} & G_{2,22} \\ G_{2,31} & G_{2,32} \\ I & 0 \\ 0 & 0 \end{bmatrix}, \quad \widetilde{G}_3 = \begin{bmatrix} G_{3,11} & I & 0 \\ G_{3,21} & 0 & 0 \\ G_{3,31} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{45}$$

which are partitioned according to the partition of $\widetilde{x}_i(t)$ in (41) and (42). The matrix blocks in (45) have the form

$$\begin{aligned} &E_1 = W_{32}'^\top W_1'^\top E, && E_2 = W_{32}^\top W_1'^\top E, && \\ &F_1 = W_{32}'^\top W_1'^\top F, && F_2 = W_{32}^\top W_1'^\top F, && F_3 = W_{11}'^\top W_1^\top F, \\ &G_{2,11} = W_{32}'^\top W_1'^\top G_2 W_2', && G_{2,21} = W_{32}^\top W_1'^\top G_2 W_2', && G_{2,31} = W_{11}'^\top W_1^\top G_2 W_2', \\ &G_{2,12} = W_{32}'^\top W_1'^\top G_2 W_2, && G_{2,22} = W_{32}^\top W_1'^\top G_2 W_2, && G_{2,32} = W_{11}'^\top W_1^\top G_2 W_2, \\ &G_{3,11} = W_{32}'^\top W_1'^\top G_3 W_3', && G_{3,21} = W_{32}^\top W_1'^\top G_3 W_3', && G_{3,31} = W_{11}'^\top W_1^\top G_3 W_3'. \end{aligned} \tag{46}$$

This leads to the following condensed form of the DAE (27):

$$
\begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix}
=
\begin{bmatrix}
E_1\Phi\big(E_1^\top x_{11}(t)+E_2^\top x_{21}(t)\big)E_1^\top \dot{x}_{11}(t)+E_1\Phi\big(E_1^\top x_{11}(t)+E_2^\top x_{21}(t)\big)E_2^\top \dot{x}_{21}(t)\\
E_2\Phi\big(E_1^\top x_{11}(t)+E_2^\top x_{21}(t)\big)E_1^\top \dot{x}_{11}(t)+E_2\Phi\big(E_1^\top x_{11}(t)+E_2^\top x_{21}(t)\big)E_2^\top \dot{x}_{21}(t)\\
0\\
0\\
0\\
W_2'^\top\Psi\big(W_2'x_{12}(t)+W_2x_{22}(t)\big)W_2'\dot{x}_{12}(t)+W_2'^\top\Psi\big(W_2'x_{12}(t)+W_2x_{22}(t)\big)W_2\dot{x}_{22}(t)\\
W_2^\top\Psi\big(W_2'x_{12}(t)+W_2x_{22}(t)\big)W_2'\dot{x}_{12}(t)+W_2^\top\Psi\big(W_2'x_{12}(t)+W_2x_{22}(t)\big)W_2\dot{x}_{22}(t)\\
0\\
0\\
0
\end{bmatrix}
$$

$$
+\begin{bmatrix}
F_1\rho\big(F_1^\top x_{11}(t)+F_2^\top x_{21}(t)+F_3^\top x_{31}(t)\big)\\
F_2\rho\big(F_1^\top x_{11}(t)+F_2^\top x_{21}(t)+F_3^\top x_{31}(t)\big)\\
F_3\rho\big(F_1^\top x_{11}(t)+F_2^\top x_{21}(t)+F_3^\top x_{31}(t)\big)\\
0\\0\\0\\0\\0\\0\\0
\end{bmatrix}
+\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & G_{2,11} & G_{2,12} & G_{3,11} & I & 0\\
0 & 0 & 0 & 0 & 0 & G_{2,21} & G_{2,22} & G_{3,21} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & G_{2,31} & G_{2,32} & G_{3,31} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-G_{2,11}^\top & -G_{2,21}^\top & -G_{2,31}^\top & -I & 0 & 0 & 0 & 0 & 0 & 0\\
-G_{2,12}^\top & -G_{2,22}^\top & -G_{2,32}^\top & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-G_{3,11}^\top & -G_{3,21}^\top & -G_{3,31}^\top & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} x_{11}(t)\\x_{21}(t)\\x_{31}(t)\\x_{41}(t)\\x_{51}(t)\\x_{12}(t)\\x_{22}(t)\\x_{13}(t)\\x_{23}(t)\\x_{33}(t) \end{bmatrix}
+\begin{bmatrix} f_{11}(t)\\f_{21}(t)\\f_{31}(t)\\f_{41}(t)\\f_{51}(t)\\f_{12}(t)\\f_{22}(t)\\f_{13}(t)\\f_{23}(t)\\f_{33}(t) \end{bmatrix}
\tag{47}
$$

The following facts can be seen from this structure:

a) The components $x_{51}(t)$ and $x_{33}(t)$ are actually not involved. As a consequence, they can be chosen freely. It follows from (40) and Lemma 4.2 c) that the vector $x_{51}(t)$ is trivial, i.e., it evolves in the zero-dimensional space, if and only if $\ker\left[E\ F\ G_2\ G_3\right]^\top = \{0\}$. Furthermore, by Lemma 4.2 d) the vector $x_{33}(t)$ is trivial if and only if $\ker G_3 = \{0\}$.
b) The components $f_{51}(t)$ and $f_{33}(t)$ have to vanish in order to guarantee solvability. Due to Lemma 4.2 c), the equation $f_{51}(t) = 0$ does not appear if and only if $\ker\left[E\ F\ G_2\ G_3\right]^\top = \{0\}$. Moreover, Lemma 4.2 d) implies that the equation $f_{33}(t) = 0$ does not appear if and only if $\ker G_3 = \{0\}$.
c) We see from a) and b) that over- and underdetermined parts occur in pairs. This is a consequence of the symmetric structure of the DAE (27).
d) The remaining components fulfil the reduced DAE

$$\begin{aligned}
0 &= \widetilde{E}_r \Phi\big(\widetilde{E}_r^\top \widetilde{x}_{1r}(t)\big)\widetilde{E}_r^\top \dot{\widetilde{x}}_{1r}(t) + \widetilde{F}_r \rho\big(\widetilde{F}_r^\top \widetilde{x}_{1r}(t)\big) + \widetilde{G}_{2r}\widetilde{x}_{2r}(t) + \widetilde{G}_{3r}\widetilde{x}_{3r}(t) + \widetilde{f}_{1r}(t),\\
0 &= \Psi\big(\widetilde{x}_{2r}(t)\big)\dot{\widetilde{x}}_{2r}(t) - \widetilde{G}_{2r}^\top \widetilde{x}_{1r}(t) + \widetilde{f}_{2r}(t),\\
0 &= -\widetilde{G}_{3r}^\top \widetilde{x}_{1r}(t) + \widetilde{f}_{3r}(t),
\end{aligned} \tag{48}$$

with the matrices, functions and vectors

$$\widetilde{E}_r = \begin{bmatrix} E_1 \\ E_2 \\ 0 \\ 0 \end{bmatrix}, \quad \widetilde{F}_r = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ 0 \end{bmatrix}, \quad \widetilde{G}_{2r} = \begin{bmatrix} G_{2,11} & G_{2,12} \\ G_{2,21} & G_{2,22} \\ G_{2,31} & G_{2,32} \\ I & 0 \end{bmatrix}, \quad \widetilde{G}_{3r} = \begin{bmatrix} G_{3,11} & I \\ G_{3,21} & 0 \\ G_{3,31} & 0 \\ 0 & 0 \end{bmatrix}, \tag{49}$$

$$\begin{aligned}
\widetilde{x}_{1r}(t) &= \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \\ x_{31}(t) \\ x_{41}(t) \end{bmatrix}, & \widetilde{f}_{1r}(t) &= \begin{bmatrix} f_{11}(t) \\ f_{21}(t) \\ f_{31}(t) \\ f_{41}(t) \end{bmatrix},\\
\widetilde{x}_{2r}(t) &= \begin{bmatrix} x_{12}(t) \\ x_{22}(t) \end{bmatrix} = \widetilde{x}_2(t), & \widetilde{f}_{2r}(t) &= \begin{bmatrix} f_{12}(t) \\ f_{22}(t) \end{bmatrix} = \widetilde{f}_2(t),\\
\widetilde{x}_{3r}(t) &= \begin{bmatrix} x_{13}(t) \\ x_{23}(t) \end{bmatrix}, & \widetilde{f}_{3r}(t) &= \begin{bmatrix} f_{13}(t) \\ f_{23}(t) \end{bmatrix}.
\end{aligned} \tag{50}$$

Note that this DAE has the same structure as (27) and (44). It is obtained from (44) by cancelling the components $x_{51}(t)$ and $x_{33}(t)$ and the equations $f_{51}(t) = 0$ and $f_{33}(t) = 0$.

We now analyze the reduced DAE (48). In particular, we show that it satisfies the preliminaries of Theorem 4.1. For this purpose, we prove the following auxiliary result.

Lemma 4.3. *Let $E \in \mathbb{R}^{n_1 \times m_1}$, $F \in \mathbb{R}^{n_1 \times m_2}$, $G_2 \in \mathbb{R}^{n_1 \times n_2}$ and $G_3 \in \mathbb{R}^{n_1 \times n_3}$ be given. Assume that the matrices W_j and W_j', $j \in J$, are of full column rank and satisfy the conditions* (C1)–(C7). *Then for the matrices in* (46)*, the following holds true:*

a) $\ker\begin{bmatrix}E_1^\top & E_2^\top\end{bmatrix} = \{0\}$;
b) $\ker\begin{bmatrix}F_3 & G_{3,31}\end{bmatrix}^\top = \{0\}$;
c) $\ker G_{3,31} = \{0\}$.

Proof.

a) First, we show that the matrix $E^\top W_1'$ has full column rank. Assume that there exists a vector x such that $E^\top W_1' x = 0$. Then

$$W_1' x \in \ker E^\top = \operatorname{im} W_1 = (\operatorname{im} W_1')^\perp,$$

and, hence, $W_1' x = 0$. Since W_1' has full column rank, we obtain that $x = 0$. Consider now an accordingly partitioned vector

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix} \in \ker\begin{bmatrix}E_1^\top & E_2^\top\end{bmatrix}.$$

From the first two relations in (46) we have

$$\begin{bmatrix}W_{32}' & W_{32}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \in \ker E^\top W_1' = \{0\}.$$

Then Lemma 4.2 a) yields $x_1 = 0$ and $x_2 = 0$.

b) Let $x \in \ker\begin{bmatrix}F_3 & G_{3,31}\end{bmatrix}^\top$. Then, $0 = G_{3,31}^\top x = W_3'^\top G_3^\top W_1 W_{11}' x$, which gives

$$G_3^\top W_1 W_{11}' x \in \ker W_3'^\top = (\operatorname{im} W_3')^\perp = (\operatorname{im} G_3^\top W_1)^\perp.$$

Hence, $G_3^\top W_1 W_{11}' x = 0$. It follows from $F_3^\top x = 0$, that $\begin{bmatrix}F & G_3\end{bmatrix}^\top W_1 W_{11}' x = 0$, and, therefore,

$$W_{11}' x \in \ker\begin{bmatrix}F & G_3\end{bmatrix}^\top W_1 = (\operatorname{im} W_1^\top\begin{bmatrix}F & G_3\end{bmatrix})^\perp = (\operatorname{im} W_{11}')^\perp.$$

This yields $W_{11}' x = 0$. Since W_{11}' has full column rank, we obtain $x = 0$.

c) Assume that $x \in \ker G_{3,31}$. Then $0 = G_{3,31} x = W_{11}'^\top W_1^\top G_3 W_3' x$, which gives

$$W_1^\top G_3 W_3' x \in \ker W_{11}'^\top = (\operatorname{im} W_{11}')^\perp = (\operatorname{im} W_1^\top\begin{bmatrix}F & G_3\end{bmatrix})^\perp \subset (\operatorname{im} W_1^\top G_3)^\perp.$$

Thus, we obtain $W_1^\top G_3 W_3' x = 0$, which is equivalent to

$$W_3' x \in \ker W_1^\top G_3 = (\operatorname{im} G_3^\top W_1)^\perp = (\operatorname{im} W_3')^\perp.$$

As a consequence, we have $W_3' x = 0$, and the property of W_3' to be of full column rank gives $x = 0$.

□

It follows from Lemma 4.3 a) and b) that $\ker \begin{bmatrix} \widetilde{E}_r & \widetilde{F}_r & \widetilde{G}_{2r} & \widetilde{G}_{3r} \end{bmatrix}^\top = \{0\}$, whereas Lemma 4.3 c) implies that $\ker \widetilde{G}_{3r} = \{0\}$. In this case, the index of the DAE (48) can be established using Theorem 4.1.

Theorem 4.4. *Let a reduced DAE* (48) *be given with matrices and functions as in* (49) *and* (50)*, respectively. Then the differentiation index $\widetilde{\mu}_d$ of* (48) *fulfils*

a) $\widetilde{\mu}_d = 0$ *if and only if* $\widetilde{E}_r = E_2$, $\widetilde{F} = F_2$, $\widetilde{G}_{2r} = G_{2,22}$ *and the matrix* $\widetilde{G}_{3r}$ *is empty.*
b) $\widetilde{\mu}_d = 1$ *if and only if it is not zero and*

$$\widetilde{E}_r = \begin{bmatrix} E_2 \\ 0 \end{bmatrix}, \quad \widetilde{F}_r = \begin{bmatrix} F_2 \\ F_3 \end{bmatrix}, \quad \widetilde{G}_{2r} = \begin{bmatrix} G_{2,22} \\ G_{2,32} \end{bmatrix}, \quad \widetilde{G}_{3r} = \begin{bmatrix} G_{3,21} \\ G_{3,31} \end{bmatrix}. \tag{51}$$

c) $\widetilde{\mu}_d = 2$ *if and only if* $\widetilde{\mu}_d \notin \{0,1\}$.

Proof. a) If $\widetilde{E}_r = E_2$ and the matrix $\widetilde{G}_{3r}$ is empty, then Lemma 4.3 a) implies that $\widetilde{E}_r$ has full row rank. Then Theorem 4.1 a) implies $\widetilde{\mu}_d = 0$. On the other hand, if $\widetilde{\mu}_d = 0$, then Theorem 4.1 a) yields that the lower two blocks of $\widetilde{E}_r$ in (49) vanish. Hence, the identity matrix in $\tilde{G}_{2r}$ has zero columns and rows meaning that the first block column in $\widetilde{G}_{2r}$ vanishes. Furthermore, the absence of $\widetilde{G}_{3r}$ implies that the first row in $\widetilde{E}_r$, $\widetilde{F}_r$ and $\widetilde{G}_{2r}$ vanishes, which gives $\widetilde{E}_r = E_2$, $\widetilde{F} = F_2$, and $\widetilde{G}_{2r} = G_{2,22}$.

b) First assume that $\widetilde{\mu}_d > 0$ and (51) holds true. Then it follows from Lemma 4.3 a) and b) that

$$\begin{bmatrix} \widetilde{E}_r & \widetilde{F}_r & \widetilde{G}_{3r} \end{bmatrix} = \begin{bmatrix} E_2 & F_2 & G_{3,21} \\ 0 & F_3 & G_{3,31} \end{bmatrix}$$

has full row rank. We can further conclude from Lemma 4.3 c) that

$$\ker \begin{bmatrix} \widetilde{E}_r & \widetilde{G}_{3r} \end{bmatrix} = \ker \begin{bmatrix} E_2 & G_{3,21} \\ 0 & G_{3,31} \end{bmatrix} = \ker E_2 \times \{0\} = \ker \widetilde{E}_r \times \{0\}.$$

Theorem 4.1 b) implies $\widetilde{\mu}_d = 1$.

To prove the converse implication, assume that $\widetilde{\mu}_d = 1$. Seeking for a contradiction, assume that the second block column of $\widetilde{G}_{3r}$ in (49) has r columns for $r > 0$. Then there exists a vector $x_3 \in \mathbb{R}^r \setminus \{0\}$. Lemma 4.3 a) implies that there exists a vector x_1 such that

$$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} x_1 = \begin{bmatrix} x_3 \\ 0 \end{bmatrix}.$$

Then using Theorem 4.1 b) we have

$$\begin{bmatrix} -x_1 \\ 0 \\ x_3 \end{bmatrix} \in \ker \begin{bmatrix} \widetilde{E}_r & \widetilde{G}_{3r} \end{bmatrix} = \ker \widetilde{E}_r \times \{0\}.$$

This is a contradiction.

It remains to prove that the forth block row of $\widetilde{E}_r$, $\widetilde{F}_r$, $\widetilde{G}_{2r}$ and $\widetilde{G}_{3r}$ vanishes. Seeking for a contradiction, assume that the forth block row has $r > 0$ rows.

Then there exists some $x_3 \in \mathbb{R}^r \setminus \{0\}$, and

$$\begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} \in \ker \begin{bmatrix} \widetilde{E}_r & \widetilde{F}_r & \widetilde{G}_{3r} \end{bmatrix}^\top = \ker \begin{bmatrix} E_2^\top & 0 & 0 \\ F_2^\top & F_3^\top & 0 \\ G_{3,21}^\top & G_{3,31}^\top & 0 \end{bmatrix}.$$

Hence, $\begin{bmatrix} \widetilde{E}_r & \widetilde{F}_r & \widetilde{G}_{3r} \end{bmatrix}$ does not have full row rank. Then Theorem 4.1 b) implies that $\widetilde{\mu}_d > 1$, which is a contradiction.

□

It follows from Lemma 4.3, Theorem 4.4 and the construction of the matrices $\widetilde{E}_r$, $\widetilde{F}_r$, $\widetilde{G}_{2r}$ and $\widetilde{G}_{3r}$ that $\widetilde{\mu}_d = 0$ if and only if

$$\operatorname{rank} \begin{bmatrix} E & F & G_3 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} E & F & G_2 & G_3 \end{bmatrix} \quad \text{and} \quad G_3 = 0.$$

Furthermore, we have $\widetilde{\mu}_d = 1$ if and only if

$$\operatorname{rank} \begin{bmatrix} E & F & G_3 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} E & F & G_2 & G_3 \end{bmatrix} \quad \text{and} \quad \ker \begin{bmatrix} E & G_3 \end{bmatrix} = \ker E \times \ker G_3.$$

Remark 4.5. *Theorem 4.4 essentially states that the blocks in* (47) *corresponding to identity matrices are responsible for the index rising to* $\widetilde{\mu}_d = 2$*. The equations in* (47) *corresponding to these blocks are algebraic constraints on variables whose derivatives are also involved in the overall DAE.* KUNKEL *and* MEHRMANN *call this phenomenon* strangeness [17, 18, 22].

5 Index Reduction for Linear DAEs of Circuit Type

In this section, we consider index reduction of the DAE (27) based on the representation (48) in which the over- and underdetermined parts are already eliminated. We restrict ourselves to linear time-invariant systems. Roughly speaking, index reduction is a manipulation of the DAE such that another DAE with lower index is obtained whose solution set does not differ from the original one. Our approach is strongly inspired by the index reduction approach by KUNKEL and MEHRMANN for linear DAEs with time-varying coefficients [17, 22] briefly described in Section 2.

Consider the DAE (27), where we assume that the functions $\Phi : \mathbb{R}^{m_1} \to \mathbb{R}^{m_1 \times m_1}$ and $\Psi : \mathbb{R}^{n_2} \to \mathbb{R}^{n_2 \times n_2}$ are constant, that is,

$$\Phi(z_1) = \Phi \ \text{ for all } z_1 \in \mathbb{R}^{m_1} \quad \text{and} \quad \Psi(z_2) = \Psi \ \text{ for all } z_2 \in \mathbb{R}^{n_2}$$

with symmetric, positive definite matrices $\Phi \in \mathbb{R}^{m_1 \times m_1}$ and $\Psi \in \mathbb{R}^{n_2 \times n_2}$. Furthermore, we assume that the function $\rho : \mathbb{R}^{m_2} \to \mathbb{R}^{m_2}$ is linear, that is, $\rho(z) = Pz$ for some $P \in \mathbb{R}^{m_2 \times m_2}$ with $P + P^\top > 0$. Then by Remark 4.5 we can apply the index reduction technique proposed in [17]. To this end, we perform the following steps:

(i) multiply the ninth equation in (47) from the left by $E_1 \Phi E_1^\top$, differentiate it and add to the first equation;
(ii) multiply the ninth equation in (47) from the left by $E_2 \Phi E_1^\top$, differentiate it and add to the second equation;
(iii) replace $x_{23}(t)$ by a new variable

$$\tilde{x}_{23}(t) = E_1 \Phi E_2^\top \dot{x}_{21}(t) + E_1 \Phi E_1^\top \dot{f}_{23}(t) + x_{23}(t).$$

(iv) multiply the fourth equation in (47) from the left by $W_2'^\top \Psi W_2'$, differentiate it and subtract from the sixth equation;
(v) multiply the fourth equation in (47) from the left by $W_2^\top \Psi W_2'$, differentiate it and subtract from the seventh equation;
(vi) replace $x_{41}(t)$ by a new variable

$$\tilde{x}_{41}(t) = -W_2'^\top \Psi W_2 \dot{x}_{22}(t) + W_2'^\top \Psi W_2' \dot{f}_{41}(t) + x_{41}(t).$$

Thereby, we obtain the DAE

$$\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\ E_2 \Phi E_2^\top \dot{x}_{21}(t)\\0\\0\\0\\0\\ W_2^\top \Psi W_2 \dot{x}_{22}(t)\\0\\0\\0\end{bmatrix} + \begin{bmatrix} F_1 P F_1^\top x_{11}(t) + F_1 P F_2^\top x_{21}(t) + F_1 P F_3^\top x_{31}(t)\\ F_2 P F_1^\top x_{11}(t) + F_2 P F_2^\top x_{21}(t) + F_2 P F_3^\top x_{31}(t)\\ F_3 P F_1^\top x_{11}(t) + F_3 P F_2^\top x_{21}(t) + F_3 P F_3^\top x_{31}(t)\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & G_{2,11} & G_{2,12} & G_{3,11} & I & 0\\ 0 & 0 & 0 & 0 & 0 & G_{2,21} & G_{2,22} & G_{3,21} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & G_{2,31} & G_{2,32} & G_{3,31} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -G_{2,11}^\top & -G_{2,21}^\top & -G_{2,31}^\top & -I & 0 & 0 & 0 & 0 & 0 & 0\\ -G_{2,12}^\top & -G_{2,22}^\top & -G_{2,32}^\top & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -G_{3,11}^\top & -G_{3,21}^\top & -G_{3,31}^\top & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11}(t)\\ x_{21}(t)\\ x_{31}(t)\\ \tilde{x}_{41}(t)\\ x_{51}(t)\\ x_{12}(t)\\ x_{22}(t)\\ x_{13}(t)\\ \tilde{x}_{23}(t)\\ x_{33}(t) \end{bmatrix} + \begin{bmatrix} f_{11}(t)\\ \tilde{f}_{21}(t)\\ f_{31}(t)\\ f_{41}(t)\\ f_{51}(t)\\ f_{12}(t)\\ \tilde{f}_{22}(t)\\ f_{13}(t)\\ f_{23}(t)\\ f_{33}(t) \end{bmatrix} \tag{52}$$

with $\tilde{f}_{21}(t) = f_{21}(t) + E_2 \Phi E_1^\top \dot{f}_{23}(t)$ and $\tilde{f}_{22}(t) = f_{22}(t) - W_2^\top \Psi W_2' \dot{f}_{41}(t)$ which is again of type (27). Furthermore, it follows from Theorem 4.1 and Lemma 4.3 that the differentiation index of the resulting DAE obtained from (52) by removing the redundant variables $x_{51}(t)$ and $x_{33}(t)$ as well as the constrained equations for the inhomogeneity components $f_{51}(t) = 0$ and $f_{33}(t) = 0$ is at most one.

Remark 5.1.

a) We note that the previously introduced index reduction heavily uses linearity. In the case where, for instance, Φ depends on $x_{11}(t)$ and $x_{21}(t)$, the transformation (ii) *would be clearly dependent on these variables as well. This causes that the unknown variables $x_{11}(t)$ and $x_{21}(t)$ enter the inhomogeneity $f_{21}(t)$.*

b) Structure-preserving index reduction for circuit equations has been considered previously in [3–5]. *An index reduction procedure presented there provides a reduced model which can be interpreted as an electrical circuit containing controlled sources. As a consequence, the index-reduced system is not a DAE of type* (27) *anymore.*

6 Consequences for Circuit Equations

In this section, we present a graph-theoretical interpretation of the previous results for circuit equations. First, we collect some basic concepts from the graph theory, which will be used in the subsequent discussion. For more details, we refer to [10].

Let $\mathcal{G} = (\mathcal{V}, \mathcal{B})$ be a directed graph with a finite set $\mathcal{V}$ of vertices and a finite set $\mathcal{B}$ of branches. For v_{k_1}, v_{k_2}, an ordered pair $b_{k_1} = \langle v_{k_1}, v_{k_2} \rangle$ denotes a branch leaving v_{k_1} and entering v_{k_2}. A tuple $(b_{k_1}, \ldots, b_{k_{s-1}})$ of branches $b_{k_j} = \langle v_{k_j}, v_{k_{j+1}} \rangle$ in $\mathcal{G}$ is called a *path* connecting v_{k_1} and v_{k_s} if all vertices $v_{k_1}, \ldots, v_{k_s}$ are different except possibly v_{k_1} and v_{k_s}. A path is *closed* if $v_{k_1} = v_{k_s}$, and *open*, otherwise. A closed path is called a *loop*. A graph $\mathcal{G}$ is called *connected* if for every two different vertices there exists an open path connecting them.

A *subgraph* $\mathcal{K} = (\mathcal{V}', \mathcal{B}')$ of $\mathcal{G} = (\mathcal{V}, \mathcal{B})$ is a graph with $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{B}' \subseteq \mathcal{B}\,|_{\mathcal{V}'} = \left\{ b_{k_1} = \langle v_{k_1}, v_{k_2} \rangle \in \mathcal{B} \,:\, v_{k_1}, v_{k_2} \in \mathcal{V}' \right\}$. A subgraph $\mathcal{K} = (\mathcal{V}', \mathcal{B}')$ is called *spanning* if $\mathcal{V}' = \mathcal{V}$. A spanning subgraph $\mathcal{K} = (\mathcal{V}, \mathcal{B}')$ is called a *cutset* of a connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{B})$ if a complementary subgraph $\mathcal{G} - \mathcal{K} = (\mathcal{V}, \mathcal{B} \setminus \mathcal{B}')$ is disconnected and $\mathcal{K}$ is minimal with this property. For a spanning subgraph $\mathcal{K}$ of $\mathcal{G}$, a subgraph $\mathcal{L}$ of $\mathcal{G}$ is called a $\mathcal{K}$*-cutset*, if $\mathcal{L}$ is a cutset of $\mathcal{K}$. Furthermore, a path ℓ of $\mathcal{G}$ is called a $\mathcal{K}$*-loop*, if ℓ is a loop of $\mathcal{K}$.

For an electrical circuit, we consider an associated graph $\mathcal{G}$ whose vertices correspond to the nodes of the circuit and whose branches correspond to the circuit elements. Let $A \in \mathbb{R}^{n_\mathrm{n}-1 \times n_\mathrm{b}}$ and $B \in \mathbb{R}^{n_\mathrm{b}-n_\mathrm{n}+1 \times n_\mathrm{b}}$ be the reduced incidence and loop matrices as defined in Section 3. For a spanning graph $\mathcal{K}$ of $\mathcal{G}$, we denote by $A_{\mathcal{K}}$ (resp. $A_{\mathcal{G}-\mathcal{K}}$) a submatrix of A formed by the columns corresponding to the branches in $\mathcal{K}$ (resp. the complementary graph $\mathcal{G} - \mathcal{K}$). Analogously, we construct the loop matrices $B_{\mathcal{K}}$ and $B_{\mathcal{G}-\mathcal{K}}$. By a suitable reordering of the branches, the reduced incidence and loop matrices can be partitioned as

$$A = \begin{bmatrix} A_{\mathcal{K}} & A_{\mathcal{G}-\mathcal{K}} \end{bmatrix}, \qquad B = \begin{bmatrix} B_{\mathcal{K}} & B_{\mathcal{G}-\mathcal{K}} \end{bmatrix}. \tag{53}$$

The following lemma from [28] characterizes the absence of $\mathcal{K}$-loops and $\mathcal{K}$-cutsets in terms of submatrices of the incidence and loop matrices. It is crucial for our considerations. Note that this result has previously been proven for incidence matrices in [30].

Lemma 6.1 (Subgraphs, incidence and loop matrices [28, Lemma 4.10])**.** *Let $\mathcal{G}$ be a connected graph with the reduced incidence and loop matrices $A \in \mathbb{R}^{n_{\mathrm{n}}-1\times n_e}$ and $B \in \mathbb{R}^{n_e-n_{\mathrm{n}}+1\times n_e}$. Further, let $\mathcal{K}$ be a spanning subgraph of $\mathcal{G}$. Assume that the branches of $\mathcal{G}$ are sorted in a way that* (53) *is satisfied.*

a) The following three assertions are equivalent:
- *(i) $\mathcal{G}$ does not contain $\mathcal{K}$-cutsets;*
- *(ii)* $\ker A_{\mathcal{G}-\mathcal{K}}^\top = \{0\}$*;*
- *(iii)* $\ker B_{\mathcal{K}} = \{0\}$.

b) The following three assertions are equivalent:
- *(i) $\mathcal{G}$ does not contain $\mathcal{K}$-loops;*
- *(ii)* $\ker A_{\mathcal{K}} = \{0\}$*;*
- *(iii)* $\ker B_{\mathcal{G}-\mathcal{K}}^\top = \{0\}$.

The next two auxiliary results are concerned with properties of subgraphs of subgraphs and give some equivalent characterizations in terms of their incidence and loop matrices. These statements have first been proven for incidence matrices in [30, Propositions 4.4 and 4.5].

Lemma 6.2 (Loops in subgraphs [28, Lemma 4.11])**.** *Let $\mathcal{G}$ be a connected graph with the reduced incidence and loop matrices $A \in \mathbb{R}^{n_{\mathrm{n}}-1\times n_e}$ and $B \in \mathbb{R}^{n_e-n_{\mathrm{n}}+1\times n_e}$. Further, let $\mathcal{K}$ be a spanning subgraph of $\mathcal{G}$, and let $\mathcal{L}$ be a spanning subgraph of $\mathcal{K}$. Assume that the branches of $\mathcal{G}$ are sorted in a way that*

$$A = \begin{bmatrix} A_{\mathcal{L}} & A_{\mathcal{K}-\mathcal{L}} & A_{\mathcal{G}-\mathcal{K}} \end{bmatrix}, \qquad B = \begin{bmatrix} B_{\mathcal{L}} & B_{\mathcal{K}-\mathcal{L}} & B_{\mathcal{G}-\mathcal{K}} \end{bmatrix}. \tag{54}$$

Then the following three assertions are equivalent:
- *(i) $\mathcal{G}$ does not contain $\mathcal{K}$-loops except for $\mathcal{L}$-loops;*
- *(ii) For some (and hence any) matrix $Z_{\mathcal{L}}$ with $\operatorname{im} Z_{\mathcal{L}} = \ker A_{\mathcal{L}}^\top$ holds*

$$\ker Z_{\mathcal{L}}^\top A_{\mathcal{K}-\mathcal{L}} = \{0\};$$

- *(iii) For some (and hence any) matrix $Y_{\mathcal{K}-\mathcal{L}}$ with $\operatorname{im} Y_{\mathcal{K}-\mathcal{L}} = \ker B_{\mathcal{K}-\mathcal{L}}^\top$ holds*

$$Y_{\mathcal{K}-\mathcal{L}}^\top B_{\mathcal{G}-\mathcal{K}} = 0.$$

Lemma 6.3 (Cutsets in subgraphs [28, Lemma 4.12])**.** *Let $\mathcal{G}$ be a connected graph with the reduced incidence and loop matrices $A \in \mathbb{R}^{n_{\mathrm{n}}-1\times n_e}$ and $B \in \mathbb{R}^{n_e-n_{\mathrm{n}}+1\times n_e}$.*

Further, let $\mathcal{K}$ be a spanning subgraph of $\mathcal{G}$, and let $\mathcal{L}$ be a spanning subgraph of $\mathcal{K}$. Assume that the branches of $\mathcal{G}$ are sorted in a way that (54) *is satisfied. Then the following three assertions are equivalent:*

(i) *$\mathcal{G}$ does not contain $\mathcal{K}$-cutsets except for $\mathcal{L}$-cutsets;*

(ii) *For some (and hence any) matrix $Y_{\mathcal{L}}$ with* $\operatorname{im} Y_{\mathcal{L}} = \ker B_{\mathcal{L}}^\top$ *holds*

$$\ker Y_{\mathcal{L}}^\top B_{\mathcal{K}-\mathcal{L}} = \{0\};$$

(iii) *For some (and hence any) matrix $Z_{\mathcal{K}-\mathcal{L}}$ with* $\operatorname{im} Z_{\mathcal{K}-\mathcal{L}} = \ker A_{\mathcal{K}-\mathcal{L}}^\top$ *holds*

$$Z_{\mathcal{K}-\mathcal{L}}^\top A_{\mathcal{G}-\mathcal{K}} = 0.$$

We use these results to analyze the condensed form (47) for the MNA equations (23). The MLA equations can be treated analogously. For a given electrical circuit whose corresponding graph is connected and has no self-loops (see [28]), we introduce the following matrices which take the role of the matrices W_i and W_i' defined in Section 4. Consider matrices of full column rank satisfying the following conditions:

(C1′) $\operatorname{im} Z_{\mathcal{C}} = \ker A_{\mathcal{C}}^\top$, $\quad \operatorname{im} Z'_{\mathcal{C}} = \operatorname{im} A_{\mathcal{C}}$,

(C2′) $\operatorname{im} Z_{\mathcal{RV}-\mathcal{C}} = \ker \left[A_{\mathcal{R}}\ A_{\mathcal{V}}\right]^\top Z_{\mathcal{C}}$, $\quad \operatorname{im} Z'_{\mathcal{RV}-\mathcal{C}} = \operatorname{im} Z_{\mathcal{C}}^\top \left[A_{\mathcal{R}}\ A_{\mathcal{V}}\right]$,

(C3′) $\operatorname{im} Z_{\mathcal{L}-\mathcal{CRV}} = \ker A_{\mathcal{L}}^\top Z_{\mathcal{C}} Z_{\mathcal{RV}-\mathcal{C}}$, $\quad \operatorname{im} Z'_{\mathcal{L}-\mathcal{CRV}} = \operatorname{im} Z_{\mathcal{RV}-\mathcal{C}}^\top Z_{\mathcal{C}}^\top A_{\mathcal{L}}$,

(C4′) $\operatorname{im} \bar{Z}_{\mathcal{L}-\mathcal{CRV}} = \ker Z_{\mathcal{RV}-\mathcal{C}}^\top Z_{\mathcal{C}}^\top A_{\mathcal{L}}$, $\quad \operatorname{im} \bar{Z}'_{\mathcal{L}-\mathcal{CRV}} = \operatorname{im} A_{\mathcal{L}}^\top Z_{\mathcal{C}} Z_{\mathcal{RV}-\mathcal{C}}$,

(C5′) $\operatorname{im} \bar{Z}_{\mathcal{V}-\mathcal{C}} = \ker Z_{\mathcal{C}}^\top A_{\mathcal{V}}$, $\quad \operatorname{im} \bar{Z}'_{\mathcal{V}-\mathcal{C}} = \operatorname{im} A_{\mathcal{V}}^\top Z_{\mathcal{C}}$,

(C6′) $\operatorname{im} \widetilde{Z}_{\mathcal{V}-\mathcal{C}} = \ker A_{\mathcal{V}} \bar{Z}_{\mathcal{V}-\mathcal{C}}$, $\quad \operatorname{im} \widetilde{Z}'_{\mathcal{V}-\mathcal{C}} = \operatorname{im} \bar{Z}_{\mathcal{V}-\mathcal{C}}^\top A_{\mathcal{V}}^\top$,

(C7′) $\operatorname{im} \widetilde{Z}_{\mathcal{CVC}} = \ker \bar{Z}_{\mathcal{V}-\mathcal{C}}^\top A_{\mathcal{V}}^\top Z'_{\mathcal{C}}$, $\quad \operatorname{im} \widetilde{Z}'_{\mathcal{CVC}} = \operatorname{im} Z_{\mathcal{C}}'^\top A_{\mathcal{V}} \bar{Z}_{\mathcal{V}-\mathcal{C}}$.

Note that the introduced matrices can be determined by computationally cheap graph search algorithms [12, 16]. We have the following correspondences to the matrices W_i and W_i':

$$Z_{\mathcal{C}} \widehat{=} W_1, \quad Z'_{\mathcal{C}} \widehat{=} W'_1, \quad Z_{\mathcal{RV}-\mathcal{C}} \widehat{=} W_{11}, \quad Z'_{\mathcal{RV}-\mathcal{C}} \widehat{=} W'_{11},$$
$$Z_{\mathcal{L}-\mathcal{CRV}} \widehat{=} W_{12}, \quad Z'_{\mathcal{L}-\mathcal{CRV}} \widehat{=} W'_{12}, \quad \bar{Z}_{\mathcal{L}-\mathcal{CRV}} \widehat{=} W_2, \quad \bar{Z}'_{\mathcal{L}-\mathcal{CRV}} \widehat{=} W'_2,$$
$$\bar{Z}_{\mathcal{V}-\mathcal{C}} \widehat{=} W_3, \quad \bar{Z}'_{\mathcal{V}-\mathcal{C}} \widehat{=} W'_3, \quad \widetilde{Z}_{\mathcal{V}-\mathcal{C}} \widehat{=} W_{31}, \quad \widetilde{Z}'_{\mathcal{V}-\mathcal{C}} \widehat{=} W'_{31},$$
$$\widetilde{Z}_{\mathcal{CVC}} \widehat{=} W_{32}, \quad \widetilde{Z}'_{\mathcal{CVC}} \widehat{=} W'_{32}.$$

Using Lemmas 4.2 and 6.1–6.3, we can characterize the absence of certain blocks in the condensed form (47) in terms of the graph structure of the circuit. Based on the definition of $\mathcal{K}$-loop and $\mathcal{K}$-cutset, we arrange the following way of speaking. An expression like "$\mathcal{CV}$-loop" indicates a loop in the circuit graph whose branch set consists only of branches corresponding to capacitors and/or voltage sources.

Likewise, an "$\mathcal{LI}$-cutset" is a cutset in the circuit graph whose branch set consists only of branches corresponding to inductors and/or current sources.

a) The matrix $Z_{\mathcal{C}}$ has zero columns if and only if the circuit does not contain any $\mathcal{RLVI}$-cutsets (Lemma 6.1 a)).
b) The matrix $Z'_{\mathcal{C}}$ has zero columns if and only if the circuit does not contain any capacitors.
c) The matrix $Z_{\mathcal{RV}-\mathcal{C}}$ has zero columns if and only if the circuit does not contain any $\mathcal{LI}$-cutsets (Lemma 4.2 b) and Lemma 6.1 a)).
d) The matrix $Z'_{\mathcal{RV}-\mathcal{C}}$ has zero columns if and only if the circuit does not contain any $\mathcal{CLI}$-cutsets except for $\mathcal{LI}$-cutsets (Lemma 6.3).
e) The matrix $Z_{\mathcal{L}-\mathcal{CRV}}$ has zero columns if and only if the circuit does not contain any $\mathcal{I}$-cutsets (Lemma 4.2 c) and Lemma 6.1 a)).
f) The matrix $Z'_{\mathcal{L}-\mathcal{CRV}}$ (and by Lemma 4.2 f) also the matrix $\bar{Z}'_{\mathcal{L}-\mathcal{CRV}}$) has zero columns if and only if the circuit does not contain any $\mathcal{CRVI}$-cutsets except for $\mathcal{I}$-cutsets (Lemma 4.2 b) and Lemma 6.3).
g) The matrix $\bar{Z}_{\mathcal{L}-\mathcal{CRV}}$ has zero columns if and only if the circuit does not contain any $\mathcal{RCVL}$-loops except for $\mathcal{RCV}$-loops (Lemma 4.2 b) and Lemma 6.2)).
h) The matrix $\bar{Z}_{\mathcal{V}-\mathcal{C}}$ has zero columns if and only if the circuit does not contain any $\mathcal{CV}$-loops except for $\mathcal{C}$-loops (Lemma 6.2).
i) The matrix $\bar{Z}'_{\mathcal{V}-\mathcal{C}}$ has zero columns if and only if the circuit does not contain any $\mathcal{RCLI}$-cutsets except for $\mathcal{RLI}$-cutsets (Lemma 6.3).
j) The matrix $\tilde{Z}_{\mathcal{V}-\mathcal{C}}$ has zero columns if and only if the circuit does not contain any $\mathcal{V}$-loops (Lemma 4.2 d) and Lemma 6.1 b)).
k) The matrix $\tilde{Z}'_{\mathcal{CVC}}$ (and by Lemma 4.2 e) also the matrix $\tilde{Z}'_{\mathcal{V}-\mathcal{C}}$) has zero columns if and only if the circuit does not contain any $\mathcal{CV}$-loops except for $\mathcal{C}$-loops and $\mathcal{V}$-loops (this can be proven analogous to Lemma 6.2).

Exemplarily, we will show a) only. Other assertions can be proved analogously. For the MNA system (23), we have $E = A_{\mathcal{C}}$. Then by definition, the matrix $Z_{\mathcal{C}}$ has zero columns if and only if $\ker A_{\mathcal{C}}^{\top} = \{0\}$. By Lemma 6.1 a), this condition is equivalent to the absence of $\mathcal{RLVI}$-cutsets.

In particular, we obtain from the previous findings that the condensed form (47) does not have any redundant variables and equations if and only if the circuit neither contains $\mathcal{I}$-cutsets nor $\mathcal{V}$-loops. We can also infer some assertions on the differentiation index of the reduced DAE (48) obtained from (47) by removing the redundant variables and equations. The DAE (48) has the differentiation index $\tilde{\mu}_d = 0$ if and only if the circuit does not contain voltage sources and $\mathcal{RLI}$-cutsets except for $\mathcal{I}$-cutsets. Furthermore, we have $\tilde{\mu}_d = 1$ if and only if and the circuit neither contains $\mathcal{CV}$-loops except for $\mathcal{C}$-loops and $\mathcal{V}$-loops nor $\mathcal{LI}$-cutsets except for $\mathcal{I}$-cutsets.

7 Conclusion

In this paper, we have presented a structural analysis for the MNA and MLA equations which are DAEs modelling electrical circuits with uncontrolled voltage and current sources, resistors, capacitors and inductors. These DAEs are shown to be of the same structure. A special condensed form under linear transformations has been introduced which allows to determine the differentiation index. In the linear case, we have presented an index reduction procedure which provides a DAE system of the differentiation index one and preserves the structure of the circuit DAE. Graph-theoretical characterizations of the condensed form have also been given.

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