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Abstract: In the present paper we investigate the problem of identifying simultaneously the diffusion matrix, source term and boundary condition as well as the state in the Neumann boundary value problem for an elliptic partial differential equation (PDE) from a measurement data, which is weaker than required of the exact state. A variational method based on energy functions with Tikhonov regularization is here proposed to treat the identification problem. We discretize the PDE with piecewise linear, continuous finite elements and prove the convergence as well as analyse error bounds of this approach.

Key words and phrases: Coefficient identification, simultaneous identification, diffusion matrix, source term, Neumann boundary condition, energy function, finite element method.

AMS Subject Classifications: 65N21, 65N12, 35J25, 35R30.

1 Introduction

Let Ω be an open bounded connected domain of $R^d, 1 \le d \le 3$ with polygonal boundary $\partial\Omega$. In this paper we study the problem of identifying simultaneously the *diffusion matrix* Q, *source term* f and *boundary condition* g as well as the *state* Φ in the Neumann boundary value problem for the elliptic PDE

$$-\nabla \cdot (Q\nabla\Phi) = f \text{ in } \Omega, \tag{1.1}$$
$$Q\nabla\Phi \cdot \vec{n} = g \text{ on } \partial\Omega \tag{1.2}$$

from a measurement $z_\delta \in L^2(\Omega)$ of the solution Φ, where $\vec{n}$ is the unit outward normal on $\partial\Omega$.

To formulate precisely our problem, let us first denote by $\mathcal{S}_d$ the set of all symmetric $d \times d$-matrices equipped with the inner product $M \cdot N := \text{trace}(MN)$ and the corresponding norm $\|M\|_{\mathcal{S}_d} = (M \cdot M)^{1/2}$. Furthermore, we denote for $1 \le p \le \infty$

$$\mathbf{L}^p_{\text{sym}}(\Omega) := \left\{ H \in L^p(\Omega)^{d\times d} \mid H(x) \in \mathcal{S}_d \text{ a.e. in } \Omega \right\}.$$

In $\mathbf{L}^2_{\text{sym}}(\Omega)$ we use the scalar product $\left(H^1, H^2\right)_{\mathbf{L}^2_{\text{sym}}(\Omega)} = \sum_{i,j=1}^d (h^1_{ij}, h^2_{ij})_{L^2(\Omega)}$ and the corresponding norm $\|H\|_{\mathbf{L}^2_{\text{sym}}(\Omega)} := \left(\sum_{i,j=1}^d \|h_{ij}\|^2_{L^2(\Omega)}\right)^{1/2}$. In a more general sense $\|H\|_{\mathbf{L}^p_{\text{sym}}(\Omega)} := \left(\int_\Omega \|H(x)\|^p_{\mathcal{S}_d}\right)^{1/p}$ with $1 \le p < \infty$ while the space $\mathbf{L}^\infty_{\text{sym}}(\Omega)$ is endowed with the norm $\|H\|_{\mathbf{L}^\infty_{\text{sym}}(\Omega)} := \max_{i,j=\overline{1,d}} \|h_{ij}\|_{L^\infty(\Omega)}$.

Let us denote by

$$\mathcal{H}_{ad} := \mathcal{Q}_{ad} \times \mathcal{F}_{ad} \times \mathcal{G}_{ad}$$

with

$$\begin{aligned} &\mathcal{Q}_{ad} := \left\{ Q \in \mathbf{L}^\infty_{\text{sym}}(\Omega) \mid \underline{q}|\xi|^2 \le Q(x)\xi\cdot\xi \le \overline{q}|\xi|^2 \text{ for all } \xi \in R^d \right\}, \\ &\mathcal{F}_{ad} := L^2(\Omega), \\ &\mathcal{G}_{ad} := L^2(\partial\Omega) \end{aligned} \tag{1.3}$$

and $\underline{q}$, $\overline{q}$ being given constants satisfying $\overline{q} \ge \underline{q} > 0$. Let

$$\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$$

be the continuous Dirichlet trace operator and $H^1_\diamond(\Omega)$ be the closed subspace of $H^1(\Omega)$ consisting all functions with zero-mean on the boundary, i.e.,

$$H^1_\diamond(\Omega) := \left\{ u \in H^1(\Omega) \;\middle|\; \int_{\partial\Omega} \gamma u = 0 \right\}$$

while C_Ω stands for the positive constant appearing in the Poincaré-Friedrichs inequality (cf. [35])

$$C_\Omega \int_\Omega \varphi^2 \le \int_\Omega |\nabla \varphi|^2 \text{ for all } \varphi \in H^1_\diamond(\Omega). \tag{1.4}$$

Then, due to the coervicity condition

$$\|\varphi\|^2_{H^1(\Omega)} \le \frac{1+C_\Omega}{C_\Omega} \int_\Omega |\nabla\varphi|^2 \le \frac{1+C_\Omega}{C_\Omega \underline{q}} \int_\Omega Q\nabla\varphi \cdot \nabla\varphi \tag{1.5}$$

holding for all $\varphi \in H^1_\diamond(\Omega), Q \in \mathcal{Q}_{ad}$ and the Lax-Milgram lemma, we conclude for each $(Q, f, g) \in \mathcal{H}_{ad}$, there exists a unique weak solution Φ of (1.1)–(1.2) in the sense that $\Phi \in H^1_\diamond(\Omega)$ and satisfies the identity

$$\int_\Omega Q\nabla\Phi \cdot \nabla\varphi dx = (f, \varphi) + \langle g, \gamma\varphi \rangle \tag{1.6}$$

for all $\varphi \in H^1_\diamond(\Omega)$. Here the expressions $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ stand for the scalar product on space $L^2(\Omega)$ and $L^2(\partial\Omega)$, respectively. Furthermore, there holds the priori estimate

$$\begin{aligned}\|\Phi\|_{H^1(\Omega)} &\le \frac{1+C_\Omega}{C_\Omega \underline{q}} \left(\|\gamma\|_{\mathcal{L}\left(H^1(\Omega), H^{1/2}(\partial\Omega)\right)} \|g\|_{L^2(\partial\Omega)} + \|f\|_{L^2(\Omega)} \right) \\ &\le C_{\mathcal{N}} \left(\|g\|_{L^2(\partial\Omega)} + \|f\|_{L^2(\Omega)} \right) \end{aligned} \tag{1.7}$$

with

$$C_{\mathcal{N}} := \frac{1+C_\Omega}{C_\Omega \underline{q}} \max\left(1, \|\gamma\|_{\mathcal{L}\left(H^1(\Omega), H^{1/2}(\partial\Omega)\right)}\right).$$

Then we can define the *non-linear coefficient-to-solution operator*

$$\mathcal{U} : \mathcal{H}_{ad} \to H^1_\diamond(\Omega)$$

which maps each $(Q, f, g) \in \mathcal{H}_{ad}$ to the unique weak solution $\mathcal{U}_{Q,f,g} := \Phi$ of the problem (1.1)–(1.2). The identification problem is now stated as follows:

Given $\Phi^\dagger := \mathcal{U}_{Q,f,g} \in H^1_\diamond(\Omega)$, find an element $(Q, f, g) \in \mathcal{H}_{ad}$ such that (1.6) is satisfied with $\Phi^\dagger$ and Q, f, g.

This problem may have more than one solution. Thus to identify, we shall use the notion of the *unique minimum norm solution* which is defined as

$$\left(Q^\dagger, f^\dagger, g^\dagger\right) := \arg \min_{(Q,f,g)\in\mathcal{I}(\Phi^\dagger)} \mathcal{R}(Q, f, g), \tag{1.8}$$

where $\mathcal{I}(\Phi^\dagger) := \left\{(Q, f, g) \in \mathcal{H}_{ad} \mid \mathcal{U}_{Q,f,g} = \Phi^\dagger\right\}$ and

$$\mathcal{R}(Q, f, g) := \|Q\|^2_{\mathbf{L}^2_{\mathrm{sym}}(\Omega)} + \|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\partial\Omega)}.$$

We mention that the set $\mathcal{I}(\Phi^\dagger)$ is non-empty, convex and weakly closed in $\mathbf{L}^2_{\mathrm{sym}}(\Omega) \times L^2(\Omega) \times L^2(\partial\Omega)$, so that the minimizer $(Q^\dagger, f^\dagger, g^\dagger)$ is defined uniquely. Furthermore, the exact data $\Phi^\dagger$ may be not known in practice, thus we assume instead of $\Phi^\dagger$ to have a measurement $z_\delta \in L^2(\Omega)$ such that

$$\left\|\Phi^\dagger - z_\delta\right\|_{L^2(\Omega)} \le \delta \tag{1.9}$$

holds for some $\delta > 0$. Our identification problem is now to reconstruct $\left(Q^\dagger, f^\dagger, g^\dagger\right)$ from z_δ.

We also note that the condition $z_\delta \in L^2(\Omega)$ is weaker than required of the exact state $\Phi^\dagger \in H^1(\Omega)$. In the numerical implementation of §6 the data z_δ is only assumed to be given at nodes of the *coarsest* triangulation grid of the domain Ω. Then the interpolation of the computed numerical state, which is followed by an algorithm presented in §5, corresponding to the coarsest grid on the next finer grid is considered as an observation of the exact state on this finer grid, and so on.

Let $\left(\mathcal{T}^h\right)_{0<h<1}$ denote a family of triangulations of the domain $\overline{\Omega}$ with the mesh size h and $\mathcal{U}^h$ be the approximation of the operator $\mathcal{U}$ on the piecewise linear, continuous finite element space associated with $\mathcal{T}^h$. Furthermore, let Π^h be the Clément's mollification interpolation operator (cf. §2). The standard method for solving the above mentioned identification problem is the output least squares one with Tikhonov regularization, i.e., one considers a minimizer of the problem

$$\min_{(Q,f,g)\in\mathcal{H}_{ad}} \left\|\mathcal{U}^h_{Q,f,g} - \Pi^h z_\delta\right\|^2_{L^2(\Omega)} + \rho\mathcal{R}(Q,f,g) \tag{1.10}$$

as a discrete approximation of the identified coefficient $\left(Q^\dagger, f^\dagger, g^\dagger\right)$, here $\rho > 0$ is the regularization parameter. However, due to the non-linearity of the coefficient-to-solution operator, we are faced with certain difficulties in holding the *non-convex* minimization problem (1.10). Thus, instead of working with the above least squares functional and following the use of energy functions (cf. [34, 32, 44]), in the present work the *convex* cost function (cf. §2)

$$(Q,f,g) \in \mathcal{H}_{ad} \mapsto \mathcal{J}^h_\delta(Q,f,g) := \int_\Omega Q\nabla\left(\mathcal{U}^h_{Q,f,g} - \Pi^h z_\delta\right)\cdot\nabla\left(\mathcal{U}^h_{Q,f,g} - \Pi^h z_\delta\right)$$

will be taken into account. We then consider a *unique* minimizer $\left(Q^h, f^h, g^h\right)$ of the *strictly convex* problem

$$\min_{(Q,f,g)\in\mathcal{H}_{ad}} \mathcal{J}^h_\delta(Q,f,g) + \rho\mathcal{R}(Q,f,g) \tag{1.11}$$

as a discrete regularized solution of the identification problem. Note that every solution of the minimization problem (1.11) automatically belongs to finite dimensional spaces. Thus, a discretization of the admissible set $\mathcal{H}_{ad}$ can be avoided.

In §3 we will show the convergence of these approximation solutions $\left(Q^h, f^h, g^h\right)$ to the identification $\left(Q^\dagger, f^\dagger, g^\dagger\right)$ in the $\mathbf{L}^2_{\mathrm{sym}}(\Omega) \times L^2(\Omega) \times L^2(\partial\Omega)$-norm as well as the convergence of corresponding approximation states $\left(\mathcal{U}^h_{Q^h,f^h,g^h}\right)$ to the exact $\Phi^\dagger$ in the $H^1(\Omega)$-norm. Under the structural source condition — but without the smallness requirement — of the general convergence theory for non-linear, ill-posed problems (cf. [15, 16]), we prove in §4 error bounds for these discrete approximations. For the numerical solution of the minimization problem (1.11) we in §5 employ a gradient projection algorithm with Armijo steplength rule. Finally, a numerical implementation will be performed to illustrate the theoretical findings.

The coefficient identification problem in PDEs arises from different contexts of applied sciences, e.g., from aquifer analysis, geophysical prospecting and pollutant detection, and attracted great attention from many scientists in the last 30 years or so. For surveys on the subject one may consult in [3, 9, 26, 39, 41, 42]. So far there is no paper devoted to such a simultaneous identification problem. The problem of identifying the scalar diffusion coefficient has been extensively studied for above theoretical research and numerical implementation, see e.g., [7, 8, 10, 11, 17, 18, 19, 25, 27, 29, 30, 33, 37, 44]. Some contributions for the problem of simultaneously identifying coefficients can be found in [2, 20, 21, 31] while some works treated the diffusion matrix case have been obtained in [14, 22, 23, 24, 36].

We conclude this introduction with the following note. By using the H-convergent concept, the convergence analysis presented in [14, 22] maybe not applied directly to the problem of identifying *scalar* diffusion coefficients. The main difficulty is that the set

$$\mathcal{D} := \left\{qI_d \;\middle|\; q \in L^\infty(\Omega) \text{ with } \underline{q} \le q(x) \le \overline{q} \text{ a.e. in } \Omega \text{ and } I_d \text{ is the unit } d\times d\text{-matrix}\right\}$$

is in general not a closed subset of $\mathcal{Q}_{ad}$ under the topology of the H-convergence (cf. [43]), i.e., if the sequence $(q_nI_d)_n \subset \mathcal{D}$ is H-convergent to $Q \in \mathcal{Q}_{ad}$, then Q is not necessarily proportional to I_d in dimension $d \ge 2$ or $Q \notin \mathcal{D}$. However, it is wroth to note that $\mathcal{D}$ is a weakly* closed subset of $\mathbf{L}^\infty_{\mathrm{sym}}(\Omega)$ (cf. Remark 2.1) and so that the technique presented in the present paper covers the scalar diffusion identification case.

Throughout the paper we write $\int_\Omega \cdots$ instead of $\int_\Omega \cdots dx$ for the convenience of relevant notations. We use the standard notion of Sobolev spaces $H^1(\Omega)$, $H^2(\Omega)$, $W^{k,p}(\Omega)$, etc from, e.g., [1].

2 Finite element discretization

2.1 Preliminaries

In product spaces $\mathbf{L}^2_{\text{sym}}(\Omega) \times L^2(\Omega) \times L^2(\partial\Omega)$ and $\mathbf{L}^\infty_{\text{sym}}(\Omega) \times L^2(\Omega) \times L^2(\partial\Omega)$ we use the norm

$$\|(H,l,s)\|_{\mathbf{L}^2_{\text{sym}}(\Omega)\times L^2(\Omega)\times L^2(\partial\Omega)} = \left(\|H\|^2_{\mathbf{L}^2_{\text{sym}}(\Omega)} + \|l\|^2_{L^2(\Omega)} + \|s\|^2_{L^2(\partial\Omega)}\right)^{1/2}$$

and

$$\|(H,l,s)\|_{\mathbf{L}^\infty_{\text{sym}}(\Omega)\times L^2(\Omega)\times L^2(\partial\Omega)} = \|H\|_{\mathbf{L}^\infty_{\text{sym}}(\Omega)} + \|l\|_{L^2(\Omega)} + \|s\|_{L^2(\partial\Omega)},$$

respectively.

We note that the coefficient-to-solution operator

$$\mathcal{U} : \mathcal{H}_{ad} \subset \mathbf{L}^\infty_{\text{sym}}(\Omega) \times L^2(\Omega) \times L^2(\partial\Omega) \to H^1_\diamond(\Omega)$$

with

$$\Gamma := (Q,f,g) \in \mathcal{H}_{ad} \to \mathcal{U}(\Gamma) := \mathcal{U}_\Gamma$$

is Fréchet differentiable on $\mathcal{H}_{ad}$. For each $\Gamma = (Q,f,g) \in \mathcal{H}_{ad}$ the action of its Fréchet derivative in direction $\lambda := (H,l,s) \in \mathbf{L}^\infty_{\text{sym}}(\Omega) \times L^2(\Omega) \times L^2(\partial\Omega)$ denoted by $\xi_\lambda := \mathcal{U}'_\Gamma(\lambda) := \mathcal{U}'(\Gamma)(\lambda)$ is the unique weak solution in $H^1_\diamond(\Omega)$ to the equation

$$\int_\Omega Q\nabla\xi_\lambda \cdot \nabla\varphi = -\int_\Omega H\nabla\mathcal{U}_\Gamma \cdot \nabla\varphi + (l,\varphi) + \langle s, \gamma\varphi\rangle \tag{2.1}$$

for all $\varphi \in H^1_\diamond(\Omega)$.

In $\mathcal{S}_d$ we introduce the convex subset

$$\mathcal{K} := \{M \in \mathcal{S}_d \mid \underline{q} \le M\xi\cdot\xi \le \overline{q} \text{ for all } \xi \in R^d\}$$

together with the orthogonal projection $P_\mathcal{K} : \mathcal{S}_d \to \mathcal{K}$ that is characterised by

$$(A - P_\mathcal{K}(A))\cdot(B - P_\mathcal{K}(A)) \le 0$$

for all $A \in \mathcal{S}_d$ and $B \in \mathcal{K}$. Furthermore, let $\xi := (\xi_1,\cdots,\xi_d)$ and $\eta := (\eta_1,\cdots,\eta_d)$ be two arbitrary vectors in R^d, we use the notation

$$(\xi\otimes\eta)_{1\le i,j\le d} \in \mathcal{S}_d \text{ with } (\xi\otimes\eta)_{ij} := \frac{1}{2}(\xi_i\eta_j + \xi_j\eta_i) \text{ for all } i,j = 1,\cdots,d.$$

We close this subsection by the following note.

Remark 2.1. Let

$$\mathbf{D} := \left\{q \in L^\infty(\Omega) \;\middle|\; \underline{q} \le q(x) \le \overline{q} \text{ a.e. in } \Omega\right\}.$$

Then $\mathbf{D}$ is a weakly* compact subset of $L^\infty(\Omega)$, i.e., for any sequence $(q_n)_n \subset \mathbf{D}$ a subsequence $(q_{n_m})_m$ and an element $\xi_\infty \in \mathbf{D}$ exist such that $(q_{n_m})_m$ is weakly* convergent in $L^\infty(\Omega)$ to ξ_∞. In other words,

$$\lim_{m\to\infty}\int_\Omega q_{n_m}\theta_1 = \int_\Omega \xi_\infty\theta_1$$

for all $\theta_1 \in L^1(\Omega)$.

Proof. Indeed, we first note that $\mathbf{D}$ is a non-empty, convex, bounded and closed subset of $L^2(\Omega)$. Thus, $\mathbf{D}$ is a weakly compact subset of $L^2(\Omega)$ and so that a subsequence $(q_{n_m})_m$ of $(q_n)_n$ and an element $\xi_2 \in \mathbf{D}$ exist such that $\lim_{m\to\infty}\int_\Omega q_{n_m}\theta_2 = \int_\Omega \xi_2\theta_2$ for all $\theta_2 \in L^2(\Omega)$. Furthermore, since the sequence $(q_{n_m})_m$ is bounded in the $L^\infty(\Omega)$-norm, a subsequence not relabelled and an element $\xi_\infty \in L^\infty(\Omega)$ exist such that $\lim_{m\to\infty}\int_\Omega q_{n_m}\theta_1 = \int_\Omega \xi_\infty\theta_1$ for all $\theta_1 \in L^1(\Omega)$. Then, for all $\theta \in L^\infty(\Omega)$ we get $\int_\Omega(\xi_\infty - \xi_2)\theta = 0$ which implies that $\xi_\infty = \xi_2 \in \mathbf{D}$. □

We also remark that any $\Psi \in L^\infty(\Omega)$ can be considered as an element in $L^\infty(\Omega)^*$ by

$$\langle \Psi, \psi \rangle_{\left(L^\infty(\Omega)^*, L^\infty(\Omega)\right)} := \int_\Omega \Psi\psi \tag{2.2}$$

for all ψ in $L^\infty(\Omega)$ and $\|\Psi\|_{L^\infty(\Omega)^*} \le |\Omega| \cdot \|\Psi\|_{L^\infty(\Omega)}$.

2.2 Discretization

Let $\left(\mathcal{T}^h\right)_{0<h<1}$ be a family of regular and quasi-uniform triangulations of the domain $\overline{\Omega}$ with the mesh size h such that each vertex of the polygonal boundary $\partial\Omega$ is a node of $\mathcal{T}_h$. For the definition of the discretization space of the state functions let us denote

$$\mathcal{V}_1^h := \left\{ \varphi^h \in C\left(\overline{\Omega}\right) \cap H^1_\diamond(\Omega) \mid {\varphi^h}_{|T} \in \mathcal{P}_1(T) \text{ for all } T \in \mathcal{T}^h \right\} \tag{2.3}$$

with $\mathcal{P}_r$ consisting all polynomial functions of degree at most r. Similar to the continuous case, we have the following result.

Lemma 2.2. *Let (Q, f, g) be in $\mathcal{H}_{ad}$. Then the variational equation*

$$\int_\Omega Q\nabla\Phi^h \cdot \nabla\varphi^h = (f, \varphi^h) + \langle g, \gamma\varphi^h \rangle \tag{2.4}$$

for all $\varphi^h \in \mathcal{V}_1^h$ admits a unique solution $\Phi^h \in \mathcal{V}_1^h$. Furthermore, the priori estimate

$$\|\Phi^h\|_{H^1(\Omega)} \le C_{\mathcal{N}} \left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \right) \tag{2.5}$$

is satisfied.

The map $\mathcal{U}^h : \mathcal{H}_{ad} \subset \mathbf{L}^\infty_{\mathrm{sym}}(\Omega) \times L^2(\Omega) \times L^2(\partial\Omega) \to \mathcal{V}_1^h$ from each $\Gamma := (Q, f, g) \in \mathcal{H}_{ad}$ to the unique solution $\mathcal{U}^h_\Gamma := \Phi^h$ of (2.4) is called the *discrete coefficient-to-solution operator.* This operator is also Fréchet differentiable on the set $\mathcal{H}_{ad}$. For each $\Gamma = (Q, f, g) \in \mathcal{H}_{ad}$ and $\lambda := (H, l, s) \in \mathbf{L}^\infty_{\mathrm{sym}}(\Omega) \times L^2(\Omega) \times L^2(\partial\Omega)$ the Fréchet differential $\xi^h_\lambda := {\mathcal{U}^h_\Gamma}'(\lambda)$ is an element of $\mathcal{V}_1^h$ and satisfies for all φ^h in $\mathcal{V}_1^h$ the equation

$$\int_\Omega Q\nabla\xi^h_\lambda \cdot \nabla\varphi^h = -\int_\Omega H\nabla\mathcal{U}^h_\Gamma \cdot \nabla\varphi^h + (l, \varphi^h) + \langle s, \gamma\varphi^h \rangle. \tag{2.6}$$

Due to the standard theory of the finite element method for elliptic problems (cf. [6, 12]), for any fixed $\Gamma = (Q, f, g) \in \mathcal{H}_{ad}$ it holds

$$\lim_{h\to 0} \left\| \mathcal{U}_\Gamma - \mathcal{U}^h_\Gamma \right\|_{H^1(\Omega)} = 0. \tag{2.7}$$

Let

$$\Pi^h : L^1(\Omega) \to \left\{ \varphi^h \in C\left(\overline{\Omega}\right) \mid {\varphi^h}_{|T} \in \mathcal{P}_1(T) \text{ for all } T \in \mathcal{T}^h \right\}$$

be the Clément's mollification interpolation operator with properties

$$\lim_{h\to 0} \left\| \phi - \Pi^h\phi \right\|_{H^k(\Omega)} = 0 \text{ for all } k \in \{0, 1\} \tag{2.8}$$

and

$$\left\| \phi - \Pi^h\phi \right\|_{H^k(\Omega)} \le Ch^{l-k} \|\phi\|_{H^l(\Omega)} \tag{2.9}$$

for $0 \le k \le l \le 2$, where C is independent of h and ϕ (cf. [13, 4, 5, 40]). Then, using the discrete operator $\mathcal{U}^h$ and the interpolation operator Π^h, we can now introduce the discrete cost functional

$$\mathcal{J}^h_\delta(Q, f, g) := \int_\Omega Q\nabla\left(\mathcal{U}^h_{Q,f,g} - \Pi^h z_\delta\right) \cdot \nabla\left(\mathcal{U}^h_{Q,f,g} - \Pi^h z_\delta\right), \tag{2.10}$$

where $(Q, f, g) \in \mathcal{H}_{ad}$.

Lemma 2.3. *Assume that the sequence* $(\Gamma_n)_n := (Q_n, f_n, g_n)_n \subset \mathcal{H}_{ad}$ *weakly converges to* $\Gamma := (Q, f, g)$ *in* $\mathbf{L}^2_{sym}(\Omega) \times L^2(\Omega) \times L^2(\partial\Omega)$. *Then for any fixed* $h > 0$ *the sequence* $\big(\mathcal{U}^h_{\Gamma_n}\big)_n \subset \mathcal{V}^h_1$ *converges to* $\mathcal{U}^h_\Gamma$ *in the* $H^1(\Omega)$*-norm.*

Proof. Due to Remark 2.1, $(Q_n)_n$ has a subsequence denoted by the same symbol which is weakly* convergent in $\mathbf{L}^\infty_{\rm sym}(\Omega)$ to Q. Furthermore, by (2.5), the corresponding state sequence $\big(\mathcal{U}^h_{\Gamma_n}\big)_n$ is bounded in the *finite* dimensional space $\mathcal{V}^h_1$. A subsequence which is not relabelled and an element $\Theta^h \in \mathcal{V}^h_1$ then exist such that $\big(\mathcal{U}^h_{\Gamma_n}\big)_n$ converges to Θ^h in the $H^1(\Omega)$-norm. It follows from the equation (2.4) that

$$\int_\Omega Q_n \nabla\big(\mathcal{U}^h_{\Gamma_n} - \mathcal{U}^h_\Gamma\big)\cdot\nabla\varphi^h = \int_\Omega (Q - Q_n)\nabla\mathcal{U}^h_\Gamma\cdot\nabla\varphi^h + \big(f_n - f, \varphi^h\big) + \big\langle g_n - g, \gamma\varphi^h\big\rangle \tag{2.11}$$

for all $\varphi^h \in \mathcal{V}^h_1$. Taking $\varphi^h = \mathcal{U}^h_{\Gamma_n} - \mathcal{U}^h_\Gamma$, by (1.5), we obtain that

$$\begin{aligned}
\frac{C_\Omega \underline{q}}{1 + C_\Omega}\left\|\mathcal{U}^h_{\Gamma_n} - \mathcal{U}^h_\Gamma\right\|^2_{H^1(\Omega)} &\le \int_\Omega (Q - Q_n)\nabla\mathcal{U}^h_\Gamma\cdot\nabla\left(\mathcal{U}^h_{\Gamma_n} - \Theta^h + \Theta^h - \mathcal{U}^h_\Gamma\right) \\
&\quad + \left(f_n - f, \mathcal{U}^h_{\Gamma_n} - \Theta^h + \Theta^h - \mathcal{U}^h_\Gamma\right) + \left\langle g_n - g, \gamma\left(\mathcal{U}^h_{\Gamma_n} - \Theta^h + \Theta^h - \mathcal{U}^h_\Gamma\right)\right\rangle \\
&\le C\left\|\mathcal{U}^h_{\Gamma_n} - \Theta^h\right\|_{H^1(\Omega)} + \int_\Omega (Q - Q_n)\nabla\mathcal{U}^h_\Gamma\cdot\nabla\left(\Theta^h - \mathcal{U}^h_\Gamma\right) \\
&\quad + \left(f_n - f, \Theta^h - \mathcal{U}^h_\Gamma\right) + \left\langle g_n - g, \gamma\left(\Theta^h - \mathcal{U}^h_\Gamma\right)\right\rangle.
\end{aligned} \tag{2.12}$$

Since $Q_n \rightharpoonup Q$ weakly* in $\mathbf{L}^\infty_{\rm sym}(\Omega)$, we get

$$\lim_{n\to\infty}\int_\Omega (Q - Q_n)\nabla\mathcal{U}^h_\Gamma\cdot\nabla\left(\Theta^h - \mathcal{U}^h_\Gamma\right) = 0.$$

Sending n to ∞, we thus obtain from the last inequality that $\lim_{n\to\infty}\left\|\mathcal{U}^h_{\Gamma_n} - \mathcal{U}^h_\Gamma\right\|_{H^1(\Omega)} = 0$, which finishes the proof. □

We now state the following useful result on the convexity of the cost functional.

Lemma 2.4. $\mathcal{J}^h_\delta$ *is convex and continuous on* $\mathcal{H}_{ad}$ *with respect to the* $\mathbf{L}^2_{sym}(\Omega) \times L^2(\Omega) \times L^2(\partial\Omega)$*-norm.*

Proof. The continuity of $\mathcal{J}^h_\delta$ follows directly from Lemma 2.3. We show that $\mathcal{J}^h_\delta$ is convex.

Let $\Gamma := (Q, f, g) \in \mathcal{H}_{ad}$ and $\lambda := (H, l, s) \in \mathbf{L}^\infty_{\rm sym}(\Omega) \times L^2(\Omega) \times L^2(\partial\Omega)$. We have that

$${\mathcal{U}^h_\Gamma}'(\lambda) = \frac{\partial\mathcal{U}^h_\Gamma}{\partial Q}H + \frac{\partial\mathcal{U}^h_\Gamma}{\partial f}l + \frac{\partial\mathcal{U}^h_\Gamma}{\partial g}s \text{ and } {\mathcal{J}^h_\delta}'(\Gamma)(\lambda) = \frac{\partial\mathcal{J}^h_\delta(\Gamma)}{\partial Q}H + \frac{\partial\mathcal{J}^h_\delta(\Gamma)}{\partial f}l + \frac{\partial\mathcal{J}^h_\delta(\Gamma)}{\partial g}s.$$

We compute for each term in the right hand side of the last equation. First we get

$$\frac{\partial\mathcal{J}^h_\delta(\Gamma)}{\partial Q}H = \int_\Omega H\nabla\left(\mathcal{U}^h_\Gamma - \Pi^h z_\delta\right)\cdot\nabla\left(\mathcal{U}^h_\Gamma - \Pi^h z_\delta\right) + 2\int_\Omega Q\nabla\left(\frac{\partial\mathcal{U}^h_\Gamma}{\partial Q}H\right)\cdot\nabla\left(\mathcal{U}^h_\Gamma - \Pi^h z_\delta\right).$$

For the second term we have

$$\frac{\partial\mathcal{J}^h_\delta(\Gamma)}{\partial f}l = 2\int_\Omega Q\nabla\left(\frac{\partial\mathcal{U}^h_\Gamma}{\partial f}l\right)\cdot\nabla\left(\mathcal{U}^h_\Gamma - \Pi^h z_\delta\right).$$

Finally, we have

$$\frac{\partial\mathcal{J}^h_\delta(\Gamma)}{\partial g}s = 2\int_\Omega Q\nabla\left(\frac{\partial\mathcal{U}^h_\Gamma}{\partial g}s\right)\cdot\nabla\left(\mathcal{U}^h_\Gamma - \Pi^h z_\delta\right).$$

Therefore,

$$\begin{aligned}
{\mathcal{J}^h_\delta}'(\Gamma)(\lambda) &= 2\int_\Omega Q\nabla\left(\frac{\partial\mathcal{U}^h_\Gamma}{\partial Q}H + \frac{\partial\mathcal{U}^h_\Gamma}{\partial f}l + \frac{\partial\mathcal{U}^h_\Gamma}{\partial g}s\right)\cdot\nabla\left(\mathcal{U}^h_\Gamma - \Pi^h z_\delta\right) + \int_\Omega H\nabla(\mathcal{U}^h_\Gamma - \Pi^h z_\delta)\cdot\nabla(\mathcal{U}^h_\Gamma - \Pi^h z_\delta) \\
&= 2\int_\Omega Q\nabla{\mathcal{U}^h_\Gamma}'(\lambda)\cdot\nabla\left(\mathcal{U}^h_\Gamma - \Pi^h z_\delta\right) + \int_\Omega H\nabla\left(\mathcal{U}^h_\Gamma - \Pi^h z_\delta\right)\cdot\nabla\left(\mathcal{U}^h_\Gamma - \Pi^h z_\delta\right) \\
&= 2\int_\Omega Q\nabla{\mathcal{U}^h_\Gamma}'(\lambda)\cdot\nabla\left(\mathcal{U}^h_\Gamma - \bar{\Pi}^h z_\delta\right) + \int_\Omega H\nabla\left(\mathcal{U}^h_\Gamma - \bar{\Pi}^h z_\delta\right)\cdot\nabla\left(\mathcal{U}^h_\Gamma - \bar{\Pi}^h z_\delta\right),
\end{aligned}$$

where

$$\bar{\Pi}^h z_\delta := \Pi^h z_\delta - |\Omega|^{-1}\left\langle 1, \gamma \Pi^h z_\delta \right\rangle \in \mathcal{V}_1^h \text{ with } \nabla \bar{\Pi}^h z_\delta = \nabla \Pi^h z_\delta. \tag{2.13}$$

By (2.6), we infer that

$$\begin{aligned}
\mathcal{J}_\delta^{h\prime}(\Gamma)(\lambda) &= -2\int_\Omega H\nabla \mathcal{U}_\Gamma^h \cdot \nabla\left(\mathcal{U}_\Gamma^h - \bar{\Pi}^h z_\delta\right) + 2\left(l, \mathcal{U}_\Gamma^h - \bar{\Pi}^h z_\delta\right) + 2\left\langle s, \gamma\left(\mathcal{U}_\Gamma^h - \bar{\Pi}^h z_\delta\right)\right\rangle \\
&\quad + \int_\Omega H\nabla\left(\mathcal{U}_\Gamma^h - \bar{\Pi}^h z_\delta\right) \cdot \nabla\left(\mathcal{U}_\Gamma^h - \bar{\Pi}^h z_\delta\right) \\
&= -\int_\Omega H\nabla \mathcal{U}_\Gamma^h \cdot \nabla \mathcal{U}_\Gamma^h + \int_\Omega H\nabla \bar{\Pi}^h z_\delta \cdot \nabla \bar{\Pi}^h z_\delta + 2\left(l, \mathcal{U}_\Gamma^h - \bar{\Pi}^h z_\delta\right) + 2\left\langle s, \gamma\left(\mathcal{U}_\Gamma^h - \bar{\Pi}^h z_\delta\right)\right\rangle. \quad (2.14)
\end{aligned}$$

Therefore, by (2.6) again, we arrive at

$$\begin{aligned}
\mathcal{J}_\delta^{h\prime\prime}(\Gamma)(\lambda,\lambda) &= -2\int_\Omega H\nabla \mathcal{U}_\Gamma^h \cdot \nabla \mathcal{U}_\Gamma^{h\prime}(\lambda) + 2\left(l, \mathcal{U}_\Gamma^{h\prime}(\lambda)\right) + 2\left\langle s, \gamma \mathcal{U}_\Gamma^{h\prime}(\lambda)\right\rangle \\
&= 2\int_\Omega Q\nabla \mathcal{U}_\Gamma^{h\prime}(\lambda) \cdot \nabla \mathcal{U}_\Gamma^{h\prime}(\lambda) \geq 2\frac{C_\Omega \underline{q}}{1+C_\Omega}\left\|\mathcal{U}_\Gamma^{h\prime}(\lambda)\right\|_{H^1(\Omega)}^2 \geq 0,
\end{aligned}$$

by (1.5), which completes the proof. □

Now we are in position to prove the main result of this section.

Theorem 2.5. *The strictly convex minimization problem*

$$\min_{(Q,f,g)\in\mathcal{H}_{ad}} \Upsilon_\delta^{\rho,h}(Q,f,g) := \mathcal{J}_\delta^h(Q,f,g) + \rho\mathcal{R}(Q,f,g) \tag{$\mathcal{P}_\delta^{\rho,h}$}$$

attains a unique minimizer. Furthermore, an element $\Gamma := (Q,f,g) \in \mathcal{H}_{ad}$ is the unique minimizer to $\left(\mathcal{P}_\delta^{\rho,h}\right)$ if and only if the system

$$\begin{aligned}
Q(x) &= P_{\mathcal{K}}\left(\frac{1}{2\rho}\left(\nabla \mathcal{U}_\Gamma^h(x) \otimes \nabla \mathcal{U}_\Gamma^h(x) - \nabla \bar{\Pi}^h z_\delta(x) \otimes \nabla \bar{\Pi}^h z_\delta(x)\right)\right), \\
f(x) &= \frac{1}{\rho}\left(\bar{\Pi}^h z_\delta(x) - \mathcal{U}_\Gamma^h(x)\right), \\
g(x) &= \frac{1}{\rho}\gamma\left(\bar{\Pi}^h z_\delta(x) - \mathcal{U}_\Gamma^h(x)\right)
\end{aligned} \tag{2.15}$$

holds for a.e. in Ω, where $\bar{\Pi}^h$ was generated from Π^h according to (2.13).

Proof. Let $(\Gamma_n)_n := (Q_n, f_n, g_n)_n \subset \mathcal{H}_{ad}$ be a minimizing sequence of $\left(\mathcal{P}_\delta^{\rho,h}\right)$, i.e.,

$$\lim_{n\to\infty} \Upsilon_\delta^{\rho,h}(\Gamma_n) = \inf_{(Q,f,g)\in\mathcal{H}_{ad}} \Upsilon_\delta^{\rho,h}(Q,f,g).$$

The sequence $(\Gamma_n)_n$ is thus bounded in the $\mathbf{L}^2_{\mathrm{sym}}(\Omega) \times L^2(\Omega) \times L^2(\partial\Omega)$-norm. A subsequence not relabelled and an element $\Gamma := (Q,f,g) \in \mathbf{L}^2_{\mathrm{sym}}(\Omega) \times L^2(\Omega) \times L^2(\partial\Omega)$ exist such that $\Gamma_n \rightharpoonup \Gamma$ weakly in $\mathbf{L}^2_{\mathrm{sym}}(\Omega) \times L^2(\Omega) \times L^2(\partial\Omega)$. On the other hand, since $\mathcal{H}_{ad}$ is a convex, closed subset of $\mathbf{L}^2_{\mathrm{sym}}(\Omega) \times L^2(\Omega) \times L^2(\partial\Omega)$, so is weakly closed, it follows that $\Gamma \in \mathcal{H}_{ad}$.

By Lemma 2.4, $\mathcal{J}_\delta^h$ and $\mathcal{R}$ are both weakly lower semi-continuous on $\mathcal{H}_{ad}$ which yields that

$$\mathcal{J}_\delta^h(\Gamma) \leq \liminf_{n\to\infty} \mathcal{J}_\delta^h(\Gamma_n) \text{ and } \mathcal{R}(\Gamma) \leq \liminf_{n\to\infty} \mathcal{R}(\Gamma_n).$$

We therefore have that

$$\begin{aligned}
\mathcal{J}_\delta^h(\Gamma) + \mathcal{R}(\Gamma) &\leq \liminf_{n\to\infty} \mathcal{J}_\delta^h(\Gamma_n) + \liminf_{n\to\infty} \mathcal{R}(\Gamma_n) \leq \liminf_{n\to\infty}\left(\mathcal{J}_\delta^h(\Gamma_n) + \mathcal{R}(\Gamma_n)\right) \\
&= \lim_{n\to\infty} \Upsilon_\delta^{\rho,h}(\Gamma_n) = \inf_{(Q,f,g)\in\mathcal{H}_{ad}} \Upsilon_\delta^{\rho,h}(Q,f,g),
\end{aligned}$$

and Γ is then a minimizer to $\left(\mathcal{P}_\delta^{\rho,h}\right)$. Since $\Upsilon_\delta^{\rho,h}$ is strictly convex, this minimizer is unique.

Next, an element $\Gamma := (Q,f,g) \in \mathcal{H}_{ad}$ is the minimizer to $\left(\mathcal{P}_\delta^{\rho,h}\right)$ if and only if the condition

$$\Upsilon_\delta^{\rho,h\prime}(\Gamma)(\overline{\Gamma}-\Gamma) \ge 0$$

for all $\overline{\Gamma} = (H,l,s) \in \mathcal{H}_{ad}$. Then, in view of (2.14), we get that

$$\begin{aligned}
0 \le& \int_\Omega (H-Q)\nabla\bar{\Pi}^h z_\delta \cdot \nabla\bar{\Pi}^h z_\delta - \int_\Omega (H-Q)\nabla\mathcal{U}_\Gamma^h \cdot \nabla\mathcal{U}_\Gamma^h + 2\rho(H-Q,Q)\\
&+ 2\left(l-f,\mathcal{U}_\Gamma^h - \bar{\Pi}^h z_\delta\right) + 2\rho(l-f,f) + 2\left\langle s-g,\gamma\left(\mathcal{U}_\Gamma^h - \bar{\Pi}^h z_\delta\right)\right\rangle + 2\rho\left\langle s-g,g\right\rangle\\
=& \int_\Omega (H-Q)\cdot\left(\nabla\bar{\Pi}^h z_\delta \otimes \nabla\bar{\Pi}^h z_\delta - \nabla\mathcal{U}_\Gamma^h \otimes \nabla\mathcal{U}_\Gamma^h + 2\rho Q\right)\\
&+ 2\left(l-f,\mathcal{U}_\Gamma^h - \bar{\Pi}^h z_\delta + \rho f\right) + 2\left\langle s-g,\gamma\left(\mathcal{U}_\Gamma^h - \bar{\Pi}^h z_\delta\right) + \rho g\right\rangle
\end{aligned}$$

for all $\overline{\Gamma} = (H,l,s) \in \mathcal{H}_{ad}$. Taking $\overline{\Gamma}_1 = (H,f,g)$, $\overline{\Gamma}_2 = (Q,l,g)$ and $\overline{\Gamma}_3 = (Q,f,s)$ into the above inequality we obtain the system (2.15). The proof is completed. □

Remark 2.6. We denote by

$$\begin{aligned}
\mathcal{V}_0^h &:= \left\{\varphi^h \in L^2(\Omega) \mid \varphi^h{}_{|T} = \text{ const for all triangulations } T \in \mathcal{T}^h\right\},\\
\mathcal{E}_1^h &:= \left\{\varphi^h \in C(\partial\Omega) \mid \varphi^h{}_{|e} \in \mathcal{P}_1 \text{ for all boundary edges } e \text{ of } \mathcal{T}^h\right\}.
\end{aligned}$$

Since $\mathcal{U}_\Gamma^h \in \mathcal{V}_1^h$ and $\bar{\Pi}^h z_\delta \in \mathcal{V}_1^h$, the system (2.15) shows that every solution of $\left(\mathcal{P}_\delta^{\rho,h}\right)$ automatically belongs to the finite dimensional space ${\mathcal{V}_0^h}^{d\times d} \times \mathcal{V}_1^h \times \mathcal{E}_1^h$. Thus the discretization of the admissible $\mathcal{H}_{ad}$ can be avoid.

3 Convergence

For abbreviation in what follows we denote by C a generic positive constant independent of the mesh size h, the noise level δ and the regularization parameter ρ. By (2.8) and (2.9), we can introduce for each $\Phi \in H^1(\Omega)$

$$\chi_\Phi^h := \left\|\Phi - \Pi^h\Phi\right\|_{H^1(\Omega)}$$

which satisfies

$$\lim_{h\to 0}\chi_\Phi^h = 0 \text{ and } 0 \le \chi_\Phi^h \le Ch$$

in case $\Phi \in H^2(\Omega)$. Likewise, by (2.7), for all $\Gamma \in \mathcal{H}_{ad}$

$$\beta_{\mathcal{U}_\Gamma}^h := \left\|\mathcal{U}_\Gamma - \mathcal{U}_\Gamma^h\right\|_{H^1(\Omega)} \to 0 \text{ as } h\to 0 \text{ and } 0 \le \beta_{\mathcal{U}_\Gamma}^h \le Ch \text{ as } \mathcal{U}_\Gamma \in H^2(\Omega).$$

Furthermore, by (2.9), we get

$$\|\Pi^h\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))} \le C \text{ and } \|\Pi^h\|_{\mathcal{L}(H^1(\Omega),H^1(\Omega))} \le C. \tag{3.1}$$

Thus, it follows from the inverse inequality (cf. [6, 12]):

$$\|\varphi^h\|_{H^1(\Omega)} \le Ch^{-1}\|\varphi^h\|_{L^2(\Omega)} \text{ for all } \varphi^h \in \left\{\varphi^h \in C\left(\overline{\Omega}\right) \mid \varphi^h{}_{|T} \in \mathcal{P}_1(T) \text{ for all } T \in \mathcal{T}^h\right\}$$

that

$$\begin{aligned}
\|\Phi^\dagger - \Pi^h z_\delta\|_{H^1(\Omega)} &\le \|\Pi^h\left(\Phi^\dagger - z_\delta\right)\|_{H^1(\Omega)} + \|\Phi^\dagger - \Pi^h\Phi^\dagger\|_{H^1(\Omega)} \le Ch^{-1}\|\Pi^h\left(\Phi^\dagger - z_\delta\right)\|_{L^2(\Omega)} + \chi_{\Phi^\dagger}^h\\
&\le Ch^{-1}\|\Pi^h\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))}\|\Phi^\dagger - z_\delta\|_{L^2(\Omega)} + \chi_{\Phi^\dagger}^h \le Ch^{-1}\delta + \chi_{\Phi^\dagger}^h.
\end{aligned} \tag{3.2}$$

The following result shows the convergence of finite element approximations to the unique minimum norm solution $\Gamma^\dagger := \left(Q^\dagger, f^\dagger, g^\dagger\right)$ of the identification problem, which is defined by (1.8).

Theorem 3.1. *Let $(h_n)_n$ be a sequence with $\lim_{n\to\infty} h_n = 0$ and $(\delta_n)_n$ and $(\rho_n)_n$ are any positive sequences such that*

$$\rho_n \to 0,\ \frac{\delta_n}{h_n\sqrt{\rho_n}} \to 0,\ \frac{\beta^{h_n}_{\mathcal{U}_{\Gamma^\dagger}}}{\sqrt{\rho_n}} \to 0 \text{ and } \frac{\chi^{h_n}_{\Phi^\dagger}}{\sqrt{\rho_n}} \to 0 \text{ as } n \to \infty.$$

Assume that $(z_{\delta_n})_n \subset L^2(\Omega)$ is a sequence satisfying $\left\| z_{\delta_n} - \Phi^\dagger \right\|_{L^2(\Omega)} \le \delta_n$ and $\Gamma_n := (Q_n, f_n, g_n)$ is the unique minimizer of the problem $\left(\mathcal{P}^{\rho_n,h_n}_{\delta_n}\right)$ for each $n \in N$. Then the sequence $(\Gamma_n)_n$ converges to $\Gamma^\dagger$ in the $\mathbf{L}^2_{sym}(\Omega) \times L^2(\Omega) \times L^2(\partial\Omega)$-norm as $n \to \infty$. Furthermore, the corresponding discrete state sequence $\left(\mathcal{U}^{h_n}_{\Gamma_n}\right)_n$ also converges to $\Phi^\dagger$ in the $H^1(\Omega)$-norm.

Remark 3.2. In case $\Phi^\dagger = \mathcal{U}_{\Gamma^\dagger} \in H^2(\Omega)$ we have $0 \le \beta^{h_n}_{\mathcal{U}_{\Gamma^\dagger}}, \chi^{h_n}_{\Phi^\dagger} \le Ch_n$. Therefore, the convergence of Theorem 3.1 is obtained if $\delta_n \sim h_n^2$ and the sequence $(\rho_n)_n$ is chosen such that

$$\rho_n \to 0, \text{ and } \frac{h_n}{\sqrt{\rho_n}} \to 0 \text{ as } n \to \infty.$$

To prove Theorem 3.1, we need the following auxiliary estimate.

Lemma 3.3. *There holds the estimate*

$$\mathcal{J}^h_\delta(\Gamma^\dagger) \le C\left(h^{-2}\delta^2 + \left(\chi^h_{\Phi^\dagger}\right)^2 + \left(\beta^h_{\mathcal{U}_{\Gamma^\dagger}}\right)^2\right). \tag{3.3}$$

Proof. We have with $\Phi^\dagger = \mathcal{U}_{\Gamma^\dagger}$ and (3.2) that

$$\begin{aligned}\mathcal{J}^h_\delta(\Gamma^\dagger) &= \int_\Omega Q^\dagger \nabla\left(\mathcal{U}^h_{\Gamma^\dagger} - \Pi^h z_\delta\right) \cdot \nabla\left(\mathcal{U}^h_{\Gamma^\dagger} - \Pi^h z_\delta\right) \le \overline{q}\left\|\mathcal{U}^h_{\Gamma^\dagger} - \Pi^h z_\delta\right\|^2_{H^1(\Omega)} \\ &= \overline{q}\left\|\mathcal{U}^h_{\Gamma^\dagger} - \mathcal{U}_{\Gamma^\dagger} + \Phi^\dagger - \Pi^h z_\delta\right\|^2_{H^1(\Omega)} \le C\left(\left\|\mathcal{U}^h_{\Gamma^\dagger} - \mathcal{U}_{\Gamma^\dagger}\right\|^2_{H^1(\Omega)} + \left\|\Phi^\dagger - \Pi^h z_\delta\right\|^2_{H^1(\Omega)}\right) \\ &\le C\left(h^{-2}\delta^2 + \left(\chi^h_{\Phi^\dagger}\right)^2 + \left(\beta^h_{\mathcal{U}_{\Gamma^\dagger}}\right)^2\right),\end{aligned}$$

which finishes the proof. □

Proof of Theorem 3.1. By the optimality of Γ_n and Lemma 3.3, we have that

$$\begin{aligned}\mathcal{J}^{h_n}_{\delta_n}(\Gamma_n) + \rho_n\mathcal{R}(\Gamma_n) &\le \mathcal{J}^{h_n}_{\delta_n}\left(\Gamma^\dagger\right) + \rho_n\mathcal{R}\left(\Gamma^\dagger\right) \\ &\le C\left(h_n^{-2}\delta_n^2 + \left(\chi^{h_n}_{\Phi^\dagger}\right)^2 + \left(\beta^{h_n}_{\mathcal{U}_{\Gamma^\dagger}}\right)^2\right) + \rho_n\mathcal{R}\left(\Gamma^\dagger\right)\end{aligned}$$

which yields

$$\lim_{n\to\infty} \mathcal{J}^{h_n}_{\delta_n}(\Gamma_n) = 0 \tag{3.4}$$

and

$$\limsup_{n\to\infty} \mathcal{R}(\Gamma_n) \le \mathcal{R}\left(\Gamma^\dagger\right). \tag{3.5}$$

A subsequence of the sequence $(\Gamma_n)_n$ denoted by the same symbol and an element $\Gamma_0 := (Q_0, f_0, g_0) \in \mathcal{H}_{ad}$ then exist such that

$$\begin{aligned}&Q_n \rightharpoonup Q_0 \text{ weakly}^* \text{ in } \mathbf{L}^\infty_{\mathrm{sym}}(\Omega),\\ &f_n \rightharpoonup f_0 \text{ weakly in } L^2(\Omega),\\ &g_n \rightharpoonup g_0 \text{ weakly in } L^2(\partial\Omega).\end{aligned}$$

We will show that $(\Gamma_n)_n$ converges to Γ_0 in the $\mathbf{L}^2_{\mathrm{sym}}(\Omega) \times L^2(\Omega) \times L^2(\partial\Omega)$-norm and $\Gamma_0 = \Gamma^\dagger$. We have from (3.2) that

$$\lim_{n\to\infty} \left\|\Pi^{h_n} z_{\delta_n} - \mathcal{U}_{\Gamma^\dagger}\right\|_{H^1(\Omega)} \le \lim_{n\to\infty}\left(Ch_n^{-1}\delta_n + \chi^{h_n}_{\Phi^\dagger}\right) = 0. \tag{3.6}$$

Combining this with $\lim_{n\to\infty} \|\mathcal{U}_{\Gamma_0} - \mathcal{U}^{h_n}_{\Gamma_0}\|_{H^1(\Omega)} = 0$ from (2.7), we arrive at

$$\lim_{n\to\infty} \mathcal{J}^{h_n}_{\delta_n}(\Gamma_0) = \lim_{n\to\infty} \int_\Omega Q_0 \nabla \left(\mathcal{U}^{h_n}_{\Gamma_0} - \Pi^{h_n} z_{\delta_n}\right) \cdot \nabla \left(\mathcal{U}^{h_n}_{\Gamma_0} - \Pi^{h_n} z_{\delta_n}\right) = \int_\Omega Q_0 \nabla \left(\mathcal{U}_{\Gamma_0} - \mathcal{U}_{\Gamma^\dagger}\right) \cdot \nabla \left(\mathcal{U}_{\Gamma_0} - \mathcal{U}_{\Gamma^\dagger}\right).$$

Now for each fixed n we consider an arbitrary subsequence $(\Gamma_{n_m})_m$ of $(\Gamma_n)_n$. By the weakly l.s.c. property of the functional $\mathcal{J}^{h_n}_{\delta_n}$ (cf. Lemma 2.4), we obtain that

$$\mathcal{J}^{h_n}_{\delta_n}(\Gamma_0) \le \liminf_{m\to\infty} \mathcal{J}^{h_n}_{\delta_n}(\Gamma_{n_m}).$$

Again, using the convexity of $\mathcal{J}^{h_n}_{\delta_n}$, we get that

$$\mathcal{J}^{h_n}_{\delta_n}(\Gamma_n) \ge \mathcal{J}^{h_n}_{\delta_n}(\Gamma_{n_m}) + {\mathcal{J}^{h_n}_{\delta_n}}'(\Gamma_{n_m})(\Gamma_n - \Gamma_{n_m}).$$

By (1.5), we thus arrive at

$$\begin{aligned}
C\|\mathcal{U}_{\Gamma_0} - \mathcal{U}_{\Gamma^\dagger}\|^2_{H^1(\Omega)} &\le \int_\Omega Q_0 \nabla \left(\mathcal{U}_{\Gamma_0} - \mathcal{U}_{\Gamma^\dagger}\right) \cdot \nabla \left(\mathcal{U}_{\Gamma_0} - \mathcal{U}_{\Gamma^\dagger}\right) = \lim_{n\to\infty} \mathcal{J}^{h_n}_{\delta_n}(\Gamma_0) \le \lim_{n\to\infty} \left(\liminf_{m\to\infty} \mathcal{J}^{h_n}_{\delta_n}(\Gamma_{n_m})\right) \\
&\le \lim_{n\to\infty} \liminf_{m\to\infty} \left(\mathcal{J}^{h_n}_{\delta_n}(\Gamma_n) + {\mathcal{J}^{h_n}_{\delta_n}}'(\Gamma_{n_m})(\Gamma_{n_m} - \Gamma_n)\right).
\end{aligned}$$

Using (3.4), we infer from the last inequality that

$$C\|\mathcal{U}_{\Gamma_0} - \mathcal{U}_{\Gamma^\dagger}\|^2_{H^1(\Omega)} \le \lim_{n\to\infty} \liminf_{m\to\infty} {\mathcal{J}^{h_n}_{\delta_n}}'(\Gamma_{n_m})(\Gamma_{n_m} - \Gamma_n). \tag{3.7}$$

In view of (2.14) we get that

$$\begin{aligned}
&{\mathcal{J}^{h_n}_{\delta_n}}'(\Gamma_{n_m})(\Gamma_{n_m} - \Gamma_n) \\
&= \int_\Omega (Q_{n_m} - Q_n) \nabla \bar{\Pi}^{h_n} z_{\delta_n} \cdot \nabla \bar{\Pi}^{h_n} z_{\delta_n} - 2\left(f_{n_m} - f_n, \bar{\Pi}^{h_n} z_{\delta_n}\right) - 2\left\langle g_{n_m} - g_n, \gamma \bar{\Pi}^{h_n} z_{\delta_n}\right\rangle \\
&\quad - \int_\Omega (Q_{n_m} - Q_n) \nabla \mathcal{U}^{h_n}_{\Gamma_{n_m}} \cdot \nabla \mathcal{U}^{h_n}_{\Gamma_{n_m}} + 2\left(f_{n_m} - f_n, \mathcal{U}^{h_n}_{\Gamma_{n_m}}\right) + 2\left\langle g_{n_m} - g_n, \gamma \mathcal{U}^{h_n}_{\Gamma_{n_m}}\right\rangle \\
&:= A_1 - 2A_2 - 2A_3 - A_4 + 2A_5 + 2A_6.
\end{aligned} \tag{3.8}$$

Since $Q_{n_m} \rightharpoonup Q_0$ weakly* in $\mathbf{L}^\infty_{\mathrm{sym}}(\Omega)$ as $m \to \infty$, we have for the first term that

$$\begin{aligned}
&\lim_{n\to\infty} \lim_{m\to\infty} A_1 \\
&:= \lim_{n\to\infty} \left(\lim_{m\to\infty} \int_\Omega (Q_{n_m} - Q_n) \nabla \bar{\Pi}^{h_n} z_{\delta_n} \cdot \nabla \bar{\Pi}^{h_n} z_{\delta_n}\right) = \lim_{n\to\infty} \int_\Omega (Q_0 - Q_n) \nabla \bar{\Pi}^{h_n} z_{\delta_n} \cdot \nabla \bar{\Pi}^{h_n} z_{\delta_n} \\
&= \underbrace{\lim_{n\to\infty} \int_\Omega (Q_0 - Q_n) \nabla \mathcal{U}_{\Gamma^\dagger} \cdot \nabla \mathcal{U}_{\Gamma^\dagger}}_{=0, \text{ since } Q_n \rightharpoonup Q_0 \text{ weakly}^* \text{ in } \mathbf{L}^\infty_{\mathrm{sym}}(\Omega)} + \lim_{n\to\infty} \int_\Omega (Q_0 - Q_n) \nabla \left(\bar{\Pi}^{h_n} z_{\delta_n} - \mathcal{U}_{\Gamma^\dagger}\right) \cdot \nabla \left(\bar{\Pi}^{h_n} z_{\delta_n} + \mathcal{U}_{\Gamma^\dagger}\right) \\
&= \lim_{n\to\infty} \int_\Omega (Q_0 - Q_n) \nabla \left(\bar{\Pi}^{h_n} z_{\delta_n} - \mathcal{U}_{\Gamma^\dagger}\right) \cdot \nabla \left(\bar{\Pi}^{h_n} z_{\delta_n} + \mathcal{U}_{\Gamma^\dagger}\right).
\end{aligned}$$

Furthermore, by (3.6), we get that

$$\begin{aligned}
&\lim_{n\to\infty} \left| \int_\Omega (Q_0 - Q_n) \nabla \left(\bar{\Pi}^{h_n} z_{\delta_n} - \mathcal{U}_{\Gamma^\dagger}\right) \cdot \nabla \left(\bar{\Pi}^{h_n} z_{\delta_n} + \mathcal{U}_{\Gamma^\dagger}\right) \right| \\
&\quad \le \lim_{n\to\infty} C \left\|\nabla(\bar{\Pi}^{h_n} z_{\delta_n} - \mathcal{U}_{\Gamma^\dagger})\right\|_{L^2(\Omega)} = \lim_{n\to\infty} C \left\|\nabla(\Pi^{h_n} z_{\delta_n} - \mathcal{U}_{\Gamma^\dagger})\right\|_{L^2(\Omega)} \\
&\quad \le C \lim_{n\to\infty} \left\|\Pi^{h_n} z_{\delta_n} - \mathcal{U}_{\Gamma^\dagger}\right\|_{H^1(\Omega)} = 0.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \lim_{m\to\infty} A_1 = 0. \tag{3.9}$$

On the other hand, we get

$$\begin{aligned}
\lim_{n\to\infty}\lim_{m\to\infty} A_2 &:= \lim_{n\to\infty}\lim_{m\to\infty}\left(f_{n_m}-f_n,\bar{\Pi}^{h_n}z_{\delta_n}\right) = \lim_{n\to\infty}\left(f_0-f_n,\bar{\Pi}^{h_n}z_{\delta_n}\right)\\
&= \underbrace{\lim_{n\to\infty}\left(f_0-f_n,\mathcal{U}_{\Gamma^\dagger}\right)}_{=0} + \lim_{n\to\infty}\left(f_0-f_n,\bar{\Pi}^{h_n}z_{\delta_n}-\mathcal{U}_{\Gamma^\dagger}\right)\\
&\le C\lim_{n\to\infty}\left\|\bar{\Pi}^{h_n}z_{\delta_n}-\mathcal{U}_{\Gamma^\dagger}\right\|_{L^2(\Omega)} \overset{\text{by (1.4)}}{\le} C\lim_{n\to\infty}\left\|\nabla\left(\bar{\Pi}^{h_n}z_{\delta_n}-\mathcal{U}_{\Gamma^\dagger}\right)\right\|_{L^2(\Omega)} = 0. \qquad (3.10)
\end{aligned}$$

We now have that

$$\begin{aligned}
\lim_{n\to\infty}\lim_{m\to\infty} A_3 &:= \lim_{n\to\infty}\lim_{m\to\infty}\left\langle g_{n_m}-g_n,\gamma\bar{\Pi}^{h_n}z_{\delta_n}\right\rangle = \lim_{n\to\infty}\left\langle g_0-g_n,\gamma\bar{\Pi}^{h_n}z_{\delta_n}\right\rangle\\
&= \lim_{n\to\infty}\left\langle g_0-g_n,\gamma\Pi^{h_n}z_{\delta_n}\right\rangle - |\partial\Omega|^{-1}\lim_{n\to\infty}\left\langle g_0-g_n,\left\langle 1,\gamma\Pi^{h_n}z_{\delta_n}\right\rangle\right\rangle
\end{aligned}$$

with

$$\begin{aligned}
\lim_{n\to\infty}\left\langle g_0-g_n,\gamma\Pi^{h_n}z_{\delta_n}\right\rangle &= \lim_{n\to\infty}\left\langle g_0-g_n,\gamma\left(\Pi^{h_n}z_{\delta_n}-\mathcal{U}_{\Gamma^\dagger}\right)\right\rangle + \underbrace{\lim_{n\to\infty}\left\langle g_0-g_n,\gamma\mathcal{U}_{\Gamma^\dagger}\right\rangle}_{=0}\\
&\le C\lim_{n\to\infty}\|g_0-g_n\|_{L^2(\partial\Omega)}\|\gamma\|_{\mathcal{L}\left(H^1(\Omega),H^{1/2}(\partial\Omega)\right)}\left\|\Pi^{h_n}z_{\delta_n}-\mathcal{U}_{\Gamma^\dagger}\right\|_{H^1(\Omega)}\\
&\le C\lim_{n\to\infty}\left\|\Pi^{h_n}z_{\delta_n}-\mathcal{U}_{\Gamma^\dagger}\right\|_{H^1(\Omega)} = 0
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty}\left\langle g_0-g_n,\left\langle 1,\gamma\Pi^{h_n}z_{\delta_n}\right\rangle\right\rangle &\le \lim_{n\to\infty}\left|\left\langle 1,\gamma\Pi^{h_n}z_{\delta_n}\right\rangle\right|\left|\langle g_0-g_n,1\rangle\right| \le C\lim_{n\to\infty}\left\|\Pi^{h_n}z_{\delta_n}\right\|_{H^1(\Omega)}\left|\langle g_0-g_n,1\rangle\right|\\
&\le C\lim_{n\to\infty}\left|\langle g_0-g_n,1\rangle\right| = 0
\end{aligned}$$

so that

$$\lim_{n\to\infty}\lim_{m\to\infty} A_3 = 0. \qquad (3.11)$$

Next, we rewrite

$$\begin{aligned}
\lim_{n\to\infty}\lim_{m\to\infty} A_4 &:= \lim_{n\to\infty}\lim_{m\to\infty}\int_\Omega \left(Q_{n_m}-Q_n\right)\nabla\mathcal{U}^{h_n}_{\Gamma_{n_m}}\cdot\nabla\mathcal{U}^{h_n}_{\Gamma_{n_m}}\\
&= \lim_{n\to\infty}\lim_{m\to\infty}\int_\Omega \left(Q_{n_m}-Q_n\right)\nabla\mathcal{U}^{h_n}_{\Gamma_0}\cdot\nabla\mathcal{U}^{h_n}_{\Gamma_0}\\
&\quad + \lim_{n\to\infty}\lim_{m\to\infty}\int_\Omega \left(Q_{n_m}-Q_n\right)\nabla\left(\mathcal{U}^{h_n}_{\Gamma_{n_m}}-\mathcal{U}^{h_n}_{\Gamma_0}\right)\cdot\nabla\left(\mathcal{U}^{h_n}_{\Gamma_{n_m}}+\mathcal{U}^{h_n}_{\Gamma_0}\right).
\end{aligned}$$

By (2.7), likewise as (3.9), we get that

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_\Omega \left(Q_{n_m}-Q_n\right)\nabla\mathcal{U}^{h_n}_{\Gamma_0}\cdot\nabla\mathcal{U}^{h_n}_{\Gamma_0} = 0.$$

Furthermore, we have

$$\left|\int_\Omega \left(Q_{n_m}-Q_n\right)\nabla\left(\mathcal{U}^{h_n}_{\Gamma_{n_m}}-\mathcal{U}^{h_n}_{\Gamma_0}\right)\cdot\nabla\left(\mathcal{U}^{h_n}_{\Gamma_{n_m}}+\mathcal{U}^{h_n}_{\Gamma_0}\right)\right| \le C\left\|\mathcal{U}^{h_n}_{\Gamma_{n_m}}-\mathcal{U}^{h_n}_{\Gamma_0}\right\|_{H^1(\Omega)}.$$

By Lemma 2.3, for each fixed n we have that the sequence $\left(\mathcal{U}^{h_n}_{\Gamma_{n_m}}\right)_m \subset \mathcal{V}^{h_n}_1$ converges to $\mathcal{U}^{h_n}_{\Gamma_0}$ in the $H^1(\Omega)$-norm as m tends to ∞. Then we deduce that

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\left|\int_\Omega \left(Q_{n_m}-Q_n\right)\nabla\left(\mathcal{U}^{h_n}_{\Gamma_{n_m}}-\mathcal{U}^{h_n}_{\Gamma_0}\right)\cdot\nabla\left(\mathcal{U}^{h_n}_{\Gamma_{n_m}}+\mathcal{U}^{h_n}_{\Gamma_0}\right)\right|\\
&\quad \le C\lim_{n\to\infty}\lim_{m\to\infty}\left\|\mathcal{U}^{h_n}_{\Gamma_{n_m}}-\mathcal{U}^{h_n}_{\Gamma_0}\right\|_{H^1(\Omega)} = C\lim_{n\to\infty}\left\|\mathcal{U}^{h_n}_{\Gamma_0}-\mathcal{U}^{h_n}_{\Gamma_0}\right\|_{H^1(\Omega)} = 0.
\end{aligned}$$

Thus, we obtain

$$\lim_{n\to\infty}\lim_{m\to\infty} A_4 = 0. \tag{3.12}$$

Finally, we also get that

$$\begin{aligned}
\lim_{n\to\infty}\lim_{m\to\infty} A_5 &:= \lim_{n\to\infty}\lim_{m\to\infty}\left(f_{n_m} - f_n, \mathcal{U}^{h_n}_{\Gamma_{n_m}}\right)\\
&= \lim_{n\to\infty}\lim_{m\to\infty}\left(f_{n_m} - f_n, \mathcal{U}^{h_n}_{\Gamma_0}\right) + \lim_{n\to\infty}\lim_{m\to\infty}\left(f_{n_m} - f_n, \mathcal{U}^{h_n}_{\Gamma_{n_m}} - \mathcal{U}^{h_n}_{\Gamma_0}\right)\\
&\le \lim_{n\to\infty}\left(f_0 - f_n, \mathcal{U}^{h_n}_{\Gamma_0}\right) + C\lim_{n\to\infty}\lim_{m\to\infty}\left\|\mathcal{U}^{h_n}_{\Gamma_{n_m}} - \mathcal{U}^{h_n}_{\Gamma_0}\right\|_{H^1(\Omega)} = 0
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
\lim_{n\to\infty}\lim_{m\to\infty} A_6 &:= \lim_{n\to\infty}\lim_{m\to\infty}\left\langle g_{n_m} - g_n, \gamma\mathcal{U}^{h_n}_{\Gamma_{n_m}}\right\rangle = \lim_{n\to\infty}\lim_{m\to\infty}\left\langle g_{n_m} - g_n, \gamma\big(\mathcal{U}^{h_n}_{\Gamma_{n_m}} - \mathcal{U}^{h_n}_{\Gamma_0}\big)\right\rangle\\
&\le C\lim_{n\to\infty}\lim_{m\to\infty}\left\|\gamma\big(\mathcal{U}^{h_n}_{\Gamma_{n_m}} - \mathcal{U}^{h_n}_{\Gamma_0}\big)\right\|_{L^2(\partial\Omega)} \le C\lim_{n\to\infty}\lim_{m\to\infty}\left\|\mathcal{U}^{h_n}_{\Gamma_{n_m}} - \mathcal{U}^{h_n}_{\Gamma_0}\right\|_{H^1(\Omega)} = 0.
\end{aligned} \tag{3.14}$$

Therefore, it follows from the equations (3.8)–(3.14) that

$$\lim_{n\to\infty}\lim_{m\to\infty} {\mathcal{J}^{h_n}_{\delta_n}}'\left(\Gamma_{n_m}\right)\left(\Gamma_{n_m} - \Gamma_n\right) = 0.$$

Combining this with (3.7), we obtain that

$$\mathcal{U}_{\Gamma_0} = \mathcal{U}_{\Gamma^\dagger}.$$

Then, by the definition of $\Gamma^\dagger$, the weakly l.s.c. property of $\mathcal{R}$ and (3.5), we get

$$\mathcal{R}(\Gamma^\dagger) \le \mathcal{R}(\Gamma_0) \le \liminf_n \mathcal{R}\left(\Gamma_n\right) \le \limsup_n \mathcal{R}\left(\Gamma_n\right) \le \mathcal{R}(\Gamma^\dagger).$$

Thus,

$$\mathcal{R}(\Gamma^\dagger) = \mathcal{R}(\Gamma_0) = \lim_{n\to\infty}\mathcal{R}\left(\Gamma_n\right).$$

By the uniqueness of $\Gamma^\dagger$, we have that $\Gamma_0 = \Gamma^\dagger$. Furthermore, since $\left(\Gamma_n\right)_n$ weakly converges in $\mathbf{L}^2_{\mathrm{sym}}(\Omega) \times L^2(\Omega) \times L^2(\partial\Omega)$ to Γ_0, we conclude from the last equation that $\left(\Gamma_n\right)_n$ converges to Γ_0 in the $\mathbf{L}^2_{\mathrm{sym}}(\Omega) \times L^2(\Omega) \times L^2(\partial\Omega)$-norm.

It remains to show that the sequence $\big(\mathcal{U}^{h_n}_{\Gamma_n}\big)_n$ converges to $\Phi^\dagger = \mathcal{U}_{\Gamma^\dagger}$ in the $H^1(\Omega)$-norm. We first get from (2.7) that

$$\lim_{n\to\infty}\big\|\mathcal{U}_{\Gamma^\dagger} - \mathcal{U}^{h_n}_{\Gamma^\dagger}\big\|_{H^1(\Omega)} = 0. \tag{3.15}$$

Furthermore, in view of (2.12) we also have that

$$\begin{aligned}
\frac{C_\Omega \underline{q}}{1 + C_\Omega}\big\|\mathcal{U}^{h_n}_{\Gamma_n} - \mathcal{U}^{h_n}_{\Gamma^\dagger}\big\|^2_{H^1(\Omega)} &\le \int_\Omega (Q^\dagger - Q_n)\nabla\mathcal{U}^{h_n}_{\Gamma^\dagger}\cdot\nabla\big(\mathcal{U}^{h_n}_{\Gamma_n} - \mathcal{U}^{h_n}_{\Gamma^\dagger}\big)\\
&\quad + \big(f_n - f^\dagger, \mathcal{U}^{h_n}_{\Gamma_n} - \mathcal{U}^{h_n}_{\Gamma^\dagger}\big) + \left\langle g_n - g^\dagger, \gamma\big(\mathcal{U}^{h_n}_{\Gamma_n} - \mathcal{U}^{h_n}_{\Gamma^\dagger}\big)\right\rangle.
\end{aligned} \tag{3.16}$$

Since $f_n \to f^\dagger$ in the $L^2(\Omega)$-norm and $g_n \to g^\dagger$ in the $L^2(\partial\Omega)$-norm together with the uniform boundedness (2.5), it follows that

$$\lim_{n\to\infty}\left(\big(f_n - f^\dagger, \mathcal{U}^{h_n}_{\Gamma_n} - \mathcal{U}^{h_n}_{\Gamma^\dagger}\big) + \left\langle g_n - g^\dagger, \gamma\big(\mathcal{U}^{h_n}_{\Gamma_n} - \mathcal{U}^{h_n}_{\Gamma^\dagger}\big)\right\rangle\right) = 0. \tag{3.17}$$

We now rewrite

$$\begin{aligned}
\int_\Omega (Q^\dagger - Q_n)\nabla\mathcal{U}^{h_n}_{\Gamma^\dagger}\cdot\nabla\big(\mathcal{U}^{h_n}_{\Gamma_n} - \mathcal{U}^{h_n}_{\Gamma^\dagger}\big) &= \int_\Omega (Q^\dagger - Q_n)\nabla\mathcal{U}_{\Gamma^\dagger}\cdot\nabla\big(\mathcal{U}^{h_n}_{\Gamma_n} - \mathcal{U}^{h_n}_{\Gamma^\dagger}\big)\\
&\quad + \int_\Omega (Q^\dagger - Q_n)\nabla\big(\mathcal{U}^{h_n}_{\Gamma^\dagger} - \mathcal{U}_{\Gamma^\dagger}\big)\cdot\nabla\big(\mathcal{U}^{h_n}_{\Gamma_n} - \mathcal{U}^{h_n}_{\Gamma^\dagger}\big)\\
&\le C\left(\int_\Omega \|Q^\dagger - Q_n\|^2_{\mathcal{S}_d}|\nabla\mathcal{U}_{\Gamma^\dagger}|^2\right)^{1/2} + C\big\|\mathcal{U}_{\Gamma^\dagger} - \mathcal{U}^{h_n}_{\Gamma^\dagger}\big\|_{H^1(\Omega)}.
\end{aligned}$$

Since $Q_n \to Q^\dagger$ in the $\mathbf{L}^2_{\rm sym}(\Omega)$-norm, up to a subsequence we assume that $(Q_n)_n$ converges to $Q^\dagger$ a.e. in Ω. Then, by the Lebesgue dominated convergence theorem, we deduce that

$$\lim_{n\to\infty}\int_\Omega \|Q^\dagger - Q_n\|^2_{\mathcal{S}_d}|\nabla \mathcal{U}_{\Gamma^\dagger}|^2 = 0.$$

Thus, together with (3.15), we have

$$\lim_{n\to\infty}\int_\Omega (Q^\dagger - Q_n)\nabla \mathcal{U}^{h_n}_{\Gamma^\dagger}\cdot\nabla\big(\mathcal{U}^{h_n}_{\Gamma_n} - \mathcal{U}^{h_n}_{\Gamma^\dagger}\big) = 0. \tag{3.18}$$

It follows from (3.16)–(3.18) that $\lim_{n\to\infty}\big\|\mathcal{U}^{h_n}_{\Gamma_n} - \mathcal{U}^{h_n}_{\Gamma^\dagger}\big\|_{H^1(\Omega)} = 0$. Then, by serving of (3.15) again, we conclude that $\lim_{n\to\infty}\big\|\mathcal{U}^{h_n}_{\Gamma_n} - \mathcal{U}_{\Gamma^\dagger}\big\|_{H^1(\Omega)} = 0$, which finishes the proof. □

4 Error bounds

In this section we investigate error bounds of discrete regularized solutions to the identification problem. For any $\Gamma := (Q,f,g) \in \mathcal{H}_{ad}$ the mapping

$$\mathcal{U}'_\Gamma : \mathbf{L}^\infty_{\rm sym}(\Omega)\times L^2(\Omega)\times L^2(\partial\Omega) \to H^1_\diamond(\Omega)$$

is linear, continuous with the dual

$$\mathcal{U}'^{\,*}_\Gamma : H^1_\diamond(\Omega)^* \to \mathbf{L}^\infty_{\rm sym}(\Omega)^*\times L^2(\Omega)\times L^2(\partial\Omega).$$

Theorem 4.1. *Assume that a function $w^* \in H^1_\diamond(\Omega)^*$ exists such that*

$$\mathcal{U}'^{\,*}_{\Gamma^\dagger} w^* = \Gamma^\dagger. \tag{4.1}$$

Then

$$\big\|\mathcal{U}^h_{\Gamma^h} - \mathcal{U}_{\Gamma^\dagger}\big\|^2_{H^1(\Omega)} + \rho\big\|\Gamma^h - \Gamma^\dagger\big\|^2_{\mathbf{L}^2_{sym}(\Omega)\times L^2(\Omega)\times L^2(\partial\Omega)} = \mathcal{O}\Big(h^{-2}\delta^2 + \big(\chi^h_{\Phi^\dagger}\big)^2 + \big(\beta^h_{\mathcal{U}_{\Gamma^\dagger}}\big)^2 + \big(\chi^h_w\big)^2 + \rho^2\Big), \tag{4.2}$$

where $\Gamma^h := (Q^h, f^h, g^h)$ is the unique solution to $\big(\mathcal{P}^{\rho,h}_\delta\big)$ and $w \in H^1_\diamond(\Omega)$ is the unique weak solution of the Neumann problem

$$-\nabla\cdot(Q^\dagger\nabla w) = f^\dagger + w^* \ \text{ in } \Omega \text{ and } Q^\dagger\nabla w\cdot\vec{n} = g^\dagger \ \text{ on } \partial\Omega. \tag{4.3}$$

Remark 4.2. Due to Remark 3.2, in case $\mathcal{U}_{\Gamma^\dagger},\ w \in H^2(\Omega)$ we have

$$0 \le \chi^h_{\Phi^\dagger},\ \beta^h_{\mathcal{U}_{\Gamma^\dagger}},\ \chi^h_w \le Ch.$$

Therefore, with $\delta \sim h^2$ and $\rho \sim h$ we obtain the following error bounds

$$\big\|\mathcal{U}^h_{\Gamma^h} - \mathcal{U}_{\Gamma^\dagger}\big\|_{H^1(\Omega)} = \mathcal{O}(h) \text{ and } \big\|\Gamma^h - \Gamma^\dagger\big\|_{\mathbf{L}^2_{\rm sym}(\Omega)\times L^2(\Omega)\times L^2(\partial\Omega)} = \mathcal{O}\big(h^{1/2}\big).$$

Proof of Theorem 4.1. The line of argument follows the proof of [22, Theorem 5.1] with a slight mollification. Due to the optimality of Γ^h, we get that

$$\mathcal{J}^h_\delta\left(\Gamma^h\right) + \rho\mathcal{R}\left(\Gamma^h\right) \le \mathcal{J}^h_\delta\left(\Gamma^\dagger\right) + \rho\mathcal{R}\left(\Gamma^\dagger\right)$$

which implies

$$\begin{aligned}\mathcal{J}^h_\delta\left(\Gamma^h\right) + \rho\big\|\Gamma^h - \Gamma^\dagger\big\|^2_{\mathbf{L}^2_{\rm sym}(\Omega)\times L^2(\Omega)\times L^2(\partial\Omega)} &\le \mathcal{J}^h_\delta\left(\Gamma^\dagger\right) + 2\rho\left\langle \Gamma^\dagger, \Gamma^\dagger - \Gamma^h\right\rangle_{\mathbf{L}^2_{\rm sym}(\Omega)\times L^2(\Omega)\times L^2(\partial\Omega)}\\ &\le C\Big(h^{-2}\delta^2 + \big(\chi^h_{\Phi^\dagger}\big)^2 + \big(\beta^h_{\mathcal{U}_{\Gamma^\dagger}}\big)^2\Big) + 2\rho\left\langle \Gamma^\dagger, \Gamma^\dagger - \Gamma^h\right\rangle_{\mathbf{L}^2_{\rm sym}(\Omega)\times L^2(\Omega)\times L^2(\partial\Omega)},\end{aligned} \tag{4.4}$$

by Lemma 3.3. Now, by (2.2) and (4.1), we infer that

$$\begin{aligned}
I &:= \left\langle \Gamma^\dagger, \Gamma^\dagger - \Gamma^h \right\rangle_{\mathbf{L}^2_{\mathrm{sym}}(\Omega)\times L^2(\Omega)\times L^2(\partial\Omega)} = \left\langle \Gamma^\dagger, \Gamma^\dagger - \Gamma^h \right\rangle_{\left(\mathbf{L}^\infty_{\mathrm{sym}}(\Omega)^*\times L^2(\Omega)\times L^2(\partial\Omega), \mathbf{L}^\infty_{\mathrm{sym}}(\Omega)\times L^2(\Omega)\times L^2(\partial\Omega)\right)} \\
&= \left\langle {\mathcal{U}'_{\Gamma^\dagger}}^* w^*, \Gamma^\dagger - \Gamma^h \right\rangle_{\left(\mathbf{L}^\infty_{\mathrm{sym}}(\Omega)^*\times L^2(\Omega)\times L^2(\partial\Omega), \mathbf{L}^\infty_{\mathrm{sym}}(\Omega)\times L^2(\Omega)\times L^2(\partial\Omega)\right)} \\
&= \left\langle w^*, \mathcal{U}'_{\Gamma^\dagger}\left(\Gamma^\dagger - \Gamma^h\right) \right\rangle_{(H^1(\Omega)^*, H^1(\Omega))}. \qquad (4.5)
\end{aligned}$$

Thus, by the definition of the weak solution to (4.3) and (2.1), we obtain

$$\begin{aligned}
I &= \int_\Omega Q^\dagger \nabla \mathcal{U}'_{\Gamma^\dagger}\left(\Gamma^\dagger - \Gamma^h\right)\cdot\nabla w \underbrace{-\left(f^\dagger, \mathcal{U}'_{\Gamma^\dagger}\left(\Gamma^\dagger - \Gamma^h\right)\right) - \left\langle g^\dagger, \gamma \mathcal{U}'_{\Gamma^\dagger}\left(\Gamma^\dagger - \Gamma^h\right)\right\rangle}_{-\int_\Omega Q^\dagger\nabla \mathcal{U}_{\Gamma^\dagger}\cdot\nabla \mathcal{U}'_{\Gamma^\dagger}(\Gamma^\dagger - \Gamma^h),\ \text{by (1.6)}} \\
&= \int_\Omega Q^\dagger \nabla \mathcal{U}'_{\Gamma^\dagger}\left(\Gamma^\dagger - \Gamma^h\right)\cdot\nabla\left(w - \mathcal{U}_{\Gamma^\dagger}\right) \\
&= -\int_\Omega \left(Q^\dagger - Q^h\right)\nabla \mathcal{U}_{\Gamma^\dagger}\cdot\nabla\left(w - \mathcal{U}_{\Gamma^\dagger}\right) + \left(f^\dagger - f^h, w - \mathcal{U}_{\Gamma^\dagger}\right) + \left\langle g^\dagger - g^h, \gamma\left(w - \mathcal{U}_{\Gamma^\dagger}\right)\right\rangle \\
&= \underbrace{-\int_\Omega Q^\dagger \nabla \mathcal{U}_{\Gamma^\dagger}\cdot\nabla\left(w - \mathcal{U}_{\Gamma^\dagger}\right) + \left(f^\dagger, w - \mathcal{U}_{\Gamma^\dagger}\right) + \left\langle g^\dagger, \gamma\left(w - \mathcal{U}_{\Gamma^\dagger}\right)\right\rangle}_{=0,\ \text{by (1.6)}} \\
&\quad + \int_\Omega Q^h \nabla \mathcal{U}_{\Gamma^\dagger}\cdot\nabla\left(w - \mathcal{U}_{\Gamma^\dagger}\right) \underbrace{-\left(f^h, w - \mathcal{U}_{\Gamma^\dagger}\right) - \left\langle g^h, \gamma\left(w - \mathcal{U}_{\Gamma^\dagger}\right)\right\rangle}_{-\int_\Omega Q^h\nabla\mathcal{U}_{\Gamma^h}\cdot\nabla\left(w - \mathcal{U}_{\Gamma^\dagger}\right)} = \int_\Omega Q^h\nabla\left(\mathcal{U}_{\Gamma^\dagger} - \mathcal{U}_{\Gamma^h}\right)\cdot\nabla\left(w - \mathcal{U}_{\Gamma^\dagger}\right)
\end{aligned}$$

which yields

$$\begin{aligned}
I &= \int_\Omega Q^h\nabla\left(\mathcal{U}_{\Gamma^\dagger} - \Pi^h z_\delta\right)\cdot\nabla\left(w - \mathcal{U}_{\Gamma^\dagger}\right) + \int_\Omega Q^h\nabla\left(\mathcal{U}^h_{\Gamma^h} - \mathcal{U}_{\Gamma^h}\right)\cdot\nabla\left(w - \mathcal{U}_{\Gamma^\dagger}\right) \\
&\quad + \int_\Omega Q^h\nabla\left(\Pi^h z_\delta - \mathcal{U}^h_{\Gamma^h}\right)\cdot\nabla\left(w - \mathcal{U}_{\Gamma^\dagger}\right) := I_1 + I_2 + I_3. \qquad (4.6)
\end{aligned}$$

For I_1 we have from (3.2) that

$$I_1 := \int_\Omega Q^h\nabla\left(\mathcal{U}_{\Gamma^\dagger} - \Pi^h z_\delta\right)\cdot\nabla\left(w - \mathcal{U}_{\Gamma^\dagger}\right) \le C\left\|\mathcal{U}_{\Gamma^\dagger} - \Pi^h z_\delta\right\|_{H^1(\Omega)} \le Ch^{-1}\delta + \chi^h_{\Phi^\dagger}. \qquad (4.7)$$

Due to (1.6) and (2.4), we get $\int_\Omega Q^h\nabla\left(\mathcal{U}^h_{\Gamma^h} - \mathcal{U}_{\Gamma^h}\right)\cdot\nabla\Pi^h\left(w - \mathcal{U}_{\Gamma^\dagger}\right) = 0$ and then infer that

$$\begin{aligned}
I_2 &:= \int_\Omega Q^h\nabla\left(\mathcal{U}^h_{\Gamma^h} - \mathcal{U}_{\Gamma^h}\right)\cdot\nabla\left(w - \mathcal{U}_{\Gamma^\dagger}\right) = \int_\Omega Q^h\nabla\left(\mathcal{U}^h_{\Gamma^h} - \mathcal{U}_{\Gamma^h}\right)\cdot\nabla\left(w - \mathcal{U}_{\Gamma^\dagger} - \Pi^h\left(w - \mathcal{U}_{\Gamma^\dagger}\right)\right) \\
&\le C\left(\left\|w - \Pi^h w\right\|_{H^1(\Omega)} + \left\|\mathcal{U}_{\Gamma^\dagger} - \Pi^h\mathcal{U}_{\Gamma^\dagger}\right\|_{H^1(\Omega)}\right) \le C\left(\chi^h_w + \chi^h_{\Phi^\dagger}\right). \qquad (4.8)
\end{aligned}$$

Finally, we have that

$$\begin{aligned}
I_3 &:= \int_\Omega Q^h\nabla\left(\Pi^h z_\delta - \mathcal{U}^h_{\Gamma^h}\right)\cdot\nabla\left(w - \mathcal{U}_{\Gamma^\dagger}\right) \\
&\le \left(\int_\Omega Q^h\nabla\left(\mathcal{U}^h_{\Gamma^h} - \Pi^h z_\delta\right)\cdot\nabla\left(\mathcal{U}^h_{\Gamma^h} - \Pi^h z_\delta\right)\right)^{1/2}\cdot\left(\int_\Omega Q^h\nabla\left(w - \mathcal{U}_{\Gamma^\dagger}\right)\cdot\nabla\left(w - \mathcal{U}_{\Gamma^\dagger}\right)\right)^{1/2} \\
&\le \underbrace{\frac{1}{4\rho}\int_\Omega Q^h\nabla\left(\mathcal{U}^h_{\Gamma^h} - \Pi^h z_\delta\right)\cdot\nabla\left(\mathcal{U}^h_{\Gamma^h} - \Pi^h z_\delta\right)}_{\mathcal{J}^h_\delta(\Gamma^h)} + \rho\int_\Omega Q^h\nabla\left(w - \mathcal{U}_{\Gamma^\dagger}\right)\cdot\nabla\left(w - \mathcal{U}_{\Gamma^\dagger}\right) \\
&\le \frac{1}{4\rho}\mathcal{J}^h_\delta\left(\Gamma^h\right) + C\rho. \qquad (4.9)
\end{aligned}$$

It follows from (4.6)–(4.9) that

$$I \le C\left(h^{-1}\delta + \chi^h_{\Phi^\dagger} + \chi^h_w + \rho\right) + \frac{1}{4\rho}\mathcal{J}^h_\delta\left(\Gamma^h\right).$$

Thus, together with (4.4)–(4.5), we get

$$\frac{1}{2}\mathcal{J}_\delta^h\left(\Gamma^h\right) + \rho\left\|\Gamma^h - \Gamma^\dagger\right\|^2_{\mathbf{L}^2_{\rm sym}(\Omega)\times L^2(\Omega)\times L^2(\partial\Omega)} \le C\left(h^{-2}\delta^2 + \left(\chi^h_{\Phi^\dagger}\right)^2 + \left(\beta^h_{\mathcal{U}_{\Gamma^\dagger}}\right)^2 + \left(\chi^h_w\right)^2 + \rho^2\right),$$

which finishes the proof. □

5 Gradient projection algorithm with Armijo steplength rule

In this section we present the gradient projection algorithm with Armijo steplength rule (cf. [28, 38]) for numerical solution of the minimization problem $\left(\mathcal{P}^{\rho,h}_\delta\right)$.

We first note that for each $\Gamma := (Q,f,g) \in \mathcal{H}_{ad}$, in view of (2.14), the $\mathcal{L}^2$-gradient of the strictly convex cost function $\Upsilon^{\rho,h}_\delta$ of the problem $\left(\mathcal{P}^{\rho,h}_\delta\right)$ is given by $\nabla\Upsilon^{\rho,h}_\delta(\Gamma) := \left(\Upsilon_Q(\Gamma), \Upsilon_f(\Gamma), \Upsilon_g(\Gamma)\right)$ with

$$\begin{cases} \Upsilon_Q(\Gamma) &= \nabla\bar\Pi^h z_\delta \otimes \nabla\bar\Pi^h z_\delta - \nabla\mathcal{U}^h_\Gamma \otimes \nabla\mathcal{U}^h_\Gamma + 2\rho Q,\\ \Upsilon_f(\Gamma) &= 2\big(\mathcal{U}^h_\Gamma - \bar\Pi^h z_\delta + \rho f\big),\\ \Upsilon_g(\Gamma) &= 2\big(\gamma\big(\mathcal{U}^h_\Gamma - \bar\Pi^h z_\delta\big) + \rho g\big) \end{cases}$$

and $\bar\Pi^h$ generating from Π^h according to (2.13).

The algorithm is then read as: given a step size control $\beta \in (0,1)$, an initial approximation (cf. Remark 2.6) $\Gamma_0 := (Q_0, f_0, g_0) \in \mathcal{H}_{ad} \cap \left({\mathcal{V}^h_0}^{d\times d} \times \mathcal{V}^h_1 \times \mathcal{E}^h_1\right)$, number of iteration N and setting $k = 0$.

1. Compute $\mathcal{U}^h_{\Gamma_k}$ from the variational equation

$$\int_\Omega Q_k \nabla\mathcal{U}^h_{\Gamma_k}\cdot\nabla\varphi^h = \left(f_k, \varphi^h\right) + \left\langle g_k, \gamma\varphi^h\right\rangle \text{ for all } \varphi^h \in \mathcal{V}^h_1 \tag{5.1}$$

   as well as

$$\Upsilon^h_{\rho,\delta}(\Gamma_k) = \int_\Omega Q_k\nabla\left(\mathcal{U}^h_{\Gamma_k} - \bar\Pi^h z_\delta\right)\cdot\nabla\left(\mathcal{U}^h_{\Gamma_k} - \bar\Pi^h z_\delta\right) + \rho\big(\|Q_k\|^2_{\mathbf{L}^2_{\rm sym}(\Omega)} + \|f_k\|^2_{L^2(\Omega)} + \|g_k\|^2_{L^2(\partial\Omega)}\big). \tag{5.2}$$

2. Compute the gradient $\nabla\Upsilon^{\rho,h}_\delta(\Gamma_k) := \left(\Upsilon_{Q_k}(\Gamma_k), \Upsilon_{f_k}(\Gamma_k), \Upsilon_{g_k}(\Gamma_k)\right)$ with

$$\begin{cases} \Upsilon_{Q_k}(\Gamma_k) &= \nabla\bar\Pi^h z_\delta \otimes \nabla\bar\Pi^h z_\delta - \nabla\mathcal{U}^h_{\Gamma_k} \otimes \nabla\mathcal{U}^h_{\Gamma_k} + 2\rho Q_k,\\ \Upsilon_{f_k}(\Gamma_k) &= 2\big(\mathcal{U}^h_{\Gamma_k} - \bar\Pi^h z_\delta + \rho f_k\big),\\ \Upsilon_{g_k}(\Gamma_k) &= 2\big(\gamma\big(\mathcal{U}^h_{\Gamma_k} - \bar\Pi^h z_\delta\big) + \rho g_k\big). \end{cases}$$

3. Set $\widetilde\Gamma_k := \big(\widetilde Q_k, \widetilde f_k, \widetilde g_k\big)$ with $\widetilde Q_k(x) := P_{\mathcal{K}}\big(Q_k(x) - \beta\Upsilon_{Q_k}(\Gamma_k)(x)\big)$, $\widetilde f_k(x) := f_k(x) - \beta\Upsilon_{f_k}(\Gamma_k)(x)$ and $\widetilde g_k(x) := g_k(x) - \beta\Upsilon_{g_k}(\Gamma_k)(x)$.
   (a) Compute $\mathcal{U}^h_{\widetilde\Gamma_k}$ according to (5.1), $\Upsilon^h_{\rho,\delta}\big(\widetilde\Gamma_k\big)$ according to (5.2), and

   $L := \Upsilon^h_{\rho,\delta}\big(\widetilde\Gamma_k\big) - \Upsilon^h_{\rho,\delta}(\Gamma_k) + \tau\beta\big(\|\widetilde Q_k - Q_k\|^2_{\mathbf{L}^2_{\rm sym}(\Omega)} + \|\widetilde f_k - f_k\|^2_{L^2(\Omega)} + \|\widetilde g_k - g_k\|^2_{L^2(\partial\Omega)}\big)$ with $\tau = 10^{-4}$.

   (b) If $L \le 0$
   go to the next step (c) below
   else
   set $\beta := \frac{\beta}{2}$ and then go back (a)

   (c) Update $\Gamma_k = \widetilde\Gamma_k$, set $k = k+1$.

4. Compute

$$\text{Tolerance} := \left\|\nabla\Upsilon^h_{\rho,\delta}(\Gamma_k)\right\|_{\mathbf{L}^2_{\rm sym}(\Omega)\times L^2(\Omega)\times L^2(\partial\Omega)} - \tau_1 - \tau_2\left\|\nabla\Upsilon^h_{\rho,\delta}(\Gamma_0)\right\|_{\mathbf{L}^2_{\rm sym}(\Omega)\times L^2(\Omega)\times L^2(\partial\Omega)} \tag{5.3}$$

   with $\tau_1 := 10^{-3}h$ and $\tau_2 := 10^{-2}h$. If Tolerance ≤ 0 or $k > N$, then stop; otherwise go back Step 1.

6 Numerical implementation

For illustrating the theoretical result we consider the Neumann problem

$$-\nabla \cdot \left(Q^\dagger \nabla \Phi^\dagger\right) = f^\dagger \text{ in } \Omega, \tag{6.1}$$

$$Q^\dagger \nabla \Phi^\dagger \cdot \vec{n} = g^\dagger \text{ on } \partial\Omega \tag{6.2}$$

with $\Omega = \{x = (x_1, x_2) \in R^2 \mid -1 < x_1, x_2 < 1\}$.

The special constants in the equation (1.3) are chosen as $\underline{q} = 0.05$ and $\overline{q} = 10$. For discretization we divide the interval $(-1, 1)$ into ℓ equal segments, and so the domain $\Omega = (-1, 1)^2$ is divided into $2\ell^2$ triangles, where the diameter of each triangle is $h_\ell = \frac{\sqrt{8}}{\ell}$.

We assume that entries of the symmetric diffusion matrix $Q^\dagger$ are discontinuous which are defined as

$$q^\dagger_{11} = 2\chi_{\Omega_{11}} + \chi_{\Omega\setminus\Omega_{11}},\ q^\dagger_{12} = q^\dagger_{21} = \chi_{\Omega_{12}},\ q^\dagger_{22} = 3\chi_{\Omega_{22}} + 2\chi_{\Omega\setminus\Omega_{22}},$$

where χ_D is the characteristic functional of the Lebesgue measurable set D and

$$\Omega_{11} := \left\{(x_1, x_2) \in \Omega \;\middle|\; |x_1| \le 3/4 \text{ and } |x_2| \le 3/4\right\},\ \Omega_{12} := \left\{(x_1, x_2) \in \Omega \;\middle|\; |x_1| + |x_2| \le 3/4\right\} \text{ and}$$
$$\Omega_{22} := \left\{(x_1, x_2) \in \Omega \;\middle|\; x_1^2 + x_2^2 \le 9/16\right\}.$$

The source functional $f^\dagger$ is assumed to be also discontinuous and defined as

$$f^\dagger = \frac{93 - 2\pi}{48}\chi_{\Omega_1} + \frac{45 - 2\pi}{48}\chi_{\Omega_2} - \frac{3 + 2\pi}{48}\chi_{\Omega\setminus(\Omega_1\cup\Omega_2)},$$

where

$$\Omega_1 := \left\{(x_1, x_2) \in \Omega \;\middle|\; 9(x_1 + 1/2)^2 + 16(x_2 - 1/2)^2 \le 1\right\} \text{ and}$$
$$\Omega_2 := \left\{(x_1, x_2) \in \Omega \;\middle|\; |x_1 - 1/2| \le 1/4 \text{ and } |x_2 + 1/2| \le 1/4\right\}.$$

The Neumann boundary condition $g^\dagger$ is chosen with

$$g^\dagger = -2\chi_{[-1,0]\times\{-1\}} + \chi_{(0,1]\times\{-1\}} - \chi_{[-1,0]\times\{1\}} + 2\chi_{(0,1]\times\{1\}}$$
$$+ 3\chi_{\{-1\}\times(-1,0]} - 4\chi_{\{-1\}\times(0,1)} + 4\chi_{\{1\}\times(-1,0]} - 3\chi_{\{1\}\times(0,1)}.$$

The exact state $\Phi^\dagger$ is then computed from the finite element equation $KU = F$, where K and F are the stiffness matrix and the load vector associated to the problem (6.1)–(6.2), respectively.

We start the computation with the coarsest level $\ell = 3$. To this end, for constructing observations with noise of the exact state $\Phi^\dagger$ on this coarsest grid we use

$$z_{\delta_\ell} := \Phi^\dagger + \mathcal{N}_{\overline{\delta_\ell}} \text{ and } \delta_\ell := \left\|z_{\delta_\ell} - \Phi^\dagger\right\|_{L^2(\Omega)},$$

where $\overline{\delta_\ell} = 10\rho_\ell^{1/2} h_\ell^{3/2}$, $\rho_\ell = 10^{-3} h_\ell$ and $\mathcal{N}_{\overline{\delta_\ell}}$ is a $M^{h_\ell} \times 1$-matrix of random numbers generated from the uniform distribution on the interval $\left(-\overline{\delta_\ell}, \overline{\delta_\ell}\right)$, $M^{h_\ell} = (\ell + 1)^2$ is the number of nodes of the triangulation $\mathcal{T}^{h_\ell}$. Therefore, the exact state $\Phi^\dagger$ is only measured at 16 nodes of $\mathcal{T}^{h_\ell}$.

We use the algorithm described in §5 for computing the numerical solution of the problem $\left(\mathcal{P}^{h_\ell}_{\rho_\ell,\delta_\ell}\right)$. The step size control is chosen with $\beta = 0.75$. As the initial approximation we choose

$$Q_0 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},\ f_0 = \chi_{[-1,0)\times[-1,1]} - \chi_{[0,1]\times[-1,1]},\ g_0 = \chi_{[-1,1]\times\{1\}} - \chi_{[-1,1]\times\{-1\}} + \chi_{\{1\}\times(-1,1)} - \chi_{\{-1\}\times(-1,1)}.$$

At each iteration k we compute Tolerance defined by (5.3). Then the iteration was stopped if Tolerance ≤ 0 or the number of iterations reached the maximum iteration count of 800.

After obtaining the numerical solution $\Gamma_\ell = (Q_\ell, f_\ell, g_\ell)$ and the computed numerical state $\mathcal{U}_\ell = \mathcal{U}^{h_\ell}_{\Gamma_\ell}$ of the first iteration process with respect to the coarsest level $\ell = 3$, we use their interpolations on the next finer mesh $\ell = 6$ as an initial approximation and an observation of the exact state for the algorithm on this finer mesh, i.e., for the next iteration process with respect to the level $\ell = 6$ we employ

$$(Q_0, f_0, g_0) = I_1^{h_6}\Gamma_3 \text{ and } z_{\delta_6} = I_1^{h_6}\mathcal{U}_3 \text{ with } \delta_6 := \left\|z_{\delta_6} - \Phi^\dagger\right\|_{L^2(\Omega)}$$

and $I_1^{h_\ell}$ being the usual node value interpolation operator on $\mathcal{T}^{h_\ell}$, and so on $\ell = 12, 24, \ldots$. We note that the computation process only requires the measurement data of the exact data for the coarsest level $\ell = 3$.

The numerical results are summarized in Table 1 and Table 2, where we present the refinement level ℓ, mesh size h_ℓ of the triangulation, regularization parameter ρ_ℓ, noise δ_ℓ, number of iterates and value of Tolerance as well as the final L^2-error in the coefficients, the final L^2 and H^1-error in the states, and their experimental order of convergence (EOC).

All figures are here presented corresponding to $\ell = 96$. Figure 1 from left to right shows the graphs of $\Phi^\dagger$, computed numerical state $\mathcal{U}_\ell$ of the algorithm at the last iteration, and the difference to $\Phi^\dagger$. In Figure 2 we display the computed numerical source term and boundary condition f_ℓ, g_ℓ at the last iteration as well as the differences $f_\ell - f^\dagger$, $g_\ell - g^\dagger$. We write the computed numerical diffusion matrix at the last iteration as

$$Q_\ell = \begin{bmatrix} q_{\ell,11} & q_{\ell,12} \\ q_{\ell,12} & q_{\ell,22} \end{bmatrix}.$$

Figure 3 then shows $q_{\ell,11}$, $q_{\ell,12}$ and $q_{\ell,22}$ while Figure 4 shows differences $q_{\ell,11} - q^\dagger_{11}$, $q_{\ell,12} - q^\dagger_{12}$ and $q_{\ell,22} - q^\dagger_{22}$. For abbreviation we denote by $\Gamma^\dagger := \left(Q^\dagger, f^\dagger, g^\dagger\right)$ and errors

$$\Delta := \left\|\Gamma_\ell - \Gamma^\dagger\right\|_{\mathbf{L}^2_{\mathrm{sym}}(\Omega)\times L^2(\Omega)\times L^2(\partial\Omega)}, \ \Sigma := \left\|\mathcal{U}_\ell - \Phi^\dagger\right\|_{L^2(\Omega)}, \ \Lambda := \left\|\mathcal{U}_\ell - \Phi^\dagger\right\|_{H^1(\Omega)}.$$

Convergence history					
ℓ	h_ℓ	ρ_ℓ	δ_ℓ	**Iterate**	**Tolerance**
3	0.9428	9.4281e-4	0.1755	800	0.1995
6	0.4714	4.7140e-4	0.3847	800	0.4252
12	0.2357	2.3570e-4	0.3334	800	0.3677
24	0.1179	1.1790e-4	0.1508	800	0.1761
48	5.8926e-2	5.8926e-5	6.5163e-2	800	6.7593e-2
96	2.9463e-2	2.9463e-5	2.9896e-2	800	2.0480e-2

Table 1: Refinement level ℓ, mesh size h_ℓ of the triangulation, regularization parameter ρ_ℓ, noise δ_ℓ, number of iterates and value of Tolerance.

Convergence history and EOC						
ℓ	Δ	Σ	Λ	**EOC**$_\Delta$	**EOC**$_\Sigma$	**EOC**$_\Lambda$
3	0.6349	6.2551e-2	0.2789	—	—	—
6	0.1974	3.7602e-2	0.1847	1.6854	0.7342	0.5946
12	8.3571e-2	1.7066e-2	0.1382	1.2400	1.1397	0.4184
24	3.1600e-2	5.4913e-3	6.1769e-2	1.4031	1.6359	1.1618
48	1.1524e-2	9.4491e-4	2.0742e-2	1.4553	2.5389	1.5743
96	4.1183e-3	2.2575e-4	8.9372e-3	1.4845	2.0655	1.2147
Mean of EOC				1.4537	1.6228	0.9928

Table 2: Errors Δ, Σ and Λ and Experimental order of convergence between finest and coarsest level.

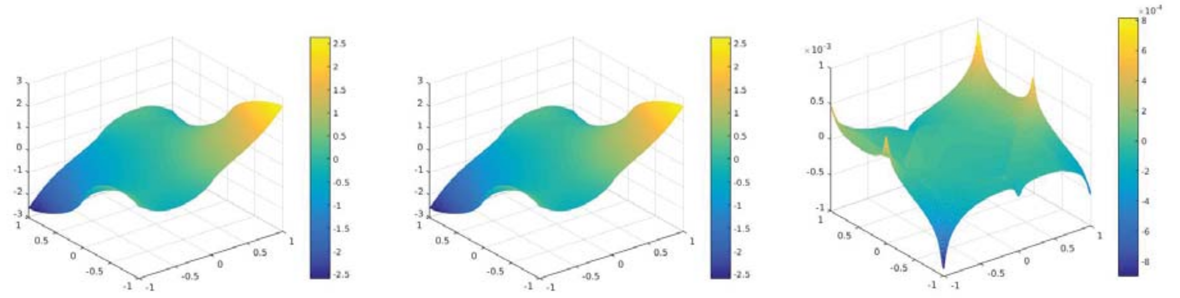

Figure 1: Graphs of $\Phi^\dagger$, computed numerical state $\mathcal{U}_\ell$ of the algorithm at the 800th iteration, and the difference to $\Phi^\dagger$.

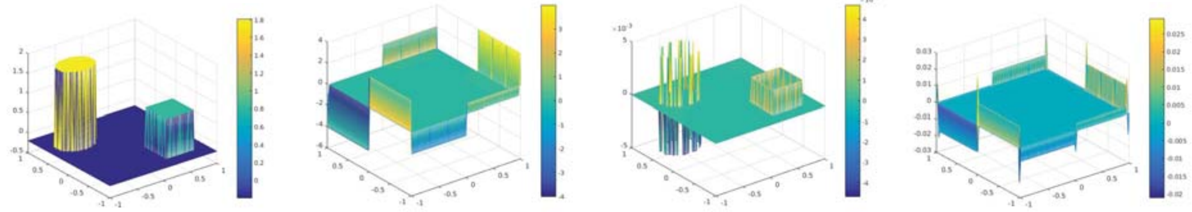

Figure 2: Graphs of f_ℓ, g_ℓ at the 800th iteration and the differences $f_\ell - f^\dagger$, $g_\ell - g^\dagger$.

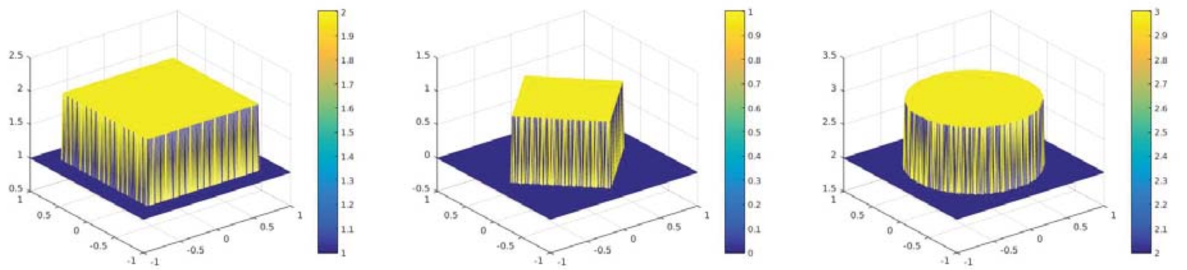

Figure 3: Graphs of $q_{\ell,11}$, $q_{\ell,12}$ and $q_{\ell,22}$ at the 800th iteration.

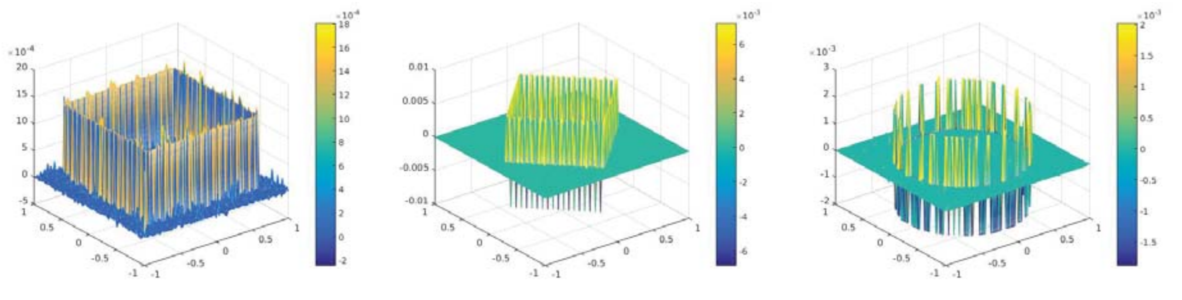

Figure 4: Differences $q_{\ell,11} - q_{11}^\dagger$, $q_{\ell,12} - q_{12}^\dagger$ and $q_{\ell,22} - q_{22}^\dagger$.

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