In absence of long chordless cycles, large tree-width becomes a local phenomenon

Daniel Weißauer

Abstract

We prove that, for all ℓ and s, every graph of sufficiently large tree-width contains either a complete bipartite graph $K_{s,s}$ or a chordless cycle of length greater than ℓ.

1 Introduction

In an effort to make the statement in the title precise, let us call a graph parameter P *global* if there is a constant c such that for all k and r there exists a graph G for which every subgraph H of order at most r satisfies $P(H) < c$, while $P(G) > k$. The intention here is that P being small, even bounded by a constant, on subgraphs of bounded order does not provide a bound on $P(G)$.

Tree-width is a global parameter (we may take $c = 2$), as is the chromatic number (with $c = 3$). Indeed, it is a classic result of Erdős [6] that for all k and r there exists a graph of chromatic number $> k$ for which every subgraph on at most r vertices is a forest.

It is well-known (see [4]) that the situation changes when we restrict ourselves to *chordal graphs*, graphs without chordless cycles of length ≥ 4:

$$\forall k: \text{ Every } K_{k+1}\text{-free chordal graph has tree-width} < k. \tag{1}$$

Hence the only obstruction for a chordal graph to have small tree-width is the presence of a large clique. Since the chromatic number of a graph is at most its tree-width plus one ([4]), the same is true for the chromatic number. In particular, tree-width and chromatic number are *local* parameters for the class of chordal graphs.

In 1985, Gyárfás [8] made a famous conjecture which implies that chromatic number is a local parameter[1] for the larger class of ℓ-*chordal* graphs, those which have no chordless cycle of length $> \ell$:

$$\forall \ell, r\ \exists k: \text{ Every } K_r\text{-free } \ell\text{-chordal graph is } k\text{-colourable.} \tag{2}$$

[1] Indeed, in terms of our earlier definition, (2) implies that given any integer c, there exists a k such that every ℓ-chordal graph of chromatic number $> k$ has a subgraph of order $\leq c$ and chromatic number $\geq c$.

This conjecture remained unresolved for 30 years and was proved only recently by Chudnovsky, Scott and Seymour [3]. In view of (1), it is tempting to think that an analogue of (2) might hold with tree-width in place of chromatic number. Complete bipartite graphs, however, are examples of triangle-free 4-chordal graphs of large tree-width. Therefore a verbatim analogue of (2) is not possible and any graph whose presence we can hope to force by assuming ℓ-chordality and large tree-width will be bipartite.

On the positive side, Bodlaender and Thilikos [2] showed that every *star* can be forced as a subgraph in ℓ-chordal graphs by assuming large tree-width (see Section 3). However, since stars have tree-width 1, this does not establish locality of tree-width in the sense of our earlier definition. Our main result is that in fact *any* bipartite graph can be forced as a subgraph:

Theorem 1. *Let $\ell \geq 4$ be an integer and F a graph. Then F is bipartite if and only if there exists an integer k such that every ℓ-chordal graph of tree-width $\geq k$ contains F as a subgraph.*

This shows that tree-width is local for ℓ-chordal graphs: Given any integer c, there exists an integer k such that every ℓ-chordal graph of tree-width $\geq k$ has a subgraph isomorphic to $K_{c,c}$, which has order $2c$ and tree-width c.

Theorem 1 also has an immediate application to an Erdős-Pósa type problem. Kim and Kwon [9] showed that chordless cycles of length > 3 have the Erdős-Pósa property:

Theorem 2 ([9])**.** *For every integer k there exists an integer m such that every graph G either contains k vertex-disjoint chordless cycles of length > 3 or a set X of at most m vertices such that $G - X$ is chordal.*

They also constructed, for every integer $\ell \geq 4$, a family of graphs showing that the analogue of Theorem 2 for chordless cycles of length $> \ell$ fails. We complement their negative result by proving that the Erdős-Pósa property *does* hold when restricting the host graphs to graphs not containing $K_{s,s}$ as a subgraph.

Corollary 3. *For all ℓ, s and k there exists an integer m such that every $K_{s,s}$-free graph G either contains k vertex-disjoint chordless cycles of length $> \ell$ or a set X of at most m vertices such that $G - X$ is ℓ-chordal.*

The paper is organised as follows. Section 2 contains some basic definitions. Theorem 1, our main result, is proved in Section 3. In Section 4 we formally introduce the Erdős-Pósa property, restate Corollary 3 in that language and give a proof thereof. Section 5 closes with some open problems.

2 Notation and definitions

All graphs considered here are finite and undirected and contain neither loops nor parallel edges. Our notation and terminology mostly follow that of [4].

For two graphs G and H, we say that G is *H-free* if G does not contain a subgraph isomorphic to H. Given a tree T and $s, t \in T$, we write sTt for the

unique s-t-path in T. Given a graph G and a set X of vertices of G, a path $P \subseteq G$ is an *X-path* if it contains at least one edge and meets X precisely in its endvertices. A *separation* of G is a tuple (A, B) with $V = A \cup B$ such that there are no edges between $A \setminus B$ and $B \setminus A$. The *order* of (A, B) is the number of vertices in $A \cap B$. We call the separation (A, B) *tight* if for all $x, y \in A \cap B$, both $G[A]$ and $G[B]$ contain an x-y-path with no internal vertices in $A \cap B$.

Given an integer k, a set X of at least k vertices of G is a *k-block* if it is inclusion-maximal with the property that for every separation (A, B) of order $< k$, either $X \subseteq A$ or $X \subseteq B$. By Menger's Theorem, G then contains k internally disjoint paths between any two non-adjacent vertices in X.

A *tree-decomposition of* G is a pair $(T, \mathcal{V})$, where T is a tree and $\mathcal{V} = (V_t)_{t \in T}$ a family of sets of vertices of G such that for every $v \in V(G)$, the set of $t \in T$ with $v \in V_t$ induces a non-empty subtree of T and for every edge $vw \in E(G)$ there is a $t \in T$ with $v, w \in V_t$. If $(T, \mathcal{V})$ is a tree-decomposition of G, then every $st \in E(T)$ induces a separation (G^t_s, G^s_t) of G, where G^y_x is the union of V_u for all $u \in T$ for which $y \notin uTx$. Note that $G^t_s \cap G^s_t = V_s \cap V_t$. We call $(T, \mathcal{V})$ *tight* if every separation induced by an edge of T is tight.

Given $t \in T$, the *torso at* t is the graph obtained from $G[V_t]$ by adding, for every neighbor s of t, an edge between any two non-adjacent vertices in $V_s \cap V_t$.

Given graphs G and H, a *subdivision of H in G* consists of an injective map $\eta : V(H) \to V(G)$ and a map P which assigns to every edge $xy \in E(H)$ an $\eta(x)$-$\eta(y)$-path $P^{xy} \subseteq G$ so that the paths $(P^{xy} \colon xy \in E(H))$ are internally disjoint and no P^{xy} has an internal vertex in $X := \eta(V(H))$. The vertices in X are called *branchvertices*. For an integer r, the subdivision is a *$(\leq r)$-subdivision* if every path P^{xy} has length at most r. When H is a complete graph, the map η is irrelevant and we only keep track of the set X of branchvertices and the family $(P^{xy} \colon x, y \in X)$.

3 Proof of Theorem 1

As observed in the introduction, the complete bipartite graphs $K_{s,s}$ show that no bound on the tree-width of F-free ℓ-chordal graphs exists if F is not bipartite. We now prove that F being bipartite is sufficient. Since every bipartite graph is a subgraph of some $K_{s,s}$, it suffices to prove Theorem 1 for the case $F = K_{s,s}$.

Our proof is a cascade with three steps. First, we show that sufficiently large tree-width forces the presence of a k-block.

Lemma 4. *Let ℓ, k and $t \geq 2(\ell - 2)(k - 1)^2$ be positive integers. Then every ℓ-chordal graph of tree-width $\geq t$ contains a k-block.*

We then prove that the existence of a k-block yields a bounded-length subdivision of a complete graph.

Lemma 5. *Let ℓ, m and $k \geq 5m^2\ell/4$ be positive integers. Then every ℓ-chordal graph that contains a k-block contains a $(\leq 2\ell - 3)$-subdivision of K_m.*

In the last step, we show that such a bounded-length subdivision gives rise to a copy of $K_{s,s}$.

Lemma 6. *For all integers ℓ and s there exists a $q > 0$ such that the following holds. Let m, r be positive integers with $m \geq qr$. Then every ℓ-chordal graph that contains a $(\leq r)$-subdivision of K_m contains $K_{s,s}$ as a subgraph.*

It is immediate that Theorem 1 follows once we have established these three lemmas.

3.1 Proof of Lemma 4

A trivial obstacle to our search for a copy of $K_{s,s}$ is the absence of vertices of high degree. Bodlaender and Thilikos [2] showed, however, that ℓ-chordal graphs of bounded degree have bounded tree-width. Their exponential bound was later improved by Kosowski, Li, Nisse and Suchan [10] and by Seymour [17].

Theorem 7 ([17])**.** *Let ℓ and Δ be positive integers and G a graph. If G is ℓ-chordal and has no vertices of degree greater than Δ, then the tree-width of G is at most $(\ell - 2)(\Delta - 1) + 1$.*

By demanding large tree-width, we can therefore guarantee a large number of vertices of high degree. We now show that these are not all just scattered about the graph. It was shown by the author in [19] that either there is a k-block or there is a tree-decomposition which separates the set of vertices of high degree into small pieces. This also follows, without explicit bounds, from a far more general result of Dvořák [5].

Theorem 8 ([19])**.** *Let $k \geq 3$ be a positive integer and G a graph. If G has no k-block, then there is a tight tree-decomposition $(T, \mathcal{V})$ of G such that every torso has fewer than k vertices of degree at least $2(k-1)(k-2)$.*

In fact, tightness of the tree-decomposition is not explicit in [19, Theorem 1], but is established in the proof as *Lemma 6*.

Now let ℓ, k and $t \geq 2(\ell - 2)(k - 1)^2$ be positive integers. Let G be an ℓ-chordal graph with no k-block. For $k = 2$, this means that G is acyclic and therefore has tree-width 1. Suppose from now on that $k \geq 3$. We show that the tree-width of G is less than t.

By Theorem 8, there is a tight tree-decomposition $(T, \mathcal{V})$ of G such that every torso has fewer than k vertices of degree at least $d := 2(k-1)(k-2)$. Let $t \in T$ arbitrary, let N be the set of neighbors of t in T and let H be the torso at t. We claim that H is ℓ-chordal.

Let $C \subseteq H$ be a chordless cycle. For every edge $xy \in E(C) \setminus E(G)$, there is some $s \in N$ with $x, y \in V_s \cap V_t$. Since $(T, \mathcal{V})$ is tight, there exists an x-y-path P^{xy} in G_s^t which meets V_t only in its endpoints. Observe that for every $s \in N$, C contains at most two vertices of V_s and these are adjacent in C. Hence we can replace every edge $xy \in E(C) \setminus E(G)$ by P^{xy} and obtain a chordless cycle C' of G with $|C'| \geq |C|$. Since G is ℓ-chordal, it follows that $|C| \leq \ell$. This proves our claim.

Now, let $A \subseteq V(H)$ be the set of all vertices of degree $\geq d$ in H. Then $H - A$ is ℓ-chordal and has no vertices of degree $> d - 1$. By Theorem 7, the tree-width of $H - A$ is at most $(\ell - 2)(d - 2) + 1$. Therefore

$$\mathrm{tw(H)} \leq \mathrm{|A|} + \mathrm{tw(H - A)} \leq \mathrm{k} + (\ell - 2)(\mathrm{d} - 2) < \mathrm{t}.$$

We have shown that every torso has tree-width $< t$. We can then take a tree-decomposition of width $< t$ of each torso and combine all these to a tree-decomposition of width $< t$ of G. □

3.2 Proof of Lemma 5

In general, the presence of a k-block does not guarantee the existence of any subdivision of K_m for $m \geq 5$. For example, take a rectangular $k^2 \times k$-grid, add $2(k + 1)$ new vertices to the outer face and make each of these adjacent to k consecutive vertices on the perimeter of the grid (see Figure 3.2). These new vertices are then a k-block in the resulting planar graph.

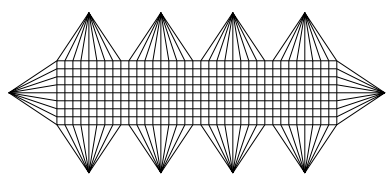

Figure 1: A planar graph with a 9-block

Our aim in this section is to show that for ℓ-chordal graphs, sufficiently large blocks do indeed yield bounded-length subdivisions of complete graphs.

Let ℓ, m and $k \geq 5m^2\ell/4$ be positive integers. Let G be an ℓ-chordal graph and $X \subseteq V(G)$ a k-block of G. Let $L := 2\ell - 3$. Assume for a contradiction that G contained no $(\leq L)$-subdivision of K_m. Let $x, y \in X$ non-adjacent. Then G contains a set $\mathcal{P}^{xy}$ of k internally disjoint x-y-paths. Taking sub-paths, if necessary, we may assume that each path in $\mathcal{P}^{xy}$ is induced. Let $p_0 := m + m^2(\ell - 2)$.

Claim: Fewer than p_0 paths in $\mathcal{P}^{xy}$ have length $> \ell/2$.

Proof of Claim. Let $\mathcal{P}_0$ be the set of all paths in $\mathcal{P}^{xy}$ of length $> \ell/2$ and $p := |\mathcal{P}_0|$. Assume for a contradiction that $p \geq p_0$. Let $P, Q \in \mathcal{P}_0$. Then $P \cup Q$ is a cycle of length $> \ell$. Since G is ℓ-chordal, $P \cup Q$ has a chord. This chord must join an internal vertex of P to an internal vertex of Q. Choose such vertices $v_P^Q \in P$ and $v_Q^P \in Q$ so that the cycle $D := xPv_P^Qv_Q^PQx$ has minimum length. Note that D is an induced cycle and therefore has length at most ℓ. In particular, the segment of P joining x to v_P^Q has length at most $\ell - 2$ and similarly for Q and v_Q^P.

For $P \in \mathcal{P}_0$, let P' be a minimal subpath of P containing every vertex v_P^Q, $Q \in \mathcal{P}_0 \setminus \{P\}$. Then $\mathcal{P} := \{P' \colon P \in \mathcal{P}_0\}$ is a family of p disjoint paths, each of length at most $\ell - 3$, and G contains an edge between any two of them. Fix an arbitrary $\mathcal{Q} \subseteq \mathcal{P}$ with $|\mathcal{Q}| = m$. Since $p \geq p_0$, every $Q \in \mathcal{Q}$ contains a vertex u_Q which has neighbors on at least m^2 different paths in $\mathcal{P} \setminus \mathcal{Q}$.

Let $U := \{u_Q \colon Q \in \mathcal{Q}\}$. We iteratively construct a $(\leq L)$-subdivision of K_m with branchvertices in U. Let $t := \binom{m}{2}$ and enumerate the pairs of vertices of U arbitrarily as $e_1, \ldots, e_t$. In the j-th step, we assume that we have constructed a family $\mathcal{R}^j = (R_i)_{i<j}$ of internally disjoint U-paths of length at most L, so that R_i joins the vertices of e_i and meets at most two paths in $\mathcal{P} \setminus \mathcal{Q}$. We now find a suitable path R_j.

Let $Q^1, Q^2 \in \mathcal{Q}$ with $e_j = u_{Q^1} u_{Q^2}$. At most $2(j-1) < m^2$ paths in $\mathcal{P} \setminus \mathcal{Q}$ meet any of the paths in $\mathcal{R}^j$. Since u_{Q^1} is adjacent to vertices on at least m^2 different paths in $\mathcal{P} \setminus \mathcal{Q}$, there is a $P^1 \in \mathcal{P} \setminus \mathcal{Q}$ which is disjoint from every R_i, $i < j$, and contains a neighbor of u_{Q^1}. We similarly find a path $P^2 \in \mathcal{P} \setminus \mathcal{Q}$ for u_{Q^2}. Since either $P^1 = P^2$ or G has an edge between P^1 and P^2, $P^1 \cup P^2 \cup \{u_{Q^1}, u_{Q^2}\}$ induces a connected subgraph of G and therefore contains a u_{Q^1}-u_{Q^2}-path R_j of length at most L, which meets only two paths in $\mathcal{P} \setminus \mathcal{Q}$.

Proceeding like this, we find the desired subdivision of K_m after t steps. This contradiction finishes the proof of the claim. □

Let $Y \subseteq X$ with $|Y| = m$. For any two non-adjacent $x, y \in Y$, let $\mathcal{Q}^{xy} \subseteq \mathcal{P}^{xy}$ be the set of all $P \in \mathcal{P}^{xy}$ of length at most $\ell/2$ which have no internal vertices in Y. By the claim above, we have

$$|\mathcal{Q}^{xy}| > k - p_0 - (m-2) \geq \binom{m}{2} \frac{\ell}{2}.$$

Pick one path $P \in \mathcal{Q}^{xy}$ for each pair of non-adjacent vertices $x, y \in Y$ in turn, disjoint from all previously chosen paths. Since $|Q^{xy}| \geq \binom{m}{2} \frac{\ell}{2}$ and each path only has at most $\ell/2 - 1$ internal vertices which future paths need to avoid, we can always find a suitable such path P. Together with all edges between adjacent vertices of Y, this yields a $(\leq \ell/2)$-subdivision of K_m in G with branchvertices in Y. □

We would like to point out that a modification of the above argument can be used to produce a $(\leq \ell/2)$-subdivision of K_m if k is significantly larger.

Indeed, suppose we find a family $\mathcal{P}$ of p disjoint paths, each of length at most $\ell - 3$, such that G contains an edge between any two of them. Then the subgraph H induced by $\bigcup_{P \in \mathcal{P}} V(P)$ has at most $(\ell-2)p$ vertices and at least $\binom{p}{2}$ edges. One can then use a classic result of Kövari, Sós and Turán [11] to show that H contains a copy of K_{m,m^2} if p is sufficiently large. Since K_{m,m^2} contains a (≤ 2)-subdivision of K_m, this establishes an upper bound on the number of paths of length $> \ell/2$ in any $\mathcal{P}^{xy}$. The rest of the proof remains the same.

3.3 Proof of Lemma 6

The combination of Lemma 4 and Lemma 5 already establishes that tree-width is a local parameter for ℓ-chordal graphs. The purpose of Lemma 6 is merely to narrow the set of bounded-order obstructions down as far as possible. We will use the following theorem of Kühn and Osthus [13].

Theorem 9 ([13])**.** *For every integer s and every graph H there exists a d so that every graph with average degree at least d either contains $K_{s,s}$ as a subgraph or contains an induced subdivision of H.*

In fact, we only need the special case $H = C_{\ell+1}$. This special case has a simpler proof which can be found in Kühn's PhD-thesis [12]. Fix an integer d so that every ℓ-chordal graph of average degree at least d contains $K_{s,s}$ as a subgraph. We prove the assertion of Lemma 6 with $q := d^2 \frac{\ell^\ell}{4(\ell-3)!}$.

Let m, r be positive integers with $m \geq qr$ and let G be an ℓ-chordal graph containing a $(\leq r)$-subdivision of K_m. Let X be the set of branchvertices and $(P^{xy} \colon x, y \in X)$ the family of paths of the subdivision. Taking subpaths, if necessary, we may assume that every path is induced.

Assume for a contradiction that G contained no copy of $K_{s,s}$. By Theorem 9, every subgraph of G contains a vertex of degree $< d$. In particular, there is an independent set $Y \subseteq X$ with $|Y| \geq m/d$. Let H be the subgraph of G induced by $\bigcup_{x,y \in Y} V(P^{xy})$. Note that $|H| \leq r\binom{|Y|}{2}$.

Call an edge of H *red* if it joins a vertex $x \in Y$ to an internal vertex of a path P^{yz} with $x \notin \{y, z\}$. Call an edge of H *blue* if it joins an internal vertex of a path P^{wx} to an internal vertex of a path P^{yz} with $\{w, x\} \neq \{y, z\}$. We will show that H must contain many edges which are either red or blue, so that the average degree of H is at least d.

Fix an arbitrary cycle R with $V(R) = Y$. For any $Z \subseteq Y$ with $|Z| = \ell$, obtain the cycle R_Z with $V(R_Z) = Z$ by contracting every Z-path of R to a single edge. We then get a cycle $C_Z \subseteq H$ by replacing every edge $xy \in R_Z$ with the path P^{xy}. Since each path P^{xy} has length at least 2 and H is ℓ-chordal, the cycle C_Z must have a chord. Since Y is independent and every path P^{xy} is induced, the chord must be a red or blue edge of H.

Consider a red edge $xv \in E(H)$ with $x \in Y$, $v \in P^{yz}$ and $x \notin \{y, z\}$. If this edge is a chord for a cycle C_Z, then $\{x, y, z\} \subseteq Z$. Hence it can only occur as a chord for at most

$$\binom{|Y| - 3}{\ell - 3} \leq \frac{|Y|^{\ell-3}}{(\ell-3)!}$$

choices of Z. Similarly, every blue edge $uv \in E(H)$ with $u \in P^{wx}$, $v \in P^{yz}$ and $\{w, x\} \neq \{y, z\}$ can only be a chord of C_Z if $\{w, x, y, z\} \subseteq Z$. This also happens for at most

$$\binom{|Y| - 3}{\ell - 3} \leq \frac{|Y|^{\ell-3}}{(\ell-3)!}$$

choices of Z. Let f be the number of edges of H which are either red or blue.

Since every $Z \subseteq Y$ with $|Z| = \ell$ gives rise to a chord, it follows that

$$\frac{|Y|^\ell}{\ell^\ell} \leq \binom{|Y|}{\ell} \leq f\frac{|Y|^{\ell-3}}{(\ell-3)!}.$$

This shows that the average degree of H is

$$d(H) \geq \frac{2f}{|H|} \geq \frac{4(\ell-3)!}{r\ell^\ell}|Y| \geq d.$$

By Theorem 9, H contains a copy of $K_{s,s}$. □

4 Erdős-Pósa for long chordless cycles

A classic theorem of Erdős and Pósa [7] asserts that for every integer k there is an integer r such that every graph either contains k disjoint cycles or a set of at most r vertices meeting every cycle. This result has been the starting point for an extensive line of research, see the survey by Raymond and Thilikos [15].

Let $\mathcal{F}, \mathcal{G}$ be classes of graphs and $\leq$ a containment relation between graphs. We say that $\mathcal{F}$ *has the Erdős-Pósa property for* $\mathcal{G}$ *with respect to* $\leq$ if there exists a function f such that for every $G \in \mathcal{G}$ and every integer k, either there are disjoint $Z_1, \ldots, Z_k \subseteq V(G)$ such that for every $1 \leq i \leq k$ there is an $F_i \in \mathcal{F}$ with $F_i \leq G[Z_i]$, or there is a $X \subseteq V(G)$ with $|X| \leq f(k)$ such that $F \not\leq G - X$ for every $F \in \mathcal{F}$. When $\mathcal{G}$ is the class of all graphs, we simply say that $\mathcal{F}$ *has the Erdős-Pósa property with respect to* $\leq$. We write $F \subseteq G$ if F is isomorphic to a subgraph of G and $F \subseteq_{\mathrm{i}} G$ if F is isomorphic to an *induced* subgraph of G.

The theorem of Erdős and Pósa then asserts that the class of cycles has the Erdős-Pósa property with respect to $\subseteq$. This implies that cycles also have the Erdős-Pósa property with respect to $\subseteq_{\mathrm{i}}$. It is known that for every ℓ, the class of cycles of length $> \ell$ has the Erdős-Pósa property with respect to $\subseteq$, see [18, 1, 14]. Recently, Kim and Kwon [9] proved that cycles of length > 3 possess the Erdős-Pósa property with respect to $\subseteq_{\mathrm{i}}$:

Theorem 10 ([9])**.** *There exists a constant c such that for every integer k, every graph G either contains k vertex-disjoint chordless cycles of length > 3 or a set X of at most $ck^2 \log k$ vertices such that $G - X$ is chordal.*

In contrast, Kim and Kwon [9] showed that, for any given $\ell \geq 4$, cycles of length $> \ell$ do *not* have the Erdős-Pósa property with respect to $\subseteq_{\mathrm{i}}$. For any given n, they constructed a graph G_n with no two disjoint chordless cycles of length $> \ell$, for which no set of fewer than n vertices meets every chordless cycle of length $> \ell$ in G_n. This graph G_n contains a copy of $K_{n,n}$. We show that this is essentially necessary:

Corollary 11. *For all integers ℓ and s, the class of cycles of length $> \ell$ has the Erdős-Pósa property for the class of $K_{s,s}$-free graphs with respect to $\subseteq_{\mathrm{i}}$.*

This follows from Theorem 1 by a standard argument. Since the proof is quite short, we provide it for the sake of completeness. First, recall the following consequence of the Grid Minor Theorem of Robertson and Seymour [16].

Theorem 12 ([16])**.** *For all positive integers p and q there exists an r such that for every graph G with tree-width $\geq r$, there are disjoint $Z_1, \ldots, Z_p \subseteq V(G)$ such that $G[Z_i]$ has tree-width $\geq q$ for every $1 \leq i \leq p$.*

Proof of Corollary 11. Let k be an integer. By Theorem 1 there exists an integer t such that every ℓ-chordal graph with tree-width $\geq t$ contains $K_{s,s}$. By Theorem 12, there exists an r such that every graph with tree-width $> r$ has k vertex-disjoint subgraphs of tree-width $\geq t$.

Let G be a $K_{s,s}$-free graph. We show that either G contains k disjoint chordless cycles of length $> \ell$ or there is a set of at most $r(k-1)$ vertices whose deletion leaves an ℓ-chordal graph.

Suppose first that the tree-width of G was greater than r. Let $Z_1, \ldots, Z_k$ be disjoint sets of vertices such that $G[Z_i]$ has tree-width $\geq t$ for every i. Then, by Theorem 1, every $G[Z_i]$ must contain a chordless cycle of length $> \ell$, since $K_{s,s} \not\subseteq G[Z_i]$. Therefore G contains k disjoint chordless cycles of length $> \ell$.

Suppose now that G had a tree-decomposition $(T, \mathcal{V})$ of width $< r$. For every chordless cycle $C \subseteq G$ of length $> \ell$, let $T_C \subseteq T$ be the subtree of all $t \in T$ with $V_t \cap V(C) \neq \emptyset$. If there are k disjoint such subtrees $T_{C^1}, \ldots, T_{C^k}$, then $C^1, \ldots, C^k$ are also disjoint and we are done. Otherwise, there exists $S \subseteq V(T)$ with $|S| < k$ which meets every subtree T_C. Then $Z := \bigcup_{s \in S} V_s$ meets every chordless cycle of length $> \ell$ in G and $|Z| \leq r(k-1)$.

□

5 Open problems

A large amount of research is dedicated to the study of *χ-boundedness* of graph classes, introduced by Gyárfás [8]. Here, a class $\mathcal{G}$ of graphs is called *χ-bounded* if there exists a function f so that for every integer k and $G \in \mathcal{G}$, either G contains a clique on $k+1$ vertices or G is $f(k)$-colourable. This is a strengthening of the statement that chromatic number is a local parameter for $\mathcal{G}$, with cliques being the only bounded-order subgraphs to look for.

As we have seen, cliques are not the only reasonable local obstruction to having small tree-width. Nontheless, we may still ask

1. For which classes of graphs is tree-width a local parameter?
2. What kind of bounded-order subgraphs can we force on these classes?
3. For which classes can we force large cliques by assuming large tree-width?

We have seen in Section 4 that long chordless cycles have the Erdős-Pósa property for the class of $K_{s,s}$-free graphs. For which other classes is this true? Kim and Kwon [9] raised this question for the class of graphs without chordless cycles of length four.

References

[1] Etienne Birmelé, J. Adrian Bondy, and Bruce A. Reed. The Erdős-Pósa property for long circuits. *Combinatorica*, 27(2):135–145, 2007.

[2] Hans L. Bodlaender and Dimitrios M. Thilikos. Treewidth for graphs with small chordality. *Discrete Appl. Math.*, 79(1-3):45–61, 1997. 4th Twente Workshop on Graphs and Combinatorial Optimization (Enschede, 1995).

[3] Maria Chudnovsky, Alex Scott, and Paul Seymour. Induced Subgraphs of Graphs with Large Chromatic Number. III. Long Holes. *Combinatorica*, 37(6):1057–1072, 2017.

[4] R. Diestel. *Graph Theory*. Springer, 5th edition, 2017.

[5] Zdenek Dvořák. A stronger structure theorem for excluded topological minors. *arXiv:1209.0129*, 2012.

[6] P. Erdős. Graph theory and probability. *Canad. J. Math.*, 11:34–38, 1959.

[7] P. Erdős and L. Pósa. On independent circuits contained in a graph. *Canad. J. Math.*, 17:347–352, 1965.

[8] A. Gyárfás. Problems from the world surrounding perfect graphs. In *Proceedings of the International Conference on Combinatorial Analysis and its Applications (Pokrzywna, 1985)*, volume 19, pages 413–441 (1988), 1987.

[9] Eun Jung Kim and O-joung Kwon. Erdős-Pósa property of chordless cycles and its applications. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1665–1684. SIAM, 2018.

[10] Adrian Kosowski, Bi Li, Nicolas Nisse, and Karol Suchan. k-chordal graphs: from cops and robber to compact routing via treewidth. In *Automata, languages, and programming. Part II*, volume 7392 of *Lecture Notes in Comput. Sci.*, pages 610–622. Springer, Heidelberg, 2012.

[11] T. Kövari, V. T. Sós, and P. Turán. On a problem of K. Zarankiewicz. *Colloquium Math.*, 3:50–57, 1954.

[12] Daniela Kühn. *Cycles, minors and trees*. PhD thesis, Hamburg University, 2001.

[13] Daniela Kühn and Deryk Osthus. Induced subdivisions in $K_{s,s}$-free graphs of large average degree. *Combinatorica*, 24(2):287–304, 2004.

[14] F. Mousset, A. Noever, N. Škorić, and F. Weissenberger. A tight Erdős-Pósa function for long cycles. *J. Combin. Theory Ser. B*, 125:21–32, 2017.

[15] Jean-Florent Raymond and Dimitrios M. Thilikos. Recent techniques and results on the Erdős-Pósa property. *Discrete Appl. Math.*, 231:25–43, 2017.

[16] N. Robertson and P.D. Seymour. Graph minors. V. Excluding a planar graph. *J. Combin. Theory (Series B)*, 41:92–114, 1986.

[17] Paul Seymour. Tree-chromatic number. *J. Combin. Theory Ser. B*, 116:229–237, 2016.

[18] Carsten Thomassen. On the presence of disjoint subgraphs of a specified type. *J. Graph Theory*, 12(1):101–111, 1988.

[19] Daniel Weißauer. On the block number of graphs. *arXiv:1702.04245*, 2017.