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NUMERICAL SOLUTION OF MATRIX EQUATIONS ARISING IN LINEAR-QUADRATIC OPTIMAL CONTROL OF DIFFERENTIAL-ALGEBRAIC EQUATIONS

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Abstract. We consider the minimization of quadratic cost functionals subject to linear differential-algebraic constraints. We show that the optimal control design can be led back to the solution of certain quadratic matrix equations, which generalize Lur'e and Riccati equations to the differential-algebraic case. The solvability of this type of equations will shown to be equivalent to feasibility of the optimal control problem. The latter corresponds to the finiteness of the infimum of the cost functional for all consistent initial values. We do not assume regularity of the underlying differential-algebraic equation (DAE). Our theory and numerical analysis also covers indefinite cost functionals and the case where the optimal control problem is singular, which means that the optimal control is not unique, or the infimum is not attained.

Thereafter we consider the numerical solution of the matrix equations arising in optimal control. The approach is based on first deflating a "critical part" from certain associated even matrix pencils. Thereafter we obtain an algebraic Riccati equation on a subspace, which is solved by the Newton-ADI iteration. Thereby, we are able to deal with large scale problems. In particular, sparsity of the involved matrices will be exploited.

Key words. differential-algebraic equations, behavior, linear-quadratic optimal control, Lur'e equation

AMS subject classifications. 15A24, 34A09, 49J15, 49N10, 65L80, 93A15, 93B60

1. Introduction.

We focus on the linear-quadratic optimal control with differential-algebraic constraints, that is, for given $\mathcal{A},\mathcal{E}\in\mathbb{K}^{g\times q}$, a Hermitian matrix $\mathcal{Q}\in\mathbb{K}^{q\times q}$ and $w^0\in\mathbb{K}^q$ (to be specified later)

(1)
$$\text{Infimize } \int_0^\infty w(t)^*\mathcal{Q}w(t)\,\mathrm{d}t \quad \text{s. t.} \quad \tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}w=\mathcal{A}w,\quad \mathcal{E}w(0)=\mathcal{E}w^0,\quad \lim_{t\to\infty}\mathcal{E}w(t)=0.$$

For the general notation, we refer to the notation list on p. 4.

Our aim is to numerically obtain the matrix determining the optimal cost for all possible (consistent) initial values. We use the behavior approach [24]: A function $w:\mathbb{R}\to\mathbb{K}^q$ is said to be a *solution* of a DAE $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}w=\mathcal{A}w$, if it belongs to the *behavior*

$$\mathfrak{B}_{[\mathcal{E},\mathcal{A}]}:=\left\{w\in\mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}_{\geq0};\mathbb{K}^q)\;\middle|\;\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}w=\mathcal{A}w\right\},$$

where the derivative in the above definition has to be understood in the weak sense. Note that this implies that $\mathcal{E}w$ is weakly differentiable, whence we are allowed to define the evaluation $\mathcal{E}w(t):=(\mathcal{E}w)(t)$ for all $t\in\mathbb{R}_{\geq0}$. In particular, the initial and end conditions in (1) are well-defined. Next we introduce some spaces related to the behavior. These will play a crucial role in our theoretical and numerical considerations.

Definition 1. *Let $\mathcal{E},\mathcal{A}\in\mathbb{K}^{g\times q}$.*

(i) The system space *is the smallest subspace $\mathcal{V}_{[\mathcal{E},\mathcal{A}]}$ with $\mathfrak{B}_{[\mathcal{E},\mathcal{A}]}\subset\mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}_{\geq0};\mathcal{V}_{[\mathcal{E},\mathcal{A}]})$.*

(ii) The space of consistent differential initial values *is*

$$\mathcal{V}^{\mathrm{diff}}_{[\mathcal{E},\mathcal{A}]}:=\left\{w^0\in\mathbb{K}^q\;\middle|\;\exists w\in\mathfrak{B}_{[\mathcal{E},\mathcal{A}]}:\mathcal{E}w(0)=\mathcal{E}w^0\right\}.$$

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The following notations are fundamental for dealing with the optimal control problem (1).

Definition 2. *Let $\mathcal{E}, \mathcal{A} \in \mathbb{K}^{g\times q}$, $\mathcal{Q} \in \mathbb{K}^{q\times q}$ with $\mathcal{Q} = \mathcal{Q}^*$ be given.*

(i) The optimal cost *is the function $V^+ : \mathcal{E}\mathcal{V}^{\mathrm{diff}}_{[\mathcal{E},\mathcal{A}]} \to \mathbb{R} \cup \{-\infty, \infty\}$, with*

$$V^+(\mathcal{E}w^0) = \inf_{\substack{w\in\mathfrak{B}_{[\mathcal{E},\mathcal{A}]}\\ \mathcal{E}w(0)=\mathcal{E}w^0\\ \lim_{t\to\infty}\mathcal{E}w(t)=0}} \int_0^\infty w(t)^*\mathcal{Q}w(t)\,\mathrm{d}t < \infty. \tag{2}$$

(ii) The optimal control problem (1) is called feasible*, if for all $w^0 \in \mathcal{V}^{\mathrm{diff}}_{[\mathcal{E},\mathcal{A}]}$ holds*

$$-\infty < V^+(\mathcal{E}w^0) < \infty. \tag{3}$$

(iii) $\hat{w} \in \mathfrak{B}_{[\mathcal{E},\mathcal{A}]}$ with $\mathcal{E}\hat{w}(0) = \mathcal{E}w^0$ and $\lim_{t\to\infty}\mathcal{E}\hat{w}(t) = 0$ is called an optimal control for *(1), if*

$$\int_0^\infty \hat{w}(t)^*\mathcal{Q}\hat{w}(t)\,\mathrm{d}t = V^+(\mathcal{E}w^0). \tag{4}$$

(iv) The optimal control problem (1) is called regular*, if for all $w^0 \in \mathcal{V}^{\mathrm{diff}}_{[\mathcal{E},\mathcal{A}]}$, there exists a unique optimal control for (1). Otherwise, we call (1)* singular*.*

Let a feasible optimal control problem (1) be given. Our aim is to numerically determine a Hermitian matrix $X \in \mathbb{K}^{g\times g}$ such that

$$\forall w^0 \in \mathcal{V}^{\mathrm{diff}}_{[\mathcal{E},\mathcal{A}]}: \quad (\mathcal{E}w^0)^*X\mathcal{E}w^0 = V^+(\mathcal{E}w^0). \tag{5}$$

In case of existence, we further aim to determine the optimal control. We will lead back this problem to the solution of a certain matrix equation on the system space. To this end, we first define what we mean be equality of two matrices on some subspace.

Definition 3. *Let $\mathcal{V} \subset \mathbb{K}^n$ be a subspace and M, $N \in \mathbb{K}^{n\times n}$ be Hermitian. Then we write $M =_{\mathcal{V}} N$ if $x^*Mx = x^*Nx$ for all $x \in \mathcal{V}$.*

We will show in Theorem 15 that $X \in \mathbb{K}^{g\times g}$ as in (5) is a *stabilizing solution of the behavioral Lur'e equation*. That is, there exists some $p \in \mathbb{N}_0$ and some $\mathcal{K} \in \mathbb{K}^{p\times q}$, such that the pair $(X, \mathcal{K})$ fulfills

$$\mathcal{A}^*X\mathcal{E} + \mathcal{E}^*X\mathcal{A} + \mathcal{Q} =_{\mathcal{V}_{[\mathcal{E},\mathcal{A}]}} \mathcal{K}^*\mathcal{K}, \quad X = X^*, \tag{6a}$$

$$\forall\lambda \in \mathbb{C}_+ : \operatorname{rank}\left[\begin{smallmatrix}\mathcal{A}-\lambda\mathcal{E}\\ \mathcal{K}\end{smallmatrix}\right] V = n + p, \tag{6b}$$

where V is a matrix with full column rank and $\operatorname{im} V = \mathcal{V}_{[\mathcal{E},\mathcal{A}]}$, and $n = \operatorname{rank}\mathcal{E}V$. We will in particular show that feasibility of the optimal control problem is equivalent to the existence a pair $(X, \mathcal{K})$ with (6). Furthermore, an optimal control will be proven to be a solution of the initial value problem

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\mathcal{E}\\ 0\end{bmatrix}\hat{w} = \begin{bmatrix}\mathcal{A}\\ \mathcal{K}\end{bmatrix}\hat{w}, \quad \mathcal{E}\hat{w}(0) = \mathcal{E}w^0. \tag{7}$$

Thereafter we will present an algorithm for the numerical determination of $(X, \mathcal{K})$. This algorithm is able to treat the case where g and q are large provided that

- $\mathcal{E}$ and $\mathcal{A}$ are sparse and $\mathcal{Q}$ has low-rank, and
- X has low numerical rank. That is, only few eigenvalues of X are not small.

In particular, our algorithm will produce a *low-rank approximation* $X \approx SS^*$, where $S \in \mathbb{K}^{g\times l}$ with $l \ll g$.

Remark 4 (Optimal control of DAEs).

a) Let $E, A, Q \in \mathbb{K}^{n\times n}$, $B, S \in \mathbb{K}^{n\times m}$ and $R \in \mathbb{K}^{m\times m}$ be given with $Q = Q^*$ and $R = R^*$. Consider the optimal control problem:

$$\begin{aligned} &\text{Infimize } \int_0^\infty \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \mathrm{d}t \\ &\text{subject to } \quad \tfrac{\mathrm{d}}{\mathrm{d}t}Ex = Ax + Bu, \quad Ex(0) = Ex^0, \quad \lim_{t\to\infty} Ex(t) = 0. \end{aligned} \tag{8}$$

By putting

$$\mathcal{E} = \begin{bmatrix} E & 0 \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} A & B \end{bmatrix}, \quad w(t) = \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}, \quad \mathcal{Q} = \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix}, \tag{9}$$

we see that (8) a special case of (1).

b) The solution of the infinite time horizon linear-quadratic optimal control problem for DAEs with constant coefficients is a well researched topic. The vast majority of literature on this topic considers the optimal control problem (8). In particular, there exists various articles on theoretical analysis of the problem [2, 13, 15, 18, 19, 21, 28]. Except for [13, 28], these article assume regularity of the optimal control problem (8) together with invertibility of $R \in \mathbb{K}^{m\times m}$. In [13], the matrix $\mathcal{Q} = \left[\begin{smallmatrix} Q & S \\ S^* & R \end{smallmatrix}\right]$ is assumed to be positive semidefinite. In [12], optimal control for controllable behavior systems is considered. The most general theoretical approach to the optimal control problem (8) has been presented in [28], where only regularity of the pencil $sE - A$ has been presumed.
Most of the aforementioned articles only treat theoretical aspects. A numerical treatment has been done in [21], where an a priori feedback transformation has been performed, which leads to optimal control of a system with index one. This allows to extract an optimal control problem for ordinary differential equation (ODE) and thereafter using appropriate numerical methods.

We note that the use of the behavioral approach (i.e., we do not distinguish between input and state) is not the main achievement of this article. We use this approach since it is both, slightly more general than (8), and our theoretical and numerical considerations do not require the distinction between input and state variables. In an abstract manner, a DAE optimal control task can be regarded as a constrained optimization problem. These constraints are given by the DAE. The distinction of input and state is therefore not necessary in optimal control. A similar justification of the behavioral approach for the consideration of optimal control of time-varying DAEs has been made in [17].

1.1. Outline. In the first four sections we present the theoretical basis for our numerical investigations. The forthcoming section 2 is about fundamentals on matrix pencils and behaviors. We further give some matrix theoretical characterizations of the system space and the space of consistent initial differential values using a *Wong sequence*. This will build the foundation for our numerical analysis. Then we show in section 3 that the existence of a stabilizing solution is equivalent to the feasibility of the linear quadratic optimal control problem. In section 4 we will give a special representation for the solutions of behavioral Lur'e equations via deflating subspaces of so-called *even matrix pencils*. We show that - via deflation of a certain part of the even matrix pencil - we are led to a projected Lur'e equation that is equivalent to a *projected algebraic Riccati equation*. The latter can thereafter be solved by the Newton-ADI iteration [8]. The tool to get the right subspaces for the deflation are the so-called *neutral Wong sequences*.

In section 5, we present the numerical algorithms which are derived from the presented theory. Our approach relies on overloading methods like the matrix multiplication and inversion with methods for projected matrices. We first give a reference implementation for these methods, which is suitable for large scale and sparse problems. Then we give algorithms for both, the Wong sequence and the neutral Wong sequences. This leads to a projected Lur'e equation which can - in theory - be reformulated as an algebraic Riccati equation. However, instead of using the methods given in [8], we avoid reformulating the equation by modifying the Newton algorithm such that the we are able to overload specific operations. This preserves in particular the sparsity. In section 6 we demonstrate the applicability of our results by means of examples from electrical circuit theory and flow dynamics.

2. Preliminaries.

In this section we recall the basics and show some auxiliary results needed in the following sections. We use the following notation.

$\mathbb{K}$	either the field $\mathbb{R}$ of real or the field $\mathbb{C}$ of complex numbers
$\mathbb{N}_0$	set of natural numbers including zero
$\mathbb{R}_{\geq 0}$,	set of non-negative real numbers
$\mathbb{C}_+$	open set of complex numbers with positive real part
$\mathbb{K}[s]$	the ring of polynomials with coefficients in $\mathbb{K}$
$R^{g\times q}$	the set of $g \times q$ matrices with entries in the ring R
$\mathrm{Gl}_n(\mathbb{K})$	the group of invertible $n \times n$ matrices with coefficients in the field $\mathbb{K}$
I_n, $0_{g,q}$	identity matrix of size $n\times n$ and zero matrix of size $g\times q$, resp. (subscript may be omitted, if clear from context)
M^*	conjugate transpose of $M \in \mathbb{K}^{m\times n}$
M^+	a pseudoinverse of $M \in \mathbb{K}^{m\times n}$, i.e., $M^+ \in \mathbb{K}^{n\times m}$ with $MM^+M = M$ and $M^+MM^+ = M^+$
$M^{-1}\mathcal{V}$	$= \{x \in \mathbb{K}^q \mid Mx \in \mathcal{V}\}$, the preimage of $\mathcal{V} \subset \mathbb{K}^g$ under $M \in \mathbb{K}^{g\times q}$
$\mathcal{L}^2_{\mathrm{loc}}(\mathcal{I};\mathcal{V})$	the set of measurable and locally square integrable functions $f : \mathcal{I} \to \mathcal{V}$, where $\mathcal{I} \subseteq \mathbb{R}$, and $\mathcal{V}$ is a finite-dimensional normed space

2.1. Matrix pencils and Wong sequences.

The analysis of linear DAEs $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}w = \mathcal{A}w$ with $\mathcal{E}, \mathcal{A} \in \mathbb{K}^{g\times q}$ leads to the study of *matrix pencils* $\mathcal{A} - s\mathcal{E} \in \mathbb{K}[s]^{g\times q}$. Next we introduce some concepts related to matrix pencils.

DEFINITION 5.

a) A matrix $V \in \mathbb{K}^{q\times \ell}$ *is called a* basis matrix *for a subspace* $\mathcal{V} \subset \mathbb{K}^q$*, if it has full column rank and* $\operatorname{im} V = \mathcal{V}$.

b) A subspace $\mathcal{V} \subset \mathbb{K}^q$ *is called* deflating subspace *for the pencil* $\mathcal{A} - s\mathcal{E} \in \mathbb{K}[s]^{g\times q}$ *if, for a basis matrix* $V \in \mathbb{K}^{q\times k}$ *of* $\mathcal{V}$*, there exists some* $\ell \in \mathbb{N}_0$*, a matrix* $W \in \mathbb{K}^{g\times \ell}$ *with full column rank, and a pencil* $\widetilde{\mathcal{A}} - s\widetilde{\mathcal{E}} \in \mathbb{K}[s]^{\ell\times k}$ *with* $\max_{\lambda\in\mathbb{C}} \operatorname{rank}(\widetilde{\mathcal{A}} - \lambda\widetilde{\mathcal{E}}) = \ell$ *such that*

$$(\mathcal{A} - s\mathcal{E})V = W(\widetilde{\mathcal{A}} - s\widetilde{\mathcal{E}}). \tag{10}$$

c) The matrix pencil $\mathcal{A} - s\mathcal{E} \in \mathbb{K}[s]^{g\times q}$ *is called* regular*, if* $q = g$ *and there exist* $\lambda \in \mathbb{K}$ *such that* $\mathcal{A} - \lambda\mathcal{E} \in \mathrm{Gl}_q(\mathbb{K})$.

d) $\lambda \in \mathbb{C}$ *is called* generalized eigenvalue of $\mathcal{A} - s\mathcal{E} \in \mathbb{K}[s]^{g\times q}$, *if* $\operatorname{rank} \mathcal{A} - \lambda\mathcal{E} < \max_{\mu\in\mathbb{C}} \operatorname{rank} \mathcal{A} - \mu\mathcal{E}$.

The definition of deflating subspaces is taken from [26, 25], since we will refer to some related results. On the other hand, for numerical methods a different definition became accepted [22]. The latter is advantageous for developing numerical algorithms, which is why we show that these definitions are equivalent.

Lemma 6. *Let a matrix pencil $\mathcal{A} - s\mathcal{E} \in \mathbb{K}[s]^{g\times q}$ be given, and assume that $V \in \mathbb{K}^{q\times k}$ is a matrix with full column rank. Then the following statements are equivalent:*

(i) There exists some $\ell \in \mathbb{N}_0$, a matrix $W \in \mathbb{K}^{g\times \ell}$ with full column rank, and a pencil $\widetilde{\mathcal{A}} - s\widetilde{\mathcal{E}} \in \mathbb{K}[s]^{\ell\times k}$ with $\max_{\lambda\in\mathbb{C}} \operatorname{rank}(\widetilde{\mathcal{A}} - \lambda\widetilde{\mathcal{E}}) = \ell$ such that $(\mathcal{A} - s\mathcal{E})V = W(\widetilde{\mathcal{A}} - s\widetilde{\mathcal{E}})$.

(ii) There exist matrices $R, L \in \mathbb{K}^{k\times k}$ such that $\mathcal{A}VL = \mathcal{E}VR$ and $L - sR \in \mathbb{K}[s]^{k\times k}$ is regular.

Proof.

(i)⇒(ii) Since $\max_{\lambda\in\mathbb{C}} \operatorname{rank}(\widetilde{\mathcal{A}} - \lambda\widetilde{\mathcal{E}}) = \ell$ there exist $\lambda_1, \lambda_2 \in \mathbb{K}$ with $0 \neq \lambda_1 \neq \lambda_2$ such that $\operatorname{rank}(\widetilde{\mathcal{A}} - \lambda_1\widetilde{\mathcal{E}}) = \operatorname{rank}(\widetilde{\mathcal{A}} - \lambda_2\widetilde{\mathcal{E}}) = \ell$. Therefore, there exist $K_1, K_2 \in \mathbb{K}^{(k-\ell)\times k}$ such that $S_i := \begin{bmatrix} \widetilde{\mathcal{A}}-\lambda_i\widetilde{\mathcal{E}} \\ K_i \end{bmatrix} \in \mathrm{Gl}_k(\mathbb{K})$ for $i \in \{1, 2\}$, and then we have

$$(\mathcal{A} - \lambda_i\mathcal{E})V = W(\widetilde{\mathcal{A}} - \lambda_i\widetilde{\mathcal{E}}) = \begin{bmatrix} W & 0 \end{bmatrix} \begin{bmatrix} \widetilde{\mathcal{A}}-\lambda_i\widetilde{\mathcal{E}} \\ K_i \end{bmatrix} \quad \text{for } i \in \{1, 2\}.$$

This gives $(\mathcal{A}V - \lambda_i\mathcal{E}V)S_i^{-1} = \begin{bmatrix} W & 0 \end{bmatrix}$, whence

$$\mathcal{A}V(S_1^{-1} - S_2^{-1}) = \mathcal{E}V(\lambda_1 S_1^{-1} - \lambda_2 S_2^{-1}).$$

Consequently, $\mathcal{A}VL = \mathcal{E}VR$ for $L = S_1^{-1} - S_2^{-1}$ and $R = \lambda_1 S_1^{-1} - \lambda_2 S_2^{-1}$. The regularity of $L - sR \in \mathbb{K}[s]^{k\times k}$ follows from $L - \frac{1}{\lambda_1}R = (\frac{\lambda_2}{\lambda_1} - 1)S_2^{-1} \in \mathrm{Gl}_k(\mathbb{K})$.

(ii)⇒(i) By regularity of $L - sR$, there exists some $\lambda \in \mathbb{K}\setminus\{0\}$ with $(L - \lambda R) \in \mathrm{Gl}_k(\mathbb{K})$. With $\mathcal{A}VL = \mathcal{E}VR$ we have

$$(\mathcal{A}-s\mathcal{E})V(L-\lambda R) = \mathcal{E}VR - \lambda\mathcal{A}VR - s(\mathcal{E}VL - \lambda\mathcal{A}L) = (\mathcal{E}V - \lambda\mathcal{A}V)(R - sL).$$

Let $W \in \mathbb{K}^{q\times \ell}$ with $\ell \le k$ be a basis matrix of $\operatorname{im}(\mathcal{E}V - \lambda\mathcal{A}V)$. Then there exists some $C \in \mathbb{K}^{\ell\times k}$ with full row rank and $\mathcal{E}V - \lambda\mathcal{A}V = WC$. This leads to $(\mathcal{A} - s\mathcal{E})V = WC(R - sL)(L - \lambda R)^{-1} = W(\widetilde{\mathcal{A}} - s\widetilde{\mathcal{E}})$ for $\widetilde{\mathcal{A}} = CR(L - \lambda R)^{-1}$ and $\widetilde{\mathcal{E}} = CL(L - \lambda R)^{-1}$. Furthermore, we have $\max_{\lambda\in\mathbb{C}} \operatorname{rank}(\widetilde{\mathcal{A}} - \lambda_2\widetilde{\mathcal{E}}) = \ell$ since C has full row rank and $\widetilde{\mathcal{A}} - \frac{1}{\lambda}\widetilde{\mathcal{E}} = C(R - \frac{1}{\lambda}L)(L - \lambda R)^{-1} = -\frac{1}{\lambda}C$. □

Wong sequences [10, 11, 31] are a tool to construct certain deflating subspaces of matrix pencils.

Definition 7. *The* Wong sequences *for the matrix pencil $\mathcal{A} - s\mathcal{E} \in \mathbb{K}[s]^{g\times q}$ are given by*

$$\mathcal{V}_0 = \mathbb{K}^q, \quad \mathcal{V}_{i+1} = \mathcal{A}^{-1}(\mathcal{E}\mathcal{V}_i) i \in \mathbb{N}_0, \tag{11}$$

and their limits *by* $\mathcal{V} = \bigcap_{i\in\mathbb{N}_0} \mathcal{V}_i$.

Since the Wong sequence consists of nested subspaces of $\mathbb{K}^q$, it is easy to see that it stagnates at a specific index. In other words, there exists some $k \in \mathbb{N}_0$ such that $\mathcal{V}_k = \mathcal{V}$. Furthermore, we have $\mathcal{A}\mathcal{V} \subseteq \mathcal{E}\mathcal{V}$ [10].

In the following lemma we point out that the limit $\mathcal{V}$ of the Wong sequence is a deflating subspace with a special property.

LEMMA 8. *Let a matrix pencil $\mathcal{A} - s\mathcal{E} \in \mathbb{K}[s]^{g\times q}$, and let $\mathcal{V} \subset \mathbb{K}^g$ as in Definition 7. Then the following holds:*

a) $\mathcal{V}$ is a deflating subspace of $\mathcal{A} - s\mathcal{E}$. In other words, there exist $n, m \in \mathbb{N}_0$, a matrix $W \in \mathbb{K}^{g\times n}$ with full column rank, and a pencil $\widetilde{\mathcal{A}} - s\widetilde{\mathcal{E}} \in \mathbb{K}[s]^{n\times(n+m)}$ with $\max_{\lambda\in\mathbb{C}} \operatorname{rank}(\widetilde{\mathcal{A}} - \lambda\widetilde{\mathcal{E}}) = n$ and $V \in \mathbb{K}^{q\times(n+m)}$ with $\mathcal{V} = \operatorname{im} V$, such that (10) holds.

b) The matrix in $\widetilde{\mathcal{E}}$ in a) has full row rank.

Proof. This statement follows from [11, Thm. 2.1]. ∎

We further consider *even matrix pencils*, that is, $\mathcal{H} - s\mathcal{G} \in \mathbb{K}[s]^{N\times N}$ with $\mathcal{G}^* = -\mathcal{G}$ and $\mathcal{H} = \mathcal{H}^*$. For the solution of behavioral Lur'e equations, it will be crucial to consider a *$\mathcal{G}$-neutral Wong sequence* of the even matrix pencil $\mathcal{H} - s\mathcal{G} \in \mathbb{K}[s]^{N\times N}$. In the following we define $\mathcal{G}$-neutrality and the needed Wong sequence.

DEFINITION 9. *Let the matrix $\mathcal{G} \in \mathbb{K}^{N\times N}$ and a subspace $\mathcal{V} \subseteq \mathbb{K}^N$ be given. Then we call a subspace $\mathcal{V} \subseteq \mathbb{K}^N$* $\mathcal{G}$-neutral, *if $\mathcal{V} \subset (\mathcal{G}\mathcal{V})^\perp$.*

DEFINITION 10. *The* $\mathcal{G}$-neutral Wong sequence $(\mathcal{Y}_i)$ *for the even matrix pencil $\mathcal{H} - s\mathcal{G} \in \mathbb{K}[s]^{g\times q}$ is given by*

$$(12)\qquad \mathcal{Y}_0 = \{0\},\ \mathcal{Z}_{i+1} = \mathcal{G}^{-1}\left(\mathcal{H}\mathcal{Y}_i\right),\ \mathcal{Y}_{i+1} = \mathcal{Y}_i + \Big(\mathcal{Z}_{i+1} \cap (\mathcal{G}cZ_{i+1})^\perp\Big),\ i \in \mathbb{N}_0,$$

and its limit *by $\mathcal{Y} := \bigcup_{i=0}^\infty \mathcal{Y}_i$.*

For the limit $\mathcal{Y}$ we have following result:

LEMMA 11. *Let an even matrix pencil $\mathcal{H} - s\mathcal{G} \in \mathbb{K}[s]^{N\times N}$ be given. Then the limit $\mathcal{Y} \subset \mathbb{K}^N$ as in Definition 10 is a $\mathcal{G}$-neutral deflating subspace of $\mathcal{H} - s\mathcal{G}$.*

Proof. This is a consequence of [25, Thm. 2.11]. ∎

2.2. Behaviors and transformations. We now consider special forms of the behavior which are achieved by coordinate transformations. We will make use of the fact that for $\mathcal{A}, \mathcal{E} \in \mathbb{K}^{g\times q}$, $V \in \mathrm{Gl}_q(\mathbb{K})$ and $W \in \mathrm{Gl}_g(\mathbb{K})$ holds

$$(13)\qquad \mathfrak{B}_{[\mathcal{E},\mathcal{A}]} = V \cdot \mathfrak{B}_{[W^{-1}\mathcal{E}V, W^{-1}\mathcal{A}V]},$$

where the multiplication of V with the functions in the behavior has to be understood pointwisely. Now we show that the system space equals to the limit of the Wong sequence. Moreover, the behavior can be compressed in some sense.

LEMMA 12. *Let $\mathcal{A}, \mathcal{E} \in \mathbb{K}^{g\times q}$. Let $\mathcal{V} \subset \mathbb{K}^g$ be as in Definition 7. Then the system space $\mathcal{V}_{[\mathcal{E},\mathcal{A}]}$ (see Definition 1) fulfills $\mathcal{V} = \mathcal{V}_{[\mathcal{E},\mathcal{A}]}$. Moreover, for $V \in \mathbb{K}^{q\times(n+m)}$, $W \in \mathbb{K}^{g\times n}$ and $\widetilde{\mathcal{E}}, \widetilde{\mathcal{A}} \in \mathbb{K}^{n\times(n+m)}$ as in Lemma 8 we have*

$$(14)\qquad \mathfrak{B}_{[\mathcal{E},\mathcal{A}]} = V \cdot \mathfrak{B}_{[\widetilde{\mathcal{E}},\widetilde{\mathcal{A}}]}.$$

Proof. The first statement follows by a combination of [11, Cor. 2.3] and [10, Thm 3.2]. It remains to prove (14): Since $\mathcal{V} = \mathcal{V}_{[\mathcal{E},\mathcal{A}]}$, for all $w \in \mathfrak{B}_{[\mathcal{E},\mathcal{A}]}$, there exists some $z \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{K}^{n+m})$ such that $w = Vz$. Then

$$(\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} - \mathcal{A})w = (\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} - \mathcal{A})Vz = W(\tfrac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}})z = 0.$$

Since W has full column rank, we obtain $z \in \mathfrak{B}_{[\widetilde{\mathcal{E}},\widetilde{\mathcal{A}}]}$, and thus $\mathfrak{B}_{[\mathcal{E},\mathcal{A}]} \subset V \cdot \mathfrak{B}_{[\widetilde{\mathcal{E}},\widetilde{\mathcal{A}}]}$. To prove the reverse inclusion, assume that $z \in \mathfrak{B}_{[\widetilde{\mathcal{E}},\widetilde{\mathcal{A}}]}$ and define $w = Vz$. Then

$$(\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} - \mathcal{A})w = (\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} - \mathcal{A})Vz = W(\tfrac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}})z = 0,$$

whence $w \in \mathfrak{B}_{[\mathcal{E},\mathcal{A}]}$. ∎

With the previous notation, we have seen from Lemma 8 that $\widetilde{\mathcal{E}}$ has full row rank. Hence, the matrices V and W can be even chosen such that $\widetilde{\mathcal{E}} = \begin{bmatrix} I_n & 0 \end{bmatrix}$. This leads to a behavior, that is described by the solution set of an ODE. This is subject of the subsequent result.

Lemma 13. *Let $\mathcal{A}, \mathcal{E} \in \mathbb{K}^{g\times q}$ and let $n, m \in \mathbb{N}_0$ as in Lemma 12. Then there exist matrices $T \in \mathbb{K}^{q\times(n+m)}$ with $\operatorname{im} T = \mathcal{V}_{[\mathcal{E},\mathcal{A}]}$, $W \in \mathbb{K}^{g\times n}$ and $A \in \mathbb{K}^{n\times n}$, $B \in \mathbb{K}^{n\times m}$ such that*

$$(\mathcal{A} - s\mathcal{E})T = W \begin{bmatrix} A - sI_n & B \end{bmatrix}.$$

Proof. Let $V \in \mathbb{K}^{q\times(n+m)}$, $W \in \mathbb{K}^{g\times n}$ and $\widetilde{\mathcal{E}}, \widetilde{\mathcal{A}} \in \mathbb{K}^{n\times(n+m)}$ be as in Lemma 8. Note that $\widetilde{\mathcal{E}}$ has full row rank. Hence, for a right inverse $\widetilde{\mathcal{E}}^+ \in \mathbb{K}^{(n+m)\times n}$ of $\widetilde{\mathcal{E}}^+$ and a basis matrix $V_B \in \mathbb{K}^{(n+m)\times m}$ of $\ker \widetilde{\mathcal{E}}$, we have $\widetilde{T} := [\,\widetilde{\mathcal{E}}^+ \; V_B\,] \in \mathrm{Gl}_{n+m}(\mathbb{K})$ and $\widetilde{\mathcal{E}}\widetilde{T} = [\,I_n \; 0\,]$. The previous findings together with Lemma 12 imply that $T := V\widetilde{T}$ has the desired properties. ∎

We further recap the property of behavioral stabilizability for DAEs, cf. [9, Def. 5.1] and [24, Def. 5.2.29].

Definition 14. *Let $\mathcal{E}, \mathcal{A} \in \mathbb{K}^{g\times q}$. The DAE $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}w = \mathcal{A}w$ is called* behavioral stabilizable*, if*

$$\forall w^0 \in \mathcal{V}^{\mathrm{diff}}_{[\mathcal{E},\mathcal{A}]}\ \exists w \in \mathfrak{B}_{[\mathcal{E},\mathcal{A}]} \,:\, \mathcal{E}w(0) = \mathcal{E}w^0 \wedge \lim_{t\to\infty} \mathcal{E}w(t) = 0.$$

It has been shown in [9, Cor. 5.1] (see also [24, Thm. 5.2.30]) that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}w = \mathcal{A}w$ is behavioral stabilizable if, and only if, all generalized eigenvalues of $\mathcal{A} - s\mathcal{E}$ have negative real part. With $m, n \in \mathbb{N}_0$ as in Lemma 8 and $V \in \mathbb{K}^{q\times(n+m)}$ being a basis matrix for the system space $\mathcal{V}_{[\mathcal{E},\mathcal{A}]}$, it can be concluded from the previous statement together with Lemmas 8 and 12 that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}w = \mathcal{A}w$ is behavioral stabilizable if, and only if, for all $\lambda \in \overline{\mathbb{C}_+}$ holds $\operatorname{rank}(\mathcal{A} - \lambda\mathcal{E})V = n$. Note that this implies that $\mathcal{A}V$ and hence $\widetilde{A}$ as in Lemma 12 has full row rank.

3. Optimal control of DAEs. In this section we present the connection between feasibility of the optimal control problem (cf. Definition 2) and the existence of a stabilizing solution of the behavioral Lur'e equation. We subsequently show that for a behavioral stabilizable DAE $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}w = \mathcal{A}w$ the existence of a solution $(X, \mathcal{K}) \in \mathbb{K}^{g\times g} \times \mathbb{K}^{p\times q}$ of the Lur'e equation (6) is equivalent to feasibility of the optimal control problem (1). Furthermore, the quadratic form with $\mathcal{E}^*X\mathcal{E}$ will be shown to express the optimal cost and $\hat{w} \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{K}^q)$ is an optimal control if, and only if, it fulfills DAE (7). All these statements are contained in the subsequent result. Note that behavioral stabilizability is necessary for the feasibility of the optimal control problem, since it is equivalent to the optimal cost fulfilling $V^+(\mathcal{E}w^0) < \infty$.

Theorem 15. *Let $\mathcal{E}, \mathcal{A} \in \mathbb{K}^{g\times q}$ such that the DAE $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}w = \mathcal{A}w$ with system space $\mathcal{V}_{[\mathcal{E},\mathcal{A}]} \subset \mathbb{K}^q$ is behavioral stabilizable; let $\mathcal{Q} \in \mathbb{K}^{q\times q}$ be Hermitian and, for $m, n \in \mathbb{N}_0$ as in Lemma 8, let $V \in \mathbb{K}^{q\times(n+m)}$ be a basis matrix for $\mathcal{V}_{[\mathcal{E},\mathcal{A}]}$. Then the subsequent statements are equivalent:*

(i) The Lur'e equation (6a) has a solution $(X, \mathcal{K})$. That is, there exists some $p \in \mathbb{N}_0$, some $\mathcal{K} \in \mathbb{K}^{p\times q}$ and Hermitian $X \in \mathbb{K}^{g\times g}$, such that

$$\mathcal{A}^*X\mathcal{E} + \mathcal{E}^*X\mathcal{A} + \mathcal{Q} =_{\mathcal{V}_{[\mathcal{E},\mathcal{A}]}} \mathcal{K}^*\mathcal{K}, \quad X = X^*.$$

(ii) The Lur'e equation (6) has a solution $(X,\mathcal{K})$. That is, there exists some $p\in\mathbb{N}_0$, some $\mathcal{K}\in\mathbb{K}^{p\times q}$ and Hermitian $X\in\mathbb{K}^{g\times g}$, such that (6a) and

$$\forall\lambda\in\mathbb{C}_+:\operatorname{rank}\left[\begin{smallmatrix}\mathcal{A}-\lambda\mathcal{E}\\ \mathcal{K}\end{smallmatrix}\right]V=n+p.$$

(iii) The optimal control problem (1) is feasible. That is, the optimal cost V^+ as in (2) fulfills $-\infty<V^+(\mathcal{E}w^0)<\infty$ for all $w^0\in\mathcal{V}^{\mathrm{diff}}_{[\mathcal{E},\mathcal{A}]}$.

Further, if $(X,\mathcal{K})$ is a solution of (6), i.e., a stabilizing solution of the Lur'e equation, then the following holds:

a) The optimal cost is quadratic with $V^+(\mathcal{E}w^0)=(\mathcal{E}w^0)^*X\mathcal{E}w^0$.

b) $\hat{w}\in\mathfrak{B}_{[\mathcal{E},\mathcal{A}]}$ is an optimal control for (1) (i.e., (4) holds) if, and only if, it fulfills the DAE

$$\tag{15}\tfrac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\mathcal{E}\\0\end{bmatrix}\hat{w}=\begin{bmatrix}\mathcal{A}\\ \mathcal{K}\end{bmatrix}\hat{w},\quad \mathcal{E}\hat{w}(0)=\mathcal{E}w^0,\quad \lim_{t\to\infty}\hat{w}(t)=0.$$

Proof. We follow the pattern (ii)⇒(i)⇒(iii)⇒(ii) to prove the equivalence:

(ii)⇒(i) This is trivial.

(i)⇒(iii) Assume that $(X,\mathcal{K})\in\mathbb{K}^{m\times m}\times\mathbb{K}^{p\times q}$ solves the Lur'e equation (6a). Consider $w^0\in\mathcal{V}^{\mathrm{diff}}_{[\mathcal{E},\mathcal{A}]}$ and $w\in\mathfrak{B}_{[\mathcal{E},\mathcal{A}]}$ with $\mathcal{E}w(0)=\mathcal{E}w^0$ and $\lim_{t\to\infty}\mathcal{E}w(t)=0$. Then, by the fundamental theorem of calculus and $w(t)\in\mathcal{V}_{[\mathcal{E},\mathcal{A}]}$ for almost all $t\in\mathbb{R}_{\geq 0}$, we obtain

(16)

$$\begin{aligned}(\mathcal{E}w^0)^*X\mathcal{E}w^0 &= -\int_0^\infty \tfrac{\mathrm{d}}{\mathrm{d}t}\left(w(t)^*\mathcal{E}^*X\mathcal{E}w(t)\right)\mathrm{d}t\\ &= -\int_0^\infty \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}w(t)\right)^*X\mathcal{E}w(t)+w(t)^*\mathcal{E}^*X\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}w(t)\,\mathrm{d}t\\ &= -\int_0^\infty w(t)^*\mathcal{A}^*X\mathcal{E}w(t)+w(t)^*\mathcal{E}^*X\mathcal{A}w(t)\,\mathrm{d}t\\ &\overset{(6a)}{=}\int_0^\infty w(t)^*\mathcal{Q}w(t)-w(t)^*\mathcal{K}^*\mathcal{K}w(t)\,\mathrm{d}t\ \leq\ \int_0^\infty w(t)^*\mathcal{Q}w(t)\,\mathrm{d}t.\end{aligned}$$

In particular, we have $V^+(\mathcal{E}w^0)\geq(\mathcal{E}w^0)^*X\mathcal{E}w^0>-\infty$. The behavioral stabilizability implies that $V^+(\mathcal{E}w^0)<\infty$ and we obtain that the optimal control problem (1) is feasible.

(iii)⇒(ii) We prove this implication by a transformation to a control system whose dynamics are described by an ODE. To this end, consider $T\in\mathbb{K}^{q\times(n+m)}$, $W\in\mathbb{K}^{g\times n}$ and $A\in\mathbb{K}^{n\times n}$, $B\in\mathbb{K}^{n\times m}$ as in Lemma 13. In particular, we have $\operatorname{im}T=\mathcal{V}_{[\mathcal{E},\mathcal{A}]}=\operatorname{im}V$. Further, let $x\in\mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{K}^n)$ and $u\in\mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}_{\geq 0};\mathbb{K}^m)$ such that $w=T\left[\begin{smallmatrix}x\\u\end{smallmatrix}\right]$. Let $Q\in\mathbb{K}^{n\times n}$, $S\in\mathbb{K}^{n\times m}$ and $R\in\mathbb{K}^{m\times m}$ such that $T^*\mathcal{Q}T=\left[\begin{smallmatrix}Q&S\\S^*&R\end{smallmatrix}\right]$ and consider the optimal cost functional

$$\tag{17}\begin{aligned}&\widetilde{V}^+:\mathbb{K}^n\to\mathbb{R}\cup\{-\infty\},\\ &x^0\mapsto \inf_{\substack{\left[\begin{smallmatrix}x\\u\end{smallmatrix}\right]\in\mathfrak{B}_{[[I,0],[A,B]]}\\ x(0)=x^0\\ \lim_{t\to\infty}x(t)=0}}\int_0^\infty\begin{pmatrix}x(t)\\u(t)\end{pmatrix}^*\begin{bmatrix}Q&S\\S^*&R\end{bmatrix}\begin{pmatrix}x(t)\\u(t)\end{pmatrix}\mathrm{d}t.\end{aligned}$$

Note that the behavioral stabilizability of $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}w=\mathcal{A}w$ implies stabilizability of $\frac{\mathrm{d}}{\mathrm{d}t}x=Ax+Bu$. Since W yields to $\operatorname{im}W=\mathcal{E}\cdot\mathcal{V}^{\mathrm{diff}}_{[\mathcal{E},\mathcal{A}]}$, we have that for all

$x^0 \in \mathbb{K}^n$ there exist $\mathcal{E}w^0 \in \mathcal{E}\mathcal{V}^{\mathrm{diff}}_{[\mathcal{E},\mathcal{A}]}$ with $\mathcal{E}w^0 = Wx^0$. This implies

$$V^+(\mathcal{E}w^0) = V^+(Wx^0) = \widetilde{V}^+(x^0). \tag{18}$$

Hence, (iii) implies that $-\infty < \widetilde{V}^+(x^0) < \infty$ for all $x^0 \in \mathbb{K}^n$. In other words, the ODE optimal control problem with $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu$ and cost functional (17) is feasible.

Now using [14, Thm. 8.3], we obtain that there exist $p \in \mathbb{N}_0$, $\widetilde{X} \in \mathbb{K}^{n\times n}$, $K \in \mathbb{K}^{p\times n}$ and $L \in \mathbb{K}^{p\times m}$ such that $(\widetilde{X}, K, L)$ is a stabilizing solution of a Lur'e equation corresponding to an ODE system, i.e.,

$$\begin{aligned} &\begin{bmatrix} A^*\widetilde{X} + \widetilde{X}A + Q & \widetilde{X}B + S \\ B^*\widetilde{X} + S^* & R \end{bmatrix} = \begin{bmatrix} K^*K & K^*L \\ L^*K & L^*L \end{bmatrix}, \quad \widetilde{X} = \widetilde{X}^*, \\ &\forall \lambda \in \mathbb{C}_+ : \operatorname{rank} \begin{bmatrix} A - \lambda I_n & B \\ K & L \end{bmatrix} = n + p. \end{aligned} \tag{19}$$

Since W and T have full column rank, there exist $X \in \mathbb{K}^{g\times g}$ and $\mathcal{K} \in \mathbb{K}^{p\times q}$ such that $\widetilde{X} = W^*XW$, $X = X^*$ and $\begin{bmatrix} K & L \end{bmatrix} = \mathcal{K}T$. Using these relations in (19) and $W\begin{bmatrix} A - sI_n & B \end{bmatrix} = (\mathcal{A} - s\mathcal{E})T$ (by Lemma 13), we obtain

$$T^*(\mathcal{A}^*X\mathcal{E} + \mathcal{E}^*X\mathcal{A} + \mathcal{Q})T = T^*\mathcal{K}^*\mathcal{K}T, \quad X = X^*.$$

Since $\operatorname{im} T = \mathcal{V}_{[\mathcal{E},\mathcal{A}]}$, the latter is equivalent to $(X, \mathcal{K})$ fulfilling (6a). Moreover, for all $\lambda \in \mathbb{C}_+$, we have

$$\begin{aligned} p + n = \operatorname{rank} \begin{bmatrix} A - \lambda I_n & B \\ K & L \end{bmatrix} = \operatorname{rank} \begin{bmatrix} W(A - \lambda I_n) & WB \\ K & L \end{bmatrix} = \\ \operatorname{rank} \begin{bmatrix} \mathcal{A} - \lambda\mathcal{E} \\ \mathcal{K} \end{bmatrix} T = \operatorname{rank} \begin{bmatrix} \mathcal{A} - \lambda\mathcal{E} \\ \mathcal{K} \end{bmatrix} TU. \end{aligned}$$

Then $V = TU$ together with $U \in \mathrm{Gl}_{n+m}(\mathbb{K})$ implies that (6b) holds.

The second part of the theorem assumes that $(X, \mathcal{K}) \in \mathbb{K}^{g\times g} \times \mathbb{K}^{p\times q}$ is a solution of the Lur'e equation (6), that is the stabilizing solution. We now we prove that for this case a) holds:

Again, let $W \in \mathbb{K}^{g\times n}$, $T \in \mathbb{K}^{q\times(n+m)}$ and $\widetilde{V}^+$ be defined as in the proof of "(iii)⇒(ii)". Let $\widetilde{X} = W^*XW$ and $K \in \mathbb{K}^{p\times n}$, $L \in \mathbb{K}^{p\times m}$ with $\begin{bmatrix} K & L \end{bmatrix} = \mathcal{K}T$. Then, by the same argumentation as in "(iii)⇒(ii)", we obtain that (19) holds. Setting $w = T\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right)$ and accordingly $w^0 = T\left(\begin{smallmatrix} x^0 \\ u^0 \end{smallmatrix}\right)$ with $u^0 \in \mathbb{K}^m$, (18) and [14, Thm. 8.3] gives rise to

$$\begin{aligned} V^+(\mathcal{E}w^0) = \widetilde{V}^+(x^0) = {x^0}^*\widetilde{X}x^0 &= \left(\begin{smallmatrix} x^0 \\ u^0 \end{smallmatrix}\right)^* \begin{bmatrix} I_n & 0 \end{bmatrix}^* W^*XW \begin{bmatrix} I_n & 0 \end{bmatrix} \left(\begin{smallmatrix} x^0 \\ u^0 \end{smallmatrix}\right) \\ &= \left(\mathcal{E}T\left(\begin{smallmatrix} x^0 \\ u^0 \end{smallmatrix}\right)\right)^* X\mathcal{E}T\left(\begin{smallmatrix} x^0 \\ u^0 \end{smallmatrix}\right) = (\mathcal{E}w^0)^*X\mathcal{E}w^0. \end{aligned}$$

Finally, we show that b) holds. To this end, let $w^0 \in V^{\mathrm{diff}}_{[\mathcal{E},\mathcal{A}]}$ and $\hat{w} \in \mathfrak{B}_{[\mathcal{E},\mathcal{A}]}$ with $\mathcal{E}\hat{w}(0) = \mathcal{E}(w^0)$ and $\lim_{t\to\infty} \mathcal{E}\hat{w}(t) = 0$. By (16), we obtain

$$\begin{aligned} V^+(\mathcal{E}w^0) + \int_0^\infty \hat{w}(t)^*\mathcal{K}^*\mathcal{K}\hat{w}(t)\,\mathrm{d}t = \\ (w^0)^*\mathcal{E}^*X\mathcal{E}w^0 + \int_0^\infty \hat{w}(t)^*\mathcal{K}^*\mathcal{K}\hat{w}(t)\,\mathrm{d}t = \int_0^\infty \hat{w}(t)^*\mathcal{Q}\hat{w}(t)\,\mathrm{d}t. \end{aligned}$$

Thus, $\hat{w}$ is an optimal control if, and only if, $\mathcal{K}\hat{w} = 0$. ◻

We can furthermore characterize regularity of an optimal control problem.

THEOREM 16. *Let $\mathcal{E}, \mathcal{A} \in \mathbb{K}^{g\times q}$ and let $\mathcal{Q} \in \mathbb{K}^{q\times q}$ be Hermitian such that the optimal control problem (1) is feasible. Let $m, n \in \mathbb{N}_0$ and $V \in \mathbb{K}^{q\times(n+m)}$ be defined as in Lemma 8 and let $(X, \mathcal{K}) \in \mathbb{K}^{g\times g} \times \mathbb{K}^{p\times q}$ be a solution of Lur'e equation (6). Then the following statements are equivalent:*

(i) The optimal control problem (1) is regular.

(ii) $\forall \lambda \in \overline{\mathbb{C}_+} : \operatorname{rank} \left[\begin{smallmatrix} \mathcal{A}-\lambda\mathcal{E} \\ \mathcal{K} \end{smallmatrix}\right] V = n + m$ *and* $\operatorname{im} \mathcal{K}V = \mathcal{K}V \ker(\mathcal{E}V)$.

Proof. By using Theorem 15, we see that the optimal control problem (1) is regular if, and only if for all $w^0 \in V^{\mathrm{diff}}_{[\mathcal{E},\mathcal{A}]}$, there exists some unique $\widehat{w} \in \mathcal{L}^2_{\mathrm{loc}}(\mathcal{I}; \mathbb{R}^q)$ with $\frac{\mathrm{d}}{\mathrm{d}t} \left[\begin{smallmatrix} \mathcal{E} \\ 0 \end{smallmatrix}\right] \widehat{w} = \left[\begin{smallmatrix} \mathcal{A} \\ \mathcal{K} \end{smallmatrix}\right] \widehat{w}$, $\mathcal{E}w^0 = \mathcal{E}\widehat{w}(0)$ and $\lim_{t\to\infty} \mathcal{E}\widehat{w}(t) = 0$. By Lemma 12 this is equivalent to the property that for all $z^0 \in \mathbb{K}^{n+m}$, there exists some unique $z \in \mathcal{L}^2_{\mathrm{loc}}(\mathcal{I}; \mathbb{R}^{n+m})$ with $\frac{\mathrm{d}}{\mathrm{d}t} \left[\begin{smallmatrix} \mathcal{E} \\ 0 \end{smallmatrix}\right] Vz = \left[\begin{smallmatrix} \mathcal{A} \\ \mathcal{K} \end{smallmatrix}\right] Vz$, $\mathcal{E}Vz^0 = \mathcal{E}Vz(0)$ and $\lim_{t\to\infty} \mathcal{E}Vz(t) = 0$. This means that the latter DAE is strongly stable in the sense of [9, Def. 5.1]. By [9, Cor. 5.2], this is equivalent to $\operatorname{rank} \left[\begin{smallmatrix} \mathcal{A}-\lambda\mathcal{E} \\ \mathcal{K} \end{smallmatrix}\right] V = n + m$ for all $\lambda \in \overline{\mathbb{C}_+}$, together with

$$\operatorname{im} \left[\begin{smallmatrix} \mathcal{A} \\ \mathcal{K} \end{smallmatrix}\right] V \subseteq \operatorname{im} \left[\begin{smallmatrix} \mathcal{E} \\ 0 \end{smallmatrix}\right] V + \left[\begin{smallmatrix} \mathcal{A} \\ \mathcal{K} \end{smallmatrix}\right] V \ker(\mathcal{E}V). \tag{20}$$

Since, by Lemma 8, $\operatorname{im} V$ is a basis matrix of the limit of a specific Wong sequence (11), we have $\operatorname{im} \mathcal{A}V \subseteq \operatorname{im} \mathcal{E}V$. Thus (20) reduces to $\operatorname{im} \mathcal{K}V \subseteq \mathcal{K}V \ker(\mathcal{E}V)$. Since the reverse inclusion holds true in any case, the proof is complete. ∎

This, in particular, translates to $\left[\begin{smallmatrix} \mathcal{A}-s\mathcal{E} \\ \mathcal{K} \end{smallmatrix}\right] V$ being regular and having Kronecker index not larger than one.

4. The behavioral Lur'e equation and even matrix pencils.

In the previous section we have seen that the optimal control problem (1) leads to the behavioral Lur'e equation (6). Due to Lemma 8 and $\mathcal{AV} \subseteq \mathcal{EV}$ it follows that (6) is equivalent to

$$P^*_{V_w}(\mathcal{A}^* P^*_{W_w} X P_{W_w} \mathcal{E} + \mathcal{E}^* P^*_{W_w} X P_{W_w} \mathcal{A} + \mathcal{Q}) P_{V_w} = P^*_{V_w}(\mathcal{K}^*\mathcal{K}) P_{V_w}, \; X = X^*, \tag{21a}$$

$$\forall \lambda \in \mathbb{C}_+ : \operatorname{rank} \begin{bmatrix} P_{W_w}(\mathcal{A}-\lambda\mathcal{E})P_{V_w} \\ \mathcal{K}P_{V_w} \end{bmatrix} = n + p, \tag{21b}$$

where P_{V_w} is a projector onto $\mathcal{V}$ and P_{W_w} is a projector onto $\mathcal{EV}$ where $\mathcal{V}$ is the limit of the Wong sequence as in Definition 7. We see from (21) that we may restrict to Further, setting

$$\mathcal{E}_w = P_{W_w} \mathcal{E} P_{V_w}, \qquad \mathcal{A}_w = P_{W_w} \mathcal{A} P_{V_w}, \qquad \mathcal{Q}_w = P^*_{V_w} \mathcal{Q} P_{V_w}, \tag{22}$$

with $n = \operatorname{rank} P_{W_w}$, (21) leads to the Lur'e equation

$$\mathcal{A}^*_w X \mathcal{E}_w + \mathcal{E}^*_w X \mathcal{A}_w + \mathcal{Q}_w = \mathcal{K}^*\mathcal{K}, \quad X = X^*, \tag{23a}$$

$$\forall \lambda \in \mathbb{C}_+ : \operatorname{rank} \left[\begin{smallmatrix} \mathcal{A}_w-\lambda\mathcal{E}_w \\ \mathcal{K} \end{smallmatrix}\right] = n + p, \tag{23b}$$

Moreover, by definition of P_{W_w}, the restriction $\mathcal{E}_w : \operatorname{im} P_{V_w} \to \operatorname{im} P_{W_w}$ is surjective. Lur'e equations of this type allow to represent the solutions by means of deflating subspaces of the even matrix pencil

$$\mathcal{H} - s\mathcal{G} := \begin{bmatrix} 0 & \mathcal{A}_w \\ \mathcal{A}^*_w & \mathcal{Q}_w \end{bmatrix} - s \begin{bmatrix} 0 & \mathcal{E}_w \\ -\mathcal{E}^*_w & 0 \end{bmatrix}. \tag{24}$$

Namely, $(X, \mathcal{K})$ fulfilling (6) defines a $\mathcal{G}$-neutral deflating subspace of (24) via

$$\begin{bmatrix} 0 & \mathcal{A}_w - s\mathcal{E}_w \\ \mathcal{A}^*_w + s\mathcal{E}^*_w & \mathcal{Q}_w \end{bmatrix} \begin{bmatrix} X\mathcal{E}_w \\ P_{V_w} \end{bmatrix} = \begin{bmatrix} P_{W_w} & 0 \\ -\mathcal{E}^*_w X & \mathcal{K}^* \end{bmatrix} \begin{bmatrix} \mathcal{A}_w - s\mathcal{E}_w \\ \mathcal{K} \end{bmatrix}.$$

In [25], a deflation procedure has been proposed which we adapt to our more general setup. Later we show that this leads to the consideration of an algebraic Riccati equation.

Note that the Lur'e equation (23) could be condensed by forming suitable basis matrices of $\operatorname{im} P_{V_w}$ and $\operatorname{im} P_{W_w}^*$. Since the restriction $\mathcal{E}_w : \operatorname{im} P_{V_w} \to \operatorname{im} P_{W_w}$ is surjective, it is in particular theoretically possible to choose these basis matrices such that a Lur'e equation with $\mathcal{E} = \begin{bmatrix} I & 0 \end{bmatrix}$ is obtained. This corresponds to a standard Lur'e equation for ODE systems, a type which has been intensively studied also from a numerical point of view in [26, 25]. Indeed, this approach could be done to solve our behavioral Lur'e equations numerically. However, such a compression will typically lead to dense matrices, whence we will prefer to work with the projected Lur'e equation (21) especially in our numerical considerations in section 5.

In our next step, we will reduce Lur'e equations considering some deflating $\mathcal{G}$-neutral subspaces of $\mathcal{H} - s\mathcal{G}$.

THEOREM 17 (Partial solution). *Let $\mathcal{E}, \mathcal{A} \in \mathbb{K}^{g\times q}$ and $\mathcal{Q} \in \mathbb{K}^{q\times q}$. Let $P_{V_w} \in \mathbb{K}^{q\times q}$ be a projector onto the system space $\mathcal{V}_{[\mathcal{E},\mathcal{A}]}$ and let $P_{W_w} \in \mathbb{K}^{g\times g}$ be a projector onto $\mathcal{E}\mathcal{V}_{[\mathcal{E},\mathcal{A}]}$. Let $(X, \mathcal{K})$ be a solution of the Lur'e equation (6) with $X = XP_{W_w}$. Let $\mathcal{E}_w$, $\mathcal{A}_w$, $\mathcal{Q}_w$ be defined as in (22). Let $V_1 \in \mathbb{K}^{g\times k}$ and $V_2 \in \mathbb{K}^{q\times k}$ such that $V = \left[\begin{smallmatrix} V_1 \\ V_2 \end{smallmatrix}\right]$ has full column rank and $\operatorname{im} V \subset \operatorname{im} \left[\begin{smallmatrix} X\mathcal{E}_w \\ P_{W_w} \end{smallmatrix}\right]$ and let $L, R \in \mathbb{K}^{k\times \ell}$ such that*

$$(25)\qquad \begin{bmatrix} 0 & \mathcal{A}_w \\ \mathcal{A}_w^* & \mathcal{Q}_w \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} L = \begin{bmatrix} 0 & \mathcal{E}_w \\ -\mathcal{E}_w^* & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} R$$

holds for some $\ell \in \mathbb{N}_0$. Let C be a basis matrix of $\ker \mathcal{H}V$ and let $(\mathcal{E}_w V_2)^+$ be a pseudoinverse of $\mathcal{E}_w V_2$ with $(\mathcal{E}_w V_2)^+ = (\mathcal{E}_w V_2)^+ P_w$ (which exists due to $\operatorname{im} \mathcal{E}_w V_2 \subset \operatorname{im} P_w$). Consider the matrices

$$(26)\qquad X_p = V_1(\mathcal{E}_w V_2)^+ + ((\mathcal{E}_w V_2)^+)^* V_1^* - ((\mathcal{E}_w V_2)^+)^* V_1^* \mathcal{E}_w V_2 (\mathcal{E}_w V_2)^+,$$

$$(27)\qquad \mathcal{Q}_s = \mathcal{Q}_w + \mathcal{A}_w^* X_p \mathcal{E}_w + \mathcal{E}_w^* X_p \mathcal{A}_w.$$

Then the following holds:

a) There exist some $p \in \mathbb{N}_0$ and $(X_s, \mathcal{K}) \in \mathbb{K}^{g\times g} \times \mathbb{K}^{p\times g}$ such that

$$(28a)\qquad \mathcal{A}_w^* X_s \mathcal{E}_w + \mathcal{E}_w^* X_s \mathcal{A}_w + \mathcal{Q}_s = \mathcal{K}^*\mathcal{K}, \quad X_s = X_s^*, \quad X_s \mathcal{E}_w V_2 = 0,$$

$$(28b)\qquad \forall \lambda \in \mathbb{C}_+ : \operatorname{rank} \left[\begin{smallmatrix} \mathcal{A}_w - \lambda \mathcal{E}_w \\ \mathcal{K} \end{smallmatrix}\right] = p + n.$$

Furthermore $(X_p + X_s, \mathcal{K})$ is a solution of the Lur'e equation (6).

b) $(\mathcal{A}_w^* X_s \mathcal{E}_w + \mathcal{E}_w^* X_s \mathcal{A}_w + \mathcal{Q}_s) V_2 \begin{bmatrix} L & C \end{bmatrix} = 0$ for all X_s with $X_s \mathcal{E}_w V_2 = 0$.

Proof.

a) Assume that $(X, \mathcal{K})$ solves the Lur'e equation (6) and consider $X_s = X - X_p$. Then simple arithmetic shows that $(X_p + X_s, \mathcal{K})$ fulfills $\mathcal{A}_w^* X_s \mathcal{E}_w + \mathcal{E}_w^* X_s \mathcal{A}_w + \mathcal{Q}_w = \mathcal{K}^*\mathcal{K}$ and (28b). Since $\operatorname{im} \left[\begin{smallmatrix} X\mathcal{E}_w \\ P_{V_w} \end{smallmatrix}\right]$ is $\mathcal{G}$-neutral and $\operatorname{im} V \subset \operatorname{im} \left[\begin{smallmatrix} X\mathcal{E}_w \\ P_{V_w} \end{smallmatrix}\right]$, $\operatorname{im} V$ is $\mathcal{G}$-neutral as well and further $X\mathcal{E}_w V_2 = V_1$. Hence, $V_1^* \mathcal{E}_w V_2$ is Hermitian, which further implies that X_p and thus also $X_s = X_p - X$ is Hermitian. Hence, it remains to show that $X_s \mathcal{E}_w V_2 = 0$:
Let N be a basis matrix of $\ker \mathcal{E}_w V_2$ and B be a basis matrix of $\operatorname{im}(\mathcal{E}_w V_2)^+$. Then $\begin{bmatrix} B & N \end{bmatrix}$ is invertible. By using $V_1 N = X\mathcal{E}_w V_2 N = 0$, we obtain

$$V_1(\mathcal{E}_w V_2)^+ \mathcal{E}_w V_2 = \begin{bmatrix} V_1(\mathcal{E}_w V_2)^+ \mathcal{E}_w V_2 B & V_1(\mathcal{E}_w V_2)^+ \mathcal{E}_w V_2 N \end{bmatrix} \begin{bmatrix} B & N \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} V_1 B & V_1 N \end{bmatrix} \begin{bmatrix} B & N \end{bmatrix}^{-1} = V_1.$$

This gives rise to

$$X_s\mathcal{E}_wV_2 = X_p\mathcal{E}_wV_2 - X\mathcal{E}_wV_2 = V_1(\mathcal{E}_wV_2)^+\mathcal{E}_wV_2 + ((\mathcal{E}_wV_2)^+)^*V_1^*\mathcal{E}_wV_2 \\ - ((\mathcal{E}_wV_2)^+)^*V_1^*\mathcal{E}_wV_2(\mathcal{E}_wV_2)^+\mathcal{E}_wV_2 - V_1 = V_1(\mathcal{E}_wV_2)^+\mathcal{E}_wV_2 - V_1 = 0.$$

b) (25) implies $\mathcal{A}_wV_2L = \mathcal{E}_wV_2R$. Moreover we have $\mathcal{A}_wV_2C = 0$. Hence, $V_1 = X_p\mathcal{E}_wV_2$ together with (25) leads to

$$(\mathcal{A}_w^*X_s\mathcal{E}_w + \mathcal{E}_w^*X_s\mathcal{A}_w + \mathcal{Q}_s)\,V_2\begin{bmatrix}L & C\end{bmatrix} = \mathcal{Q}_sV_2\begin{bmatrix}L & C\end{bmatrix} = \\ \begin{bmatrix}\mathcal{A}_w^* & \mathcal{Q}_w\end{bmatrix}\begin{bmatrix}V_1\\V_2\end{bmatrix}\begin{bmatrix}L & C\end{bmatrix} + \begin{bmatrix}\mathcal{E}_w^*V_1R & 0\end{bmatrix} = 0. \qquad \square$$

Remark 18. If $\ker V_1^* \cap \operatorname{im}\mathcal{E}_wV_2 = \{0\}$, then a possible choice for a pseudoinverse of $\mathcal{E}_wV_2$ is $(\mathcal{E}_wV_2)^+ = (V_1^*\mathcal{E}_WV_2)^\dagger V_1^*$, where $(V_1^*\mathcal{E}_wV_2)^\dagger$ is the Moore–Penrose inverse of $V_1^*\mathcal{E}_wV_2$. This choice leads to $X_p = V_1(V_1^*\mathcal{E}_wV_2)^\dagger V_1^*$. Note that the condition $\ker V_1^* \cap \operatorname{im}\mathcal{E}_wV_2 = \{0\}$ is for instance fulfilled, if X is semidefinite, see [25].

With the notation of Theorem 17 and projectors P_{V_s} and P_{W_s} along $\operatorname{im}V_2\begin{bmatrix}L & C\end{bmatrix}$ and $\operatorname{im}\mathcal{E}_wV_2$, respectively, we can reformulate (28) with

$$\mathcal{E}_s = P_{W_s}\mathcal{E}_wP_{V_s}, \qquad \mathcal{A}_s = P_{W_s}\mathcal{A}_wP_{V_s}, \tag{29}$$

as

$$\begin{aligned} &\mathcal{A}_s^*X_s\mathcal{E}_s + \mathcal{E}_s^*X_s\mathcal{A}_s + \mathcal{Q}_s = \mathcal{K}^*\mathcal{K}, \\ &X_s = X_s^*, \quad X_s = X_sP_{W_w} = X_sP_{W_s} \end{aligned} \tag{30a}$$

$$\forall\lambda\in\mathbb{C}_+ : \operatorname{rank}\begin{bmatrix} P_{W_s}(\mathcal{A}-\lambda\mathcal{E})P_{V_s} \\ \mathcal{K}P_{V_s}\end{bmatrix} = p+n. \tag{30b}$$

Hence we can further reduce our Lur'e equation in the same way as presented in (21)–(23). Furthermore, if we use the limit of the $\mathcal{G}$-neutral Wong sequence (12) we end up with a Lur'e equation which can be reformulated to an algebraic Riccati equation. This is indicated by the following result.

LEMMA 19. *With the assumptions and notations of Theorem 17 and the the limit of the even Wong sequence $\mathcal{Y}$ from Definition 10. Assume that (25) holds with invertible R, let V be a basis matrix of $\begin{bmatrix} P_{W_w}^* & 0 \\ 0 & P_{V_w}\end{bmatrix}\mathcal{Y}$, let W be a basis matrix of $\operatorname{im}W \cap \ker(\mathcal{E}V_2)^* \cap \operatorname{im}P_{W_w}^*$, and let $\widetilde{V}_B$ a basis matrix of $\ker W^*\mathcal{E} \cap \operatorname{im}P_{V_s} \cap \operatorname{im}P_{V_w}$, where P_{V_s} is a projector along $\operatorname{im}V_2\begin{bmatrix}L & C\end{bmatrix}$. Then the following holds:*
a) For X_s as in (28b), there exists some Hermitian matrix Y, such that $X_s = WYW^*$.
b) $\widetilde{V}_B^*\mathcal{Q}_s\widetilde{V}_B > 0$.

Proof. Due to the surjectivity of the restriction $\mathcal{E}_w : \mathcal{V}_{[E,A]} \to \mathcal{E}\mathcal{V}_{[E,A]}$, a direct deflation of the even matrix pencil with a suitable basis matrix of $\operatorname{im}\begin{bmatrix} P_{W_w}^* & 0 \\ 0 & P_{V_w}\end{bmatrix}$ will result in an even matrix pencil as in [25] implying that $\operatorname{im}V \subset \operatorname{im}\begin{bmatrix} X\mathcal{E}_w \\ P_{V_w}\end{bmatrix}$. Hence we can apply Theorem 17.

a) Since X_s is Hermitian, $X_s\mathcal{E}_wV_2 = 0$ implies that the image of X_s is a subset of $\ker(\mathcal{E}_wV_2)^*$. Further $X_sP_{W_w} = X_s$ implies that $\operatorname{im}X_s \subset \operatorname{im}P_{W_w}^*$. Hence $\operatorname{im}X_s \subset \operatorname{im}W$ and the assertion holds.

b) **Step 1:** We show that $\operatorname{im} V_2 = \ker W^*\mathcal{E}_w \cap \operatorname{im} P_{V_w}$:
The inclusion "$\subset$" follows from $(\mathcal{E}_w V_2)^* W = ((W^*\mathcal{E}_w)V_2)^* = 0$, where the latter holds since $\operatorname{im} W$ is a subset $\ker(\mathcal{E}_w V_2)^*$. To show the reserve inclusion, we first observe that the construction of the even Wong sequence (12) yields $\left[\begin{smallmatrix} P_{W_w}^* & 0 \\ 0 & P_{V_w} \end{smallmatrix}\right] \ker \mathcal{G} \subset \operatorname{im} V$, and thus $\ker \mathcal{E}_w \cap \operatorname{im} P_{V_w} \subset \operatorname{im} V_2$. Now assume that $x \in \ker W^*\mathcal{E}_w \cap \operatorname{im} P_{V_w}$. Then $\mathcal{E}_w x \in P_{W_w} \ker W^* = P_{W_w}(\operatorname{im} W)^\perp = P_{W_w}(\ker(\mathcal{E}V_2)^* \cap \operatorname{im} P_{W_w}^*)^\perp = \operatorname{im} \mathcal{E}_w V_2$. Hence, we have $\mathcal{E}_w x = \mathcal{E}_w V_2 z$ for some vector z of appropriate dimension. This gives $x - V_2 z \in \ker \mathcal{E}_w \cap \operatorname{im} P_{V_w} \subset \operatorname{im} V_2$, and thus $x \in \operatorname{im} V_2$.
Step 2: We conclude assertion b):
It suffices to show that for all $x \in \operatorname{im} \widetilde{V}_B \setminus \{0\}$ holds $x^*\mathcal{Q}_s x > 0$. Assume that $x \in \operatorname{im} \widetilde{V}_B \setminus \{0\}$. Then, by step 1, there exist a nonzero vector z such that $x = V_2 z$. Furthermore $z \notin \operatorname{im} \begin{bmatrix} L & C \end{bmatrix}$, since $\operatorname{im} V_2 \begin{bmatrix} L & C \end{bmatrix} \cap \operatorname{im} P_{V_s} = \{0\}$. Using $V_1 = X_p \mathcal{E}_w V_2$ we obtain

$$\mathcal{H}Vz = \left[\begin{smallmatrix} \mathcal{A}_w \\ \mathcal{A}_w^* X_p \mathcal{E}_w + \mathcal{Q}_w \end{smallmatrix}\right] V_2 z \notin \operatorname{im} \left[\begin{smallmatrix} \mathcal{A}_w \\ \mathcal{A}_w^* X_p \mathcal{E} - w + \mathcal{Q}_w \end{smallmatrix}\right] V_2 \begin{bmatrix} L & C \end{bmatrix} = \operatorname{im} \mathcal{G}VR = \operatorname{im} \mathcal{G}.$$

On the other hand, a combination of [26, Thm. 8] with [25, Thm. 2.11] and the fact that $\operatorname{im} V$ is the projection of limit of the even Wong sequence consequences implies that for all $y \in \operatorname{im} V$ with $\mathcal{H}y \notin \mathcal{G} \operatorname{im} V$ holds $y^*\mathcal{H}y > 0$. This implies $x^*\mathcal{Q}_s x = (V_2 z)^* \mathcal{Q}_s V_2 z = (Vz)^* \mathcal{H} V z > 0$. □

The previous result assumes that the matrix R in (25) is invertible. In fact, there even exists a basis matrix V such that (25) holds with $R = I$. Indeed, Algorithm 3 will produce a matrix V such that (25) holds with $R = I$.

Moreover Lemma 19 allows to define a basis matrix $\widetilde{V} = [\,\widetilde{V}_A \ \widetilde{V}_B\,]$ such that $\operatorname{im} \widetilde{V} = \operatorname{im} P_{V_s} \cap \operatorname{im} P_{V_w}$ and $\widetilde{V}_B$ is a basis matrix of $\ker W^*\mathcal{E} \cap \operatorname{im} P_{V_s} \cap \operatorname{im} P_{V_w}$. The construction of W and $\widetilde{V}_A$ further implies that $\hat{E} := W^*\mathcal{E}\widetilde{V}_A$ is square and invertible. Now with

$$\begin{aligned} &\hat{A} := \hat{E}^{-1} W^* \mathcal{A} \widetilde{V}_A, && \hat{B} := \hat{E}^{-1} W^* \mathcal{A} \widetilde{V}_B, \\ &\hat{Q} := (\hat{E}^*)^{-1} \widetilde{V}_A^* \widetilde{Q} \widetilde{V}_A \hat{E}^{-1}, && \hat{S} := (\hat{E}^*)^{-1} \widetilde{V}_A^* \widetilde{Q} \widetilde{V}_B, && \hat{R} := \widetilde{V}_B^* \widetilde{Q} \widetilde{V}_B \end{aligned}$$

(30) is equivalent to

$$\begin{aligned} &\begin{bmatrix} \hat{A}^*Y + Y\hat{A} + \hat{Q} & Y\hat{B} + \hat{S} \\ \hat{B}^*Y + \hat{S} & \hat{R} \end{bmatrix} = \begin{bmatrix} K^*K & K^*L \\ L^*K & L^*L \end{bmatrix}, \quad Y^* = Y \\ &\forall \lambda \in \mathbb{C}_+ : \operatorname{rank} \left[\begin{smallmatrix} \hat{A} - \lambda I & \hat{B} \\ K & L \end{smallmatrix}\right] = \hat{n} + p, \end{aligned}$$

for some $K \in \mathbb{K}^{p \times \hat{n}}$ and $L \in \mathbb{K}^{p \times p}$, where $\hat{n}$ is defined by $\hat{A} \in \mathbb{K}^{\hat{n} \times \hat{n}}$. Note that $\hat{R} > 0$ implies that L is square and invertible. The matrices K and L can therefore be eliminated, which yields that Y is the stabilizing solution of the algebraic Riccati equation

$$\hat{A}^*Y + Y\hat{A} + \hat{Q} - (Y\hat{B} + \hat{S})\hat{R}^{-1}(Y\hat{B} + \hat{S})^* = 0. \tag{32}$$

In subsection 5.4, we will present an iterative method for solving (30). This iterative method is theoretically based on the reformulation as an algebraic Riccati equation. It however does not form the matrices $\hat{A}$, $\hat{B}$, $\hat{Q}$, $\hat{S}$ and $\hat{R}$. Instead, this method will work with the original coordinates such that possible sparsity can be exploited.

Similar as in [25], it is further possible to construct partial solutions which are obtained by the so-called *shifted $\mathcal{G}$-neutral Wong sequences*

(33)

$$\mathcal{Y}_0^\lambda = \{0\},\ \mathcal{Z}_{i+1}^\lambda = (\mathcal{H} - \lambda\mathcal{G})^{-1}(\mathcal{G}\mathcal{Y}_i)\,,\ \mathcal{Y}_{i+1}^\lambda = \mathcal{Y}_i^\lambda + \left(\mathcal{Z}_{i+1}^\lambda \cap \left(\mathcal{G}\mathcal{Z}_{i+1}^\lambda\right)^\perp\right),\ i \in \mathbb{N}_0.$$

where λ is a generalized eigenvalue of the even matrix pencil $\mathcal{H} - s\mathcal{G}$ on the closed left half-plane. As we illustrate in section 6, an incorporation of the generalized eigenvalues of $\mathcal{H} - s\mathcal{G}$ on the imaginary axis will improve the convergence of our iterative method to solve (30).

5. Numerical solution.

Now we present methods and algorithms whose aim is to solve the optimal control problem and exploit the sparsity of matrices in order to be applicable to large scale DAEs.

The overall procedure is theoretically based on three consecutive steps:

1. We determine a projector onto the system space $\mathcal{V}_{[\mathcal{E},\mathcal{A}]}$ using Wong sequences (subsection 5.2).
2. By applying the theory from section 4, we use the $\mathcal{G}$-neutral Wong sequence and shifted $\mathcal{G}$-neutral Wong sequences of the associated even matrix pencil to reduce the problem to the solution of a Lur'e equation which can be reformulated as an algebraic Riccati equation (subsection 5.3).
3. We iteratively solve the reduced Lur'e equation by an extension of the Newton-Kleinman method [16] presented in subsection 5.4. The Newton steps consist of the solution of a generalized Lyapunov equation. The latter will be solved by an extension of the ADI (alternating direction implicit) method [23] presented in subsection 5.5.

This leads to Algorithm 1, where we avoid the explicit formulation of the equation by the use of suitable projectors.

As already pointed out, step 1 defines projectors which we use to introduce the surjective restriction $\mathcal{E}_w : \operatorname{im} P_{V_w} \to \operatorname{im} P_{W_w}$. This allows the usage of Theorem 17 and in particular Lemma 19 in step 2. We obtain a projected Lur'e equation which can indeed be reformulated as an algebraic Riccati equation. This allows us to use methods in step 3, which are modifications of methods for numerical solution of algebraic Riccati equations. As we see, every step gives a new projected systems, which suffices the requirements of the next step.

This section is organized as follows: In subsection 5.1, we present some implementation details on operations of linear mappings on subspaces, which are formed by the multiplication of matrices from the left and right with projectors. Subsection 5.2 is devoted to the numerical determination of the system space $\mathcal{V}_{[\mathcal{E},\mathcal{A}]}$. Thereafter, in subsection 5.3, we present algorithms for determining partial solutions according to the theory from section 4. This leads to a projected Lur'e equation, which will be solved by the Newton-ADI method. The algorithms for the latter are presented in subsection 5.4 and subsection 5.5.

An approach based on partial solutions has been presented in [25] for the case $\mathcal{E} = \begin{bmatrix} I & 0 \end{bmatrix}$. The solution of projected algebraic Riccati equations has been considered in [8] for the case $\mathcal{A} - s\mathcal{E} = \begin{bmatrix} A - sE & B \end{bmatrix}$, where $A - sE$ is regular. The approach in [8] has been formulated for the deflating subspace of $A - sE$ corresponding to the generalized eigenvalues of $A - sE$. We note that the projected algebraic Riccati equation that we obtain after step 2 is of different nature.

5.1. Projectors and mappings on subspaces.

Assume that $P_V \in \mathbb{K}^{q\times q}$ and $P_W \in \mathbb{K}^{g\times g}$ are projectors, let $\mathcal{A} \in \mathbb{K}^{g\times q}$. First we present an implementation

Algorithm 1 Lur'e solver

Input: $\mathcal{A}, \mathcal{E} \in \mathbb{K}^{g\times q}$, $\mathcal{Q} \in \mathbb{K}^{q\times q}$
Output: $X \in \mathbb{K}^{g\times g}$, $\mathcal{K} \in \mathbb{K}^{p\times q}$ such that $||\mathcal{A}^*X\mathcal{E} + \mathcal{E}^*X\mathcal{A} + \mathcal{Q} - \mathcal{K}^*\mathcal{K}|| <$ tol

compute (W_w, V_w) with Algorithm 2 using $(\mathcal{A}, \mathcal{E})$
$P_W \leftarrow I - (W_w^+)^* W_w^*$, $P_V \leftarrow I - (V_w^+)^* V_w^*$, $\mathcal{Q}_w = P_V^* \mathcal{Q} P_V$.
compute $(\left[\begin{smallmatrix} V_1 \\ V_2 \end{smallmatrix}\right], L, C)$ with Algorithm 3 using $(P_W\mathcal{A}P_V, P_W\mathcal{E}P_V, \mathcal{Q}_w)$
$X_p \leftarrow V_1(\mathcal{E}V_2)^+ + ((\mathcal{E}V_2)^+)^*V_1^* - ((\mathcal{E}V_2)^+)^*V_1^*\mathcal{E}V_2(\mathcal{E}V_2)^+$
with $(\mathcal{E}V_2)^+ = (\mathcal{E}V_2)^+ P_W$
$\mathcal{Q}_s \leftarrow \mathcal{Q}_w + \mathcal{A}^*X_p\mathcal{E} + \mathcal{E}^*X_p\mathcal{A}$
compute a basis matrix V_s of $\operatorname{im} V_2 \begin{bmatrix} L & C \end{bmatrix}$
compute a basis matrix W_s of $\operatorname{im} P_W\mathcal{E}P_V V_2$
$P_W \leftarrow (I - W_sW_s^+)P_W$
$P_V \leftarrow P_V(I - V_sV_s^+)$ with $V_s^+ = V_s^+P_V$
for all generalized eigenvalues λ_i of $\left[\begin{smallmatrix} 0 & \mathcal{A}-s\mathcal{E} \\ \mathcal{A}^*+s\mathcal{E}^* & \mathcal{Q} \end{smallmatrix}\right]$ in $i\mathbb{R}$ **do**
  compute $(\left[\begin{smallmatrix} V_1 \\ V_2 \end{smallmatrix}\right])$ with Algorithm 4 using $(P_W\mathcal{A}P_V, P_W\mathcal{E}P_V, \mathcal{Q}_s, \lambda_i)$
  $X_p \leftarrow X_p + V_1(\mathcal{E}V_2)^+ + ((\mathcal{E}V_2)^+)^*V_1^* - ((\mathcal{E}V_2)^+)^*V_1^*\mathcal{E}V_2(\mathcal{E}V_2)^+$
  with $(\mathcal{E}V_2)^+ = (\mathcal{E}V_2)^+ P_W$
  $\mathcal{Q}_s \leftarrow \mathcal{Q}_w + \mathcal{A}^*X_p\mathcal{E} + \mathcal{E}^*X_p\mathcal{A}$
  compute a basis matrix W_s of $\operatorname{im} P_W\mathcal{E}P_V V_2$
  $P_W \leftarrow (I - W_sW_s^+)P_W$
  $P_V \leftarrow P_V(I - V_2V_2^+)$ with $V_2^+ = V_2^+P_V$
end for
compute $(X_s, \mathcal{K})$ with Algorithm 5 or Algorithm 6 using $(P_W\mathcal{A}P_V, P_W\mathcal{E}P_V, \mathcal{Q}_s)$
$X = X_p + X_s$

for certain operations including the linear mapping $\mathcal{A}_M : \operatorname{im} P_V \to \operatorname{im} P_W$, $x \mapsto P_W\mathcal{A}P_V x$. The operations that we have to implement are the following:

1. Evaluate $\mathcal{A}_M X$ for some matrix X of suitable size;
2. Solve $\mathcal{A}_M X = B$ for a matrix X with $\operatorname{im} X \subset \operatorname{im} P_V$, where B is a matrix with $\operatorname{im} B \subseteq \operatorname{im} P_W$;
3. Compute a basis matrix of $\ker \mathcal{A}_M = \{x \in \operatorname{im} P_V \mid P_W\mathcal{A}P_V x = 0\}$;

We call $\ker \mathcal{A}_M$ the *projected kernel*. The evaluation translates straightforward to three consecutive matrix multiplications. For the solution of linear systems as well as the closely related computation of the kernel we first point out the most advantageous way to write down the projectors.

Given a basis matrix $V \in \mathbb{K}^{q\times k}$, then $P_V = I - VV^+$ is a projector onto $\ker V^+$ and along $\operatorname{im} V$. With an accordingly defined projector $P_W = I - WW^+ \in \mathbb{K}^{g\times g}$, problem 2 is equivalent to the matrix equation

$$P_W\mathcal{A}P_V X = P_W B, \quad P_V X = X. \tag{34}$$

Explicitly forming the product $P_W\mathcal{A}P_V$ in general leads to a dense and singular matrix. In particular when $k \ll q$, it is numerically more convenient to consider the *extended system*

$$\begin{bmatrix} \mathcal{A} & W \\ V^+ & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} B \\ 0 \end{bmatrix}. \tag{35}$$

If X and Y fulfill (35), then $V^+X = 0$ leads to $P_V X = X$ and a multiplication of the first row in (35) from the left with P_W gives rise to $P_W\mathcal{A}P_V X = P_W B$. Furthermore

(34) implies that (35) holds with $Y = W^+B - W^+AX$. Consequently, (34) and (35) are equivalent. Since (35) is a simple matrix equation we are able to use variants of Gaussian elimination to solve the system as well as determine basis matrix of the *projected kernel* and, further, exploit sparsity by this approach.

Since $\mathcal{A}_M^* = P_W^* \mathcal{A}^* P_V^*$ and $P_V^* = I - (V^+)^* V^*$, $P_W^* = I - (W^+)^* W^*$, the above considerations can be also applied to evaluations, solution of equations and kernel determination in which $\mathcal{A}_M^*$ is involved. In particular, the solution of the equation $\mathcal{A}_M^* X = B$ for a matrix X with $\operatorname{im} X \subset \operatorname{im} P_W^*$, where B is a matrix with $\operatorname{im} B \subseteq \operatorname{im} P_V^*$, leads to a linear system which involves the conjugate transpose of the extended matrix in (35).

5.2. Projector onto the system space. Now we present a numerical approach to determine a projector P_{V_w} onto the system space $\mathcal{V}_{[\mathcal{E},\mathcal{A}]}$. By Lemma 8, $\mathcal{V}_{[\mathcal{E},\mathcal{A}]}$ is the limit of Wong sequence defined by $\mathcal{V}_0 = \mathbb{K}^q$, $\mathcal{V}_{i+1} = \mathcal{A}^{-1}(\mathcal{E}\mathcal{V}_i)$. Since for many practically relevant DAEs, the dimension of this space is typically only slightly smaller than q, a determination of a basis matrix is costly and memory consuming. Therefore we prefer to determine a basis matrix of the orthogonal complement of $\mathcal{V}_{[\mathcal{E},\mathcal{A}]}$. We use the following lemma to obtain a feasible numerical scheme as well as matrices which fit nicely into the notation of a projector as in subsection 5.1.

LEMMA 20. *[10, Lem. 4.5] Let $s\mathcal{E} - \mathcal{A} \in \mathbb{K}[s]^{g\times q}$ be a matrix pencil and consider the sequences*

$$\mathcal{V}_0 = \mathbb{K}^q, \quad \mathcal{V}_{i+1} = \mathcal{A}^{-1}(\mathcal{E}\mathcal{V}_i), \qquad \mathcal{W}_0 = \{0\}, \quad \mathcal{W}_{i+1} = (\mathcal{E}^*)^{-1}(\mathcal{A}^*\mathcal{W}_i), \qquad i \in \mathbb{N}_0.$$

Then the respective limits $\mathcal{V}$ and $\mathcal{W}$ are related by $\mathcal{V} = (\mathcal{A}^*\mathcal{W})^\perp$ and $\mathcal{W} = (\mathcal{E}\mathcal{V})^\perp$.

Algorithm 2 determines a basis matrix W_w with $\operatorname{im} W_w = \mathcal{W}$. A projector onto $\mathcal{V}_{[\mathcal{E},\mathcal{A}]}$ is then given by $P_{V_w} = I - (V_w^+)^* V_w^*$ where V_w is a basis matrix of $\operatorname{im} \mathcal{A}^* W_w$. Further, $P_{W_w} = I - (W_w^+)^* W_w^*$ is a projector onto $\mathcal{E}\mathcal{V}_{[\mathcal{E},\mathcal{A}]}$

Algorithm 2 Wong sequence

Input: $\mathcal{A}, \mathcal{E} \in \mathbb{K}^{g\times q}$

Output: A basis matrices W_w of the limit of $\mathcal{W}_0 = \{0\}$, $\mathcal{W}_{i+1} = (\mathcal{E}^*)^{-1}(\mathcal{A}^*\mathcal{W}_i)$ and a basis matrix V_w of $\operatorname{im} \mathcal{A}^* W_w$

compute a basis matrix U of $\ker(\mathcal{E})$
compute a basis matrix W_0 of $\ker(\mathcal{E}^*)$
$W_w \leftarrow W_0$, $i \leftarrow 0$
repeat
  $T \leftarrow \mathcal{A}^* W_i$
  compute a basis matrix S of $\ker(U^*T)$
  $i \leftarrow i + 1$
  solve $\mathcal{E}^* W_i = TS$ for W_i
  choose C such that $\begin{bmatrix} W_w & W_i C \end{bmatrix}$ is a basis matrix of $\operatorname{im} \begin{bmatrix} W_w & W_i \end{bmatrix}$
  $W_i \leftarrow W_i C$
  $W_w \leftarrow \begin{bmatrix} W_w & W_i C_i \end{bmatrix}$
until $\operatorname{im} C = \{0\}$
compute a basis matrix V_w of $\operatorname{im} \mathcal{A}^* W_w$

Algorithm 2 may be sensitive to rounding errors due to the rank decision during the kernel computation. If possible, the physical structure should be exploited to determine $\ker \mathcal{E}$, $\ker \mathcal{E}^*$ and the successive nullspaces.

5.3. Partial solutions of the Lur'e equation. With projectors P_{V_w} and P_{W_w} introduced right before Algorithm 2, it follows from the argumentation at the beginning of section 4 that the behavioral Lur'e equation (6) is equivalent to (21). It can be further concluded from Lemma 8 b) that the restriction $P_{W_w}\mathcal{E}P_{V_w} : \mathcal{V}_{[\mathcal{E},\mathcal{A}]} \to \mathcal{E}\mathcal{V}_{[\mathcal{E},\mathcal{A}]}$ is surjective. This allows the construction of partial solutions according to Theorem 17. This eliminates the parts of the Lur'e equation leading to singularities of the optimal control problem. The $\mathcal{G}$-neutral Wong sequence (12) and the $\mathcal{G}$-neutral shifted Wong sequences (33) associated to purely imaginary generalized eigenvalues of $\mathcal{H} - s\mathcal{G}$ are used to obtain these partial solutions. However the Wong sequences are in general restricted to specific subspaces. This is enforced by a suitable projector.

From Algorithm 3 we obtain the matrices which are needed to apply Theorem 17. In particular, with $V = \left[\begin{smallmatrix} V_1 \\ V_2 \end{smallmatrix}\right]$ partitioned according the structure of $\mathcal{H} - s\mathcal{G}$, we determine the partial solution X_p by (26), and we compute basis matrices V_s of $\operatorname{im} V_2 \begin{bmatrix} L & C \end{bmatrix}$ and W_s of $\operatorname{im} \mathcal{E}V_2$. The projectors

$$P_{V_s} = I - V_s V_s^+, \qquad P_{W_s} = I - W_s W_s^+, \tag{36}$$

lead to the projected Lur'e equation (30). Since the matrix V determined by Algorithm 3 fits the prerequisites of Lemma 19, it is possible to reformulate the projected Lur'e equation as an algebraic Riccati equation, such as explained at the end of section 4.

Algorithm 3 Wong sequence for the even matrix pencil

Input: $P_W\mathcal{A}P_V, P_W\mathcal{E}P_V \in \mathbb{K}^{g\times q}$, $\mathcal{Q} \in \mathbb{K}^{q\times q}$

Output: $V \in \mathbb{K}^{(g+q)\times k}$, $L \in \mathbb{K}^{k\times \ell_1}$, $C \in \mathbb{K}^{k\times \ell_2}$ such that $\mathcal{H}VL = \mathcal{G}V$, $\operatorname{im} V$ is limit of the $\mathcal{G}$-neutral Wong sequence (12) intersected by $\operatorname{im} P_{\mathcal{H}}$ and C is a basis matrix of $\ker \mathcal{H}V \cap \operatorname{im} P_{\mathcal{H}}$

$\mathcal{H} = \begin{bmatrix} 0 & P_W\mathcal{A}P_V \\ (P_W\mathcal{A}P_V)^* & \mathcal{Q} \end{bmatrix}$, $\quad \mathcal{G} = \begin{bmatrix} 0 & P_W\mathcal{E}P_V \\ -(P_W\mathcal{E}P_V)^* & 0 \end{bmatrix}$, $\quad P_{\mathcal{H}} = \begin{bmatrix} P_W^* & 0 \\ 0 & P_V \end{bmatrix}$

compute a basis matrix V_0 of $\ker(\mathcal{G}) \cap \operatorname{im} P_{\mathcal{H}}$

$W \leftarrow [\,]$, $c \leftarrow [\,]$, $U \leftarrow [\,]$, $i \leftarrow 0$

repeat

  $W_i = \mathcal{H}V_i$

  choose U_i such that $\begin{bmatrix} W & W_iU_i \end{bmatrix}$ is a basis matrix of $\operatorname{im} \begin{bmatrix} W & W_i \end{bmatrix}$

  choose C_i^1 and C_i^2 such that $\left[\begin{smallmatrix} C_i^1 \\ C_i^2 \end{smallmatrix}\right]$ is a basis matrix of $\ker \begin{bmatrix} W & W_i \end{bmatrix}$

  $W \leftarrow \begin{bmatrix} W & W_iC_i \end{bmatrix}$, $C \leftarrow \left[\begin{smallmatrix} C & UC_i^1 \\ 0 & C_i^2 \end{smallmatrix}\right]$, $U \leftarrow \left[\begin{smallmatrix} U & 0 \\ 0 & U_i \end{smallmatrix}\right]$

  compute a basis matrix S_i of $\ker(C_i^* V_i^* W_i C_i)$

  solve $\mathcal{G}V_{i+1} = W_iC_iS_i$ with $P_{\mathcal{H}}V_{i+1} = V_{i+1}$

  $i \leftarrow i+1$

until $\operatorname{im} S_{i-1} = \{0\}$

$V = \begin{bmatrix} V_0, \dots, V_{i-1} \end{bmatrix}$, $L = \begin{bmatrix} 0 & C_0S_0 & & \\ & \ddots & \ddots & \\ & & 0 & C_{i-1}S_{i-1} \end{bmatrix}$

compute a basis matrix C of $\ker(\mathcal{H}V) \cap \operatorname{im} P_{\mathcal{H}}$

Algorithm 3 leads to a projected Lur'e equation (30). Thereafter, Algorithm 4 can be applied to this projected Lur'e equation to obtain projected Lur'e equation which is easier to solve numerically. We note that the difficulty is to find the generalized eigenvalues of $\mathcal{H} - s\mathcal{G}$ on the imaginary axis. Sometimes, it is possible to use physical knowledge to find such eigenvalues. For instance, it is possible to characterize the presence of a generalized eigenvalue at zero for electrical circuits [29].

We emphasize that this step is optional and can be omitted in the presented framework. However, generalized eigenvalues on the imaginary axis cause worse convergence in the subsequent step of the Newton iteration.

Algorithm 4 Shifted Wong sequence for the regular even matrix pencil

---

Input: $(P_W\mathcal{A}P_V), (P_W\mathcal{E}P_V) \in \mathbb{K}^{g\times q}$, $\mathcal{Q} \in \mathbb{K}^{q\times q}$, a generalized eigenvalue $\lambda \in i\mathbb{R}$ of the even matrix pencil $\mathcal{H} - s\mathcal{G}$ as in (24)

Output: $V \in \mathbb{C}^{(g+q)\times k}$, $R \in \mathbb{C}^{k\times \ell}$ such that $\mathcal{H}_\lambda V = \mathcal{G}VR$ and $\operatorname{im} V$ is limit of (33) intersected with $\operatorname{im} P_{\mathcal{H}}$

1: $\mathcal{H}_\lambda = \begin{bmatrix} 0 & P_W(\mathcal{A}-\lambda\mathcal{E})P_V \\ P_V^*(\mathcal{A}^*+\lambda\mathcal{E}^*)P_W^* & \mathcal{Q} \end{bmatrix}$, $\mathcal{G} = \begin{bmatrix} 0 & P_W\mathcal{E}P_V \\ -(P_W\mathcal{E}P_V)^* & 0 \end{bmatrix}$, $P_{\mathcal{H}} = \begin{bmatrix} P_W^* & 0 \\ 0 & P_V \end{bmatrix}$

2: compute a basis matrix V_0 of $\ker(\mathcal{H}_\lambda) \cap \operatorname{im} P_{\mathcal{H}}$

3: $W_0 = \mathcal{G}V_0 \quad i \leftarrow 0$

4: **repeat**

5: solve $\mathcal{H}_\lambda V_{i+1} = W_i$ with $P_{\mathcal{H}}V_{i+1} = V_{i+1}$

6: $i \leftarrow i+1$

7: $W_i \leftarrow \mathcal{G}V_i$

8: compute a basis matrix Y_i of $\ker(V_{i-1}^* W_i)$

9: $V_i \leftarrow V_iY_i$, $W_i \leftarrow W_iY_i$

10: **until** $\operatorname{im} Y_i = \{0\}$

11: $V = \begin{bmatrix} V_0, \dots, V_i \end{bmatrix}$, $R = \begin{bmatrix} 0 & Y_1 & & \\ & \ddots & \ddots & \\ & & 0 & Y_i \end{bmatrix}$

To preserve sparsity, we overload the matrix operations by the according method of the linear mapping. Whereas the replacement of the of matrix multiplications of $\mathcal{G}$ and $\mathcal{H}_\lambda$ is straightforward, the solution of the systems needs more consideration. We show the method for solving the system with $\mathcal{H}$. We apply this method in Algorithm 3, Line 10 and Algorithm 4, Line 5. We have equations of type

$$\begin{bmatrix} 0 & P_W\mathcal{A}P_V \\ (P_W\mathcal{A}P_V)^* & \mathcal{Q} \end{bmatrix}\begin{bmatrix} X_u \\ X_x \end{bmatrix} = \begin{bmatrix} B_u \\ B_x \end{bmatrix}, \quad P_W^*X_u = X_u, \quad P_VX_x = X_x \tag{37}$$

where the restriction $\mathcal{A}_m : \operatorname{im} P_V \to \operatorname{im} P_W$ is surjective due to stabilizability of the underlying DAE. Additional we have a Hermitian matrix $\mathcal{Q}$. With a basis matrix K of the projected kernel of $\mathcal{A}$, $X_x = Y_x + KS$, where Y_x is a solution of $P_W\mathcal{A}P_VY_x = B_u$, $P_VY_x = Y_x$ for some jet unknown S. The solvability of the second row of (37) implies that $K^*B_x = K^*\mathcal{Q}(X_x) = K^*\mathcal{Q}(Y_x + KS)$. Hence S is the solution of the (small) linear equation $(K^*\mathcal{Q}K)S = K^*B_x - K^*\mathcal{Q}Y_x$. X_u is the solution of $(P_W\mathcal{A}P_V)^*X_u = B_x - \mathcal{Q}X_x$, $P_W^*X_u = X_u$ and is unique due to the surjectivity of the restriction. Note that all necessary operation are covert by the operation we defined for the linear mapping which makes these algorithm suitable for sparse systems.

5.4. Iterative solution of the Lur'e equation. In this section we consider the solution $(X, \mathcal{K}) \in \mathbb{K}^{g\times g} \times \mathbb{K}^{p\times q}$ of the Lur'e equation

$$P_V(\mathcal{A}^*X\mathcal{E} + \mathcal{E}^*X\mathcal{A} + \mathcal{Q})P_V = \mathcal{K}^*\mathcal{K}, \quad XP_W = X \tag{38}$$

where $\mathcal{A}, \mathcal{E} \in \mathbb{K}^{g\times q}$, $\mathcal{Q} \in \mathbb{K}^{q\times q}$ is Hermitian, the restriction $P_W\mathcal{E}P_V : \operatorname{im} P_V \to \operatorname{im} P_W$ is surjective, for the a basis matrix V_B of the projected kernel of $P_W\mathcal{E}P_V$

holds $V_B^* \mathcal{Q} V_B > 0$, and the underlying DAE is stabilizable. As mentioned before, see section 4, this implies that the Lur'e equation can be reformulated as an algebraic Riccati equation and requirements are fulfilled after we applied Algorithms 2 and 3. For these type of equation exists a variety of solver which are also able to handle large systems. A survey of state of the art solver is given in [7]. To avoid the reformulation we propose two implementations that make use of a projector and hence fit in the suggested framework. We will show that if we explicitly project the matrices the iteration steps in Algorithms 5 and 6 we obtain the steps in [16] and [1], respectively. This implies the mathematical equivalence. Note that we need to start the Newton iteration with a stabilizing feedback $\mathcal{K}_0$ as defined in Definition 21 to ensure the convergence of the solution to the stabilizing solution.

Definition 21 (Stabilizing feedback). *For $\mathcal{E}, \mathcal{A} \in \mathbb{K}^{g\times q}$ and the projectors $P_W \in \mathbb{K}^{g\times g}$, $P_V \in \mathbb{K}^{q\times q}$ a matrix $\mathcal{K} \in \mathbb{K}^{m\times n}$ is called* stabilizing feedback *if for all $\lambda \in \overline{\mathbb{C}_+}$: $\operatorname{rank} \left[\begin{smallmatrix} P_W(\mathcal{A}-\lambda\mathcal{E})P_V \\ \mathcal{K} \end{smallmatrix}\right] = \operatorname{rank} P_V$ and $\operatorname{im} \mathcal{K} = \mathcal{K}(\ker \mathcal{E} \cap \operatorname{im} P_V)$ hold.*

Algorithm 5 Feedback iteration

Input: $P_W \mathcal{A} P_V, P_W \mathcal{E} P_V \in \mathbb{K}^{g\times q}$, $\mathcal{Q} \in \mathbb{K}^{q\times q}$, $\mathcal{K}_0 \in \mathbb{K}^{m\times(n+m)}$
Output: $X \in \mathbb{K}^{g\times g}$, $\mathcal{K} \in \mathbb{K}^{p\times q}$ such that $||P_V^*(\mathcal{A}^* X \mathcal{E} + \mathcal{E}^* X \mathcal{A} + \mathcal{Q} - \mathcal{K}^*\mathcal{K})P_V|| < \text{tol}$

1: compute a basis matrix V_B of $\ker \mathcal{E} \cap \operatorname{im} P_V$
2: compute L such that $V_B^* \mathcal{Q} V_B = L^* L$
3: $B = AV_B L^{-1}$
4: $\mathcal{S} = (V_B L^{-1})^* \mathcal{Q}$
5: $\mathcal{K} \leftarrow \mathcal{K}_0$
6: **repeat**
7:    choose projector P_K onto $\ker \mathcal{K} \cap \operatorname{im} P_V$
8:    solve $P_K^* \left(\mathcal{A}^* X \mathcal{E} + \mathcal{E}^* X \mathcal{A} + \mathcal{Q}\right) P_K = 0$ with $XP_W = X$
9:    $\mathcal{K} \leftarrow B^* X \mathcal{E} + \mathcal{S}$
10: **until** $||P_V^*(\mathcal{A}^* X \mathcal{E} + \mathcal{E}^* X \mathcal{A} + \mathcal{Q} - \mathcal{K}^*\mathcal{K})P_V|| < \text{tol}$

Algorithm 5 is the reformulation of the Newton-Kleinman method [16]. The algorithm solves in every step a Lyapunov equation on a subspace determined by the feedback $\mathcal{K}_i$ given in the i-th iteration. As shown in [16], we have monotonic convergence. To show the equivalence of Algorithm 5 to the algorithm proposed by Kleinman we look at the equation which define the iteration of the latter:

$$(A - BK)^* Y + Y(A - BK) + (Q - S^* S) + (K - S)^*(K - S) = 0 \qquad (39)$$

Note that the solution in Line 8 is independent of the along direction of projector P_K. We use the projector $P_K = I - V_B L^{-1}\mathcal{K}$, where V_B is a basis matrix of the projected kernel of $\mathcal{E}$ and also fulfills $V_B \mathcal{Q} V_B = I$. We have

$$\mathcal{A}P_K = \mathcal{A} - \mathcal{A}V_B L^{-1}\mathcal{K} = \mathcal{A} - B\mathcal{K}, \quad \mathcal{E}P_K = \mathcal{E}, \quad \mathcal{Q}P_K = \mathcal{Q} - \mathcal{Q}V_B L^{-1}\mathcal{K} = \mathcal{Q} - \mathcal{S}^*\mathcal{K}$$

and constant term of the equation in Algorithm 5, Line 8 is given by

$$P_{K_i}^* \mathcal{Q} P_{K_i} = \mathcal{Q} - \mathcal{S}^*\mathcal{K} - \mathcal{K}^*\mathcal{S} + \mathcal{K}^*\mathcal{K} = \mathcal{Q} - \mathcal{S}^*\mathcal{S} + (\mathcal{K} - \mathcal{S})^*(\mathcal{K} - \mathcal{S}).$$

Now with the matrix W as a basis of $\operatorname{im} P_W^*$ and V_A such that $\operatorname{im} \begin{bmatrix} V_A & V_B \end{bmatrix} = \operatorname{im} P_V$ and $W^* P_W \mathcal{E} P_V V_A = I$, we substitute $X = WYW*$ and multiply Line 8 from the left with V_A^* and right with V_A. With $A := W^* \mathcal{A} V_A$, $Q := V_A^* \mathcal{Q} V_A$, $S := \mathcal{S} V_A$ and $K := \mathcal{K} V_A$ this leads to (39) and thus showing the equivalence.

Algorithm 6 is a modification of the Newton method for a special case. This algorithm computes updates of the solution in every step and is mathematical equivalent to the Newton method presented in [1, 8]. However this algorithm can only applied to Lur'e equation in the form of

$$P_V^*(\mathcal{A}^* X \mathcal{E} + \mathcal{E}^* X \mathcal{A} + \mathcal{J}^* \mathcal{J} - \mathcal{C}^* \mathcal{C}) P_V = \mathcal{K}^* \mathcal{K}, \quad X P_W = X$$

with $\mathcal{C} V_B = 0$ and invertible $\mathcal{J} V_B$, where V_B is a basis matrix of the projected kernel of $\mathcal{E}$. Moreover $\mathcal{J}$ has to be a stabilizing feedback. This seems like a hard condition but it can be shown that there exist a decomposition of the residual $P_V^*(\mathcal{A}^* X \mathcal{E} + \mathcal{E}^* X \mathcal{A} + \mathcal{Q} - \mathcal{K}^* \mathcal{K}) P_V$ in Algorithm 5 that fulfills these conditions. Hence we can use Algorithm 6 to obtain updates of the solution after a step of Algorithm 5 using the residual as right hand side.

Algorithm 6 Update iteration

Input: $P_W \mathcal{A} P_V, P_W \mathcal{E} P_V \in \mathbb{K}^{g \times q}$, $\mathcal{C} \in \mathbb{K}^{\ell \times q}$, $\mathcal{J} \in \mathbb{K}^{p \times q}$
Output: $X \in \mathbb{K}^{g \times g}$, $\mathcal{K} \in \mathbb{K}^{p \times q}$ such that
$||P_V^*(\mathcal{A}^* X \mathcal{E} + \mathcal{E}^* X \mathcal{A} - \mathcal{C}^* \mathcal{C} + \mathcal{J}^* \mathcal{J} - \mathcal{K}^* \mathcal{K}) P_V|| < \mathrm{tol}$
1: compute a basis matrix V_B of $\ker \mathcal{E} \cap \operatorname{im} P_V$
2: $L = \mathcal{J} V_B$
3: $B = \mathcal{A} V_B L^{-1}$
4: $\Delta\mathcal{K} \leftarrow \mathcal{C}$, $\quad \mathcal{K} \leftarrow \mathcal{J}$
5: **while** $||\Delta\mathcal{K}^* \Delta\mathcal{K}|| \geq \mathrm{tol}$ **do**
6: choose projector P_K onto $\ker \mathcal{K} \cap \operatorname{im} P_V$
7: solve $P_K^* \left(\mathcal{A}^* \Delta X \mathcal{E} + \mathcal{E}^* \Delta X \mathcal{A} - \Delta\mathcal{K}^* \Delta\mathcal{K}\right) P_K = 0$ with $\Delta X P_W = \Delta X$
8: $\Delta\mathcal{K} \leftarrow B^* \Delta X \mathcal{E}$
9: $X \leftarrow X + \Delta X$; $\quad \mathcal{K} \leftarrow \mathcal{K} + \Delta\mathcal{K}$
10: **end while**

ΔX can be written as an approximate low-rank factorisation. The solution of Line 7 is expected to be negative semidefinite due to the stabilizing feedback. This give rise to the substitution $\Delta X = -\Delta Z^* \Delta Z$. The low-rank factors Z of the approximate solution $X = -Z^* Z$ is then given by $Z \leftarrow \left[\begin{smallmatrix} Z \\ \Delta Z \end{smallmatrix}\right]$. This also implies that number of rows of the matrix Z grows with each iteration step. To avoid to much memory consumption we may apply compression methods based of rank-revealing QR-factorizations or the singular value decompositions of Z.

To show the equivalence of Algorithm 6 to the algorithm presented in [1, Sec. 3] we look at the equations which define the iteration of the latter:

$$\begin{aligned} &(A - BK)^* Y E + E^* Y (A - BK) - D^* D = 0, \\ &D = B^* Y, \qquad K \leftarrow K + D \end{aligned} \tag{40}$$

The projector $P_{K_i} = I - V_B L^{-1} \mathcal{K}_i$ gives rise to $\mathcal{A} P_{K_i} = \mathcal{A} - B \mathcal{K}_i$ and $\mathcal{E} P_{K_i} = \mathcal{E}$. Then, with W and V_A as defined as before, we substitute $\Delta X = W Y W*$ and multiply Line 7 from the left with V_A^* and right with V_A. Futhermore we multiply Lines 8 and 9 with V_A from the right. With $D := \Delta\mathcal{K} V_A$, $S := \mathcal{S} V_A$ and $K := \mathcal{K} V_A$ we obtain (39) and thus showing the equivalence.

The advantage of these two implementation is that these are easily adapted to the projector driven framework we established before. For the equation

$$P_V^* \left(\mathcal{A}^* X \mathcal{E} + \mathcal{E}^* X \mathcal{A} + \mathcal{Q}\right) P_V = \mathcal{K}^* \mathcal{K}, \quad X = X^*, \quad X P_W = X$$

we merge the projector $P_V = I - V^+ V$ with the projector P_K leading to P_{VK} onto $\ker \left[\begin{smallmatrix} V \\ \mathcal{K} \end{smallmatrix}\right]$.

5.5. Solving the projected Lyapunov equation. In this section we present an algorithm for the solution $X \in \mathbb{K}^{g \times g}$ of the projected Lyapunov equation

$$P_V^* (\mathcal{A}^* X \mathcal{E} + \mathcal{E}^* X \mathcal{A} + \mathcal{C}^* \mathcal{C}) P_V = 0, \quad X = X^*, \quad X P_W = X \tag{41}$$

where $\mathcal{A}, \mathcal{E} \in \mathbb{K}^{g \times q}$ and assumed to be stable, i.e., $\operatorname{rank} P_W (A - \lambda E) P_V = \operatorname{rank} P_V$ for all $\lambda \in \overline{\mathbb{C}_+}$. Further $\mathcal{C} \in \mathbb{K}^{\ell \times q}$ and the projector $P_V \in \mathbb{K}^{q \times q}$ and $P_W \in \mathbb{K}^{g \times g}$. A simple algorithm for this task is the *Alternating Directions Implicit* (ADI) algorithm in its low-rank version [20, 23]. Although it is an older algorithm, it is able to compete so far with other methods for solving the Lyapunov equation with respect to the runtime [30].

The shifts p_i are crucial for the convergence of the ADI and have to be chosen properly. An overview of several choices of shifts for the ADI algorithm with numerical examples can be found in [6]. Recent results have improved the implementation of the ADI significantly. First the handling of complex shifts was improved [4], then a implementation with the explicit residual factors was developed [5].

Algorithm 7 is a modification of the implementation in [5], that solves the projected Lyapunov equation (41) and can handle complex data as well as complex shifts in real arithmetic.

Algorithm 7 Low-rank ADI

Input: $P_W \mathcal{A} P_V, P_W \mathcal{E} P_V \in \mathbb{K}^{g \times q}$, $\mathcal{C} \in \mathbb{K}^{\ell \times q}$
Output: $X \in \mathbb{K}^{g \times g}$ such that $||P_V^* (\mathcal{A}^* X \mathcal{E} + \mathcal{E}^* X \mathcal{A} + \mathcal{C}^* \mathcal{C}) P_V|| < \text{tol}$, $X P_W = X$

compute p_i with $\operatorname{Re} p_i < 0$
$i \leftarrow 1$
while $||\mathcal{C} P_V||^2 \geq \text{tol}$ **do**
  solve $V P_W (\mathcal{A} + p_i \mathcal{E}) P_V = \mathcal{C}$ with $V P_W = V$
  if $\operatorname{Im} p_i = 0 \vee \mathbb{K} = \mathbb{C}$ **then**
    $\mathcal{C} \leftarrow \mathcal{C} - 2 \operatorname{Re}(p_i) V P_W \mathcal{E} P_V$
    $Z \leftarrow \begin{bmatrix} Z \\ \sqrt{-2 \operatorname{Re}(p_i)}\, V \end{bmatrix}$
  else
    $\gamma \leftarrow 2\sqrt{- \operatorname{Re} p_i}$, $\delta \leftarrow \frac{\operatorname{Re} p_i}{\operatorname{Im} p_i}$
    $\mathcal{C} \leftarrow \mathcal{C} - \gamma^2 (\operatorname{Re} V + \delta \operatorname{Im} V) V P_W \mathcal{E} P_V$
    $Z \leftarrow \begin{bmatrix} Z \\ \gamma(\operatorname{Re} V + \delta \operatorname{Im} V) \\ \gamma \sqrt{\delta^2 + 1} \operatorname{Im} V \end{bmatrix}$
  end if
  $i \leftarrow i + 1$
end while
$X = Z^* Z$

The ADI is the crucial part of the overall procedure. Most of the computational time is usually spent here. Moreover, the ADI is not applicable, if the solution cannot be approximated with a low-rank matrix or the pencil $A - sE$ has eigenvalues close to the imaginary axis. The latter typically leads to a slow convergence. Furthermore the system in Line 4 has to be solved in a reasonable amount of time.

If we use the extended systems given in subsection 5.1 to solve Line 4, the according extended matrix equation is given by

$$\begin{bmatrix} V & Y_w & Y_s & Y_K \end{bmatrix} \begin{bmatrix} \mathcal{A} + p_i\mathcal{E} & W_w & W_s \\ V_w & 0 & 0 \\ V_s^+ & 0 & 0 \\ \mathcal{K} & 0 & 0 \end{bmatrix} = \begin{bmatrix} C & 0 & 0 \end{bmatrix}, \tag{42}$$

which we solve for V, Y_w, Y_s and Y_K. V_w and W_w are hereby the output of Algorithm 2, V_s^+ and W_s are given by the projectors P_{V_s} and P_{W_s} as in (36) and $\mathcal{K}$ is the feedback from either Algorithm 5 or Algorithm 6. Note that we have some freedom in choosing $\operatorname{im} W_w$ and $\ker V_s^+$ since we do not give a explicit "along" direction for P_{W_w} and we do not specify the image of P_{V_s}. Hence we are able to use "sparse" matrices and improve the efficiency of the sparse linear solvers.

6. Numerical examples. Our first example is an electrical circuit containing resistors, capacities and voltage sources. The system is modelled by the matrices

$$\mathcal{E} = \begin{bmatrix} A_C C A_C^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathcal{A} = \begin{bmatrix} -A_R G A_R^* & -A_V & 0 \\ A_V^* & 0 & -I \end{bmatrix}, \qquad w = \begin{pmatrix} e \\ i_V \\ u_V \end{pmatrix}$$

with positive definite conductance matrix G, positive definite capacitance matrix C and incidence matrices A_C, A_R, A_V describing the circuit topology from the modified nodal analysis [27]. The function w in the DAE $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}w = \mathcal{A}w$ is composed of the node potentials as well as currents and voltages at the voltage sources. Our circuit has the property that it does not contain any cutsets only consisting of capacitances. This implies that $[A_R A_V]$ has full row rank [27], which, on the other hand yields that all generalized eigenvalues of $\mathcal{A} - s\mathcal{E}$ have negative real part. The latter implies behavioral stabilizability by the comment after Definition 14.

The considered cost will be the negative of the real part of the $\mathcal{L}^2$ inner product of the voltages and currents at the voltage sources; it is therefore represented by the matrix

$$\mathcal{Q} = \tfrac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -I \\ 0 & -I & 0 \end{bmatrix}.$$

We note that a similar example has been treated in [8]. The example in [8] can however turned into an algebraic Riccati equation without using particular solutions, and is therefore simpler than the one which is considered here.

It can be shown that for all $w \in \mathfrak{B}_{[\mathcal{E},\mathcal{A}]}$ with $\lim_{t\to\infty} \mathcal{E}w(t) = 0$ holds

$$\int_0^\infty w(t)^* \mathcal{Q} w(t)\,\mathrm{d}t = \int_0^\infty e(t)^* A_R G A_R^* e(t)\,\mathrm{d}t - e(0)^* A_C C A_C^* e(0) \geq -e(0)^* A_C C A_C^* e(0).$$

This inequality together with stabilizability of $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}w = \mathcal{A}w$ implies feasibility of the optimal control problem. Theorem 15 then implies that the Lur'e equation (6) has a solution $(X, \mathcal{K})$. We note that a basis matrix of the system space can be determined by physical considerations [29]. Nevertheless, we determine this matrix by the presented algorithms for demonstration issues.

Our example is chosen such that the associated control problem is singular. After determining a partial solution, we end up with an indefinite matrix $\mathcal{Q}_s$ that is suitable for Algorithm 6. Figure 1 shows the effect of the deflation using the subspace of the shifted $\mathcal{G}$-neutral Wong sequences for $\lambda = 0$. We observe better convergence behavior and we need fewer iterations in the ADI iteration. On the other hand, various experiments did not show any advantages of one over the other method with respect to runtime. Note that Algorithm 6 allows crude error tolerances in later steps for solving the Lyapunov equation. Some further data and quantitative results for this example are gathered in Table 1. As a measure to the quality and accuracy of our numerical solutions we use the relative residual

$$\|(P_{V_w}^*(\mathcal{A}^* X \mathcal{E} + \mathcal{E}^* X \mathcal{A} + \mathcal{Q} - \mathcal{K}^* \mathcal{K}) P_{V_w}\| / \|P_{V_w}^* \mathcal{Q} P_{V_w}\|. \tag{43}$$

For this example we choose the orthogonal projector for P_{V_w}.

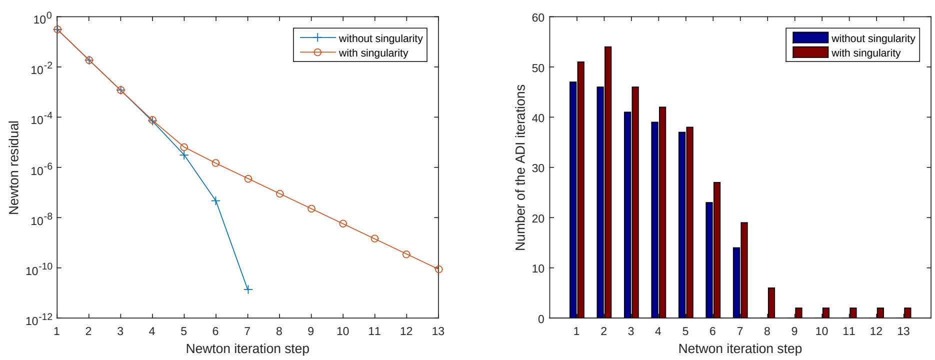

FIG. 1. *(left) The convergence history of the updated iteration for with and without the deflation with the limit of the G-neutral shifted Wong sequence at* 0*; (right) the number of ADI iterations required for solving the Lyapunov equations at each Newton iteration.*

For the second example we consider the linearized semi-discretetized Navier-Stokes equations modeling the evolution of the velocity v and the pressure p in an incompressible laminar flow as considered in [3]. The model is given by the DAE $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}w = \mathcal{A}w$, where

$$\mathcal{E} = \begin{bmatrix} M & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathcal{A} = \begin{bmatrix} A & J^* & B \\ J & 0 & 0 \end{bmatrix}, \qquad w = \begin{pmatrix} v \\ p \\ u \end{pmatrix}.$$

In particular, M is the positive definite mass matrix, and the matrix J referring to the discrete divergence operator has full row rank. The term Bu models the distributed control input. Our aim is to minimize the cost functional $\int_0^\infty \|u(t)\|^2 + \|Cv(t)\|^2 \mathrm{d}t$ subject to the above equation with some given consistent initial value. Here, $Cv(t)$ corresponds to spatially averaged velocities in some observation domain. For a deep study of this optimal control in the context of LQG-controller reduction, we refer to [3].

The above optimal control problem leads to the solution of the Lur'e equation

$$\mathcal{A}^* X \mathcal{E} + \mathcal{E}^* X \mathcal{A} + \mathcal{C}^* \mathcal{C} =_{\mathcal{V}_{[\mathcal{E},\mathcal{A}]}} \mathcal{K}^* \mathcal{K}, \quad \mathcal{C} = \begin{bmatrix} C & 0 & 0 \\ 0 & 0 & I \end{bmatrix}, \tag{44}$$

This optimal control problem can be shown to be regular. As a consequence, the matrix obtained by Algorithm 3 is trivial. We further have that $\left[\begin{smallmatrix} A-sM & J^* \\ J & 0 \end{smallmatrix}\right]$ has no generalized eigenvalues with positive real part. Consequently, $\mathcal{K}_0 = \begin{bmatrix} 0 & 0 & I \end{bmatrix} P_V$ is a stabilizing feedback. Some further data and quantitative results for this example are as well contained in Table 1.

All calculations were done in Matlab 15a and the implementation is available as supplementary material.

TABLE 1
Sizes and results of the numerical Examples

Example	RC-Circuit	Navier-Stokes
Size of $\mathcal{A}$	2007×2010	10645×10651
Rank of $\mathcal{Q}$	6	14
i_{max} of Algorithm 2	1	2
Number of columns of V_w	5	2578
Number of columns of W_w	5	2578
i_{max} of Algorithm 3	2	1
Number of columns of V_s	1	0
Number of columns of W_s	1	0
i_{max} of Algorithm 4	1	0
Number of columns of V_s^0	1	0
Number of columns of W_s^0	1	0
Relative Lur'e residual (43)	$1.333 \cdot 10^{-11}$	$1.101 \cdot 10^{-10}$
Newton-Steps	7	3
Rank of X	114	343

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