Algebraically grid-like graphs have large tree-width

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Abstract

By the Grid Minor Theorem of Robertson and Seymour, every graph of sufficiently large tree-width contains a large grid as a minor. Tree-width may therefore be regarded as a measure of 'grid-likeness' of a graph.

The grid contains a long cycle on the perimeter, which is the $\mathbb{F}_2$-sum of the rectangles inside. Moreover, the grid distorts the metric of the cycle only by a factor of two. We prove that every graph that resembles the grid in this algebraic sense has large tree-width:

Let k, p be integers, γ a real number and G a graph. Suppose that G contains a cycle of length at least $2\gamma pk$ which is the $\mathbb{F}_2$-sum of cycles of length at most p and whose metric is distorted by a factor of at most γ. Then G has tree-width at least k.

1 Introduction

For a positive integer n, the $(n \times n)$-grid is the graph G_n whose vertices are all pairs (i, j) with $1 \leq i, j \leq n$, where two points are adjacent when they are at Euclidean distance 1. The cycle C_n, which bounds the outer face in the natural drawing of G_n in the plane, has length $4(n-1)$ and is the $\mathbb{F}_2$-sum of the rectangles bounding the inner faces. This is by itself not a distinctive feature of graphs with large tree-width: The situation is similar for the n-wheel W_n, the graph consisting of a cycle D_n of length n and a vertex $x \notin D_n$ which is adjacent to every vertex of D_n. There, D_n is the $\mathbb{F}_2$-sum of all triangles xyz for $yz \in E(D_n)$. Still, W_n only has tree-width 3.

The key difference is the fact that in the wheel, the metric of the cycle is heavily distorted: any two vertices of D_n are at distance at most two within W_n, even if they are far apart within D_n. In the grid, however, the distance between two vertices of C_n within G_n is at least half of their distance within C_n.

In order to incorporate this factor of two and to allow for more flexibility, we equip the edges of our graphs with lengths. For a graph G, a *length-function on G* is simply a map $\ell : E(G) \to \mathbb{R}_{>0}$. We then define the *$\ell$-length* $\ell(H)$ of a subgraph $H \subseteq G$ as the sum of the lengths of all edges of H. This naturally induces a notion of distance between two vertices of G, where we define d_G^ℓ as the minimum ℓ-length of a path containing both. A subgraph $H \subseteq G$ is *ℓ-geodesic* if it contains a path of length $d_G^\ell(a, b)$ between any two vertices $a, b \in V(H)$.

When no length-function is specified, the notions of length, distance and geodecity are to be read with respect to $\ell \equiv 1$ constant.

On the grid-graph G_n, consider the length-function ℓ which is equal to 1 on $E(C_n)$ and assumes the value 2 elsewhere. Then C_n is ℓ-geodesic of length $\ell(C_n) = 4(n-1)$ and the sum of cycles of ℓ-length at most 8. We show that any graph which shares this algebraic feature has large tree-width.

Theorem 1. *Let k be a positive integer and $r > 0$. Let G be a graph with rational-valued length-function ℓ. Suppose G contains an ℓ-geodesic cycle C with $\ell(C) \geq 2rk$, which is the $\mathbb{F}_2$-sum of cycles of ℓ-length at most r. Then the tree-width of G is at least k.*

The starting point of Theorem 1 was a similar result of Matthias Hamann and the author [2]. There, it is assumed that not only the fixed cycle C, but the whole cycle space of G is generated by short cycles.

Theorem 2 ([2, Corollary 3])**.** *Let k, p be positive integers. Let G be a graph whose cycle space is generated by cycles of length at most p. If G contains a geodesic cycle of length at least kp, then the tree-width of G is at least k.*

It should be noted that Theorem 2 is not implied by Theorem 1, as the constant factors are different. In fact, the proofs are also quite different, although Lemma 5 below was inspired by a similar parity-argument in [2].

It is tempting to think that, conversely, Theorem 1 could be deduced from Theorem 2 by adequate manipulation of the graph G, but we have not been successful with such attempts.

2 Proof of Theorem 1

The relation to tree-width is established via a well-known separation property of graphs of bounded tree-width, due to Robertson and Seymour [3].

Lemma 3 ([3])**.** *Let k be a positive integer, G a graph and $A \subseteq V(G)$. If the tree-width of G is less than k, then there exists $X \subseteq V(G)$ with $|X| \leq k$ such that every component of $G - X$ contains at most $|A \setminus X|/2$ vertices of A.*

It is not hard to see that Theorem 1 can be reduced to the case where $\ell \equiv 1$. This case is treated in the next theorem.

Theorem 4. *Let k, p be positive integers. Let G be a graph containing a geodesic cycle C of length at least $4\lfloor p/2 \rfloor k$, which is the $\mathbb{F}_2$-sum of cycles of length at most p. Then for every $X \subseteq V(G)$ of order at most k, some component of $G - X$ contains at least half the vertices of C.*

Proof of Theorem 1, assuming Theorem 4. Let $\mathcal{D}$ be a set of cycles of length at most r with $C = \bigoplus \mathcal{D}$.

Since ℓ is rational-valued, we may assume that $r \in \mathbb{Q}$, as the premise also holds for r' the maximum ℓ-length of a cycle in $\mathcal{D}$. Take an integer M so that rM and $\ell'(e) := M\ell(e)$ are natural numbers for every $e \in E(G)$.

Obtain the subdivision G' of G by replacing every $e \in E(G)$ by a path of length $\ell'(e)$. Denote by C', D' the subdivisions of C and $D \in \mathcal{D}$, respectively. Then $C' = \bigoplus_{D \in \mathcal{D}} D'$ and $|C'| = M\ell(C) \geq 2(Mr)k$, while $|D'| = M\ell(D) \leq Mr$ for every $D \in \mathcal{D}$. By Theorem 4, for every $X \subseteq V(G')$ with $|X| \leq k$ there exists a component of $G' - X$ that contains at least half the vertices of C'. By Lemma 3, G' has tree-width at least k. Since tree-width is invariant under subdivision, the tree-width of G is also at least k. $\square$

Our goal is now to prove Theorem 4. The proof consists of two separate lemmas. The first lemma involves separators and $\mathbb{F}_2$-sums of cycles.

Lemma 5. *Let G be a graph, $C \subseteq G$ a cycle and $\mathcal{D}$ a set of cycles in G such that $C = \bigoplus \mathcal{D}$. Let $\mathcal{R}$ be a set of disjoint vertex-sets of G such that for every $R \in \mathcal{R}$, $R \cap V(C)$ is either empty or induces a connected subgraph of C. Then either some $D \in \mathcal{D}$ meets two distinct $R, R' \in \mathcal{R}$ or there is a component Q of $G - \bigcup \mathcal{R}$ with $V(C) \subseteq V(Q) \cup \bigcup \mathcal{R}$.*

Proof. Suppose that no $D \in \mathcal{D}$ meets two distinct $R, R' \in \mathcal{R}$. Then C has no edges between the sets in $\mathcal{R}$: Any such edge would have to lie in at least one $D \in \mathcal{D}$. Let $Y := \bigcup \mathcal{R}$ and let $\mathcal{Q}$ be the set of components of $G - Y$.

Let $Q \in \mathcal{Q}$, $R \in \mathcal{R}$ and $D \in \mathcal{D}$ arbitrary. If D has an edge between Q and R, then D cannot meet $Y \setminus R$. Therefore, all edges of D between Q and $V(G) \setminus Q$ must join Q to R. As D is a cycle, it has an even number of edges between Q and $V(G) \setminus Q$ and thus between Q and R. As $C = \bigoplus \mathcal{D}$, we find

$$e_C(Q, R) \equiv \sum_{D \in \mathcal{C}} e_D(Q, R) \equiv 0 \mod 2.$$

For every $R \in \mathcal{R}$ which intersects C, there are precisely two edges of C between R and $V(C) \setminus R$, because $R \cap C$ is connected. As mentioned above, C contains no edges between R and $Y \setminus R$, so both edges join R to $V(G) \setminus Y$. But C has an even number of edges between R and each component of $V(G) \setminus Y$, so it follows that both edges join R to the same $Q(R) \in \mathcal{Q}$.

Since every component of $C - (C \cap Y)$ is contained in a component of $G - Y$, it follows that there is a $Q \in \mathcal{Q}$ containing all vertices of C not contained in Y. $\square$

To deduce Theorem 4, we want to apply Lemma 5 to a suitable family $\mathcal{R}$ with $\bigcup \mathcal{R} \supseteq X$ to deduce that some component of $G - X$ contains many vertices of C. Here, $\mathcal{D}$ consists of cycles of length at most ℓ, so if the sets in $\mathcal{R}$ are at pairwise distance $> \lfloor \ell/2 \rfloor$, then no $D \in \mathcal{D}$ can pass through two of them. The next lemma ensures that we can find such a family $\mathcal{R}$ with a bound on $|\bigcup \mathcal{R}|$, when the cycle C is geodesic.

Lemma 6. *Let d be a positive integer, G a graph, $X \subseteq V(G)$ and $C \subseteq G$ a geodesic cycle. Then there exists a family $\mathcal{R}$ of disjoint sets of vertices of G with $X \subseteq \bigcup \mathcal{R} \subseteq X \cup V(C)$ and $|\bigcup \mathcal{R} \cap V(C)| \leq 2d|X|$ such that for each $R \in \mathcal{R}$, the set $R \cap V(C)$ induces a (possibly empty) connected subgraph of C and the distance between any two sets in $\mathcal{R}$ is greater than d.*

Proof. Let $Y \subseteq V(G)$ and $y \in Y$. For $j \geq 0$, let $B^j_Y(y)$ be the set of all $z \in Y$ at distance at most jd from y. Since $|B^0_Y(y)| = 1$, there is a maximum number j for which $|B^j_Y(y)| \geq 1 + j$, and we call this $j = j_Y(y)$ the *range of y in Y*. Observe that every $z \in Y \setminus B^{j_Y(y)}$ has distance greater than $(j_Y(y) + 1)d$ from y.

Starting with $X_1 := X$, repeat the following procedure for $k \geq 1$. If $X_k \cap V(C)$ is empty, terminate the process. Otherwise, pick an $x_k \in X_k \cap V(C)$ of maximum range in X_k. Let $j_k := j_{X_k}(x_k)$ and $B_k := B^{j_k}_{X_k}(x_k)$. Let $X_{k+1} := X_k \setminus B_k$ and repeat.

Since the size of X_k decreases in each step, there is a smallest integer m for which $X_{m+1} \cap V(C)$ is empty, at which point the process terminates. By construction, the distance between B_k and X_{k+1} is greater than d for each $k \leq m$. For each $1 \leq k \leq m$, there are two edge-disjoint paths $P^1_k, P^2_k \subseteq C$, starting at x_k, each of length at most $j_k d$, so that $B_k \cap V(C) \subseteq S_k := P^1_k \cup P^2_k$. Choose these paths minimal, so that the endvertices of S_k lie in B_k. Note that every vertex of S_k has distance at most $j_k d$ from x_k. Therefore, the distance between $R_k := B_k \cup S_k$ and X_{k+1} is greater than d.

We claim that the distance between R_k and $R_{k'}$ is greater than d for any $k < k'$. Since $B_{k'} \subseteq X_{k+1}$, it is clear that every vertex of $B_{k'}$ has distance greater than d from R_k. Take a vertex $q \in S_{k'} \setminus R_{k'}$ and assume for a contradiction that its distance to R_k was at most d. Then the distance between x_k and q is at most $(j_k + 1)d$. Let $a, b \in B_{k'}$ be the endvertices of $S_{k'}$. If $x_k \notin S_{k'}$, then one of a and b lies on the shortest path from x_k to q within C and therefore has distance at most $(j_k + 1)d$ from x_k. But then, since j_k is the range of x_k in X_k, that vertex would already lie in B_k, a contradiction. Suppose now that $x_k \in S_{k'}$. Then x_k lies on the path in $S_{k'}$ from x_k to one of a or b, so the distance between x_k and $x_{k'}$ is at most $j_{k'} d$. Since $x_{k'} \in X_k \cap V(C)$, it follows from our choice of x_k that

$$j_k = j_{X_k}(x_k) \geq j_{X_k}(x_{k'}) \geq j_{X_{k'}}(x_{k'}) = j_{k'},$$

where the second inequality follows from the fact that $X_{k'} \subseteq X_k$ and $j_Y(y) \geq j_{Y'}(y)$ whenever $Y \supseteq Y'$. But then $x_{k'} \in B_k$, a contradiction. This finishes the proof of the claim.

Finally, let $\mathcal{R} := \{R_k \colon 1 \leq k \leq m\} \cup \{X_{m+1}\}$. The distance between any two sets in $\mathcal{R}$ is greater than d. For $k \leq m$, $R_k \cap V(C) = S_k$ is a connected subgraph of C, while $X_{m+1} \cap V(C)$ is empty. Moreover,

$$\begin{aligned}
|\bigcup \mathcal{R} \cap V(C)| = \sum_{k=1}^{m} |S_k| &\leq \sum_{k=1}^{m} (1 + 2j_k d) \\
&\leq \sum_{k=1}^{m} (1 + 2(|B_k| - 1)d) \\
&\leq \sum_{k=1}^{m} 2|B_k|d \leq 2d|X|.
\end{aligned}$$

□

Proof of Theorem 4. Let $X \subseteq V(G)$ of order at most k and let $d := \lfloor p/2 \rfloor$. By Lemma 6, there exists a family $\mathcal{R}$ of disjoint sets of vertices of G with $X \subseteq \bigcup \mathcal{R} \subseteq X \cup V(C)$ and $|\bigcup \mathcal{R} \cap V(C)| \leq 2dk$ so that for each $R \in \mathcal{R}$, the set $R \cap V(C)$ induces a (possibly empty) connected subgraph of C and the distance between any two sets in $\mathcal{R}$ is greater than d.

Let $\mathcal{D}$ be a set of cycles of length at most p with $C = \bigoplus \mathcal{D}$. Then no $D \in \mathcal{D}$ can meet two distinct $R, R' \in \mathcal{R}$, since the diameter of D is at most d. By Lemma 5, there is a component Q of $G - \bigcup \mathcal{R}$ which contains every vertex of $C \setminus \bigcup \mathcal{R}$. This component is connected in $G - X$ and therefore contained in some component Q' of $G - X$, which then satisfies

$$|Q' \cap V(C)| \geq |C| - |\bigcup \mathcal{R} \cap V(C)| \geq |C| - 2dk.$$

Since $|C| \geq 4dk$, the claim follows. □

3 Remarks

We have described the content of Theorem 1 as an *algebraic* criterion for a graph to have large tree-width. The reader might object that the cycle C being ℓ-geodesic is a metric property and not an algebraic one. Karl Heuer has pointed out to us, however, that geodecity of a cycle can be expressed as an algebraic property after all. This is a consequence of a more general lemma of Gollin and Heuer [1], which allowed them to introduce a meaningful notion of geodecity for cuts.

Proposition 7 ([1]). *Let G be a graph with length-function ℓ and $C \subseteq G$ a cycle. Then C is ℓ-geodesic if and only if there do not exist cycles D_1, D_2 with $\ell(D_1), \ell(D_2) < \ell(C)$ such that $C = D_1 \oplus D_2$.*

Finally, we'd like to point out that Theorem 1 does not only offer a 'one-way criterion' for large tree-width, but that it has a qualitative converse. First, we recall the Grid Minor Theorem of Robertson and Seymour [4], phrased in terms of walls . For a positive integer t, an *elementary t-wall* is the graph obtained from the $2t \times t$-grid as follows. Delete all edges with endpoints $(i, j), (i, j + 1)$

when i and j have the same parity. Delete the two resulting vertices of degree one. A *t-wall* is any subdivision of an elementary t-wall. Note that the $(2t \times 2t)$-grid has a subgraph isomorphic to a t-wall.

Theorem 8 (Grid Minor Theorem [4])**.** *For every t there exists a k such that every graph of tree-width at least k contains a t-wall.*

Here, then, is our qualitative converse to Theorem 1, showing that the algebraic condition in the premise of Theorem 1 in fact captures tree-width.

Corollary 9. *For every L there exists a k such that for every graph G the following holds. If G has tree-width at least k, then there exists a rational length-function on G so that G contains a ℓ-geodesic cycle C with $\ell(C) \geq L$ which is the $\mathbb{F}_2$-sum of cycles of ℓ-length at most 1.*

Proof. Let $s := 3L$. By the Grid Minor Theorem, there exists an integer k so that every graph of tree-width at least k contains an s-wall. Suppose G is a graph of tree-width at least k. Let W be an elementary s-wall so that G contains some subdivision W' of W, where $e \in E(W)$ has been replaced by some path $P^e \subseteq G$ of length $m(e)$.

The outer cycle C of W satisfies $d_C(u, v) \leq 3d_W(u, v)$ for all $u, v \in V(C)$. Moreover, C is the $\mathbb{F}_2$-sum of cycles of length at most six.

Define a length-function ℓ on G as follows. Let $e \in E(G)$. If $e \in P^f$ for $f \in E(C)$, let $\ell(e) := 1/m(f)$. Then $\ell(P^f) = 1$ for every $f \in E(C)$. If $e \in P^f$ for $f \in E(W) \setminus E(C)$, let $\ell(e) := 3/m(f)$. Then $\ell(P^f) = 3$ for every $f \in E(W) \setminus E(C)$. If $e \notin E(W')$, let $\ell(e) := 10s^3$, so that $\ell(e) > \ell(W')$.

It is easy to see that the subdivision $C' \subseteq G$ of C is ℓ-geodesic in G. It has length $\ell(C') = |C| \geq 6s$ and is the $\mathbb{F}_2$-sum of the subdivisions of 6-cycles of W. Each of these satisfies $\ell(D) \leq 18$. Rescaling all lengths by a factor of $1/18$ yields the desired result.

□

References

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