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### Funnel control for nonlinear systems with known strict relative degree

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#### Abstract

We consider tracking control for nonlinear multi-input, multi-output systems which have arbitrary strict relative degree and stable internal dynamics. For a given sufficiently smooth reference signal, our aim is to design a controller which achieves that the tracking error evolves within a prespecified performance funnel. To this end, we introduce a new controller which involves the first r - 1 derivatives of the tracking error, where r is the strict relative degree of the system. We further present some simulations where our funnel controller is applied to a mechanical system with higher relative degree and compare it with other approaches.

Keywords: nonlinear systems, relative degree, adaptive control, funnel control.

#### 1. Introduction

In the present article we consider output regulation for nonlinear systems by funnel control. We assume knowledge of the strict relative degree of the system and that the internal dynamics are stable. The concept of funnel control has been developed in [8] for systems with relative degree one, see also the survey [6] and the references therein. The funnel controller is an output-error feedback of high-gain type; it is an adaptive controller since the gain is adapted to the actual needed value by a time-varying adaptation scheme. Controllers of high-gain type have various advantages when it comes to "real world" applications; we like to quote from [3]:

"Since only structural assumptions on the system are required, high-gain adaptive control is inherently robust and makes it attractive for industrial application."

In particular, the funnel controller proved to be the appropriate tool for tracking problems in various applications, such as chemical reactor models [12], industrial servo-systems [4, 11] and rigid, revolute joint robotic manipulators [5].

An obstacle for high-gain adaptive controllers are systems of relative degree larger than one. In [1], a "Prescribed Performance Controller" for systems with higher strict relative degree has been introduced by Bechlioulis and Rovithakis. Though this controller is applicable to a large class of systems, its drawback is that it uses not only the system output but it requires the full information of the state. Ilchmann et al. [9, 10] developed a funnel controller for systems with higher strict relative degree by introducing a "backstepping" procedure in conjunction with a precompensator. This controller achieves tracking with prescribed transient behavior for a large class of systems governed by nonlinear (functional) differential equations. Unfortunately, this backstepping procedure is quite impractical, especially since it involves high powers of a gain function which typically takes very large values. Backstepping is also used for an adaptive  $\lambda$ -tracker in an earlier work by Ye [15].

For systems with relative degree two, a proportionalderivative (PD) funnel controller has been introduced in [4] (see also the modification in [3]), where the backstepping procedure is avoided. However, a generalization of the PD funnel controller to the case of higher relative degree is not available.

In the present paper we introduce a simple funnel controller for systems with arbitrary known relative degree r and stable internal dynamics. The controller is based on a simple recursion law and involves the first r - 1 derivatives of the tracking error.

#### 1.1. Nomenclature

$= [0,\infty)$
the Euclidean norm of $x \in \mathbb{R}^n$
the set of locally essentially bounded
functions $f: I \to \mathbb{R}^n, I \subseteq \mathbb{R}$ an interval
the set of essentially bounded functions
$f: I \to \mathbb{R}^n$ with norm
$= \operatorname{ess} \sup_{t \in I} \ f(t)\ $
the set of k-times weakly differen-
tiable functions $f: I \to \mathbb{R}^n$ such that
$f, \ldots, f^{(k)} \in \mathscr{L}^{\infty}(I \to \mathbb{R}^n)$
the set of k-times continuously differen-
tiable functions $f: V \to \mathbb{R}^n$ , $V \subseteq \mathbb{R}^m$ ;
$\mathscr{C}(V \to \mathbb{R}^n) = \mathscr{C}^0(V \to \mathbb{R}^n)$
restriction of the function $f: V \to \mathbb{R}^n$ to
$W \subseteq V$

#### 1.2. System class

In the present paper we consider a class of non-linear systems described by functional differential equations of the form

$$\begin{aligned} y^{(r)}(t) &= f(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t)) \\ &+ \Gamma(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t))u(t) \\ y|_{[-h,0]} &= y^0 \in \mathscr{W}^{r-1,\infty}([-h,0] \to \mathbb{R}^m), \end{aligned}$$

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where h > 0 is the "memory" of the system,  $r \in \mathbb{N}$  is the strict relative degree, and

- (P1): the "disturbance" satisfies  $d \in \mathscr{L}^{\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^p), p \in \mathbb{N};$
- (P2):  $f \in \mathscr{C}(\mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^m), q \in \mathbb{N},$
- (P3): the "high-frequency gain matrix function"  $\Gamma \in \mathscr{C}^1(\mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^{m \times m}$  takes values in the set of positive (negative) definite matrices;
- (P4):  $T: \mathscr{C}([-h,\infty) \to \mathbb{R}^m)^r \to \mathscr{L}^{\infty}_{loc}(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$  is an operator with the following properties:
  - a) *T* maps bounded trajectories to bounded trajectories, i.e, for all c<sub>1</sub> > 0, there exists c<sub>2</sub> > 0 such that for all ζ ∈ 𝔅([−h,∞) → ℝ<sup>m</sup>)<sup>r</sup>,

$$\sup_{t\in[-h,\infty)} \|\zeta(t)\| \le c_1 \Rightarrow \sup_{t\in[0,\infty)} \|T(\zeta)(t)\| \le c_2,$$

b) T is causal, i.e, for all  $t \ge 0$  and all  $\zeta, \xi \in \mathscr{C}([-h,\infty) \to \mathbb{R}^m)^r$ ,

$$\zeta|_{[-h,t)} = \xi|_{[-h,t)} \Rightarrow T(\zeta)|_{[0,t)} \stackrel{\text{a.a.}}{=} T(\xi)|_{[0,t)},$$

where "a.a." stands for "almost all".

c) *T* is locally Lipschitz continuous in the following sense: for all  $t \ge 0$  there exist  $\tau, \delta, c > 0$  such that for all  $\zeta, \Delta \zeta \in \mathscr{C}([-h,\infty) \to \mathbb{R}^m)^r$  with  $\Delta \zeta|_{[-h,t)} = 0$  and  $\|\Delta \zeta|_{[t,t+\tau]}\|_{\infty} < \delta$  we have

$$\left\| \left( T(\zeta + \Delta \zeta) - T(\zeta) \right) |_{[t,t+\tau]} \right\|_{\infty} \le c \|\Delta \zeta|_{[t,t+\tau]} \|_{\infty}.$$

The functions  $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$  and  $y : [-h, \infty) \to \mathbb{R}^m$  are called *input* and *output* of the system (1), resp. Systems similar to (1) have been studied e.g. in [4, 7, 8, 10]. In the aforementioned references it is shows that the class of systems (1) encompasses linear and nonlinear systems with strict relative degree and (exponentially) stable internal dynamics (zero dynamics in the linear case) and the operator *T* allows for infinite-dimensional linear systems, systems with hysteretic effects or nonlinear delay elements, input-to-state stable systems, and combinations thereof. One important subclass of systems (1) are minimum-phase linear time-invariant systems

$$\dot{x}(t) = Ax(t) + Bu(t),$$
  
$$y(t) = Cx(t),$$

where  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $C \in \mathbb{R}^{m \times n}$ , which have *strict relative degree*  $r \in \mathbb{N}$  *and positive (negative) definite high-frequency gain matrix*, i.e,  $CB = CAB = \ldots = CA^{r-2}B = 0$  and  $\Gamma := CA^{r-1}B \in \mathbb{R}^{m \times m}$  is positive (negative) definite. The minimumphase assumption (equivalently, asymptotic stability of the zero dynamics, see [13]) is characterized by the condition

$$\forall \lambda \in \mathbb{C} \text{ with } \operatorname{Re} \lambda \geq 0 : \operatorname{det} \left[ \begin{array}{cc} \lambda I_n - A & B \\ C & 0 \end{array} \right] \neq 0.$$

It is known that systems of this type can be transformed in to *Byrnes-Isidori normal form*, see [10],

$$y^{(r)}(t) = \sum_{i=1}^{r} R_i y^{(i-1)}(t) + S\eta(t) + \Gamma u(t), \dot{\eta}(t) = Py(t) + Q\eta(t)$$

where  $R_i \in \mathbb{R}^{m \times m}$  for i = 1, 2, ..., r,  $S^{\top}, P \in \mathbb{R}^{(n-rm) \times m}$ , and  $Q \in \mathbb{R}^{(n-rm) \times (n-rm)}$  is a Hurwitz matrix, i.e., all eigenvalues of Q have negative real part. This is a system of type (1) with  $\Gamma \equiv CA^{r-1}B$  and

$$f(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t)) = T(y, \dot{y}, \dots, y^{(r-1)})(t)$$
  
=  $\sum_{i=1}^{r} R_i y^{(i-1)}(t) + Se^{Qt} \eta(0) + \int_{0}^{t} Se^{Q(t-\tau)} Py(\tau) d\tau.$ 

*T* is clearly causal, locally Lipschitz, and the Hurwitz property of *Q* implies that *T* has the bounded-input-bounded-output property a). Note that *T* is parameterized by  $\eta(0) \in \mathbb{R}^{n-rm}$ .

#### 1.3. Control objective

The objective is to design an output error feedback  $u(t) = F(t, e(t), \dot{e}(t), \dots, e^{(r-1)}(t))$ , where  $e(t) = y(t) - y_{ref}(t)$  for some reference trajectory  $y_{ref} \in \mathcal{W}^{r,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ , such that in the closed-loop system the tracking error e(t) evolves within a prescribed performance funnel

$$\mathscr{F}_{\boldsymbol{\varphi}} := \{ (t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \mid \boldsymbol{\varphi}(t) \| e \| < 1 \}, \qquad (2)$$

which is determined by a function  $\varphi$  belonging to

$$\Phi_{r} := \left\{ \varphi \in \mathscr{C}^{r}(\mathbb{R}_{\geq 0} \to \mathbb{R}) \middle| \begin{array}{l} \varphi, \dot{\varphi}, \dots, \varphi^{(r)} \text{ are bounded,} \\ \varphi(\tau) > 0 \text{ for all } \tau > 0, \\ \text{and } \liminf_{\tau \to \infty} \varphi(\tau) > 0 \end{array} \right\}.$$
(3)

Furthermore, all signals involved should remain bounded.

The funnel boundary is given by the reciprocal of  $\varphi$ , see Fig. 1. It is explicitly allowed that  $\varphi(0) = 0$ , meaning that no restriction on the initial value is imposed since  $\varphi(0) || e(0) || < 1$ ; the funnel boundary  $1/\varphi$  has a pole at t = 0 in this case.



Figure 1: Error evolution in a funnel  $\mathscr{F}_{\varphi}$  with boundary  $\varphi(t)^{-1}$  for t > 0.

An important property of the class  $\Phi_r$  is that each performance funnel  $\mathscr{F}_{\varphi}$  with  $\varphi \in \Phi_r$  is bounded away from zero, i.e., because of boundedness of  $\varphi$  there exists  $\lambda > 0$  such that  $1/\varphi(t) \ge \lambda$  for all t > 0. The funnel boundary is not necessarily monotonically decreasing, while in most situations it is convenient to choose a monotone funnel. However, there are

situations where widening the funnel over some later time interval might be beneficial, e.g., when the reference trajectory changes strongly or the system is perturbed by some calibration so that a large tracking error would enforce a large input action.

#### 1.4. Organization of the present paper

The paper is structured as follows. In section 2, we introduce the funnel controller for the system class presented in Section 1.2. Feasibility of the control is proved in the main result in Section 3; in particular we show that our proposed funnel controller achieves the control objective described in Section 1.3. The performance of the funnel controller is illustrated by means of several examples in Section 4. This part also contains comparisons with the approaches in [4, 9, 10].

#### 2. Controller structure

We introduce the below funnel controller for systems of type (1):

$$e_{0}(t) = e(t) = y(t) - y_{ref}(t),$$

$$e_{1}(t) = \dot{e}_{0}(t) + k_{0}(t) \cdot e_{0}(t),$$

$$e_{2}(t) = \dot{e}_{1}(t) + k_{1}(t) \cdot e_{1}(t),$$

$$\vdots$$

$$e_{r-1}(t) = \dot{e}_{r-2}(t) + k_{r-2}(t) \cdot e_{r-2}(t),$$

$$u(t) = \begin{cases} -k_{r-1}(t) \cdot e_{r-1}(t), & \text{if } \Gamma \text{ is pointwise pos. def.}, \\ k_{r-1}(t) \cdot e_{r-1}(t), & \text{if } \Gamma \text{ is pointwise neg. def.}, \end{cases}$$

$$k_{i}(t) = \frac{1}{1 - \varphi_{i}^{2}(t) ||e_{i}(t)||^{2}}, \quad i = 0, \dots, r-1$$

where, for  $\Phi_i$  as in (3), the reference signal and funnel functions have the following properties:

$$y_{\text{ref}} \in \mathscr{W}^{r,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m),$$
  
$$\varphi_0 \in \Phi_r, \ \varphi_1 \in \Phi_{r-1}, \ \dots, \ \varphi_{r-1} \in \Phi_1.$$
(5)

In the sequel we investigate existence of solutions of the initial value problem resulting from the application of the funnel controller (4) to a system (1). By a *solution* of (1), (4) on  $[-h, \omega)$  we mean a function  $y \in \mathscr{C}^{r-1}([-h, \omega) \to \mathbb{R}^m)$ ,  $\omega \in (0, \infty]$ , with  $y|_{[-h,0]} = y^0$  such that  $y^{(r-1)}|_{[0,\omega)}$  is absolutely continuous and satisfies the differential equations in (1), (4) for almost all  $t \in [0, \omega)$ ; *y* is called *maximal*, if it has no right extension that is also a solution. Existence of solutions of functional differential equations has been investigated in [8] for instance.

**Remark 2.1** (Funnel control for systems with  $r \in \{1,2,3\}$ ). In the following we determine the funnel controllers explicitly for the cases r = 1, 2, 3. We assume for convenience that the high-frequency gain matrix function  $\Gamma$  is pointwise positive definite.

*r* = 1: The control law (4) reduces to the "classical" funnel controller u(t) = -k(t)e(t) with  $k(t) = \frac{1}{1-\varphi^2(t)\|e(t)\|^2}$ . Moreover, our assumptions on the reference signal and the funnel function  $\varphi$  reduce to those made in [8].

r = 2: We obtain the controller

$$\begin{split} u(t) &= -k_1(t)(\dot{e}(t) + k_0(t)e(t)),\\ k_0(t) &= \frac{1}{1 - \varphi_0^2(t) \|e(t)\|^2},\\ k_1(t) &= \frac{1}{1 - \varphi_1^2(t) \|\dot{e}(t) + k_0(t)e(t)\|^2}. \end{split}$$

r = 3: Here the controller (4) takes the form

$$\begin{split} u(t) &= -k_2(t) \cdot \left[ \ddot{e}(t) + 2k_0(t)^2 (\varphi_0^2(t) e^\top(t) \dot{e}(t) \right. \\ &+ \varphi_0(t) \dot{\varphi}_0(t) \| e(t) \|^2 ) e(t) \\ &+ k_0(t) \dot{e}(t) + k_1(t) (\dot{e}(t) + k_0(t) e(t)) \right], \\ k_0(t) &= \frac{1}{1 - \varphi_0^2(t) \| \dot{e}(t) \|^2}, \\ k_1(t) &= \frac{1}{1 - \varphi_1^2(t) \| \dot{e}(t) + k_0(t) e(t) \|^2}, \\ k_2(t) &= \frac{1}{1 - \varphi_2^2(t) \| \ddot{e}(t) + 2k_0(t)^2 (\varphi_0^2(t) e^\top(t) \dot{e}(t) + \varphi_0(t) \dot{\varphi}_0(t) \| e(t) \|^2) e(t)} \\ &- \frac{1}{1 - \varphi_0^2(t) \| \dot{e}(t) + k_1(t) (\dot{e}(t) + k_0(t) e(t)) \|^2}. \end{split}$$

**Remark 2.2** (The intuition behind the funnel controller (4)). The classical funnel controller for systems with relative degree one and stable internal dynamics uses the "high gain property" [2], which states that such systems can can be stabilized by a proportional feedback law u(t) = -ky(t) with a sufficiently large constant k > 0. This gives rise to the intuition of the funnel controller for relative degree one: If the error approaches the funnel boundary at  $t_0$ , then  $k(t_0)$  takes a large value which stabilizes the error system.

To illustrate the functioning of the controller (4), we employ the following thought experiment for the single-input, singleoutput case m = 1, see Fig. 2: Assume that the error  $e = e_0$  approaches the upper funnel boundary  $1/\varphi_0$  at time  $t_0 > 0$ . Then  $k_0(t_0)$ , and consequently  $k_0(t_0) \cdot e(t_0)$  will be very large. Since  $e_1 = \dot{e} + k_0 \cdot e$  evolves in the performance funnel  $\mathscr{F}_{\varphi_1}$ , we may infer that  $\dot{e}(t_0) = e_1(t_0) - k_0(t_0) \cdot e(t_0)$  will take a large negative value. In other words, e will be decreasing enormously. That is, whenever the error e approaches the funnel boundary  $1/\varphi_0$ , the controller ensures a repelling effect.

This argumentation can be repeated for the functions  $e_1, \ldots, e_{r-2}$ . Finally, since  $e_{r-1}$  includes the first r-1 derivatives of e, the system with artificial output  $e_{r-1}$  has relative degree one, and the classical high gain property applies to  $e_{r-1}$ .



Figure 2: Error in the performance funnel  $\mathscr{F}_{\varphi_1}$ 

**Remark 2.3** (Funnel control by backstepping). The works [9, 10] introduce a funnel controller based on a filter and backstepping construction for systems with higher relative degree. First consider a filter with

$$\dot{\xi}_i(t) = -\xi_i(t) + \xi_{i+1}(t), \quad i = 1, \dots, r-2$$
  
$$\dot{\xi}_{r-1}(t) = -\xi_{r-1}(t) + u(t).$$

Introduce the projections

$$\pi_i: \mathbb{R}^{(r-1)m} \to \mathbb{R}^{im}, \ \xi = (\xi_1, \dots, \xi_{r-1}) \mapsto (\xi_1, \dots, \xi_i)$$

for  $i = 1, \ldots, r - 1$  and functions

$$\begin{split} \gamma_{1}(k,e) &= -k \cdot e \\ \gamma_{i}(k,e,\pi_{i-1}\xi) &:= \gamma_{i-1}(k,e,\pi_{i-2}\xi) \\ &+ \|D\gamma_{i-1}(k,e,\pi_{i-2}\xi)\|^{2}k^{4} \cdot (1 + \|\pi_{i-1}\xi\|^{2}) \\ &\cdot (\xi_{i-1} + \gamma_{i-1}(k,e,\pi_{i-2}\xi)) \end{split}$$

The controller in [10] takes the form

$$u(t) = -\gamma_r(k(t), e(t), \xi(t))$$
  
$$k(t) = \frac{1}{1 - \varphi(t)^2 \|e(t)\|^2}.$$

We stress that in [10] a much smaller class of systems than introduced in Section 1.2 is considered; in [10] *T* may only depend on *y* and  $\Gamma$  is assumed to be constant. The above presented controller works provided that  $\Gamma \in \mathbb{R}^{m \times m}$  is positive definite. However, this approach can be modified such that it also works for systems in which it is not known whether  $\Gamma$  is positive or negative definite. In this case, the function  $\gamma_1$  has to be modified by  $\gamma_1(k, e) = v(k) \cdot e$ , where  $v : \mathbb{R}_{\geq 0} \to \mathbb{R}$  is smooth and satisfies the "Nussbaum property" [10]. In the following we discuss the cases of relative degree two and three.

r = 2: Here the controller takes the form

$$u = -ke + (||e||^2 + k^2) \cdot k^4 (1 + ||\xi||^2) (\xi + ke),$$

where we omit the argument *t*. This feedback law is dynamic and the gain occurs with  $k(t)^7$ . The presence of such a large power of the funnel gain k(t) is problematic in practice; the controller produces inputs which might be impractical. We compare the backstepping approach to our controller (4) in Section 4.

r = 3: Here the controller reads

$$u = ke - k^{4}(e^{2} + k^{2})(1 + \xi_{1}^{2})(\xi_{1} - ke) - \left\{ \left[ -e + (1 + \xi_{1}^{2}) \right]^{2} + \left[ 2k^{5}(\xi_{1} - ke) + 4k^{3}(e^{2} + k^{2})(\xi_{1} - ke) - k^{4}(e^{2} + k^{2})e \right]^{2} + \left[ -k + k^{4}(1 + \xi_{1}^{2})\left[ 2e(\xi_{1} - ke) - k(e^{2} + k^{2})\right]^{2} + \left[ k^{4}(e^{2} + k^{2})\left[ 2\xi_{1}(\xi_{1} - ke) + (1 + \xi_{1}^{2})\right]^{2} \right\} k^{4}(1 + \xi_{1}^{2} + \xi_{2}^{2}) + \left[ \xi_{2} - ke + k^{4}(e^{2} + k^{2})(1 + \xi_{1}^{2})(\xi_{1} - ke)\right].$$
(6)

An expansion of the above product gives that this controller contains the 25th power (!) of the funnel gain k(t), and the problems depicted for r = 2 are present here a fortiori.

**Remark 2.4** (Proportional-derivative funnel control for relative degree two). Consider a system (1) with the properties (P1)–(P4) as in Section 1.2. Further assume that m = 1 and the high-frequency gain function  $\Gamma$  is pointwise positive definite. The work [4] introduces a funnel controller which feeds back the error *e* and its derivative. More precise, this controller reads

$$u(t) = -k_0^2(t)e(t) - k_1(t)\dot{e}(t),$$
  

$$k_0(t) = \frac{\varphi_0}{1 - \varphi_0(t)|e(t)|}, \quad k_1(t) = \frac{\varphi_1}{1 - \varphi_1(t)|\dot{e}(t)|}.$$
(7)

The funnel functions  $\varphi_0$  for the error and  $\varphi_1$  for the derivative of the error have to satisfy  $\varphi_0 \in \Phi_2$ ,  $\varphi_1 \in \Phi_1$ , and they have to fulfill the compatibility condition

$$\forall t > 0 \exists \delta > 0 : 1/\varphi_1(t) \ge \delta - \frac{\mathrm{d}}{\mathrm{d}t}(1/\varphi_0(t)) \qquad \forall t > 0.$$
(8)

This controller is simple and its practicability has been verified experimentally. However, there is no straightforward extension to systems with relative degree larger than two. We further emphasize that the funnel functions  $\varphi_0, \ldots, \varphi_{r-1}$  in the funnel controller (4) do not have to satisfy any compatibility condition.

#### 3. Main result

We show feasibility of the funnel controller (4).

**Theorem 3.1.** Consider a system (1) with strict relative degree  $r \in \mathbb{N}$  and properties (P1)-(P4). For  $\Phi_i$  as defined in (3), let

$$\varphi_i \in \Phi_{r-i} \text{ for } i = 0, ..., r-1.$$

Let  $y_{\text{ref}} \in \mathscr{W}^{r,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$  be a reference signal, and  $y|_{[-h,0]} = y^0 \in \mathscr{W}^{r-1,\infty}([-h,0] \to \mathbb{R}^m)$  an initial value such that  $e_0, \ldots, e_{r-1}$  as defined in (4) fulfill

$$\varphi_i(0) \| e_i(0) \| < 1 \text{ for } i = 0, \dots, r-1.$$
 (9)

Then the application of the funnel controller (4) to (1) yields an initial-value problem, which has a solution, and every maximal solution  $y : [-h, \omega) \to \mathbb{R}^m$ ,  $\omega \in (0, \infty]$ , has the following properties:

- (i) The solution is global (i.e,  $\omega = \infty$ ).
- (ii) The input  $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ , the gain functions  $k_0, \dots, k_{r-1} : \mathbb{R}_{\geq 0} \to \mathbb{R}$  and  $y, \dots, y^{(r-1)} : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$  are bounded.
- (iii) The functions  $e_0, \ldots, e_{r-1} : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$  evolve in their respective performance funnels, i.e.,

$$(t,e_i) \in \mathscr{F}_{\varphi_i}$$
 for all  $i = 0, \ldots, r-1$  and  $t \ge 0$ .

Furthermore, the signals  $e_i(\cdot)$  are uniformly bounded away from the funnel boundaries in the following sense:

$$\forall i = 0, \dots, r-1 \ \exists \varepsilon_i > 0 \ \forall t > 0 : \\ \|e_i(t)\| \le \varphi_i(t)^{-1} - \varepsilon_i.$$
 (10)

In particular, the error  $e(t) = y(t) - y_{ref}(t)$  evolves in the funnel  $\mathscr{F}_{\varphi_0}$  and stays uniformly away from its boundary.

*Proof.* We may, without loss of generality, assume that the high-frequency gain matrix function  $\Gamma$  of system (1) is pointwisely positive definite. We proceed in several steps.

Step 1: We show that a maximal solution  $y : [-h, \omega) \to \mathbb{R}^m$  exists with  $\omega \in (0, \infty]$ . Consider the sets

$$\mathscr{D}_{i} = \left\{ (t, e_{0}, \dots, e_{i}) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{m} \times \dots \times \mathbb{R}^{m} \middle| \begin{array}{c} (t, e_{j}) \in \mathscr{F}_{\varphi_{j}}, \\ j = 0, \dots, i \end{array} \right\}$$

for i = 0, ..., r - 1. Using the relations in the funnel controller (4), we obtain that, for some interval  $I \subseteq \mathbb{R}_{\geq 0}$ ,  $(e_0, ..., e_{r-1}) : I \to \mathbb{R}^{rm}$  with  $(t, e_0, ..., e_{r-1}) \in \mathcal{D}_{r-1}$  for all  $t \in I$  satisfies (4) if, and only if,  $e = e_0$  satisfies

$$e^{(i)} = e_i - \sum_{j=0}^{i-1} \frac{d^{i-1-j}}{dt^{i-1-j}} (k_j e_j) \quad \text{for all } i = 0, \dots, r-1.$$
(11)

Define the functions  $K_i : \mathscr{D}_i \to \mathbb{R}^m$  by

$$\begin{split} K_0(t,e_0) &:= \frac{e_0}{1-\varphi_0^2(t)\|e_0\|^2},\\ K_i(t,e_0,\ldots,e_i) &:= \frac{e_i}{1-\varphi_i^2(t)\|e_i\|^2} + \frac{\partial K_{i-1}}{\partial t}(t,e_0,\ldots,e_{i-1}) \\ &+ \sum_{j=0}^{i-1} \frac{\partial K_{i-1}}{\partial e_j}(t,e_0,\ldots,e_{i-1}) \left(e_{j+1} - \frac{e_j}{1-\varphi_j^2(t)\|e_j\|^2}\right) \end{split}$$

for i = 1, ..., r - 1. We show by induction that

$$\forall t > 0: \sum_{j=0}^{i} \frac{d^{i-j}}{dt^{i-j}} \left( k_j(t) e_j(t) \right) = K_i \left( t, e_0(t), \dots, e_i(t) \right) \quad (12)$$

for i = 0, ..., r - 1. Eq. (12) is obviously true for i = 0. Assume that  $i \in \{1, ..., r - 1\}$  and the statement holds for i - 1. Then

$$\sum_{j=0}^{i} \frac{d^{i-j}}{dt^{i-j}} (k_j(t)e_j(t)) = k_i(t)e_i(t) + \frac{d}{dt} \left( \sum_{j=0}^{i-1} \frac{d^{i-1-j}}{dt^{i-1-j}} (k_j(t)e_j(t)) \right)$$
$$= k_i(t)e_i(t) + \frac{d}{dt}K_{i-1}(t,e_0(t),\dots,e_{i-1}(t))$$
$$= K_i(t,e_0(t),\dots,e_i(t)).$$

In particular, (11) yields that, for all  $t \in I$  and all i = 0, ..., r - 1,

$$e^{(i)}(t) = e_i(t) - K_{i-1}(t, e_0(t), \dots, e_{i-1}(t)) \text{ and}$$
  

$$e^{(r)}(t) = \dot{e}_{r-1}(t) - K_{r-1}(t, e_0(t), \dots, e_{r-1}(t)) + k_{r-1}(t)e_{r-1}(t).$$

Now, define

$$\begin{aligned} \mathscr{K} := & \{ (e_0, e_1, \dots, e_{r-1}) \in \mathscr{C}([-h, \infty) \to \mathbb{R}^m)^r \mid \\ & \forall t \ge 0 : \ (t, e_0(t), \dots, e_{r-2}(t)) \in \mathscr{D}_{r-2} \}, \end{aligned}$$

and the operator  $\hat{T}: \mathscr{K} \to \mathscr{L}^{\infty}_{\text{loc}}([0,\infty) \to \mathbb{R}^q)$  by

$$\hat{T}(e_0, e_1, \dots, e_{r-1}) = T\left(e_0 + y_{\text{ref}}, e_1 + \dot{y}_{\text{ref}} - K_0(\cdot, e_0), \dots, e_{r-1} + y_{\text{ref}}^{(r-1)} - K_{r-2}(\cdot, e_0, \dots, e_{r-2})\right),$$

where  $y_{\text{ref}}$  is extended to [-h,0) such that  $y_{\text{ref}} \in \mathcal{W}^{r,\infty}([-h,\infty) \to \mathbb{R}^m)$ . Define  $x = (e_0, e_2, \dots, e_{r-1})$  and

$$F: \mathscr{D}_{r-1} \times \mathbb{R}^{q} \to \mathbb{R}^{r}, (t, e_{0}, e_{1}, \dots, e_{r-1}, \eta) \mapsto \left(e_{1} - \frac{e_{0}}{1 - \varphi_{i0}^{2}(t) \|e_{0}\|^{2}}, \dots, e_{r-1} - \frac{e_{r-2}}{1 - \varphi_{r-2}^{2}(t) \|e_{r-2}\|^{2}}, f\left(d(t), \eta\right) - \frac{\Gamma(d(t), \eta)e_{r-1}}{1 - \varphi_{r-1}^{2}(t) \|e_{r-1}\|^{2}} - y_{\text{ref}}^{(r)}(t) -K_{r-1}\left(t, e_{0}, \dots, e_{r-1}\right) + \frac{e_{r-1}}{1 - \varphi_{r-1}^{2}(t) \|e_{r-1}\|^{2}}\right).$$

Then the initial value problem (1), (4) is equivalent to

$$\dot{x}(t) = F(t, x(t), \hat{T}(x)(t)), \quad x|_{[-h,0]} = (e_0, e_2 \dots, e_{r-1})|_{[-h,0]}.$$
(13)

In particular,  $(0,x(0)) \in \mathcal{D}_{r-1}$  and *F* is measurable in *t* and locally Lipschitz continuous in  $(e_0, e_1, \ldots, e_{r-1}, \eta)$ . Hence an application of Theorem A.1 yields existence of solutions to (13). Let  $x = (e_0, e_2, \ldots, e_{r-1}) : [-h, \omega) \to \mathbb{R}^{rm}$ ,  $\omega \in (0, \infty]$  be a maximal solution of (13), then the closure of the graph of this solution is not a compact subset of  $\mathcal{D}_{r-1}$ .

Step 2: We show that  $k_0, \ldots, k_{r-1}$  as in (4) are bounded on  $[0, \omega)$ . For all  $i \in \{0, \ldots, r-1\}$ , set  $\psi_i(t) = \varphi_i^{-1}(t)$  for  $t \in (0, \omega)$ , let  $T_i \in (0, \omega)$  be arbitrary but fixed and set  $\lambda_i = \inf_{t \in (0, \omega)} \psi_i(t) > 0$ . Since  $\phi_i$  is bounded and  $\liminf_{t \to \infty} \varphi_i(t) > 0$  we find that  $\frac{d}{dt} \psi_i|_{[T_i, \infty)}$  is bounded and hence there exists a Lipschitz bound  $L_i > 0$  of  $\psi_i|_{[T_i, \infty)}$ .

Step 2a: We show that  $k_i$  is bounded for  $i \in \{0, ..., r-2\}$ . Choose  $\varepsilon_i > 0$  small enough so that

$$\varepsilon_{i} \leq \min\left\{\frac{\lambda_{i}}{2}, \inf_{t \in [0,T_{i}]}(\psi_{i}(t) - \|e_{i}(t)\|)\right\}$$
  
and  $L_{i} \leq \frac{\lambda_{i}^{2}}{4\varepsilon_{i}} - \sup_{t \in [T_{i},\infty)} |\psi_{i+1}(t)|.$  (14)

We show that

$$\forall t \in (0, \boldsymbol{\omega}): \psi_i(t) - \|e_i(t)\| \geq \varepsilon_i$$

By definition of  $\varepsilon_i$  this holds on  $(0, T_i]$ . Seeking a contradiction suppose that

$$\exists t_{i1} \in [T_i, \boldsymbol{\omega}): \ \boldsymbol{\psi}_i(t_{i1}) - \|\boldsymbol{e}_i(t_{i1})\| < \boldsymbol{\varepsilon}_i.$$

Set  $t_{i0} = \max \{ t \in [T_i, t_{i1}) | \psi_i(t) - ||e_i(t)|| = \varepsilon_i \}$ . Then, for all  $t \in [t_{i0}, t_{i1}]$ , we have that

$$egin{aligned} \psi_i(t) - \|e_i(t)\| &\leq arepsilon_i, \ \|e_i(t)\| &\geq \psi_i(t) - arepsilon_i &\geq rac{\lambda_i}{2}, \ k_i(t) &= rac{1}{1 - arphi_i^2(t) \|e_i(t)\|^2} = rac{\psi_i(t)}{2arepsilon_i} &\geq rac{\lambda_i}{2arepsilon_i}. \end{aligned}$$

Therefore, we find that by (4)

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|e_i(t)\|^2 = e_i^\top(t) \left(e_{i+1}(t) - k_i(t)e_i(t)\right) \\ = -k_i(t) \|e_i(t)\|^2 + e_i^\top(t)e_{i+1}(t)$$

$$\leq \left( -\frac{\lambda_i^2}{4\varepsilon_i} + \sup_{t \in [T_i,\infty)} |\psi_{i+1}(t)| \right) \|e_i(t)\|$$

$$\stackrel{(14)}{\leq} -L_i \|e_i(t)\|$$

for all  $t \in [t_{i0}, t_{i1}]$ . Then

$$\begin{aligned} \|e_i(t_{i1})\| - \|e_i(t_{i0})\| &= \int_{t_{i0}}^{t_{i1}} \frac{1}{2} \|e_i(t)\|^{-1} \frac{\mathrm{d}}{\mathrm{d}t} \|e_i(t)\|^2 \, \mathrm{d}t \\ &\leq -L_i(t_{i1} - t_{i0}) \\ &\leq -|\psi_i(t_{i1}) - \psi_i(t_{i0})| \\ &\leq \psi_i(t_{i1}) - \psi_i(t_{i0}), \end{aligned}$$

and thus we obtain  $\varepsilon_i = \psi_i(t_{i0}) - ||e_i(t_{i0})|| \le \psi_i(t_{i1}) - ||e_i(t_{i1})|| < \varepsilon_i$ , a contradiction.

*Step 2b:* We show that  $k_{r-1}$  is bounded. By (11) and Step 1 we have, invoking  $x = (e_0, \dots, e_{r-1})$ ,

$$\dot{e}_{r-1}(t) = f(d(t), \hat{T}(x)(t)) - k_{r-1}(t)\Gamma(d(t), \hat{T}(x)(t))e_{r-1}(t) - y_{\text{ref}}^{(r)}(t) + \sum_{i=0}^{r-2} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{r-1-i} [k_i(t)e_i(t)].$$

In the following we will prove by induction that there exist constants  $M_{i,j}, N_{i,j}, K_{i,j} > 0$  such that, for all  $t \in [0, \omega)$ ,

$$\begin{split} \left\| \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{j} [k_{i}(t)e_{i}(t)] \right\| &\leq M_{i,j}, \qquad \left\| \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{j}e_{i}(t) \right\| \leq N_{i,j}, \\ \left| \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{j}k_{i}(t) \right| &\leq K_{i,j}, \end{split}$$
(15)  
for  $i = 0, \dots, r-2, j = 0, \dots, r-1-i.$ 

First, we may infer from Step 2a that  $k_0, \ldots, k_{r-2}$  are bounded. Furthermore,  $e_0, \ldots, e_{r-1}$  are bounded since they evolve in the respective performance funnels, cf. (4). Therefore, (15) is true whenever j = 0. We prove (15) for i = r - 2 and j = 1: We find that

$$\begin{split} \dot{e}_{r-2}(t) &= e_{r-1}(t) - k_{r-2}(t)e_{r-2}(t), \\ \dot{k}_{r-2}(t) &= 2k_{r-2}^2(t) \left( \varphi_{r-2}^2(t) e_{r-2}^\top(t) \dot{e}_{r-2}(t) \right. \\ &+ \varphi_{r-2}(t) \dot{\varphi}_{r-2}(t) \| e_{r-2}(t) \|^2 \right), \\ \frac{\mathrm{d}}{\mathrm{d}t} \left[ k_{r-2}(t) e_{r-2}(t) \right] &= \dot{k}_{r-2}(t) e_{r-2}(t) + k_{r-2}(t) \dot{e}_{r-2}(t), \end{split}$$

and all of these signals are bounded since  $k_{r-2}, \varphi_{r-2}, \varphi_{r-2}, e_{r-2}, e_{r-1}$  are bounded. Now let  $p \in \{0, \dots, r-3\}$  and  $q \in \{0, \dots, r-1-q\}$  and assume that (15) is true for all  $i = p + 1, \dots, r-2$  and all  $j = 0, \dots, r-1-i$  as well as for i = p and all  $j = 0, \dots, q-1$ . We show that it is true for i = p and j = q:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{q} e_{p}(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{q-1} \left[e_{p+1}(t) - k_{p}(t)e_{p}(t)\right]$$
$$= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{q-1} e_{p+1}(t) - \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{q-1} \left[k_{p}(t)e_{p}(t)\right],$$

$$\begin{split} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{q}k_{p}(t) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{q-1} \left(2k_{p}^{2}(t)\left(\varphi_{p}^{2}(t)e_{p}^{\top}(t)\dot{e}_{p}(t)\right)\right.\\ &+ \varphi_{p}(t)\dot{\varphi}_{p}(t)\|e_{p}(t)\|^{2}\right) \bigg),\\ \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{q}\left[k_{p}(t)e_{p}(t)\right] &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{q-1} \left(\dot{k}_{p}(t)e_{p}(t) + k_{p}(t)\dot{e}_{p}(t)\right). \end{split}$$

Then, successive application of the product rule and using the induction hypothesis as wells as the fact that  $\varphi_p, \dot{\varphi}_p, \dots, \varphi_p^{(r-p)}$  are bounded, yields that the above terms are bounded. Therefore, the proof of (15) is complete.

By (15) and (11) it follows that  $e^{(i)}$  is bounded on  $[0, \omega)$  for all i = 0, ..., r-1. Therefore, also  $K_i(\cdot, e_0(\cdot), ..., e_i(\cdot))$  is bounded on  $[0, \omega)$  for all i = 0, ..., r-1. Further invoking boundedness of  $y_{\text{ref}}, ..., y_{\text{ref}}^{(r)}$  and the bounded-input, bounded-output property a) of the operator *T* it follows that  $\hat{T}(x)$  is bounded, where  $x = (e_0, ..., e_{r-1})$ . Since *f* is continuous and *d* is bounded, we may further infer that  $f(d(\cdot), \hat{T}(x)(\cdot))$  is bounded on  $[0, \omega)$ , i.e., there exists  $M_F > 0$  such that

for almost all 
$$t \in [0, \omega)$$
:  $\|f(d(t), \hat{T}(x)(t))\| \le M_F$ .

Define the compact set

$$\mathscr{M} := \left\{ \left(\delta, \eta, e\right) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m \middle| \begin{array}{l} \|\delta\| \le \|d|_{[0,\omega)}\|_{\infty} \\ \|\eta\| \le \|\hat{T}(x)|_{[0,\omega)}\|_{\infty} \\ \|e\| = 1. \end{array} \right\},$$

then, since  $\Gamma$  is pointwise positive definite and the map

$$\mathscr{M} \ni (\delta, \eta, e) \mapsto e^{\top} \Gamma(\delta, \eta) e \in \mathbb{R}_{>0}$$

is continuous, it follows that there exists  $\gamma > 0$  such that

$$\forall (\delta, \eta, e) \in \mathscr{M} : e^{\top} \Gamma(\delta, \eta) e \geq \gamma.$$

Therefore, we have

$$e_{r-1}(t)^{\top} \Gamma(d(t), \hat{T}(x)(t)) e_{r-1}(t) \\ = \left( \frac{e_{r-1}(t)^{\top}}{\|e_{r-1}(t)\|} \Gamma(d(t), \hat{T}(x)(t)) \frac{e_{r-1}(t)}{\|e_{r-1}(t)\|} \right) \|e_{r-1}(t)\|^{2} \\ \ge \gamma \|e_{r-1}(t)\|^{2}$$

for all  $t \in [0, \omega)$ . Now, choose  $\varepsilon_{r-1} > 0$  small enough so that

$$\varepsilon_{r-1} \le \min\left\{\frac{\lambda_{r-1}}{2}, \inf_{t \in (0, T_{r-1}]}(\psi_{r-1}(t) - ||e_{r-1}(t)||)\right\}$$

and

$$L_{r-1} \le \frac{\lambda_{r-1}^2}{4\varepsilon_{r-1}} \gamma - M_F - \sup_{t \in [0,\omega)} \|y_{\text{ref}}^{(r)}(t)\| - \sum_{i=0}^{r-2} M_{i,r-1-i}.$$
 (16)

We show that

$$\forall t \in (0, \boldsymbol{\omega}): \ \boldsymbol{\psi}_{r-1}(t) - \|\boldsymbol{e}_{r-1}(t)\| \geq \boldsymbol{\varepsilon}_{r-1}.$$

By definition of  $\varepsilon_{r-1}$  this holds on  $(0, T_{r-1}]$ . Seeking a contra-

diction suppose that

$$\exists t_{r-1,1} \in [T_{r-1}, \omega): \ \psi_{r-1}(t_{r-1,1}) - \|e_{r-1}(t_{r-1,1})\| < \varepsilon_{r-1}.$$

Define

$$t_{r-1,0} = \max \{ t \in [T_{r-1}, t_{r-1,1}) \mid \psi_{r-1}(t) - ||e_{r-1}(t)|| = \varepsilon_{r-1} \},$$

then, for all  $t \in [t_{r-1,0}, t_{r-1,1}]$ , we have that

$$\begin{split} \psi_{r-1}(t) - \|e_{r-1}(t)\| &\leq \varepsilon_{r-1}, \\ \|e_{r-1}(t)\| &\geq \psi_{r-1}(t) - \varepsilon_{r-1} \geq \frac{\lambda_{r-1}}{2}, \\ k_{r-1}(t) &= \frac{1}{1 - \varphi_{r-1}^2(t) \|e_{r-1}(t)\|^2} = \frac{\psi_{r-1}(t)}{2\varepsilon_{r-1}} \geq \frac{\lambda_{r-1}}{2\varepsilon_{r-1}}. \end{split}$$

We obtain, for all  $t \in [t_{r-1,0}, t_{r-1,1}]$ , that

$$\begin{split} &\frac{1}{2} \frac{d}{dt} \| e_{r-1}(t) \|^2 = e_{r-1}^{\perp}(t) \dot{e}_{r-1}(t) \\ &= e_{r-1}^{\top}(t) \left( f\left(d(t), \hat{T}(x)(t)\right) - k_{r-1}(t) \Gamma\left(d(t), \hat{T}(x)(t)\right) e_{r-1}(t) \right) \\ &- y_{\text{ref}}^{(r)}(t) + \sum_{i=0}^{r-2} \left( \frac{d}{dt} \right)^{r-1-i} [k_i(t) e_i(t)] \right) \\ &\leq \left( M_F - \frac{\lambda_{r-1}^2}{4\varepsilon_{r-1}} \gamma + \sup_{t \in (0, \omega)} \| y_{\text{ref}}^{(r)}(t) \| + \sum_{i=0}^{r-2} M_{i,r-1-i} \right) \| e_{r-1}(t) \| \\ &\leq -L_{r-1} \| e_{r-1}(t) \|, \end{split}$$

and therefore,

$$\begin{aligned} \|e_{r-1}(t_{r-1,1})\| - \|e_{r-1}(t_{r-1,0})\| &= \int_{t_{r-1,0}}^{t_{r-1,1}} \frac{1}{2} \|e_{r-1}(t)\|^{-1} \frac{d}{dt} \|e_{r-1}(t)\|^{2} dt \\ &\leq -L_{r-1}(t_{r-1,1} - t_{r-1,0}) \\ &\leq -|\psi_{r-1}(t_{r-1,1}) - \psi_{r-1}(t_{r-1,0})| \\ &\leq \psi_{r-1}(t_{r-1,1}) - \psi_{r-1}(t_{r-1,0}), \end{aligned}$$

and thus we obtain  $\varepsilon_{r-1} = \psi_{r-1}(t_{r-1,0}) - ||e_{r-1}(t_{r-1,0})|| \le \psi_{r-1}(t_{r-1,1}) - ||e_{r-1}(t_{r-1,1})|| < \varepsilon_{r-1}$ , a contradiction.

Step 3: We show that  $\omega = \infty$ . Assume that  $\omega < \infty$ . Then, since  $e_i, k_i, i = 0, ..., r-1$  are bounded by Step 2, it follows that the closure of the graph of  $x = (e_0, e_2, ..., e_{r-1})$  is a compact subset of  $\mathcal{D}_{r-1}$ , a contradiction. Hence  $\omega = \infty$  which shows (i). Statements (ii) and (iii) are then immediate consequences of Step 2.

#### 4. Simulations

To demonstrate the application of our controller, we consider an example of a mass-spring system mounted on a car from [14], see Fig. 3. The mass  $m_2$  moves on a ramp which is inclined by the angle  $\alpha$  and mounted on a car with mass  $m_1$ , for which it is possible to control the force u = F acting on it. The equations of motion for the system are given by

$$\begin{bmatrix} m_1 + m_2 & m_2 \cos \alpha \\ m_2 \cos \alpha & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x}(t) \\ \ddot{s}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ ks(t) + d\dot{s}(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ 0 \end{pmatrix},$$
(17)

where *x* is the horizontal car position and *s* the relative position of the mass on the ramp. The constants k, d > 0 are the coefficients of the spring and damper, resp. The output of the system is given by the horizontal position of the mass on the ramp,

$$y(t) = x(t) + s(t) \cos \alpha.$$



Figure 3: Mass on car system.

For the simulation, we choose the parameters  $m_1 = 4, m_2 = 1, k = 2, d = 1$  and the initial values  $x(0) = 0, \dot{x}(0) = 0, s(0) = 0, \dot{s}(0) = 0$ . The reference trajectory is  $y_{ref}(t) = \cos t$ . System (17) can be reformulated such that it belongs to the class (1), see [14], with a relative degree *r* depending on the angle  $\alpha$  and the damping *d*. We consider two cases.

**Case 1:** If  $0 < \alpha < \frac{\pi}{2}$ , see Fig. 3, then system (17) has relative degree r = 2 and the high-frequency gain matrix reads  $\Gamma = -\frac{1}{m_1 + m_2 \sin^2 \alpha} \sin^2 \alpha < 0$ . For the controller (4) we choose the funnel functions

$$\varphi_0(t) = (5e^{-2t} + 0.1)^{-1}, \qquad \varphi_1(t) = (10e^{-2t} + 0.5)^{-1},$$

and obviously the initial errors lie within the respective funnel boundaries, i.e., (9) is satisfied, thus Theorem 3.1 yields that funnel control is feasible. We compare the controller (4) with the proportional-derivative funnel controller (7) proposed in [4], which has been explained in Remark 2.4, and choose the same funnel functions  $\varphi_0$ ,  $\varphi_1$  for it. These functions satisfy the compatibility condition (8) and hence the controller (7) may be applied to (17) by [4].

The simulation of the controllers (4) and (7) applied to (17) over the time interval [0, 10] has been performed in MATLAB (solver: ode45, rel. tol.:  $10^{-14}$ , abs. tol.:  $10^{-10}$ ) and is depicted in Fig. 4. Fig. 4a shows the tracking errors corresponding to the two different controllers applied to the system, while Fig. 4b shows the respective input functions generated by them. It can be seen that our proposed funnel controller (4) requires less input action than the controller (7), both in magnitude and over time. For instance, in the time interval [3,5.5] there is no input action generated by (4), but several (large) oscillations generated by (7). It seems that our controller (4) is better able to use the inherent system properties and thus requires less input action than the controller generated by (4).

**Case 2:** If  $\alpha = 0$  and  $d \neq 0$ , see Fig. 5, then system (17) has relative degree r = 3 and high gain matrix  $\Gamma = -\frac{1}{m_1m_2}d < 0$ .



Fig. 4a: Funnel and tracking errors



Fig. 4b: Input functions

Figure 4: Simulation of the controllers (4) and (7) for the mass on car system (17).

For the controller (4) we choose the funnel functions

$$\varphi_0(t) = (5e^{-2t}+2)^{-1}, \quad \varphi_1(t) = \varphi_2(t) = (10e^{-2t}+5)^{-1},$$

and obviously the initial errors lie within the respective funnel boundaries, i.e., conditions (9) are satisfied, thus Theorem 3.1 yields that funnel control is feasible. We compare the controller (4) with the backstepping funnel controller (6) proposed in [10], which has been explained in Remark 2.3, and choose the funnel function  $\varphi = \varphi_0$  as well as  $\zeta_1(0) = \zeta_2(0) = 0$  for the initial values of the filter. Hence the controller (6) may be applied to (17) by [10]. Note that the chosen funnels for the tracking error are "wider" compared to Case 1; the reason are numerical issues with the controller (6), see the explanation below.

The simulation of the controllers (4) and (6) applied to (17) over the time interval [0,10] has been performed in MATLAB (solver: ode45, rel. tol.:  $10^{-14}$ , abs. tol.:  $10^{-10}$ ) and is depicted in Fig. 6. Fig. 6a shows the tracking errors corresponding to the two different controllers applied to the system, while Fig. 6b shows the respective input functions generated by them and Fig. 6c shows a zoom. It can be seen that our



Figure 5: Mass on car system with  $\alpha = 0$ .

proposed funnel controller (4) generates a maximal control action of approximately 5, while for the controller (6) the value is around  $1.5 \cdot 10^6$ . Obviously, our controller (4) achieves a better performance than the controller proposed in [10].

We stress that the controller (6) took a very long running time and only small changes in the system or controller parameters may induce severe problems in the numerical solution. The reason are the high powers of the gain function appearing in the control law, as explained in Remark 2.3. In the present simulation, although theoretically proved, the controller (6) is barely feasible and takes huge computational effort. Furthermore, the enormous values generated for the input function (which is a force) are not practically feasible in general. It seems that for systems with even higher relative degree, the controller from [10] is not practically realizable.

#### 5. Conclusion

In the present paper, we proposed a new funnel controller for nonlinear systems with arbitrary known relative degree and stable internal dynamics. We proved that this controller, which involves derivatives of the tracking error, achieves tracking of a sufficiently smooth reference trajectory with prescribed transient performance. We have illustrated the performance of our controller in comparison with other approaches by a simulation of a practical relevant mechanical system. We stress that, although the backstepping funnel controller presented in [10] is proved to work for systems with arbitrary relative degree, it does not seem to be practically realizable for system with relative degree larger than three. Therefore, our controller (4) seems to be the only choice for tracking with prescribed transient behavior in these cases.

#### Appendix A

Let  $\mathscr{D}\subseteq \mathbb{R}_{\geq 0}\times \mathbb{R}^n$  be non-empty, connected and open. Define

$$K := \{ \zeta \in \mathscr{C}([-h,\infty) \to \mathbb{R}^n) \mid \forall t \ge 0 : (t,\zeta(t)) \in \mathscr{D} \},\$$

and assume that *K* is non-empty. Consider an operator  $T: K \to \mathscr{L}^{\infty}_{loc}(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$  which satisfies the properties









Fig. 6c: Input functions - zoom

Figure 6: Simulation of the controllers (4) and (6) for the mass on car system (17).

a') *T* maps bounded trajectories to bounded trajectories, i.e., for all  $c_1 > 0$  there exists  $c_2 > 0$  such that for all  $\zeta \in K$ :

$$\|\zeta\|_{\infty} \leq c_1 \implies \|T(\zeta)\|_{\infty} \leq c_2.$$

b') *T* is causal, i.e., for all  $t \ge 0$  and all  $\zeta, \xi \in K$ :

$$\zeta|_{[-h,t)} = \xi|_{[-h,t)} \implies T(\zeta)|_{[0,t]} \stackrel{\text{a.a.}}{=} T(\xi)|_{[0,t]};$$

c') *T* is "locally Lipschitz" continuous in the following sense: for all  $t \ge 0$  there exist  $\tau, \delta, c > 0$  such that for all  $\zeta \in K$ ,  $\Delta \zeta \in \mathscr{C}([-h,\infty) \to \mathbb{R}^n)$  with  $\zeta + \Delta \zeta \in K$ ,  $\Delta \zeta|_{[-h,t]} = 0$  and  $\|\Delta \zeta|_{[t,t+\tau]}\|_{\infty} < \delta$  we have

$$\left\| \left( T(\zeta + \Delta \zeta) - T(\zeta) \right) \right\|_{[t,t+\tau]} \right\|_{\infty} \le c \| \Delta \zeta \|_{[t,t+\tau]} \|_{\infty}$$

Let  $F : \mathscr{D} \times \mathbb{R}^q \to \mathbb{R}^n$  be a Carathéodory function<sup>1</sup> and consider the initial value problem

$$\dot{x}(t) = F\left(t, x(t), T(x)(t)\right),$$
  

$$x|_{[-h,0]} = x^0 \in \mathscr{C}([-h,0] \to \mathbb{R}^n), \quad (0, x^0(0)) \in \mathscr{D}.$$
(18)

A function  $x \in \mathscr{C}([-h,\infty) \to \mathbb{R}^n)$  is called *solution* of (18) on  $[-h,\omega), \omega \in (0,\infty]$ , if  $x|_{[-h,0]} = x^0$  and  $x|_{[0,\omega)}$  is absolutely continuous, with  $(t,x(t)) \in \mathscr{D}$  for all  $t \in [0,\omega)$ , and satisfies the differential equation in (18) for almost all  $t \in [0,\omega)$ . A solution *x* is called *maximal*, if it has no right extension that is also a solution.

**Theorem A.1.** For all initial trajectories  $x^0 \in \mathscr{C}([-h, 0] \to \mathbb{R}^n)$ with  $(0, x^0(0)) \in \mathscr{D}$ 

- (i) the initial value problem (18) has a solution,
- (ii) every solution can be extended to a maximal solution,
- (iii) if *F* is locally essentially bounded and  $x \in \mathscr{C}([-h, \omega) \to \mathbb{R}^n)$  is a maximal solution, then the closure of graph  $x|_{[0,\omega)}$  is not a compact subset of  $\mathscr{D}$ .

*Proof.* The proof is a straightforward modification of those of [7, Thm. B.1] and [8, Thm. 5].

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<sup>1</sup>That is, for all  $[a,b] \times \overline{\mathscr{B}_{\delta}(z^0)} \subseteq \mathscr{D}$  and every compact  $K \subseteq \mathbb{R}^q$  we have:

- (i)  $F(t, \cdot, \cdot) : \overline{\mathscr{B}_{\delta}(z^0)} \times K \to \mathbb{R}^n$  is continuous for all  $t \in [a, b]$ ;
- (ii)  $F(\cdot, x, \zeta) : [a, b] \to \mathbb{R}^n$  is measurable for all  $(x, \zeta) \in \overline{\mathscr{B}_{\delta}(z^0)} \times K$ ;
- (iii) there exists an integrable function  $\gamma : [a,b] \to \mathbb{R}_{\geq 0}$  such that  $||F(t,x,\zeta)|| \leq \gamma(t)$  for almost all  $t \in [a,b]$  and all  $(x,\zeta) \in \overline{\mathscr{B}_{\delta}(z^0)} \times K$ .

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