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Abstract

On the torus $\mathbb{T}^m$ of dimension $m \geq 2$ we prove the existence of a real-analytic weak mixing diffeomorphism preserving a measurable Riemannian metric. The proof is based on a real-analytic version of the approximation by conjugation-method with explicitly defined conjugation maps and partition elements.

1 Introduction

Until 1970 it was an open question if there exists an ergodic area-preserving smooth diffeomorphism on the disc $\mathbb{D}^2$. This problem was solved by the so-called "approximation by conjugation"-method developed by D. Anosov and A. Katok in [AK70]. In fact, on every smooth compact connected manifold M of dimension $m \geq 2$ admitting a non-trivial circle action $\mathcal{R} = \{R_t\}_{t\in\mathbb{S}^1}$ preserving a smooth volume μ this method enables the construction of smooth diffeomorphisms with specific ergodic properties (e.g. weak mixing ones in [AK70], section 5) or non-standard smooth realizations of measure-preserving systems (e.g. [AK70], section 6, [Be13] and [FSW07]). See also [FK04] for more details and other results of this method. These diffeomorphisms are constructed as limits of conjugates $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$, where $\alpha_{n+1} = \alpha_n + \frac{1}{k_n \cdot l_n \cdot q_n^2} \in \mathbb{Q}$, $H_n = H_{n-1} \circ h_n$ and h_n is a measure-preserving diffeomorphism satisfying $R_{\frac{1}{q_n}} \circ h_n = h_n \circ R_{\frac{1}{q_n}}$. In each step the conjugation map h_n and the parameter l_n are chosen such that the diffeomorphism f_n imitates the desired property with a certain precision. Then the parameter k_n is chosen large enough to guarantee closeness of f_n to f_{n-1} in the C^∞-topology and so the convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ to a limit diffeomorphism is provided. It is even possible to keep this limit diffeomorphism within any given C^∞-neighbourhood of the initial element S_{α_1} or, by applying a fixed diffeomorphism g first, of $g \circ S_{\alpha_1} \circ g^{-1}$. So the construction can be carried out in a neighbourhood of any diffeomorphism conjugate to an element of the action. Thus, $\mathcal{A}(M) = \overline{\{h \circ R_t \circ h^{-1} \ : \ t \in \mathbb{S}^1, h \in \mathrm{Diff}^\infty(M,\mu)\}}^{C^\infty}$ is a natural space for the produced diffeomorphisms.

In their influential paper [AK70] Anosov and Katok proved amongst others that in $\mathcal{A}(M)$ the set of weak mixing diffeomorphisms is generic (i. e. it is a dense G_δ-set) in the $C^\infty(M)$-topology. For it they used the "approximation by conjugation"-method. In [GKa00] the conjugation maps are constructed more explicitly such that they can be equipped with the additional structure of being locally very close to an isometry. Hereby, it is shown that there exists a weak mixing smooth diffeomorphism preserving a smooth measure and a measurable Riemannian metric. Actually, it follows from the respective proofs that both results are true in the restricted space $\mathcal{A}_\alpha(M) = \overline{\{h \circ R_\alpha \circ h^{-1} \ : h \in \mathrm{Diff}^\infty(M,\mu)\}}^{C^\infty}$ for a G_δ-set of $\alpha \in \mathbb{S}^1$. However, both proofs do not give a full description of the set of $\alpha \in \mathbb{S}^1$ for which the particular result holds in $\mathcal{A}_\alpha(M)$. Such a result is the content of [GKu15]: If $\alpha \in \mathbb{R}$ is Liouville, the set of volume-preserving

diffeomorphisms, that are weak mixing and preserve a measurable Riemannian metric, is dense in the C^∞-topology in $\mathcal{A}_\alpha(M)$.
While the "approximation by conjugation"-method is one of the most powerful tools of constructing smooth diffeomorphisms with prescribed ergodic or topological properties, there are great challenging differences in the real-analytic case as discussed in [FK04], §7.1: Since maps with very large derivatives in the real domain or its inverses are expected to have singularities in a small complex neighbourhood, for a real analytic family S_t, $0 \leq t \leq t_0$, $S_0 = \mathrm{id}$, the family $h \circ S_t \circ h^{-1}$ is expected to have singularities very close to the real domain for any $t > 0$. So, the domain of analycity for maps of our form $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ will shrink at any step of the construction and the limit diffeomorphism will not be analytic. Thus, it is necessary to find conjugation maps of a special form which may be inverted more or less explicitly in such a way that one can guarantee analycity of the map and its inverse in a large complex domain.
Using very explicit conjugation maps M. Saprykina was able to construct examples of volume-preserving uniquely ergodic real-analytic diffeomorphims on $\mathbb{T}^2$ ([Sa03]). Fayad and Katok designed such examples on any odd-dimensional sphere in [FK14]. By a similar approach as Saprykina we can prove the existence of weak mixing real-analytic diffeomorphisms on $\mathbb{T}^2$ that are uniformly rigid with respect to a prescribed sequence satisfying a growth condition ([Ku15]). Recently, S. Banerjee constructed non-standard real-analytic realizations on $\mathbb{T}^2$ of some irrational circle rotations ([Ba15]). His key idea is to use entire functions that approximate certain carefully chosen step functions, which is the important mechanism in our constructions in this paper as well. We will prove the following main theorem:

Theorem. *Let $\rho > 0$, $m \geq 2$ and $\mathbb{T}^m$ be the torus with Lebesgue measure μ. There exists a weak mixing real-analytic diffeomorphism $f \in \mathit{Diff}^\omega_\rho(\mathbb{T}^m, \mu)$ preserving a measurable Riemannian metric.*

Hereby, we solve [GKa00], Problem 3.9., about the existence of real-analytic volume-preserving IM-diffeomorphisms (i. e. diffeomorphisms preserving an absolutely continuous probability measure and a measurable Riemannian metric) in the case of tori $\mathbb{T}^m$, $m \geq 2$. See [GKa00], section 3, for a comprehensive consideration of IM-diffeomorphisms and IM-group actions. In particular, the existence of a measurable invariant metric for a diffeomorphism is equivalent to the existence of an invariant measure for the projectivized derivative extension which is absolutely continuous in the fibers. In [K1] the ergodic behaviour of the derivative extension with respect to such a measure is examined. It provides the only known examples of measure-preserving diffeomorphisms whose differential is ergodic with respect to a smooth measure in the projectivization of the tangent bundle.

2 Preliminaries

2.1 Analytic topology

Real-analytic diffeomorphisms of $\mathbb{T}^m$ homotopic to the identity have a lift of type

$$F(x_1, ..., x_m) = (x_1 + f_1(x_1, ..., x_m), ..., x_m + f_m(x_1, ..., x_m)),$$

where the functions $f_i : \mathbb{R}^m \to \mathbb{R}$ are real-analytic and $\mathbb{Z}^m$-periodic for $i = 1, ..., m$. For these functions we introduce the subsequent definition:

Definition 2.1. For any $\rho > 0$ we consider the set of real-analytic $\mathbb{Z}^m$-periodic functions on $\mathbb{R}^m$, that can be extended to a holomorphic function on

$$A^\rho := \{(z_1, ..., z_m) \in \mathbb{C}^m \ : \ |\mathrm{im}(z_i)| < \rho \text{ for } i = 1, ..., m\}.$$

1. For these functions let $\|f\|_\rho := \sup_{(z_1,...,z_m)\in A^\rho} |f(z_1,...,z_m)|$.
2. The set of these functions satisfying the condition $\|f\|_\rho < \infty$ is denoted by $C^\omega_\rho(\mathbb{T}^m)$.

Furthermore, we consider the space $\mathrm{Diff}^\omega_\rho(\mathbb{T}^m,\mu)$ of those volume-preserving diffeomorphisms homotopic to the identity, for whose lift we have $f_i \in C^\omega_\rho(\mathbb{T}^m)$ for $i = 1,...,m$.

Definition 2.2. For $f, g \in \mathrm{Diff}^\omega_\rho(\mathbb{T}^m,\mu)$ we define

$$\|f\|_\rho = \max_{i=1,...,m} \|f_i\|_\rho$$

and the distance

$$d_\rho(f,g) = \max_{i=1,...,m} \left\{ \inf_{p\in\mathbb{Z}} \|f_i - g_i - p\|_\rho \right\}.$$

Remark 2.3. $\mathrm{Diff}^\omega_\rho(\mathbb{T}^m,\mu)$ is a Banach space (see [Sa03] for a more extensive treatment of these spaces).

Moreover, for a diffeomorphism T with lift $\tilde{T}(x_1,...,x_m) = (T_1(x_1,...,x_m),...,T_m(x_1,...,x_m))$ we define

$$\|DT\|_\rho = \max\left\{ \left\| \frac{\partial T_i}{\partial x_j} \right\|_\rho \text{ for } i,j = 1,...,m \right\}$$

2.2 Outline of the proof

We consider the torus $\mathbb{T}^m$ equipped with Lebesgue measure μ and the circle action $\mathcal{R} = \{R_t\}_{t\in\mathbb{S}^1}$ comprising of the diffeomorphisms $R_t(x_1,...,x_m) = (x_1+t, x_2,...,x_m)$. According to the "approximation by conjugation-method" the aimed weak mixing diffeomorphism f preserving a measurable invariant Riemannian metric is constructed as the limit of volume-preserving real-analytic diffeomorphisms f_n defined by $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$. Here, the rational numbers $\alpha_{n+1} \in \mathbb{S}^1$ and the conjugation maps $H_n \in \mathrm{Diff}^\omega(\mathbb{T}^m,\mu)$ are constructed inductively:

$$\alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} = \alpha_n + \frac{1}{k_n \cdot l_n \cdot q_n} \quad \text{and} \quad H_n = h_1 \circ ... \circ h_n,$$

where the conjugation map $h_n \in \mathrm{Diff}^\omega(\mathbb{T}^m,\mu)$ has to satisfy $h_n \circ R_{\alpha_n} = R_{\alpha_n} \circ h_n$ and $k_n, l_n \in \mathbb{N}$ are parameters that have to be chosen appropriately.
In our constructions, $h_n = g_n \circ \phi_n$ is the composition of two real-analytic diffeomorphisms, which are defined explicitly in subsection 3.3 and 3.4 respectively. Moreover, we define two types of partial partitions η_n and ζ_n in subsection 3.2. The elements of η_n have to be (γ,ε)-distributed under the map $\Phi_n := \phi_n \circ R^{m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}$, where the numbers $m_n \in \mathbb{N}$ are defined in subsection 3.1. This concept of (γ,ε)-distribution is introduced in section 4: Descriptively, it says that the partition elements, which are contained in a cuboid of small edge length $\frac{1}{l_n}$, are mapped in a almost uniformly distributed way onto a set of almost full volume in the $x_2,...,x_m$-coordinates and x_1-width smaller than γ. This property is the central notion in the criterion for weak mixing deduced in section 5. At this juncture, the map g_n is required to introduce some kind of shear into the x_1-coordinate. On the other hand, h_n has to act as an "almost isometry" on the elements of the partial partition ζ_n in order to enable us to construct the invariant measurable Riemannian metric.

Definition 2.4. For a diffeomorphism f defined on a compact subset U of a smooth Riemannian manifold we define the deviation from being an isometry by

$$\mathrm{dev}_U(f) := \max_{v\in TU, \|v\|=1} |\log \|df(v)\||$$

Remark 2.5. We observe that this quantity has the following properties:

- $\mathrm{dev}_U(f) = 0$ if and only if f is a smooth isometry of U.
- $\mathrm{dev}_U(f) = \mathrm{dev}_{f(U)}\left(f^{-1}\right)$
- $\mathrm{dev}_U\left(\tilde{f} \circ f\right) \leq \mathrm{dev}_{f(U)}\left(\tilde{f}\right) + \mathrm{dev}_U(f)$

Hereby, the invariant measurable Riemannian metric is constructed by the same approach as in [GKa00]. Finally, by choosing k_n large enough we can prove the convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{Diff}^{\omega}_{\rho}(\mathbb{T}^m, \mu)$ in section 6.

3 Explicit constructions

We present step n in our inductive process of construction. Hence, we assume that we have already defined $H_{n-1} = h_1 \circ ... \circ h_{n-1}$ and the rational numbers $\alpha_1, ..., \alpha_n \in \mathbb{S}^1$. Let l_n be a large enough integer satisfying condition A and

$$l_n > m \cdot n^2 \cdot q_n \cdot \|DH_{n-1}\|_0 . \tag{1}$$

We will use this parameter to ensure that the sequence of partial partitions under consideration converge to the decomposition into points (see the proof of Lemma 5.3) and that the constructed form ω_∞ is positive definite (see Lemma 7.3). In this connection, the parameters

$$\delta_n = \frac{1}{10 \cdot n^2 \cdot q_n \cdot l_n^{m+1}} \tag{2}$$

and

$$\varepsilon_n = \frac{1}{400 \cdot m \cdot n^4 \cdot q_n^2 \cdot l_n^{2m+2}} \tag{3}$$

are important as well.

3.1 Choice of the mixing sequence $(m_n)_{n\in\mathbb{N}}$

By condition 13 our chosen sequence $(q_n)_{n\in\mathbb{N}}$ satisfies

$$q_{n+1} = k_n \cdot l_n \cdot q_n > 40n^2 \cdot q_n \cdot l_n^{m+1}. \tag{4}$$

Define

$$\begin{aligned} m_n &= \min\left\{ m \leq q_{n+1} \quad : \quad m \in \mathbb{N}, \ \inf_{k\in\mathbb{Z}} \left| m \cdot \frac{p_{n+1}}{q_{n+1}} - \frac{1}{2 \cdot q_n} + \frac{k}{q_n} \right| \leq \frac{1}{q_{n+1}} \right\} \\ &= \min\left\{ m \leq q_{n+1} \quad : \quad m \in \mathbb{N}, \ \inf_{k\in\mathbb{Z}} \left| m \cdot \frac{q_n \cdot p_{n+1}}{q_{n+1}} - \frac{1}{2} + k \right| \leq \frac{q_n}{q_{n+1}} \right\} \end{aligned}$$

Lemma 3.1. *The set* $\left\{ m \leq q_{n+1} : \ m \in \mathbb{N}, \ \inf_{k\in\mathbb{Z}} \left| m \frac{q_n \cdot p_{n+1}}{q_{n+1}} - \frac{1}{2} + k \right| \leq \frac{q_n}{q_{n+1}} \right\}$ *is nonempty for every* $n \in \mathbb{N}$, *i.e.* m_n *exists.*

Proof. We construct the sequence $\alpha_n = \frac{p_n}{q_n}$ in such a way, that $\alpha_{n+1} = \alpha_n + \frac{1}{l_n \cdot k_n \cdot q_n}$. In particular, p_{n+1} and q_{n+1} are relatively prime. Therefore, the set $\left\{ j \cdot \frac{q_n \cdot p_{n+1}}{q_{n+1}} \ : \ j = 1, ..., q_{n+1} \right\}$ contains $\frac{q_{n+1}}{\gcd(q_n, q_{n+1})}$ different equally distributed points on $\mathbb{S}^1$. Hence, there are at least $\frac{q_{n+1}}{q_n}$ different such points and so for every $x \in \mathbb{S}^1$ there is a $j \in \{1, ..., q_{n+1}\}$ such that

$$\inf_{k \in \mathbb{Z}} \left| x - j \cdot \frac{q_n \cdot p_{n+1}}{q_{n+1}} + k \right| \leq \frac{q_n}{q_{n+1}}.$$

In particular, this is true for $x = \frac{1}{2}$. □

Remark 3.2. We define

$$a_n = \left(m_n \cdot \frac{p_{n+1}}{q_{n+1}} - \frac{1}{2 \cdot q_n} \right) \mod \frac{1}{q_n}$$

By the above construction of m_n it holds that $|a_n| \leq \frac{1}{q_{n+1}}$. By equation 4 we get:

$$|a_n| \leq \frac{1}{40n^2 \cdot q_n \cdot l_n^{m+1}} = \frac{\delta_n}{4}.$$

3.2 Sequences of partial partitions

In this subsection we define the two announced sequences of partial partitions $(\eta_n)_{n \in \mathbb{N}}$ and $(\zeta_n)_{n \in \mathbb{N}}$ of $\mathbb{T}^m$.

3.2.1 Partial partition η_n

Remark 3.3. For convenience we will use the notation $\prod_{i=2}^m [a_i, b_i]$ for $[a_2, b_2] \times ... \times [a_m, b_m]$.

Initially, η_n will be constructed on the fundamental sector $\left[0, \frac{1}{q_n}\right] \times \mathbb{T}^{m-1}$. With a view to the piecewise definition of the conjugation map ϕ_n in the following subsection we divide the fundamental sector in two sections:

- On $\left[0, \frac{1}{2 \cdot q_n}\right] \times \mathbb{T}^{m-1}$ we consider the following sets:

$$I_{j_1, ..., j_m} := \bigcup \left[\frac{j_1}{2q_n \cdot l_n} + \frac{t^{(1)}}{2q_n l_n^2} + ... + \frac{t^{(m-1)}}{2q_n l_n^m} + \delta_n, \frac{j_1}{2q_n l_n} + \frac{t^{(1)}}{2q_n l_n^2} + ... + \frac{t^{(m-1)} + 1}{2q_n l_n^m} - \delta_n \right] \times \prod_{i=2}^m \left[\frac{j_i}{l_n} + \delta_n, \frac{j_i + 1}{l_n} - \delta_n \right],$$

  where the union is taken over $t^{(s)} \in \mathbb{Z}$, $0 \leq t^{(s)} \leq l_n - 1$, for $s = 1, ..., m-1$.
  The partial partition η_n consists of all such sets $I_{j_1, ..., j_{m-1}}$, at which $j_i \in \mathbb{Z}$, $1 \leq j_1 \leq l_n - 3$ and $1 \leq j_i \leq l_n - 2$ for $i = 2, ..., m$.

- On $\left[\frac{1}{2\cdot q_n}, \frac{1}{q_n}\right] \times \mathbb{T}^{m-1}$ we consider sets of the following form:

$$\bar{I}_{j_1,\dots,j_m} := \bigcup \left[\frac{1}{2q_n} + \frac{j_1}{2q_n \cdot l_n} + \frac{t^{(1)}}{2q_n \cdot l_n^2} + \dots + \frac{t^{(m-1)}}{2q_n \cdot l_n^m} + \delta_n, \frac{1}{2q_n} + \frac{j_1}{2q_n \cdot l_n} + \frac{t^{(1)}}{2q_n \cdot l_n^2} + \dots + \frac{t^{(m-1)}+1}{2q_n \cdot l_n^m} - \delta_n\right] \times \prod_{i=2}^{m} \left[\frac{j_i}{l_n} + \delta_n, \frac{j_i+1}{l_n} - \delta_n\right],$$

  where the union is taken over $t^{(s)} \in \mathbb{Z}$, $0 \le t^{(s)} \le l_n - 1$, for $s = 1, \dots, m-1$.
  The partial partition η_n consists of all such sets $\bar{I}_{j_1,\dots,j_{m-1}}$, at which $j_i \in \mathbb{Z}$, $1 \le j_1 \le l_n - 3$ and $1 \le j_i \le l_n - 2$ for $i = 2, \dots, m$.

As the image under R_{l/q_n} with $l \in \mathbb{Z}$ this partial partition of $\left[0, \frac{1}{q_n}\right] \times \mathbb{T}^{m-1}$ is extended to a partial partition of $\mathbb{T}^m$.

Remark 3.4. By construction this sequence of partial partitions converges to the decomposition into points.

3.2.2 Partial partition ζ_n

On the fundamental sector $\left[0, \frac{1}{q_n}\right] \times \mathbb{T}^{m-1}$ the partial partition ζ_n consists of all multidimensional intervals of the following form:

$$\left[\frac{j_1^{(1)}}{2q_n \cdot l_n} + \frac{j_1^{(2)}}{2q_n l_n^2} + \dots + \frac{j_1^{(m)}}{2q_n l_n^m} + \delta_n, \frac{j_1^{(1)}}{2q_n l_n} + \frac{j_1^{(2)}}{2q_n l_n^2} + \dots + \frac{j_1^{(m)}+1}{2q_n l_n^m} - \delta_n\right] \times \prod_{i=2}^{m} \left[\frac{j_i^{(1)}}{l_n} + \frac{j_i^{(2)}}{10n^2 \cdot q_n \cdot l_n^{m+1}} + \frac{\delta_n}{10n^2 \cdot q_n \cdot l_n^{m+1}}, \frac{j_i^{(1)}}{l_n} + \frac{j_i^{(2)}+1}{10n^2 \cdot q_n \cdot l_n^{m+1}} - \frac{\delta_n}{10n^2 \cdot q_n \cdot l_n^{m+1}}\right],$$

where $j_1^{(1)} \in \mathbb{Z}$, $0 \le j_1^{(1)} \le 2l_n - 1$, and $j_1^{(s)} \in \mathbb{Z}$, $1 \le j_1^{(s)} \le l_n - 2$, for $s = 2, \dots, m$ as well as for $i = 2, \dots, m$: $j_i^{(1)} \in \mathbb{Z}$, $1 \le j_i^{(1)} \le l_n - 2$, and $j_i^{(2)} \in \mathbb{Z}$, $1 \le j_i^{(2)} \le 10n^2 \cdot q_n \cdot l_n^m - 2$.
As above we extend it to a partial partition of $\mathbb{T}^m$ as the image under R_{l/q_n} with $l \in \mathbb{Z}$.

Remark 3.5. For every $n \ge 3$ the partial partition ζ_n consists of disjoint sets, covers a set of measure at least $1 - \frac{1}{n^2}$ and the sequence $(\zeta_n)_{n\in\mathbb{N}}$ converges to the decomposition into points.

3.3 The conjugation map ϕ_n

First of all, we consider the following step functions for $d = 2, \dots, m$:

$$\tilde{\psi}_{1,n}^{(d)} : [0,1) \to \mathbb{R} \qquad \text{defined by } \tilde{\psi}_{1,n}^{(d)}(x) = \sum_{i=1}^{l_n-1} \frac{l_n - i}{2q_n \cdot l_n^d} \cdot \chi_{[\frac{i}{l_n}, \frac{i+1}{l_n})}(x)$$

$$\tilde{\psi}_{3,n}^{(d)} : [0,1) \to \mathbb{R} \qquad \text{defined by } \tilde{\psi}_{3,n}^{(d)}(x) = \sum_{i=1}^{l_n-1} \frac{i}{2q_n \cdot l_n^d} \cdot \chi_{[\frac{i}{l_n}, \frac{i+1}{l_n})}(x)$$

Furthermore, we require another type of step function: For $i \in \mathbb{Z}$, $0 \leq i \leq l_n^d - 1$, we put $\beta_i^{(2)} := \frac{j}{l_n}$ if $i \equiv j \mod l_n$. For $i \in \mathbb{Z}$, $l_n^d \leq i \leq 2l_n^d - 1$, we put $\beta_i^{(2)} := 0$. Then we consider

$$\tilde{\psi}_{2,n}^{(d)} : \left[0, \frac{1}{q_n}\right) \to \mathbb{R} \quad \text{defined by } \tilde{\psi}_{2,n}^{(d)}(x) = \sum_{i=0}^{2l_n^d-1} \beta_i^{(2)} \cdot \chi_{\left[\frac{i}{2q_n \cdot l_n^d}, \frac{i+1}{2q_n \cdot l_n^d}\right)}(x)$$

and extend it to a map on $[0,1)$ periodically.
Hereby, we define

$$\begin{aligned}
&\tilde{\phi}_{1,n}^{(d)} : \mathbb{T}^m \to \mathbb{T}^m, && \tilde{\phi}_{1,n}^{(d)}(x_1, ..., x_m) = \left(x_1 + \tilde{\psi}_{1,n}^{(d)}(x_d) \mod 1, x_2, ..., x_m\right) \\
&\tilde{\phi}_{2,n}^{(d)} : \mathbb{T}^m \to \mathbb{T}^m, && \tilde{\phi}_{2,n}^{(d)}(x_1, ..., x_m) = \left(x_1, ..., x_{d-1}, x_d + \tilde{\psi}_{2,n}^{(d)}(x_1) \mod 1, x_{d+1}, ..., x_m\right) \\
&\tilde{\phi}_{3,n}^{(d)} : \mathbb{T}^m \to \mathbb{T}^m, && \tilde{\phi}_{3,n}^{(d)}(x_1, ..., x_m) = \left(x_1 - \tilde{\psi}_{3,n}^{(d)}(x_d) \mod 1, x_2, ..., x_m\right)
\end{aligned}$$

and $\tilde{\phi}_n^{(d)} := \tilde{\phi}_{3,n}^{(d)} \circ \tilde{\phi}_{2,n}^{(d)} \circ \tilde{\phi}_{1,n}^{(d)}$. Moreover, let $\tilde{\phi}_n = \tilde{\phi}_n^{(2)} \circ ... \circ \tilde{\phi}_n^{(m)}$. These maps will help us to understand the combinatorics in the proof (see the proof of Lemma 4.3). Unfortunately, they are discontinuous. In order to construct entire conjugation maps we will use the subsequent Lemma about approximation of step functions by real-analytic diffeomorphisms inspired by [Ba15], Lemma 4.1, where we call an entire function real if it maps the real line into itself:

Lemma 3.6. *Let $l, N \in \mathbb{N}$ and $\beta = (\beta_0, ..., \beta_{l-1}) \in [0,1]^l$. We consider a step function of the form*

$$\tilde{s}_{\beta,N} : [0,1) \to \mathbb{R} \text{ defined by } \tilde{s}_{\beta,N}(x) = \sum_{i=0}^{lN-1} \tilde{\beta}_i \cdot \chi_{\left[\frac{i}{lN}, \frac{i+1}{lN}\right)}(x),$$

where $\tilde{\beta}_i := \beta_j$ in case of $j \equiv i \mod l$. Then, given any $\varepsilon > 0$ and $\delta > 0$, there exists a $\frac{1}{N}$-periodic real entire function $s_{\beta,N,\varepsilon,\delta}$ satisfying

$$\sup_{x \in [0,1) \setminus F} |s_{\beta,N,\varepsilon,\delta}(x) - \tilde{s}_{\beta,N}(x)| < \varepsilon \quad \text{and} \quad \sup_{x \in [0,1) \setminus F} \left|s'_{\beta,N,\varepsilon,\delta}(x)\right| < \varepsilon, \tag{5}$$

where $F = \bigcup_{i=0}^{lN-1} I_i \subset [0,1)$ is a union of intervals centered around $\frac{i}{lN}$, $i = 1, ..., lN-1$, $I_0 = \left[0, \frac{\delta}{2lN}\right] \cup \left[1 - \frac{\delta}{2lN}, 1\right)$ and $\lambda(I_i) = \frac{\delta}{lN}$ for every i.

Proof. By the same approach as in [Ba15], Lemma 4.1., we define the function

$$s_{\beta,N,\varepsilon,\delta}(x) = \sum_{n=-\infty}^{\infty} \left(\sum_{i=0}^{l-1} \beta_i \cdot \left(\exp^{-\exp^{-A \cdot \left(x - \frac{nl+i}{lN}\right)}} - \exp^{-\exp^{-A \cdot \left(x - \frac{nl+i+1}{lN}\right)}} \right) \right).$$

We point out that $s_{\beta,N,\varepsilon,\delta}$ is a $\frac{1}{N}$-periodic real entire function. After choosing a large enough constant A, we can guarantee that $s_{\beta,N,\varepsilon,\delta}$ satisfies the conditions 5. □

Recall ε_n and δ_n, that were defined in equation 3 and 2 respectively. With the aid of Lemma 3.6 we construct entire functions approximating the step functions defined above:

$$\begin{aligned}
&\psi_{1,n}^{(d)} = s_{\beta^{(1)},N^{(1)},\varepsilon_n,\delta_n}, \text{ where } \beta_0^{(1)} = 0,\ \beta_i^{(1)} = \frac{l_n - i}{2q_n \cdot l_n^d} \text{ for } i = 1, ..., l_n - 1,\ N^{(1)} = 1 \\
&\psi_{2,n}^{(d)} = s_{\beta^{(2)},N^{(2)},\varepsilon_n,\delta_n}, \text{ where } \beta_i^{(2)} \text{as above for } i = 0, ..., 2l_n^{d-1} - 1,\ N^{(2)} = q_n \\
&\psi_{3,n}^{(d)} = s_{\beta^{(3)},N^{(3)},\varepsilon_n,\delta_n}, \text{ where } \beta_0^{(3)} = 0,\ \beta_i^{(3)} = \frac{i}{2q_n \cdot l_n^d} \text{ for } i = 1, ..., l_n - 1,\ N^{(3)} = 1
\end{aligned}$$

Hereby, we define

$$\phi_{1,n}^{(d)} : \mathbb{T}^m \to \mathbb{T}^m, \qquad \phi_{1,n}^{(d)}(x_1, ..., x_m) = \left(x_1 + \psi_{1,n}^{(d)}(x_d) \mod 1, x_2, ..., x_m\right)$$
$$\phi_{2,n}^{(d)} : \mathbb{T}^m \to \mathbb{T}^m, \qquad \phi_{2,n}^{(d)}(x_1, ..., x_m) = \left(x_1, ..., x_{d-1}, x_d + \psi_{2,n}^{(d)}(x_1) \mod 1, x_{d+1}, ..., x_m\right)$$
$$\phi_{3,n}^{(d)} : \mathbb{T}^m \to \mathbb{T}^m, \qquad \phi_{3,n}^{(d)}(x_1, ..., x_m) = \left(x_1 - \psi_{3,n}^{(d)}(x_d) \mod 1, x_2, ..., x_m\right)$$

Let $\phi_n^{(d)} := \phi_{3,n}^{(d)} \circ \phi_{2,n}^{(d)} \circ \phi_{1,n}^{(d)}$. Since $\psi_{2,n}^{(d)}$ is $\frac{1}{q_n}$-periodic, we have $\phi_n^{(d)} \circ R_{\alpha_n} = R_{\alpha_n} \circ \phi_n^{(d)}$. Finally, we define

$$\phi_n = \phi_n^{(2)} \circ ... \circ \phi_n^{(m)}$$

and observe $\phi_n \circ R_{\alpha_n} = R_{\alpha_n} \circ \phi_n$.

Remark 3.7. We compute

$$\phi_n^{(d)}(x_1, ..., x_m) = \Bigg(x_1 + \psi_{1,n}^{(d)}(x_d) - \psi_{3,n}^{(d)}\left(x_d + \psi_{2,n}^{(d)}\left(x_1 + \psi_{1,n}^{(d)}(x_d)\right)\right), x_2, ..., x_{d-1}, x_d + \psi_{2,n}^{(d)}\left(x_1 + \psi_{1,n}^{(d)}(x_d)\right), x_{d+1}, ..., x_m\Bigg).$$

By the choice $4 \cdot m \cdot \varepsilon_n < \delta_n$, the exact positioning of the partition elements of η_n as well as ζ_n and since the $\tilde{\psi}_{i,n}^{(d)}$ are step functions, we have for a point z contained in one of the partition elements $\left|\left[\phi_n^{(d)}\right]_1(z) - \left[\tilde{\phi}_n^{(d)}\right]_1(z)\right| < 2\varepsilon_n$ and $\left|\left[\phi_n^{(d)}\right]_d(z) - \left[\tilde{\phi}_n^{(d)}\right]_d(z)\right| < \varepsilon_n$. Continuing in this way we conclude $\left|[\phi_n]_1(z) - \left[\tilde{\phi}_n\right]_1(z)\right| < 2 \cdot (m-1) \cdot \varepsilon_n$ and $\left|[\phi_n]_i(z) - \left[\tilde{\phi}_n\right]_i(z)\right| < \varepsilon_n$ in case of $i = 2, ..., m$. For the inverse $\phi_n^{(-1)}$ the same observations hold true.

We introduce the so-called "good set" $J_n \subset \mathbb{T}^{m-1}$ in the $x_2, ..., x_m$-coordinates:

$$J_n = \bigcup \prod_{i=2}^{m} \left[\frac{j_i}{l_n} + \delta_n + 2\varepsilon_n, \frac{j_i + 1}{l_n} - \delta_n - 2\varepsilon_n\right], \tag{6}$$

where the union is taken over $j_i \in \mathbb{Z}$, $0 \le j_i \le l_n - 1$, for $i = 2, ..., m$.

3.4 The conjugation map g_n

We aim at a real-analytic map, which introduces shear into the x_1-coordinate similar to the map $\bar{g}_{[nq_n^\sigma]}(x_1, ..., x_m) = (x_1 + [nq_n^\sigma] \cdot x_2, x_2, ..., x_m)$, but acts as an almost-isometry on the elements of the partial partition ζ_n. For this purpose, we consider the following step function

$$\tilde{\psi}_n : [0,1) \to \mathbb{R} \text{ defined by } \tilde{\psi}_n(x) = \sum_{i=0}^{10n^2 \cdot q_n \cdot l_n^{m+1} - 1} \frac{i}{10n^2 \cdot q_n \cdot l_n^{m+1}} \cdot \chi_{\left[\frac{i}{10n^2 \cdot q_n \cdot l_n^{m+1}}, \frac{i+1}{10n^2 \cdot q_n \cdot l_n^{m+1}}\right)}(x)$$

and the discontinuous map $\tilde{g}_n(x_1, ..., x_m) = \left(x_1 + [nq_n^\sigma] \cdot \tilde{\psi}_n(x_2), x_2, ..., x_m\right)$.
In order to find a real-analytic approximation of this map we use the subsequent result similar to Lemma 3.6.

Lemma 3.8. *Let $a \in \mathbb{N}$. We consider a step function of the form*

$$\tilde{s}_a : [0,1) \to \mathbb{R} \text{ defined by } \tilde{s}_a(x) = \sum_{i=0}^{a-1} \frac{i}{a} \cdot \chi_{\left[\frac{i}{a},\frac{i+1}{a}\right)}(x).$$

Then, given any $\varepsilon > 0$ and $\delta > 0$, there exists a 1-periodic real entire function $\bar{s}_{a,\varepsilon,\delta}$ satisfying

$$\sup_{x\in[0,1)\setminus F} |\bar{s}_{a,\varepsilon,\delta}(x) - \tilde{s}_a(x)| < \varepsilon \quad \text{and} \quad \sup_{x\in[0,1)\setminus F} \left|\bar{s}'_{a,\varepsilon,\delta}(x)\right| < \varepsilon, \tag{7}$$

where $F = \bigcup_{i=0}^{l-1} I_i \subset [0,1)$ is a union of intervals centered around $\frac{i}{a}$, $i = 1,...,a-1$, $I_0 = \left[0,\frac{\delta}{2a}\right] \cup \left[1-\frac{\delta}{2a},1\right)$ and $\lambda(I_i) = \frac{\delta}{a}$ for every i.

Proof. By the same approach as in Lemma 3.6 we define the function

$$s_{a,N,\varepsilon,\delta}(x) = \sum_{n=-\infty}^{\infty} \left(\sum_{i=1}^{a-1} \frac{1}{a} \cdot \left(\exp^{-\exp^{-A\cdot\left(x-\frac{na+i}{a}\right)}} - \exp^{-\exp^{-A\cdot(x-n-1)}} \right) \right).$$

We point out that $\bar{s}_{a,\varepsilon,\delta}$ is a 1-periodic real entire function. After choosing a large enough constant A, we can guarantee that $\bar{s}_{a,\varepsilon,\delta}$ satisfies the conditions 7. □

With the aid of Lemma 3.8 we can approximate the step function by an entire map:

$$\psi_n = \bar{s}_{10n^2\cdot q_n\cdot l_n^{m+1},\varepsilon_n,\delta_n}.$$

Hereby, we define the real-analytic diffeomorphism

$$g_n : \mathbb{T}^m \to \mathbb{T}^m, \quad g_n(x_1,...,x_m) = (x_1 + [nq_n^\sigma]\cdot\psi_n(x_2), x_2, ..., x_m)$$

and observe $g_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ g_n$.

4 (γ,ϵ)-distribution

For the sake of convenience, we denote the coordinates on $\mathbb{T}^m$ by $(\theta, r_1, ..., r_{m-1})$ below.
We introduce the central notion in the proof of the criterion for weak mixing deduced in the next section:

Definition 4.1. Let $\Phi : \mathbb{T}^m \to \mathbb{T}^m$ be a diffeomorphism and $J \subset \mathbb{T}^{m-1}$. We say that an element $\hat{I}$ of a partial partition is (γ,ϵ)-distributed on J under Φ, if the following properties are satisfied:

- $\Phi\left(\hat{I}\right)$ is contained in a set of the form $[c, c+\gamma] \times \mathbb{T}^{m-1}$ for some $c \in \mathbb{S}^1$.
- $\pi_{\vec{r}}\left(\Phi\left(\hat{I}\right)\right) \supseteq J$.
- For every $(m-1)$-dimensional interval $\tilde{J} \subseteq J$ it holds:

$$\left| \frac{\mu\left(\hat{I} \cap \Phi^{-1}\left(\mathbb{S}^1 \times \tilde{J}\right)\right)}{\mu\left(\hat{I}\right)} - \frac{\tilde{\mu}\left(\tilde{J}\right)}{\tilde{\mu}(J)} \right| \le \epsilon \cdot \frac{\tilde{\mu}\left(\tilde{J}\right)}{\tilde{\mu}(J)},$$

  at which $\tilde{\mu}$ is the Lebesgue measure on $\mathbb{T}^{m-1}$.

Remark 4.2. Analogous to [FS05] we will call the third property "almost uniform distribution" of $\hat{I}$ in the $r_1,..,r_{m-1}$-coordinates. In the following we will often write it in the form of

$$\left|\mu\left(\hat{I}\cap\Phi^{-1}\left(\mathbb{S}^1\times\tilde{J}\right)\right)\cdot\tilde{\mu}\left(J\right)-\mu\left(\hat{I}\right)\cdot\tilde{\mu}\left(\tilde{J}\right)\right|\leq\epsilon\cdot\mu\left(\hat{I}\right)\cdot\tilde{\mu}\left(\tilde{J}\right).$$

Our constructions are done in such a way that the following property is satisfied:

Lemma 4.3. *We consider the "good set" J_n defined in equation 6 as well as the diffeomorphism $\Phi_n:=\phi_n\circ R^{m_n}_{\alpha_{n+1}}\circ\phi_n^{-1}$ with the conjugating maps ϕ_n defined in section 3.3 and the numbers m_n as in section 3.1. Then the elements of the partition η_n are $\left(\frac{3}{q_n\cdot l_n},\frac{1}{n}\right)$-distributed on J_n under Φ_n.*

Proof. For $I_{j_1,\dots,j_m}\in\eta_n$ we compute $\Phi_n\left(I_{j_1,\dots,j_m}\right)$. By the choice of m_n and Remark 3.2 we obtain modulo $\frac{1}{q_n}$ in the x_1-coordinate:

$$\begin{aligned}&R^{m_n}_{\alpha_{n+1}}\circ\tilde{\phi}_n^{-1}\left(I_{j_1,\dots,j_m}\right)=\\&\bigcup\left[\frac{1}{2q_n}+\frac{j_1}{2q_nl_n}+\frac{j_2}{2q_n\cdot l_n^2}+...+\frac{j_m}{2q_n\cdot l_n^m}+\delta_n+a_n,\frac{1}{2q_n}+\frac{j_1}{2q_nl_n}+...+\frac{j_m+1}{2q_n\cdot l_n^m}-\delta_n+a_n\right]\\&\times\prod_{i=2}^{m}\left[1-\frac{t^{(i-1)}}{l_n}+\delta_n,1-\frac{t^{(i-1)}-1}{l_n}-\delta_n\right].\end{aligned}$$

The application of $\tilde{\phi}_n$ on this set yields:

$$\begin{aligned}&\bigcup\left[\frac{1}{2q_n}+\frac{j_1}{2q_n\cdot l_n}+\frac{j_2+2\cdot t^{(1)}-l_n}{2q_n\cdot l_n^2}+...+\frac{j_m+2\cdot t^{(m-1)}-l_n}{2q_n\cdot l_n^m}+\delta_n+a_n,\right.\\&\left.\frac{1}{2q_n}+\frac{j_1}{2q_n\cdot l_n}+\frac{j_2+2\cdot t^{(1)}-l_n}{2q_n\cdot l_n^2}+...+\frac{j_m+1+2\cdot t^{(m-1)}-l_n}{2q_n\cdot l_n^m}-\delta_n+a_n\right]\\&\times\prod_{i=2}^{m}\left[1-\frac{t^{(i-1)}}{l_n}+\delta_n,1-\frac{t^{(i-1)}-1}{l_n}-\delta_n\right]\end{aligned}$$

(apart from the case $t^{(i)}=0$ where we get j_{i+1} instead of $j_{i+1}+2t^{(i)}-l_n$).
In the same way we compute $\tilde{\phi}_n\circ R^{m_n}_{\alpha_{n+1}}\circ\tilde{\phi}_n^{-1}\left(\bar{I}_{j_1,\dots,j_m}\right)$:

$$\begin{aligned}&\bigcup\left[\frac{j_1}{2q_n\cdot l_n}-\frac{j_2}{2q_n\cdot l_n^2}-...-\frac{j_m}{2q_n\cdot l_n^m}+\delta_n+a_n,\frac{j_1}{2q_n\cdot l_n}-...-\frac{j_m-1}{2q_n\cdot l_n^m}-\delta_n+a_n\right]\\&\times\prod_{i=2}^{m}\left[\frac{2j_i+t^{(i-1)}}{l_n}+\delta_n,\frac{2j_i+t^{(i-1)}+1}{l_n}-\delta_n\right]\end{aligned}$$

regarded as a subset of $\mathbb{T}^m$.

We have to take the approximation error into account. By Remark 3.7 we observe for every $\hat{I}_n\in\eta_n$ $\pi_{\vec{r}}\left(\Phi_n\left(\hat{I}_n\right)\right)\supseteq J_n$ and that every of the l_n^{m-1} cuboids belonging to $\Phi_n\left(\hat{I}_n\right)$ is contained in a cuboid of θ-width $\frac{1}{2q_n\cdot l_n^m}-2\delta_n+8m\cdot\varepsilon_n$ and contains a cuboid of θ-width $\frac{1}{2q_n\cdot l_n^m}-2\delta_n-8m\cdot\varepsilon_n$. In particular, we can choose $\gamma=\frac{3}{q_n\cdot l_n}$. Let $\tilde{J}\subseteq J_n\subset\mathbb{T}^{m-1}$ be a multidimensional interval of

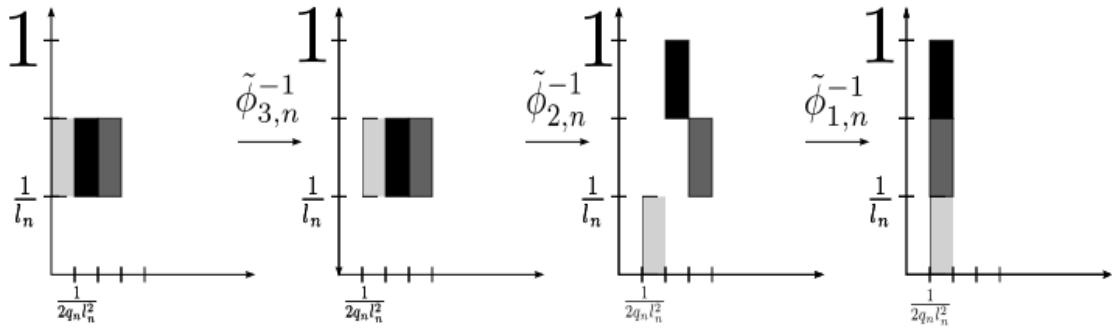

Figure 1: Qualitative shape of the action of $\tilde{\phi}_n^{-1}$ on $I_{j_1,\ldots,j_m} \in \eta_n$ in case of dimension $m = 2$.

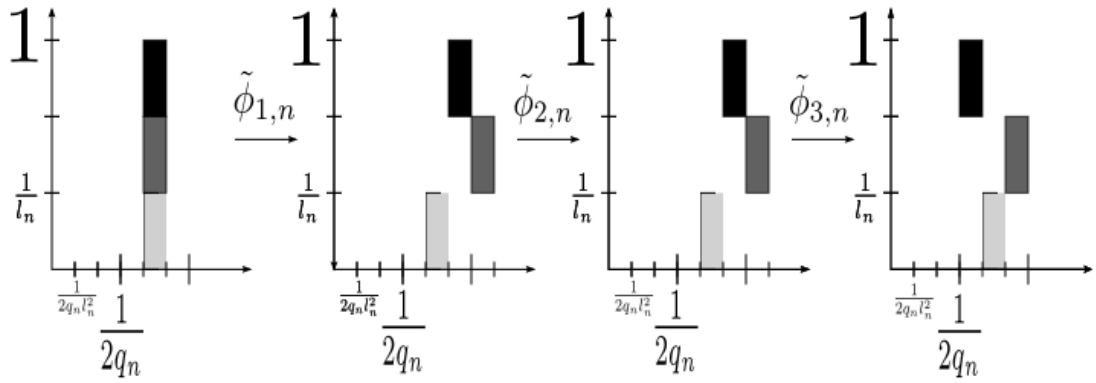

Figure 2: Qualitative shape of the action of $\tilde{\phi}_n$ on $R_{\alpha_{n+1}}^{m_n} \circ \tilde{\phi}_n^{-1}(I_{j_1,\ldots,j_m})$ in case of dimension $m = 2$.

length d_i in coordinate x_i. Then we can estimate:

$$\frac{\mu\left(\hat{I}_n \cap \Phi_n^{-1}\left(\mathbb{S}^1 \times \tilde{J}\right)\right)}{\mu\left(\hat{I}_n\right)} \leq \frac{\left(\frac{1}{2q_n \cdot l_n^m} - 2\delta_n + 8m \cdot \varepsilon_n\right) \cdot d_2 \cdot \ldots \cdot d_m}{l_n^{m-1} \cdot \left(\frac{1}{2q_n \cdot l_n^m} - 2\delta_n\right) \cdot \left(\frac{1}{l_n} - 2\delta_n\right)^{m-1}}$$

$$= \left(1 + \frac{8m \cdot \varepsilon_n \cdot 2q_n \cdot l_n^m}{1 - 4\delta_n \cdot l_n^m \cdot q_n}\right) \cdot \frac{\left(\frac{1}{l_n} - 2\delta_n - 4\varepsilon_n\right)^{m-1}}{\left(\frac{1}{l_n} - 2\delta_n\right)^{m-1}} \cdot \frac{d_2 \cdot \ldots \cdot d_m}{l_n^{m-1} \cdot \left(\frac{1}{l_n} - 2\delta_n - 4\varepsilon_n\right)^{m-1}}$$

$$\leq (1 + 32m \cdot \varepsilon_n \cdot q_n \cdot l_n^m) \cdot \frac{\tilde{\mu}\left(\tilde{J}\right)}{\tilde{\mu}\left(J_n\right)}.$$

Analogously we estimate

$$\frac{\mu\left(\hat{I}_n \cap \Phi_n^{-1}\left(\mathbb{S}^1 \times \tilde{J}\right)\right)}{\mu\left(\hat{I}_n\right)} \geq \frac{\left(\frac{1}{2q_n \cdot l_n^m} - 2\delta_n - 8m \cdot \varepsilon_n\right) \cdot d_2 \cdot \ldots \cdot d_m}{l_n^{m-1} \cdot \left(\frac{1}{2q_n \cdot l_n^m} - 2\delta_n\right) \cdot \left(\frac{1}{l_n} - 2\delta_n\right)^{m-1}}$$

$$\geq \left(1 - \frac{8m \cdot \varepsilon_n \cdot 2q_n l_n^m}{1 - 4\delta_n l_n^m \cdot q_n}\right) \cdot (1 - 8\varepsilon_n l_n)^{m-1} \cdot \frac{d_2 \cdot \ldots \cdot d_m}{l_n^{m-1} \cdot \left(\frac{1}{l_n} - 2\delta_n - 4\varepsilon_n\right)^{m-1}}$$

$$\geq (1 - 32m \cdot \varepsilon_n \cdot q_n \cdot l_n^m) \cdot (1 - (m-1) \cdot 8\varepsilon_n \cdot l_n) \cdot \frac{\tilde{\mu}\left(\tilde{J}\right)}{\tilde{\mu}\left(J_n\right)}$$

$$\geq (1 - 40m \cdot \varepsilon_n \cdot q_n \cdot l_n^m) \cdot \frac{\tilde{\mu}\left(\tilde{J}\right)}{\tilde{\mu}\left(J_n\right)}.$$

By our assumption on the number ε_n from equation 3 we conclude

$$\left|\frac{\mu\left(\hat{I}_n \cap \Phi^{-1}\left(\mathbb{S}^1 \times \tilde{J}\right)\right)}{\mu\left(\hat{I}_n\right)} - \frac{\mu^{(m-1)}\left(\tilde{J}\right)}{\mu^{(m-1)}\left(J_n\right)}\right| \leq \frac{1}{n} \cdot \frac{\mu^{(m-1)}\left(\tilde{J}\right)}{\mu^{(m-1)}\left(J_n\right)}.$$

□

5 Criterion for weak mixing

In this section we will prove a criterion for weak mixing on $M = \mathbb{T}^m$ in the setting of the beforehand constructions. It is inspired by the criterion in [FS05], but modified in many places because of the new conjugation map g_n and the new type of partitions. For the derivation we need a couple of lemmas. The first one expresses the weak mixing property on the elements of a partial partition η_n generally:

Lemma 5.1. *Let $f \in \mathit{Diff}_\rho^\omega(\mathbb{T}^m, \mu)$, $(m_n)_{n\in\mathbb{N}}$ be a sequence of natural numbers and $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of partial partitions, where $\nu_n \to \varepsilon$ and for every $n \in \mathbb{N}$ ν_n is the image of a partial partition η_n under a measure-preserving diffeomorphism F_n, satisfying the following property: For every m-dimensional cube $A \subseteq \mathbb{T}^m$ and for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $n \geq N$ and for every $\Gamma_n \in \nu_n$ we have*

$$\left|\mu\left(\Gamma_n \cap f^{-m_n}(A)\right) - \mu\left(\Gamma_n\right) \cdot \mu\left(A\right)\right| \leq 3 \cdot \epsilon \cdot \mu\left(\Gamma_n\right) \cdot \mu\left(A\right). \tag{8}$$

Then f is weak mixing.

Proof. By [Skl67] a diffeomorphism f is weak mixing if for all measurable sets $A, B \subseteq M$ it holds:

$$\lim_{n\to\infty} \left|\mu\left(B \cap f^{-m_n}(A)\right) - \mu(B)\cdot\mu(A)\right| = 0.$$

Since every measurable set in $M = \mathbb{T}^m$ can be approximated by a countable disjoint union of m-dimensional cubes in $\mathbb{T}^m$ in arbitrary precision, we only have to prove the statement in case that A is a m-dimensional cube in $\mathbb{T}^m$.
Hence, we consider an arbitrary m-dimensional cube $A \subset \mathbb{T}^m$. Moreover, let $B \subseteq M$ be a measurable set. Since $\nu_n \to \varepsilon$ for every $\epsilon \in (0,1]$ there are $n \in \mathbb{N}$ and a set $\hat{B} = \bigcup_{i\in\Lambda} \Gamma_n^i$, where $\Gamma_n^i \in \nu_n$ and Λ is a countable set of indices, such that $\mu\left(B \triangle \hat{B}\right) < \epsilon\cdot\mu(B)\cdot\mu(A)$. We obtain for sufficiently large n:

$$\begin{aligned}
&\left|\mu\left(B \cap f^{-m_n}(A)\right) - \mu(B)\cdot\mu(A)\right| \\
&\le \left|\mu\left(B \cap f^{-m_n}(A)\right) - \mu\left(\hat{B} \cap f^{-m_n}(A)\right)\right| + \left|\mu\left(\hat{B} \cap f^{-m_n}(A)\right) - \mu\left(\hat{B}\right)\cdot\mu(A)\right| \\
&\quad + \left|\mu\left(\hat{B}\right)\cdot\mu(A) - \mu(B)\cdot\mu(A)\right| \\
&= \left|\mu\left(B \cap f^{-m_n}(A)\right) - \mu\left(\hat{B} \cap f^{-m_n}(A)\right)\right| \\
&\quad + \left|\mu\left(\bigcup_{i\in\Lambda}\left(\Gamma_n^i \cap f^{-m_n}(A)\right)\right) - \mu\left(\bigcup_{i\in\Lambda}\Gamma_n^i\right)\cdot\mu(A)\right| + \mu(A)\cdot\left|\mu\left(\hat{B}\right) - \mu(B)\right| \\
&\le \mu\left(\hat{B}\triangle B\right) + \left|\sum_{i\in\Lambda}\mu\left(\Gamma_n^i \cap f^{-m_n}(A)\right) - \mu\left(\Gamma_n^i\right)\cdot\mu(A)\right| + \mu(A)\cdot\mu\left(\hat{B}\triangle B\right) \\
&\le \epsilon\cdot\mu(B)\cdot\mu(A) + \sum_{i\in\Lambda}\left(\left|\mu\left(\Gamma_n^i \cap f^{-m_n}(A)\right) - \mu\left(\Gamma_n^i\right)\cdot\mu(A)\right|\right) + \epsilon\cdot\mu(A)^2\cdot\mu(B) \\
&\le \sum_{i\in\Lambda}\left(3\cdot\epsilon\cdot\mu\left(\Gamma_n^i\right)\cdot\mu(A)\right) + 2\cdot\epsilon\cdot\mu(A)\cdot\mu(B) = 3\cdot\epsilon\cdot\mu(A)\cdot\mu\left(\bigcup_{i\in\Lambda}\hat{I}_n^i\right) + 2\cdot\epsilon\cdot\mu(A)\cdot\mu(B) \\
&= 3\cdot\epsilon\cdot\mu(A)\cdot\mu\left(\hat{B}\right) + 2\cdot\epsilon\cdot\mu(A)\cdot\mu(B) \le 3\epsilon\cdot\mu(A)\cdot\left(\mu(B) + \mu\left(\hat{B}\triangle B\right)\right) + 2\epsilon\cdot\mu(A)\cdot\mu(B) \\
&\le 5\cdot\epsilon\cdot\mu(A)\cdot\mu(B) + 3\cdot\epsilon^2\cdot\mu(A)^2\cdot\mu(B).
\end{aligned}$$

This estimate shows $\lim_{n\to\infty}\left|\mu\left(B \cap f^{-m_n}(A)\right) - \mu(B)\cdot\mu(A)\right| = 0$, because ϵ can be chosen arbitrarily small. □

In property (8) we want to replace f by f_n:

Lemma 5.2. *Let $f = \lim_{n\to\infty} f_n$ be a diffeomorphism obtained by the constructions in the preceding sections and $(m_n)_{n\in\mathbb{N}}$ be a sequence of natural numbers fulfilling $d_0\left(f^{m_n}, f_n^{m_n}\right) < \frac{1}{2^n}$. Furthermore, let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of partial partitions, where $\nu_n \to \varepsilon$ and for every $n \in \mathbb{N}$ ν_n is the image of a partial partition η_n under a measure-preserving diffeomorphism F_n, satisfying the following property: For every m-dimensional cube $A \subseteq \mathbb{T}^m$ and for every $\epsilon \in (0,1]$ there exists $N \in \mathbb{N}$ such that for every $n \ge N$ and for every $\Gamma_n \in \nu_n$ we have*

$$\left|\mu\left(\Gamma_n \cap f_n^{-m_n}(A)\right) - \mu(\Gamma_n)\cdot\mu(A)\right| \le \epsilon\cdot\mu(\Gamma_n)\cdot\mu(A). \tag{9}$$

Then f is weak mixing.

Proof. We want to show that the requirements of Lemma 5.1 are fulfilled. This implies that f is weak mixing. For it let $A \subseteq \mathbb{T}^m$ be an arbitrary m-dimensional cube and $\epsilon \in (0,1]$. We consider two m-dimensional cubes $A_1, A_2 \subset \mathbb{T}^m$ with $A_1 \subset A \subset A_2$ as well as $\mu(A \triangle A_i) < \epsilon \cdot \mu(A)$ and for sufficiently large n: $\operatorname{dist}(\partial A, \partial A_i) > \frac{1}{2^n}$ for $i = 1, 2$.

If n is sufficiently large, we obtain for $\Gamma_n \in \nu_n$ and for $i = 1, 2$ by the assumptions of this Lemma:

$$\left|\mu\left(\Gamma_n \cap f_n^{-m_n}(A_i)\right) - \mu(\Gamma_n) \cdot \mu(A_i)\right| \leq \epsilon \cdot \mu(\Gamma_n) \cdot \mu(A_i).$$

Herefrom we conclude $(1-\epsilon) \cdot \mu(\Gamma_n) \cdot \mu(A_1) \leq \mu(\Gamma_n \cap f_n^{-m_n}(A_1))$ on the one hand and $\mu(\Gamma_n \cap f_n^{-m_n}(A_2)) \leq (1+\epsilon) \cdot \mu(\Gamma_n) \cdot \mu(A_2)$ on the other hand. Because of $d_0(f^{m_n}, f_n^{m_n}) < \frac{1}{2^n}$ the following relations are true:

$$f_n^{m_n}(x) \in A_1 \Longrightarrow f^{m_n}(x) \in A,$$
$$f^{m_n}(x) \in A \Longrightarrow f_n^{m_n}(x) \in A_2.$$

Thus: $\mu(\Gamma_n \cap f_n^{-m_n}(A_1)) \leq \mu(\Gamma_n \cap f^{-m_n}(A)) \leq \mu(\Gamma_n \cap f_n^{-m_n}(A_2))$.
Altogether, it holds: $(1-\epsilon) \cdot \mu(\Gamma_n) \cdot \mu(A_1) \leq \mu(\Gamma_n \cap f^{-m_n}(A)) \leq (1+\epsilon) \cdot \mu(\Gamma_n) \cdot \mu(A_2)$.
Therewith, we obtain the following estimate from above:

$$\begin{aligned}
&\mu\left(\Gamma_n \cap f^{-m_n}(A)\right) - \mu(\Gamma_n) \cdot \mu(A) \\
&\leq (1+\epsilon) \cdot \mu(\Gamma_n) \cdot \mu(A_2) - \mu(\Gamma_n) \cdot \mu(A_2) + \mu(\Gamma_n) \cdot (\mu(A_2) - \mu(A)) \\
&\leq \epsilon \cdot \mu(\Gamma_n) \cdot \mu(A_2) + \mu(\Gamma_n) \cdot \mu(A_2 \triangle A) \leq \epsilon \cdot \mu(\Gamma_n) \cdot (\mu(A) + \mu(A_2 \triangle A)) + \epsilon \cdot \mu(\Gamma_n) \cdot \mu(A) \\
&\leq 2 \cdot \epsilon \cdot \mu(\Gamma_n) \cdot \mu(A) + \epsilon^2 \cdot \mu(\Gamma_n) \cdot \mu(A) \leq 3 \cdot \epsilon \cdot \mu(\Gamma_n) \cdot \mu(A).
\end{aligned}$$

Furthermore, we deduce the following estimate from below in an analogous way:

$$\mu\left(\Gamma_n \cap f^{-m_n}(A)\right) - \mu(\Gamma_n) \cdot \mu(A) \geq -3 \cdot \epsilon \cdot \mu(\Gamma_n) \cdot \mu(A).$$

Hence, we get: $|\mu(\Gamma_n \cap f^{-m_n}(A)) - \mu(\Gamma_n) \cdot \mu(A)| \leq 3 \cdot \epsilon \cdot \mu(\Gamma_n) \cdot \mu(A)$, i.e. the requirements of Lemma 5.1 are met. □

Now we concentrate on the setting of our explicit constructions:

Lemma 5.3. *Consider the sequence of partial partitions $(\eta_n)_{n\in\mathbb{N}}$ constructed in section 3.2.1 and the diffeomorphisms g_n from chapter 3.4. Furthermore, we define the partial partitions $\nu_n = \left\{\Gamma_n = H_{n-1} \circ g_n\left(\hat{I}_n\right) \ : \ \hat{I}_n \in \eta_n\right\}$.*
Then we get $\nu_n \to \varepsilon$.

Proof. By construction $\eta_n = \left\{\hat{I}_n^i : i \in \Lambda_n\right\}$, where Λ_n is a countable set of indices. Because of $\eta_n \to \varepsilon$ it holds $\lim_{n\to\infty} \mu\left(\bigcup_{i\in\Lambda_n} \hat{I}_n^i\right) = 1$. Since $H_{n-1} \circ g_n$ is measure-preserving, we conclude:

$$\lim_{n\to\infty} \mu\left(\bigcup_{i\in\Lambda_n} \Gamma_n^i\right) = \lim_{n\to\infty} \mu\left(\bigcup_{i\in\Lambda_n} H_{n-1} \circ g_n\left(\hat{I}_n^i\right)\right) = \lim_{n\to\infty} \mu\left(H_{n-1} \circ g_n\left(\bigcup_{i\in\Lambda_n} \hat{I}_n^i\right)\right) = 1.$$

Taking the approximation error of the map g_n into account, $g_n\left(\hat{I}_n\right)$ is contained in a cuboid with θ-width $\frac{1}{2q_n l_n} + \frac{[nq_n^\sigma]}{l_n}$ and edge length $\frac{1}{l_n} - 2\delta_n$ in the $r_1, ..., r_{m-1}$-coordinates. Hence, the diameter of $g_n\left(\hat{I}_n\right)$ is bounded by $\frac{m \cdot [nq_n^\sigma]}{l_n} + \frac{1}{2q_n \cdot l_n}$. Then, we conclude for every $\Gamma_n^i = H_{n-1} \circ g_n\left(\hat{I}_n^i\right)$:

$$\operatorname{diam}\left(\Gamma_n^i\right) \leq \|DH_{n-1}\|_0 \cdot \operatorname{diam}\left(g_n\left(\hat{I}_n^i\right)\right) \leq \|DH_{n-1}\|_0 \cdot \left(\frac{m \cdot [nq_n^\sigma]}{l_n} + \frac{1}{2q_n \cdot l_n}\right).$$

Because of $\sigma < 1$ and the requirement on l_n in equation 1 we conclude $\lim_{n\to\infty}\mathrm{diam}\left(\Gamma_n^i\right) = 0$ and consequently $\nu_n \to \varepsilon$. □

In the following the Lebesgue measures on $\mathbb{S}^1$, $\mathbb{T}^{m-2}$, $\mathbb{T}^{m-1}$ are denoted by $\tilde{\lambda}$, $\mu^{(m-2)}$ and $\tilde{\mu}$ respectively. The next technical result is needed in the proof of Lemma 5.5. For the sake of convenience, we introduce the notation $a = 10n^2 \cdot q_n \cdot l_n^{m+1}$.

Lemma 5.4. *Given an interval K on the r_1-axis and a $(m-2)$-dimensional interval Z in the $(r_2, ..., r_{m-1})$-coordinates $K_{c,\gamma}$ denotes the cuboid $[c, c+\gamma] \times K \times Z$ for some $\gamma > 0$. We consider the diffeomorphism g_n constructed in subsection 3.4 and an interval $L = [l_1, l_2]$ of $\mathbb{S}^1$ satisfying $\tilde{\lambda}(L) \geq \frac{4\cdot[nq_n^\sigma]}{a}$.*
If $[nq_n^\sigma] \cdot \lambda(K) > 2$, then for the set $Q := \pi_{\vec{r}}\left(K_{c,\gamma} \cap g_n^{-1}(L \times K \times Z)\right)$ we have:

$$\begin{aligned}&\left|\tilde{\mu}(Q) - \lambda(K) \cdot \tilde{\lambda}(L) \cdot \mu^{(m-2)}(Z)\right| \\ &\leq \left(\frac{2}{[nq_n^\sigma]} \cdot \tilde{\lambda}(L) + \frac{2\cdot\gamma}{[nq_n^\sigma]} + \gamma \cdot \lambda(K) + \frac{[nq_n^\sigma] \cdot \lambda(K) \cdot 4}{a} + \frac{8}{a}\right) \cdot \mu^{(m-2)}(Z).\end{aligned}$$

Proof. We consider the diffeomorphism $\bar{g}_b : M \to M$, $(\theta, r_1, ..., r_{m-1}) \mapsto (\theta + b \cdot r_1, r_1, ..., r_{m-1})$ and the set:

$$\begin{aligned}Q_b &:= \pi_{\vec{r}}\left(K_{c,\gamma} \cap \bar{g}_b^{-1}(L \times K \times Z)\right) \\ &= \{(r_1, r_2, ..., r_{m-1}) \in K \times Z \ : \ (\theta + b \cdot r_1, \vec{r}) \in L \times K \times Z, \theta \in [c, c+\gamma]\} \\ &= \{(r_1, r_2, ..., r_{m-1}) \in K \times Z \ : \ b \cdot r_1 \in [l_1 - c - \gamma, l_2 - c] \mod 1\}.\end{aligned}$$

The interval $b \cdot K$ seen as an interval in $\mathbb{R}$ does not intersect more than $b \cdot \lambda(K) + 2$ and not less than $b \cdot \lambda(K) - 2$ intervals of the form $[i, i+1]$ with $i \in \mathbb{Z}$.
Claim: A resulting interval on the r_1-axis of $K_{c,\gamma} \cap \bar{g}_{[nq_n^\sigma]}^{-1}(L \times K \times Z)$ and the corresponding r_1-projection of $K_{c,\gamma} \cap g_n^{-1}(L \times K \times Z)$ can differ by a length of at most $\frac{4}{a}$.
Proof: Recall that g_n is constructed as the approximation of the step function $\tilde{g}_n$. Obviously, $\tilde{g}_n(K_{c,\gamma})$ may hit (respectively leave) $L \times K \times Z$ at most one $\frac{1}{a}$-domain on the r_1-axis later than $\bar{g}_{[nq_n^\sigma]}(K_{c,\gamma})$ (see figure 3).
Moreover, the approximation error between g_n and $\tilde{g}_n$ can cause an additional deviation of at most one $\frac{1}{a}$-domain on the r_1-axis and can cause an additional deviation of at most $[nq_n^\sigma] \cdot \varepsilon_n$ on the θ-axis. Since $[nq_n^\sigma] \cdot \varepsilon_n < \frac{1}{a}$ this discrepancy will be equalised after at most one $\frac{1}{a}$-domain on the r_1-axis. This last difference can occur on both ends of the resulting interval on the r_1-axis. □

Therefore, we compute on the one side:

$$\begin{aligned}\tilde{\mu}(Q) &\leq ([nq_n^\sigma] \cdot \lambda(K) + 2) \cdot \left(\frac{l_2 - (l_1 - \gamma)}{[nq_n^\sigma]} + \frac{4}{a}\right) \cdot \mu^{(m-2)}(Z) \\ &= \left(\lambda(K) \cdot \tilde{\lambda}(L) + 2 \cdot \frac{\tilde{\lambda}(L)}{[nq_n^\sigma]} + \lambda(K) \cdot \gamma + \frac{2\cdot\gamma}{[nq_n^\sigma]} + \frac{[nq_n^\sigma] \cdot \lambda(K) \cdot 4}{a} + \frac{8}{a}\right) \cdot \mu^{(m-2)}(Z)\end{aligned}$$

and on the other side

$$\begin{aligned}\tilde{\mu}(Q) &\geq ([nq_n^\sigma] \cdot \lambda(K) - 2) \cdot \left(\frac{l_2 - (l_1 - \gamma)}{[nq_n^\sigma]} - \frac{4}{a}\right) \cdot \mu^{(m-2)}(Z) \\ &= \left(\lambda(K) \cdot \tilde{\lambda}(L) - 2 \cdot \frac{\tilde{\lambda}(L)}{[nq_n^\sigma]} + \lambda(K) \cdot \gamma - \frac{2\cdot\gamma}{[nq_n^\sigma]} - \frac{[nq_n^\sigma] \cdot \lambda(K) \cdot 4}{a} + \frac{8}{a}\right) \cdot \mu^{(m-2)}(Z).\end{aligned}$$

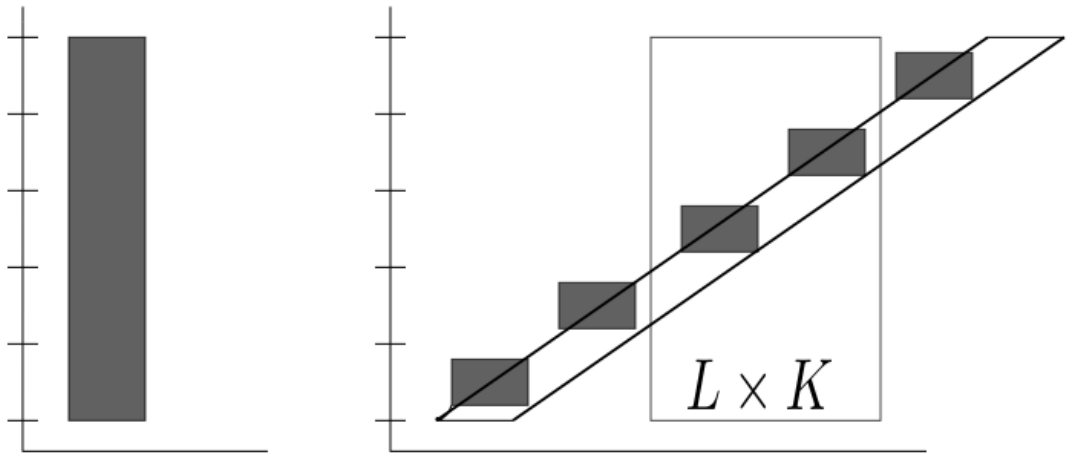

Figure 3: Qualitative shape of the action of g_n as well as $\tilde{g}_n$ on $K_{c,\gamma}$.

Both equations together yield:

$$\left|\tilde{\mu}\left(Q\right) - \lambda\left(K\right)\cdot\tilde{\lambda}\left(L\right)\cdot\mu^{(m-2)}\left(Z\right) - \gamma\cdot\lambda\left(K\right)\cdot\mu^{(m-2)}\left(Z\right) - \frac{8}{a}\cdot\mu^{(m-2)}\left(Z\right)\right|$$
$$\leq\left(\frac{2}{[nq_n^{\sigma}]}\cdot\tilde{\lambda}\left(L\right) + \frac{2\cdot\gamma}{[nq_n^{\sigma}]} + \frac{[nq_n^{\sigma}]\cdot\lambda(K)\cdot 4}{a}\right)\cdot\mu^{(m-2)}\left(Z\right).$$

The claim follows because

$$\left|\tilde{\mu}\left(Q\right) - \lambda\left(K\right)\cdot\tilde{\lambda}\left(L\right)\cdot\mu^{(m-2)}\left(Z\right)\right| - \gamma\cdot\lambda\left(K\right)\cdot\mu^{(m-2)}\left(Z\right) - \frac{8}{a}\cdot\mu^{(m-2)}\left(Z\right)$$
$$\leq\left|\tilde{\mu}\left(Q\right) - \lambda\left(K\right)\cdot\tilde{\lambda}\left(L\right)\cdot\mu^{(m-2)}\left(Z\right) - \gamma\cdot\lambda\left(K\right)\cdot\mu^{(m-2)}\left(Z\right) - \frac{8}{a}\cdot\mu^{(m-2)}\left(Z\right)\right|.$$

□

Lemma 5.5. *Let $n \geq 5$, g_n as in section 3.4 and $\hat{I}_n \in \eta_n$, where η_n is the partial partition constructed in section 3.2.1. For the diffeomorphism ϕ_n constructed in section 3.3 and m_n as in section 3.1 we consider $\Phi_n = \phi_n \circ R_{\alpha_{n+1}}^{m_n} \circ \phi_n^{-1}$ and $J_n \subset \mathbb{T}^{m-1}$ defined in equation 6.*
Then for every m-dimensional cube S of side length $q_n^{-\sigma}$ lying in $\mathbb{T}^m$ we get

$$\left|\mu\left(\hat{I}\cap\Phi_n^{-1}\circ g_n^{-1}\left(S\right)\right)\cdot\tilde{\mu}\left(J_n\right) - \mu\left(\hat{I}\right)\cdot\mu\left(S\right)\right| \leq \frac{22}{n}\cdot\mu\left(\hat{I}\right)\cdot\mu\left(S\right). \tag{10}$$

In other words this Lemma tells us that a partition element is "almost uniformly distributed" under $g_n \circ \Phi_n$ on the whole manifold $M = \mathbb{T}^m$.

Proof. Let S be a m-dimensional cube with sidelength $q_n^{-\sigma}$ lying in $\mathbb{T}^m$. Furthermore, we denote:

$$S_\theta = \pi_\theta\left(S\right) \qquad S_{r_1} = \pi_{r_1}\left(S\right) \qquad S_{\vec{r}} = \pi_{(r_2,\ldots,r_{m-1})}\left(S\right) \qquad S_r = S_{r_1}\times S_{\vec{r}} = \pi_{\vec{r}}\left(S\right)$$

Obviously: $\tilde{\lambda}(S_\theta) = \lambda(S_{r_1}) = q_n^{-\sigma}$ and $\tilde{\lambda}(S_\theta) \cdot \lambda(S_{r_1}) \cdot \mu^{(m-2)}\left(S_{\tilde{\vec{r}}}\right) = \mu(S) = q_n^{-m\sigma}$.
According to Lemma 4.3 $\Phi_n\left(\frac{3}{q_n \cdot l_n}, \frac{1}{n}\right)$-distributes the partition element $\hat{I} \in \eta_n$ on J_n, in particular $\Phi_n\left(\hat{I}\right) \subseteq [c, c+\gamma] \times \mathbb{T}^{m-1}$ for some $c \in \mathbb{S}^1$ and some $\gamma \leq \frac{3}{q_n \cdot l_n}$.
We introduce the set $\tilde{S}_r := S_r \cap J_n$ and therewith $\tilde{S} := S_\theta \times \tilde{S}_r$. In order to estimate $\mu\left(S \setminus \tilde{S}\right)$ we observe that in each coordinate $r_1, ..., r_{m-1}$ there is a "bad domain" of ϕ_n of length $2\delta_n + 4\varepsilon_n$ in each $\frac{1}{l_n}$-domain. Hence, S_r contains at most $(l_n \cdot q_n^{-\sigma} + 2)^{m-1}$ "bad domains" of measure $\frac{2\delta_n + 4\varepsilon_n}{l_n^{m-2}}$ in $\mathbb{T}^{m-1}$. Then:

$$\mu\left(S \setminus \tilde{S}\right) \leq \frac{2\delta_n + 4\varepsilon_n}{l_n^{m-2}} \cdot \left(l_n \cdot q_n^{-\sigma} + 2\right)^{m-1} \cdot q_n^{-\sigma} \leq (4\delta_n + 8\varepsilon_n) \cdot l_n \cdot \mu(S) < 5\delta_n \cdot l_n \cdot \mu(S).$$

Using the triangle inequality we obtain

$$\begin{aligned}
&\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(g_n^{-1}(S)\right)\right) \cdot \tilde{\mu}(J_n) - \mu\left(\hat{I}\right) \cdot \mu(S)\right| \\
\leq &\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(g_n^{-1}(S)\right)\right) - \mu\left(\hat{I} \cap \Phi_n^{-1}\left(g_n^{-1}\left(\tilde{S}\right)\right)\right)\right| \cdot \tilde{\mu}(J_n) \\
&+ \left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(g_n^{-1}\left(\tilde{S}\right)\right)\right) \cdot \tilde{\mu}(J_n) - \mu\left(\hat{I}\right) \cdot \mu\left(\tilde{S}\right)\right| + \mu\left(\hat{I}\right) \cdot \left|\mu\left(\tilde{S}\right) - \mu(S)\right|.
\end{aligned}$$

Since Φ_n and g_n are measure-preserving, we observe by our choice of δ_n in equation 2:

$$\begin{aligned}
&\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(g_n^{-1}(S)\right)\right) - \mu\left(\hat{I} \cap \Phi_n^{-1}\left(g_n^{-1}\left(\tilde{S}\right)\right)\right)\right| \cdot \tilde{\mu}(J_n) \leq \mu\left(S \setminus \tilde{S}\right) \cdot \tilde{\mu}(J_n) \\
&\leq 5\delta_n \cdot l_n \cdot \mu(S) \cdot \tilde{\mu}(J_n) \leq \frac{1}{n} \cdot \mu(S) \cdot \mu\left(\hat{I}\right).
\end{aligned}$$

Thus, we obtain:

$$\begin{aligned}
&\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(g_n^{-1}(S)\right)\right) \cdot \tilde{\mu}(J_n) - \mu\left(\hat{I}\right) \cdot \mu(S)\right| \\
&\leq \left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(g_n^{-1}\left(\tilde{S}\right)\right)\right) \cdot \tilde{\mu}(J_n) - \mu\left(\hat{I}\right) \cdot \mu\left(\tilde{S}\right)\right| + \frac{2}{n} \cdot \mu(S) \cdot \mu\left(\hat{I}\right).
\end{aligned} \tag{11}$$

Next, we want to estimate the first summand. By construction of the map g_n and the definition of $\tilde{S}$ it holds: $\Phi_n\left(\hat{I}\right) \cap g_n^{-1}\left(\tilde{S}\right) \subseteq [c, c+\gamma] \times \tilde{S}_r =: K_{c,\gamma}$. Because of Lemma 4.3 we have $2\gamma \leq \frac{6}{q_n \cdot l_n} < q_n^{-\sigma}$. So we can define a cuboid $S_1 \subseteq \tilde{S}$, where $S_1 := [s_1 + \gamma, s_2 - \gamma] \times \tilde{S}_r$ using the notation $S_\theta = [s_1, s_2]$. We examine the two sets

$$Q := \pi_{\vec{r}}\left(K_{c,\gamma} \cap g_n^{-1}\left(S_\theta \times \tilde{S}_r\right)\right) \qquad Q_1 := \pi_{\vec{r}}\left(K_{c,\gamma} \cap g_n^{-1}\left([s_1 + \gamma, s_2 - \gamma] \times \tilde{S}_r\right)\right).$$

As seen above $\Phi_n\left(\hat{I}\right) \cap g_n^{-1}\left(\tilde{S}\right) \subseteq K_{c,\gamma}$. Hence $\Phi_n\left(\hat{I}\right) \cap g_n^{-1}\left(\tilde{S}\right) \subseteq \Phi_n\left(\hat{I}\right) \cap g_n^{-1}\left(\tilde{S}\right) \cap K_{c,\gamma}$, which implies $\Phi_n\left(\hat{I}\right) \cap g_n^{-1}\left(\tilde{S}\right) \subseteq \Phi_n\left(\hat{I}\right) \cap \left(\mathbb{S}^1 \times Q\right)$.
Claim: On the other hand: $\Phi_n\left(\hat{I}\right) \cap \left(\mathbb{S}^1 \times Q_1\right) \subseteq \Phi_n\left(\hat{I}\right) \cap g_n^{-1}\left(\tilde{S}\right)$.
Proof of the claim: For $(\theta, \vec{r}) \in \Phi_n\left(\hat{I}\right) \cap \left(\mathbb{S}^1 \times Q_1\right)$ arbitrary it holds $(\theta, \vec{r}) \in \Phi_n\left(\hat{I}\right)$, i.e. $\theta \in [c, c+\gamma]$, and $\vec{r} \in \pi_{\vec{r}}\left(K_{c,\gamma} \cap g_n^{-1}\left([s_1 + \gamma, s_2 - \gamma] \times \tilde{S}_r\right)\right)$. This implies the existence of $\bar{\theta} \in [c, c+\gamma]$ satisfying $\left(\bar{\theta}, \vec{r}\right) \in K_{c,\gamma} \cap g_n^{-1}(S_1)$. Hence, there is $\beta \in [s_1 + \gamma, s_2 - \gamma]$ such that $g_n\left(\bar{\theta}, \vec{r}\right) = (\beta, \vec{r})$. Additionally, we observe that g_n maps sets of the form $I \times \vec{r}$, where

$I \subset \mathbb{S}^1$ is an interval, on a set of the form $\tilde{I} \times \vec{r}$ with an interval $\tilde{I} \subset \mathbb{S}^1$ and preserves the length of the interval. Since $\left|\theta - \bar{\theta}\right| \leq \gamma$ there is $\bar{\beta} \in [s_1, s_2]$ satisfying $g_n(\theta, \vec{r}) = \left(\bar{\beta}, \vec{r}\right)$. Thus, $(\theta, \vec{r}) \in \Phi_n\left(\hat{I}\right) \cap g_n^{-1}\left(\tilde{S}\right)$. □

Altogether, the following inclusions are true:

$$\Phi_n\left(\hat{I}\right) \cap \left(\mathbb{S}^1 \times Q_1\right) \subseteq \Phi_n\left(\hat{I}\right) \cap g_n^{-1}\left(\tilde{S}\right) \subseteq \Phi_n\left(\hat{I}\right) \cap \left(\mathbb{S}^1 \times Q\right)$$

Thus, we obtain:

$$\begin{aligned} &\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(g_n^{-1}\left(\tilde{S}\right)\right)\right) \cdot \tilde{\mu}\left(J_n\right) - \mu\left(\hat{I}\right) \cdot \mu\left(\tilde{S}\right)\right| \\ \leq \max\Bigg(&\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(\mathbb{S}^1 \times Q\right)\right) \cdot \tilde{\mu}\left(J_n\right) - \mu\left(\hat{I}\right) \cdot \mu\left(\tilde{S}\right)\right|, \\ &\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(\mathbb{S}^1 \times Q_1\right)\right) \cdot \tilde{\mu}\left(J_n\right) - \mu\left(\hat{I}\right) \cdot \mu\left(\tilde{S}\right)\right| \Bigg) \end{aligned} \tag{12}$$

We want to apply Lemma 5.4 for $K = \tilde{S}_{r_1}$, $L = S_\theta$, $Z = S_{\tilde{\vec{r}}}$ and $b = [n \cdot q_n^\sigma]$ (note that $\frac{4 \cdot [nq_n^\sigma]}{10n^2 \cdot q_n \cdot l_n^{m+1}} < \frac{1}{q_n^\sigma} = \tilde{\lambda}(L)$ and for $n > 4$: $b \cdot \lambda(K) = [nq_n^\sigma] \cdot q_n^{-\sigma} \geq \frac{1}{2} nq_n^\sigma \cdot q_n^{-\sigma} > 2$):

$$\begin{aligned} &\left|\tilde{\mu}(Q) - \mu\left(\tilde{S}\right)\right| \\ &\leq \left(\frac{2}{[n \cdot q_n^\sigma]} \cdot \tilde{\lambda}(S_\theta) + \frac{2\gamma}{[n \cdot q_n^\sigma]} + \gamma \cdot \lambda\left(\tilde{S}_{r_1}\right) + \frac{[nq_n^\sigma] \cdot \lambda\left(\tilde{S}_{r_1}\right) \cdot 4}{a} + \frac{8}{a}\right) \cdot \mu^{(m-2)}\left(S_{\tilde{\vec{r}}}\right) \\ &\leq \left(\frac{4}{n \cdot q_n^\sigma} \cdot \tilde{\lambda}(S_\theta) + \frac{4}{n \cdot q_n^\sigma \cdot q_n^\sigma} + \frac{1}{n \cdot q_n^\sigma} \cdot \lambda(S_{r_1}) + \frac{1}{n \cdot q_n^{2\sigma}}\right) \cdot \mu^{(m-2)}\left(S_{\tilde{\vec{r}}}\right) \\ &\leq \frac{14}{n} \cdot \mu(S). \end{aligned}$$

In particular, we receive from this estimate: $\frac{14}{n} \cdot \mu(S) \geq \tilde{\mu}(Q) - \mu\left(\tilde{S}\right) \geq \tilde{\mu}(Q) - \mu(S)$, hence: $\tilde{\mu}(Q) \leq \left(1 + \frac{14}{n}\right) \cdot \mu(S) \leq 4 \cdot \mu(S)$.
Analogously we obtain: $\tilde{\mu}(Q_1) \leq 4 \cdot \mu(S)$ as well as $|\tilde{\mu}(Q_1) - \mu(S_1)| \leq \frac{14}{n} \cdot \mu(S)$.
Since Q as well as Q_1 are a finite union of disjoint $(m-1)$-dimensional intervals contained in J_n and $\Phi_n\left(\frac{3}{q_n \cdot l_n}, \frac{1}{n}\right)$-distributes the interval $\hat{I}$ on J_n, we get:

$$\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(\mathbb{S}^1 \times Q\right)\right) \cdot \tilde{\mu}\left(J_n\right) - \mu\left(\hat{I}\right) \cdot \tilde{\mu}(Q)\right| \leq \frac{1}{n} \cdot \mu\left(\hat{I}\right) \cdot \tilde{\mu}(Q) \leq \frac{4}{n} \cdot \mu\left(\hat{I}\right) \cdot \mu(S)$$

as well as

$$\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(\mathbb{S}^1 \times Q_1\right)\right) \cdot \tilde{\mu}\left(J_n\right) - \mu\left(\hat{I}\right) \cdot \tilde{\mu}(Q_1)\right| \leq \frac{1}{n} \cdot \mu\left(\hat{I}\right) \cdot \tilde{\mu}(Q_1) \leq \frac{4}{n} \cdot \mu\left(\hat{I}\right) \cdot \mu(S).$$

Now we can proceed

$$\begin{aligned} &\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(\mathbb{S}^1 \times Q\right)\right) \cdot \tilde{\mu}\left(J_n\right) - \mu\left(\hat{I}\right) \cdot \mu\left(\tilde{S}\right)\right| \\ &\leq \left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(\mathbb{S}^1 \times Q\right)\right) \cdot \tilde{\mu}\left(J_n\right) - \mu\left(\hat{I}\right) \cdot \tilde{\mu}(Q)\right| + \mu\left(\hat{I}\right) \cdot \left|\tilde{\mu}(Q) - \mu\left(\tilde{S}\right)\right| \\ &\leq \frac{4}{n} \cdot \mu\left(\hat{I}\right) \cdot \mu(S) + \mu\left(\hat{I}\right) \cdot \frac{14}{n} \cdot \mu(S) = \frac{18}{n} \cdot \mu\left(\hat{I}\right) \cdot \mu(S). \end{aligned}$$

Noting that $\mu(S_1) = \mu\left(\tilde{S}\right) - 2\gamma \cdot \tilde{\mu}\left(\tilde{S}_r\right)$ and so $\mu\left(\tilde{S}\right) - \mu(S_1) \leq 2 \cdot \frac{1}{n \cdot q_n^{\sigma}} \cdot \tilde{\mu}\left(\tilde{S}_r\right) \leq \frac{2}{n} \cdot \mu(S)$ we obtain in the same way as above:

$$\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(\mathbb{S}^1 \times Q_1\right)\right) \cdot \tilde{\mu}(J_n) - \mu\left(\hat{I}\right) \cdot \mu\left(\tilde{S}\right)\right| \leq \frac{20}{n} \cdot \mu\left(\hat{I}\right) \cdot \mu(S).$$

Using equation 12 this yields:

$$\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(g_n^{-1}\left(\tilde{S}\right)\right)\right) \cdot \tilde{\mu}(J_n) - \mu\left(\hat{I}\right) \cdot \mu\left(\tilde{S}\right)\right| \leq \frac{20}{n} \cdot \mu\left(\hat{I}\right) \cdot \mu(S).$$

Finally, we conclude with the aid of equation 11:

$$\left|\mu\left(\hat{I} \cap \Phi_n^{-1}\left(g_n^{-1}(S)\right)\right) \cdot \tilde{\mu}(J_n) - \mu\left(\hat{I}\right) \cdot \mu(S)\right| \leq \frac{22}{n} \cdot \mu\left(\hat{I}\right) \cdot \mu(S).$$

□

Now we are able to prove the aimed criterion for weak mixing.

Proposition 5.6 (Criterion for weak mixing)**.** *Let $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ and the sequence $(m_n)_{n\in\mathbb{N}}$ be constructed as in the previous sections. Suppose additionally that $d_0\left(f^{m_n}, f_n^{m_n}\right) < \frac{1}{2^n}$ and $\|DH_{n-1}\|_0 < \ln(q_n)$ for every $n \in \mathbb{N}$ and that the limit $f = \lim_{n\to\infty} f_n$ exists.*
Then f is weak mixing.

Proof. To apply Lemma 5.2 we consider the partial partitions $\nu_n := H_{n-1} \circ g_n(\eta_n)$. As proven in Lemma 5.3 these partial partitions satisfy $\nu_n \to \varepsilon$. We have to establish equation 9. For this purpose, let $\varepsilon > 0$ and a m-dimensional cube $A \subseteq \mathbb{T}^m$ be given.
Furthermore, we note $f_n^{m_n} = H_n \circ R_{\alpha_{n+1}}^{m_n} \circ H_n^{-1} = H_{n-1} \circ g_n \circ \Phi_n \circ g_n^{-1} \circ H_{n-1}^{-1}$.
Let S_n be a m-dimensional cube of side length $q_n^{-\sigma}$ contained in $\mathbb{T}^m$. We look at $C_n := H_{n-1}(S_n)$, $\Gamma_n \in \nu_n$, and compute (since g_n and H_{n-1} are measure-preserving):

$$\begin{aligned}
&\left|\mu\left(\Gamma_n \cap f_n^{-m_n}(C_n)\right) - \mu(\Gamma_n) \cdot \mu(C_n)\right| = \left|\mu\left(\hat{I}_n \cap \Phi_n^{-1} \circ g_n^{-1}(S_n)\right) - \mu\left(\hat{I}_n\right) \cdot \mu(S_n)\right| \\
&\leq \frac{1}{\tilde{\mu}(J_n)} \cdot \left|\mu\left(\hat{I}_n \cap \Phi_n^{-1} \circ g_n^{-1}(S_n)\right) \cdot \tilde{\mu}(J_n) - \mu\left(\hat{I}_n\right) \cdot \mu(S_n)\right| + \frac{1 - \tilde{\mu}(J_n)}{\tilde{\mu}(J_n)} \cdot \mu\left(\hat{I}_n\right) \cdot \mu(S_n)
\end{aligned}$$

Bernoulli's inequality yields: $\tilde{\mu}(J_n) \geq \left(1 - \frac{1}{n}\right)^{m-1} \geq 1 + (m-1) \cdot \left(-\frac{1}{n}\right) = 1 - \frac{m-1}{n}$. Hence, we obtain for $n > 2 \cdot (m-1)$: $\tilde{\mu}(J_n) \geq \frac{1}{2}$ and so: $\frac{1-\tilde{\mu}(J_n)}{\tilde{\mu}(J_n)} \leq 2 \cdot (1 - \tilde{\mu}(J_n)) \leq \frac{2\cdot(m-1)}{n}$. We continue by applying Lemma 5.5:

$$\begin{aligned}
\left|\mu\left(\Gamma_n \cap f_n^{-m_n}(C_n)\right) - \mu(\Gamma_n) \cdot \mu(C_n)\right| &\leq 2 \cdot \frac{22}{n} \cdot \mu\left(\hat{I}_n\right) \cdot \mu(S_n) + \frac{2 \cdot (m-1)}{n} \cdot \mu\left(\hat{I}_n\right) \cdot \mu(S_n) \\
&= \frac{42 + 2 \cdot m}{n} \cdot \mu\left(\hat{I}_n\right) \cdot \mu(S_n)
\end{aligned}$$

Moreover, by our assumptions it holds $\operatorname{diam}(C_n) \leq \|DH_{n-1}\|_0 \cdot \operatorname{diam}(S_n) \leq \ln(q_n) \cdot \frac{\sqrt{m}}{q_n^{\sigma}}$, i. e. $\operatorname{diam}(C_n) \to 0$ as $n \to \infty$. Thus, we can approximate A by a countable disjoint union of sets $C_n = H_{n-1}(S_n)$ with $S_n \subseteq \mathbb{T}^m$ a m-dimensional cube of sidelength $q_n^{-\sigma}$ in given precision, when n is chosen large enough. Consequently for n sufficiently large there are sets $A_1 = \dot{\bigcup}_{i\in\Sigma_n^1} C_n^i$ and $A_2 = \dot{\bigcup}_{i\in\Sigma_n^2} C_n^i$ with countable sets Σ_n^1 and Σ_n^2 of indices satisfying $A_1 \subseteq A \subseteq A_2$ as well as $|\mu(A) - \mu(A_i)| \leq \frac{\epsilon}{3} \cdot \mu(A)$ for $i = 1, 2$.
Additionally we choose n such that $\frac{42+2\cdot m}{n} < \frac{\epsilon}{3}$ holds. It follows:

$$
\begin{aligned}
&\mu\left(\Gamma_n \cap f_n^{-m_n}(A)\right) - \mu(\Gamma_n)\cdot\mu(A)\\
&\leq \mu\left(\Gamma_n \cap f_n^{-m_n}(A_2)\right) - \mu(\Gamma_n)\cdot\mu(A_2) + \mu(\Gamma_n)\cdot(\mu(A_2)-\mu(A))\\
&\leq \sum_{i\in\Sigma_n^2}\left(\mu\left(\Gamma_n \cap f_n^{-m_n}\left(C_n^i\right)\right) - \mu(\Gamma_n)\cdot\mu\left(C_n^i\right)\right) + \frac{\epsilon}{3}\cdot\mu(\Gamma_n)\cdot\mu(A)\\
&\leq \sum_{i\in\Sigma_n^2}\left(\frac{42+2\cdot m}{n}\cdot\mu\left(\hat{I}_n\right)\cdot\mu\left(S_n^i\right)\right) + \frac{\epsilon}{3}\cdot\mu(\Gamma_n)\cdot\mu(A)\\
&= \frac{42+2\cdot m}{n}\cdot\mu(\Gamma_n)\cdot\mu\left(\bigcup_{i\in\Sigma_n^2} C_n^i\right) + \frac{\epsilon}{3}\cdot\mu(\Gamma_n)\cdot\mu(A) \leq \frac{\epsilon}{3}\cdot\mu(\Gamma_n)\cdot\mu(A_2) + \frac{\epsilon}{3}\cdot\mu(\Gamma_n)\cdot\mu(A)\\
&= \frac{\epsilon}{3}\cdot\mu(\Gamma_n)\cdot\mu(A) + \frac{\epsilon}{3}\cdot\mu(\Gamma_n)\cdot(\mu(A_2)-\mu(A)) + \frac{\epsilon}{3}\cdot\mu(\Gamma_n)\cdot\mu(A) \leq \epsilon\cdot\mu(\Gamma_n)\cdot\mu(A).
\end{aligned}
$$

Analogously we estimate: $\mu(\Gamma_n \cap f_n^{-m_n}(A)) - \mu(\Gamma_n)\cdot\mu(A) \geq -\epsilon\cdot\mu(\Gamma_n)\cdot\mu(A)$. Both estimates enable us to conclude: $|\mu(\Gamma_n \cap f_n^{-m_n}(A)) - \mu(\Gamma_n)\cdot\mu(A)| \leq \epsilon\cdot\mu(\Gamma_n)\cdot\mu(A)$. □

6 Proof of convergence of $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{Diff}^{\omega}_{\rho}(\mathbb{T}^m,\mu)$

Let $\varepsilon > 0$ and $(\epsilon_n)_{n\in\mathbb{N}}$ be a monotone decreasing sequence of positive real numbers satisfying $\sum_{n=1}^{\infty}\epsilon_n < \varepsilon$. We recall the relations $\alpha_{n+1} = \alpha_n + \frac{1}{k_n\cdot l_n\cdot q_n}$ and $h_n \circ R_{\alpha_n} = R_{\alpha_n}\circ h_n$. Hereby, we observe for any $m\in\mathbb{N}$

$$H_n\circ R^m_{\alpha_{n+1}}\circ H_n^{-1} = H_{n-1}\circ h_n\circ R^m_{\alpha_n}\circ R^m_{\frac{1}{k_n\cdot l_n\cdot q_n}}\circ h_n^{-1}\circ H_{n-1}^{-1} = H_{n-1}\circ R^m_{\alpha_n}\circ h_n\circ R_{\frac{m}{k_n\cdot l_n\cdot q_n}}\circ h_n^{-1}\circ H_{n-1}^{-1}.$$

Since the construction of the conjugation map h_n was independent of the number k_n, we can obtain

$$d_\rho(f_{n-1},f_n) = d_\rho\left(H_{n-1}\circ R_{\alpha_n}\circ H_{n-1}^{-1}, H_{n-1}\circ R_{\alpha_n}\circ h_n\circ R_{\frac{1}{k_n\cdot l_n\cdot q_n}}\circ h_n^{-1}\circ H_{n-1}^{-1}\right) < \epsilon_n$$

as well as for every $m \leq q_n$

$$d_0\left(f_{n-1}^m, f_n^m\right) = d_0\left(H_{n-1}\circ R^m_{\alpha_n}\circ H_{n-1}^{-1}, H_{n-1}\circ R^m_{\alpha_n}\circ h_n\circ R_{\frac{m}{k_n\cdot l_n\cdot q_n}}\circ h_n^{-1}\circ H_{n-1}^{-1}\right) < \frac{1}{2^n}$$

by choosing $k_n\in\mathbb{N}$ large enough under the additional conditions

$$k_n > 40n^2\cdot l_n^m \tag{13}$$

and

$$\ln(q_{n+1}) = \ln(k_n\cdot l_n\cdot q_n) > \|DH_n\|_0. \tag{14}$$

By

$$d_\rho(f_m,f_n) \leq \sum_{k=n+1}^{m} d_\rho(f_{k-1},f_k) < \sum_{k=n+1}^{m}\epsilon_k$$

we can show that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{Diff}^{\omega}_{\rho}(\mathbb{T}^m,\mu)$. Since $\mathrm{Diff}^{\omega}_{\rho}(\mathbb{T}^m,\mu)$ is a complete space, we obtain convergence $\lim_{n\to\infty} f_n = f \in \mathrm{Diff}^{\omega}_{\rho}(\mathbb{T}^m,\mu)$.

Remark 6.1. Moreover, we estimate for every $m \leq q_{n+1}$:

$$d_0\left(f^m, f_n^m\right) \leq \sum_{k=n+1}^{\infty} d_0\left(f_{k-1}^m, f_k^m\right) < \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^n}.$$

By construction of the sequence $(m_n)_{n\in\mathbb{N}}$ in subsection 3.1 we have $m_n \leq q_{n+1}$. Hence, the condition $d_0\left(f^{m_n}, f_n^{m_n}\right) < \frac{1}{2^n}$ from Proposition 5.6 is satisfied as well. Then we can apply the deduced criterion for weak mixing and conclude that f is weak mixing.

7 Construction of the f-invariant measurable Riemannian metric

In the following we construct the f-invariant measurable Riemannian metric. This construction parallels the approach in [GKa00], section 4.8.. For it we put $\omega_n := \left(H_n^{-1}\right)^* \omega_0$, where ω_0 is the standard Riemannian metric on $\mathbb{T}^m$. Each ω_n is a smooth Riemannian metric because it is the pullback of a smooth metric via a $\mathrm{Diff}_\rho^\omega\left(\mathbb{T}^m, \mu\right)$-diffeomorphism. Since $R_{\alpha_{n+1}}^* \omega_0 = \omega_0$ the metric ω_n is f_n-invariant:

$$\begin{aligned} f_n^* \omega_n &= \left(H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}\right)^* \left(H_n^{-1}\right)^* \omega_0 = \left(H_n^{-1}\right)^* R_{\alpha_{n+1}}^* H_n^* \left(H_n^{-1}\right)^* \omega_0 = \left(H_n^{-1}\right)^* R_{\alpha_{n+1}}^* \omega_0 \\ &= \left(H_n^{-1}\right)^* \omega_0 = \omega_n. \end{aligned}$$

With the succeeding Lemmas we show that the limit $\omega_\infty := \lim_{n\to\infty} \omega_n$ exists μ-almost everywhere and is the aimed f-invariant Riemannian metric.

Lemma 7.1. *On any partition element $\check{I}_n \in \zeta_n$ we have $dev_{\check{I}_n}\left(h_n\right) < \frac{\delta_n}{l_n^2}$.*

Proof. First of all, we observe for a vector $\vec{v} = (v_1, ..., v_m)$ with $\|v\| = 1$ and for maps of the form $J\left(x_1, ..., x_m\right) = \left(x_1, ..., x_{d-1}, x_d + s\left(x_j\right), x_{d+1}, ..., x_m\right)$ with $\sup_x |s'(x)| < \varepsilon < 1$:

$$\|DJ\left(\vec{v}\right)\| \leq \sqrt{1 + 2\varepsilon \cdot v_d \cdot v_j + \varepsilon^2 \cdot v_d^2} \leq 1 + \frac{1}{2} \cdot \left(2\varepsilon + \varepsilon^2\right) < 1 + 2\varepsilon.$$

Then we have $\log \|DJ\left(\vec{v}\right)\| < 2\varepsilon$.
By the exact positioning of the partition elements and Remark 3.7 every occurring conjugation map is applied on a domain, where the associated step function $s_{\beta,N,\varepsilon,\delta}$ satisfies $\left|s'_{\beta,N,\varepsilon,\delta}\right| < \varepsilon$. With the aid of Remark 2.5 and the above observations we obtain

$$\mathrm{dev}_{\check{I}_n}\left(h_n\right) \leq \mathrm{dev}_{\phi_n\left(\check{I}_n\right)}\left(g_n\right) + \mathrm{dev}_{\check{I}_n}\left(\phi_n\right) \leq 2 \cdot \left[n q_n^\sigma\right] \cdot \varepsilon_n + 3 \cdot (m-1) \cdot 2\varepsilon_n.$$

By our choice of ε_n in equation 3 we proved the claim. □

Lemma 7.2. *The sequence $(\omega_n)_{n\in\mathbb{N}}$ converges μ-a.e. to a limit ω_∞.*

Proof. On the union of the partition elements of ζ_n we conclude by Lemma 7.1:

$$\begin{aligned} d\left(\omega_n, \omega_{n-1}\right) &= d\left(\left(h_n^{-1} \circ H_{n-1}^{-1}\right)^* \omega_0, \left(H_{n-1}^{-1}\right)^* \omega_0\right) \leq \left\|H_{n-1}^*\right\| \cdot d\left(\left(h_n^{-1}\right)^* \omega_0, \omega_0\right) \\ &\leq \|DH_{n-1}\|_0^2 \cdot \frac{\delta_n}{l_n^2} < \delta_n. \end{aligned}$$

Since the elements of the partition ζ_n cover $\mathbb{T}^m$ except a set of measure at most $\frac{1}{n^2}$ by Remark 3.5 for every $n \geq 3$, this calculation shows $d(\omega_{N+k}, \omega_{N-1}) \leq \sum_{n=N}^{N+k} d(\omega_n, \omega_{n-1}) < \sum_{n=N}^{N+k} \delta_n$ on a set of measure at least $1 - \sum_{n=N}^{N+k} \frac{1}{n^2} \geq 1 - \sum_{n=N}^{\infty} \frac{1}{n^2}$. As this measure approaches 1 for $N \to \infty$, the sequence $(\omega_n)_{n\in\mathbb{N}}$ converges on a set of full measure. □

Lemma 7.3. *The limit ω_∞ is a measurable Riemannian metric.*

Proof. The limit ω_∞ is a measurable map because it is the pointwise limit of the smooth metrics ω_n, which in particular are measurable. By the same reasoning $\omega_\infty|_p$ is symmetric for μ-almost every $p \in M$. Furthermore, ω_n is positive definite for every $n \in \mathbb{N}$ and ω_∞ is $\sum_{k=n}^{\infty} \delta_k$-close to ω_{n-1} on $T_1M \otimes T_1M$ minus a set of measure at most $\sum_{k=n}^{\infty} \frac{1}{k^2}$. By choosing δ_k, $k \geq n$, small enough (depending on ω_{n-1}),

(A) which can be satisfied by choosing l_n large enough,

we can guarantee that ω_∞ is positive definite on $T_1M \otimes T_1M$ minus a set of measure at most $\sum_{k=n}^{\infty} \frac{1}{k^2}$. Since this is true for every $n \in \mathbb{N}$, ω_∞ is positive definite on a set of full measure. □

Remark 7.4. In the proof of the subsequent Lemma we will need Egoroff's theorem (for example [Ha65], §21, Theorem A): Let (N, d) denote a separable metric space. Given a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of N-valued measurable functions on a measure space (X, Σ, μ) and a measurable subset $A \subseteq X$, $\mu(A) < \infty$, such that $(\varphi_n)_{n\in\mathbb{N}}$ converges μ-a.e. on A to a limit function φ. Then for every $\varepsilon > 0$ there exists a measurable subset $B \subset A$ such that $\mu(B) < \varepsilon$ and $(\varphi_n)_{n\in\mathbb{N}}$ converges to φ uniformly on $A \setminus B$.

Lemma 7.5. *ω_∞ is f-invariant, i.e. $f^*\omega_\infty = \omega_\infty$ μ-a.e..*

Proof. By Lemma 7.2 the sequence $(\omega_n)_{n\in\mathbb{N}}$ converges in the C^∞-topology pointwise almost everywhere. Hence, we obtain using Egoroff's theorem: For every $\delta > 0$ there is a set $C_\delta \subseteq M$ such that $\mu(M \setminus C_\delta) < \delta$ and the convergence $\omega_n \to \omega_\infty$ is uniform on C_δ.
The function f was constructed as the limit of the sequence $(f_n)_{n\in\mathbb{N}}$ in the $\mathrm{Diff}^\omega_\rho(\mathbb{T}^m, \mu)$-topology. Thus, $\tilde{f}_n := f_n^{-1} \circ f \to id$ in the $\mathrm{Diff}^\omega_\rho(\mathbb{T}^m, \mu)$-topology. Since $\mathbb{T}^m$ is compact, this convergence is uniform, too.
Furthermore, the smoothness of f implies $f^*\omega_\infty = f^* \lim_{n\to\infty} \omega_n = \lim_{n\to\infty} f^*\omega_n$. Therewith we compute on C_δ: $f^*\omega_\infty = \lim_{n\to\infty} \left(\left(f_n \tilde{f}_n \right)^* \omega_n \right) = \lim_{n\to\infty} \left(\tilde{f}_n^* f_n^* \omega_n \right) = \lim_{n\to\infty} \tilde{f}_n^* \omega_n = \omega_\infty$, where we used the uniform convergence on C_δ in the last step. As this holds on every set C_δ with $\delta > 0$, it also holds on the set $\bigcup_{\delta>0} C_\delta$. This is a set of full measure and, therefore, the claim follows. □

Hence, the aimed f-invariant measurable Riemannian metric ω_∞ is constructed. Since f is also weak mixing by Remark 6.1, the main theorem is proven.

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